Open problems on Numerical range and functional calculus

Michel Crouzeix

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Abstract

Functional calculus based on the numerical range is an almost uncharted field of research, with plenty of open questions. In this paper we propose a list of such problems.

1 Introduction.

In this paper we are concerned with the functional calculus based on the numerical range. The main result on this subject is (cf. [4]) :

There exists a best constant $\mathcal{Q}$ such that

$$\|p(A)\| \leq \mathcal{Q} \sup_{z\in W(A)} |p(z)|, \tag{1}$$

• *for all polynomial functions $p : \mathbb{C} \to \mathbb{C}$,*
• *for all square matrices $A \in \mathbb{C}^{d,d}$, for all values of d.*
Furthermore we have the estimate $2 \leq \mathcal{Q} \leq 11.08$.

Let us precise our notation. The set $W(A) := \{v^*Av\,; v \in \mathbb{C}^d, \|v\|^2 = v^*v = 1\}$ is the numerical range of $A \in \mathbb{C}^{d,d}$, $\|v\| = (v^*v)^{1/2}$ is the usual Euclidean norm of the column vector v, $\|M\| := \sup\{\|M\,v\|\,;\ v \in \mathbb{C}^n,\ \|v\| = 1\}$ is the operator norm of the matrix $M \in \mathbb{C}^{m,n}$.

It is remarkable that the inequality (1) admits a completely bounded version. More precisely : *There exists a best constant $\mathcal{Q}_{cb}$ such that*

$$\|P(A)\| \leq \mathcal{Q}_{cb} \sup_{z\in W(A)} \|P(z)\|, \tag{2}$$

• *for all polynomial functions $P : \mathbb{C} \to \mathbb{C}^{m,n}$, for all values of m and n,*
• *for all square matrices $A \in \mathbb{C}^{d,d}$, for all values of d.*
Furthermore we have the estimates $2 \leq \mathcal{Q} \leq \mathcal{Q}_{cb} \leq 11.08$.

Here P is matrix-valued $P(z) = (p_{ij}(z))$, with each entry $p_{ij} \in \mathbb{C}[z]$ being a polynomial ; the matrix $P(A) \in \mathbb{C}^{md,nd}$ is constituted of $m \times n$ blocks of size $d \times d$, the (i,j)-th block being $p_{ij}(A)$.

Note that, in the case of a normal matrix A, we have better estimates $\|p(A)\| \leq \sup_{z\in\sigma(A)} |p(z)|$ and $\|P(A)\| \leq \sup_{z\in\sigma(A)} \|P(z)\|$, where $\sigma(A)$ denotes the spectrum of A (and it is well known that $\sigma(A) \subset W(A)$). The interest of (1) is to provide estimates for non-normal matrices.

The surprising thing is that these constants $\mathcal{Q}$ and $\mathcal{Q}_{cb}$ are universal. There is no dependence on the matrix A, on its size, on the degree of polynomials used, nor on m and n for $\mathcal{Q}_{cb}$. This universality allows us to extend the inequalities to any bounded linear operator $A \in \mathcal{L}(H)$ on a complex Hilbert space H (and even to unbounded operators), and also to any continuous function p (resp. P) on $\overline{W(A)}$ which is holomorphic in the interior of the numerical range. We refer to [4] for these extensions and for some applications.

Some open problems. A challenging question is to obtain the exact values of $\mathcal{Q}$ and $\mathcal{Q}_{cb}$, or at least to essentially improve the upper bound 11.08. Our proof of this estimate is quite involved and clearly not optimal. We conjecture that $\mathcal{Q} = \mathcal{Q}_{cb} = 2$. Two sub-problems are the following:
is $\mathcal{Q} = 2$ *?* and *is* $\mathcal{Q} = \mathcal{Q}_{cb}$ *?*
In case of a positive answer to the second question, it will be interesting to understand what the difference is in our situation with respect to the context of the Halmos conjecture: *polynomially bounded implies completely bounded* (which is now known to be false [11]).

We are afraid that our conjecture is a too difficult problem. In order to formulate easier questions to consider we introduce the constants

$$\mathcal{Q}(d) := \sup_{A,p}\{\|p(A)\|\,; A \in \mathbb{C}^{d,d}, p : \mathbb{C} \to \mathbb{C} \text{ polynomial, } |p(z)| \le 1 \text{ in } W(A)\},$$

$$\mathcal{Q}_{cb}(d) := \sup_{A,P,m,n}\{\|P(A)\|\,; A \in \mathbb{C}^{d,d}, P : \mathbb{C} \to \mathbb{C}^{m,n} \text{ polynomial, } \|P(z)\| \le 1 \text{ in } W(A)\}.$$

It is easily verified that these constants increase with d; furthermore $\mathcal{Q} = \sup_d \mathcal{Q}(d)$ and $\mathcal{Q}_{cb} = \sup_d \mathcal{Q}_{cb}(d)$. We have succeeded to show [1] that $\mathcal{Q}(2) = \mathcal{Q}_{cb}(2) = 2$, but failed with the questions $\mathcal{Q}(3) = \mathcal{Q}_{cb}(3)$ and $\mathcal{Q}(3) = 2$; a fortiori the analogue questions are open for $d > 3$. (The numerical experiments seem to confirm that $\mathcal{Q}(3) = 2$).

More generally it would be interesting to find a proof really different of our for the estimate $\mathcal{Q} \le 11.08$.

What are the corresponding constants if we restrict our matrices A to be real, $A \in \mathbb{R}^{d,d}$?
Same question with Toeplitz or Hankel type matrices ?
Same question with nilpotent matrices ? (the answer is 2 for matrices such that $A^2 = 0$).
Does there exist some extension of our inequalities to two commuting matrices ?

2 Problems related to a matrix

Let $A \in \mathbb{C}^{d,d}$ be a square matrix and Ω be a bounded convex domain of the complex plane such that $\overline{\Omega} \supset \sigma(A)$ (spectrum of A). We introduce the quantities

$$\psi_\Omega(A) := \sup_p\{\|p(A)\|\,; p : \mathbb{C} \to \mathbb{C} \text{ polynomial, } |p(z)| \le 1 \text{ in } \Omega\},$$

$$\psi_{cb,\Omega}(A) := \sup_{P,m,n}\{\|P(A)\|\,; P : \mathbb{C} \to \mathbb{C}^{m,n} \text{ polynomial, } \|P(z)\| \le 1 \text{ in } \Omega\},$$

$$\psi(A) := \sup_p\{\|p(A)\|\,; p : \mathbb{C} \to \mathbb{C} \text{ polynomial, } |p(z)| \le 1 \text{ in } W(A)\},$$

$$\psi_{cb}(A) := \sup_{P,m,n}\{\|P(A)\|\,; P : \mathbb{C} \to \mathbb{C}^{m,n} \text{ polynomial, } \|P(z)\| \le 1 \text{ in } W(A)\}.$$

Note that these functions are constant on the orbit

$$Ob(A) := \{U^*AU\,; U \in \mathbb{C}^{d,d}, U^*U = I\}.$$

Clearly the functions $\psi_\Omega(A)$ and $\psi_{cb,\Omega}(A)$ are decreasing with respect to Ω (for the embedding order), and

$$\psi(A) = \sup_\Omega\{\psi_\Omega(A)\,; \Omega \supset W(A)\}, \quad \psi_{cb}(A) = \sup_\Omega\{\psi_{cb,\Omega}(A)\,; \Omega \supset W(A)\}.$$

Furthermore, according to [13], the values of $\psi_{cb,\Omega}(A)$ nor of $\psi_{cb}(A)$ do not change if in the corresponding definition we restrict the values of m and n to be $m = n = d$.

We have

$$\mathcal{Q}(d) = \max\{\psi(A)\,;A \in \mathbb{C}^{d,d}\}, \quad \mathcal{Q}_{cb}(d) = \max\{\psi_{cb}(A)\,;A \in \mathbb{C}^{d,d}\}.$$

This is clearly true if we replace *max* by *sup*, but it is not difficult to see that the bounds are effectively attained.

Remark. The presented definitions are still valid if A is a bounded operator on a Hilbert space. Gilles Pisier has shown [11] that the Halmos conjecture: "$\psi_D(A) < +\infty$ implies $\psi_{cb,D}(A) < +\infty$", where D denotes the unit disk, is false. Consequently the relation $\psi_\Omega(A) = \psi_{cb,\Omega}(A)$ cannot be true for all matrices. However the problem $\psi(A) = \psi_{cb}(A)$ is, to our knowledge, open.

The use of a conformal map a from Ω onto the unit disk D allows us to restrict our studies to the case of the unit disk. Indeed, we then have (cf. for instance [3]) $\psi_\Omega(A) = \psi_D(a(A))$ and $\psi_{cb,\Omega}(A) = \psi_{cb,D}(a(A))$; furthermore if the eigenvalues of a matrix B are in the interior of the unit disk, then $\psi_D(B)$ is attained by a Blaschke product with at most $d-1$ terms. More precisely we have

$$\psi_D(B) := \sup_{\zeta_j}\{\|g(B)\|\,;g(z) = \prod_{j=1}^{r}\frac{z-\zeta_j}{1-\bar{\zeta}_j z},\ \zeta_1,\dots,\zeta_r \in D,\ r \leq q-1\}.$$

For the completely bounded analogue quantity, a Paulsen theorem [9] provides the characterisation

$$\psi_{cb,D}(B) := \min_S\{\|S\|\,\|S^{-1}\|\,;S \in \mathbb{C}^{d,d}, \|S^{-1}BS\| \leq 1\}.$$

If the numerical range of A is a disk, it is known [1] that $\psi(A) \leq \psi_{cb}(A) \leq 2$ (this is in particular the case if $A^2 = 0$).

If A is a 2×2 matrix then [1] $\psi(A) = \psi_{cb}(A) \leq 2$; furthermore $\psi(A) = 2$ implies that $W(A)$ is a disk. More generally, if A is a quadratic matrix (of any size), then $\psi(A) = \psi_{cb}(A) \leq 2$, and $\psi(A) = 2$ implies that $W(A)$ is a disk. Indeed, using [14] Theorem 1.1, A is then unitarily similar to a direct sum of 1×1 and 2×2 matrices.

Some open problems. Find an efficient method for the computation of $\psi_D(B)$. For 2×2 matrices there exists an explicit formula, but even for 3×3 matrices, we have only succeeded to use optimisation algorithms, without guarantee of convergence towards the global maximum.

Similarly we do not know a reliable algorithm for computing the matrix S in the Paulsen characterisation of $\psi_{cb,D}(B)$. To my knowledge there is no constructive proof of the corresponding theorem.

Do we have $\psi(A) \leq \psi_{cb}(A) \leq 2$, if $A^3 = 0$?

Do we have $\psi(A) \leq \psi_{cb}(A) \leq \mathcal{Q}(3)$, if A is a cubic matrix ?

Is the map $A \mapsto \psi(A)$ continuous, if we assume $A \neq \lambda I$? It is easy to verify that this map is lower semicontiuous, and continuous if all the eigenvalues of A are in the interior of the numerical range $W(A)$ (then it suffices to use the Cauchy formula on the boundary). It is not continuous if the matrix A tends to λI; indeed

$$\psi\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} = 1 \text{ and } \psi\begin{pmatrix}\lambda & \varepsilon\\ 0 & \lambda\end{pmatrix} = 2.$$

A fortiori the same question is open for $\psi_{cb}(\cdot)$

We define

$$\psi_{D,k}(B) := \sup_p\{\|p(B)\|\,; p\,:\mathbb{C}\to\mathbb{C},\ |p(z)|\le 1 \text{ in } D,\ \text{with at most } k \text{ zeros in } D\},$$

where p denotes a generic holomorphic function in D. Is this bound $\psi_{D,k}(B)$ (with $k\ge 1$) attained by a Blaschke product p with at most k factors ?

3 Constants related to a convex domain

Let $\Omega\neq\mathbb{C}$ be a (non-empty) convex domain in the complex plane, not necessarily bounded. We define the constants

$$C(\Omega,d) := \sup_{A,r}\{\|r(A)\|; A\in\mathbb{C}^{d,d},\ W(A)\subset\Omega,\ r:\mathbb{C}\to\mathbb{C},\ |r(z)|\le 1, \forall z\in\Omega\},$$

$$C_{cb}(\Omega,d) := \sup_{A,R,m,n}\{\|R(A)\|; A\in\mathbb{C}^{d,d},\ W(A)\subset\Omega,\ R:\mathbb{C}\to\mathbb{C}^{m,n},\ \|R(z)\|\le 1, \forall z\in\Omega\}.$$

$$C(\Omega) := \sup_d C(\Omega,d), \qquad C_{cb}(\Omega) := \sup_d C_{cb}(\Omega,d).$$

In these definitions, r and R denote rational functions. (This choice has been made for treating together the bounded and unbounded domain cases, but for a bounded Ω it would have sufficed to only consider polynomials r and R without change of the values. Similarly, the condition $W(A)\subset\Omega$ could be replaced by $W(A)\subset\overline{\Omega}$).

We have

$$\mathcal{Q}(d)=\sup_\Omega C(\Omega,d), \quad \mathcal{Q}=\sup_\Omega C(\Omega), \quad \mathcal{Q}_{cb}(d)=\sup_\Omega C_{cb}(\Omega,d), \quad \mathcal{Q}_{cb}=\sup_\Omega C_{cb}(\Omega)$$

Remarks.

1) The previous constants depend only on d and on the shape of Ω. More precisely, if φ is a similarity : $\varphi(z)=a+b\,z$, or an anti-similarity : $\varphi(z)=a+b\,\bar{z}$, $a,b\in\mathbb{C}$, $b\neq 0$, we have $C(\Omega)=C(\varphi(\Omega)),\dots,C_{cb}(\Omega,d)=C_{cb}(\varphi(\Omega),d))$.

2) A classical result of J. von Neumann [12] asserts that $C(\Omega)=1$, if Ω is a half-plane ; as soon as the notion of “completely bounded” has appeared, it has been remarked that in this case we also have $C_{cb}(\Omega)=1$.

3) Obviously $C(\Omega)\le C_{cb}(\Omega)$ and $C(\Omega,d)\le C_{cb}(\Omega,d)$. Furthermore, the last two constants are increasing (perhaps not strictly) functions of d.

Except of the old half-plane inequality, the first result on this subject is quite recent. In the nice paper [7] an estimate $C(\Omega)<+\infty$ is given for any bounded convex domain Ω ; in [1] we have shown that this result is still valid in completely bounded form, and improved the estimate to

$$C_{cb}(\Omega)\le 2+\pi+\inf_{\omega\in\partial\Omega}\mathrm{TV}(\log|\sigma-\omega|)\,;$$

here $\mathrm{TV}(\log|\sigma-\omega|)$ is the total variation of $\log(|\sigma-\omega|)$ as σ runs on $\partial\Omega$. (A slightly better estimate can be deduced from Lemma 9 in [4]).

A similar approach provides the inequality

$$C_{cb}(\Omega) \le 1 + \frac{2}{\pi}\int_{\alpha}^{\pi/2} \frac{\pi - x + \sin x}{\sin x}\, dx,$$

if Ω contains a sector with angle 2α, $0 < \alpha \le \frac{\pi}{2}$. For a sector $\Omega = S_\alpha$ (with angle $2\alpha \le \pi$) we have obtained the more precise estimates [5], [1], [2],

$$\frac{\pi \sin\alpha}{2\alpha} \le C(S_\alpha) \le C_{cb}(S_\alpha) \le \frac{\pi-\alpha}{\pi}\Big(2 - \frac{2}{\pi}\log\tan\big(\frac{\alpha\,\pi}{4(\pi-\alpha)}\big)\Big), \quad \text{for} \quad \alpha \in (0, \pi/2],$$

and

$$C(S_\alpha) \le C_{cb}(S_\alpha) \le 2 - \frac{2\alpha}{\pi} + \frac{2\,\cos\alpha}{\pi\sqrt{1+2\cos 2\alpha}} \arccos\Big(\frac{\cos(\pi-2\alpha)}{\cos\alpha}\Big), \quad \text{for} \quad \alpha \in [0, \pi/3].$$

The second bound is better than the first if $\alpha \le .22\,\pi$ and is still valid if we replace the sector S_α by (a domain limited by) a branch of hyperbola of angle 2α. In [2] we derive the bound $C_{cb}(\mathcal{E}) \le 2 + 2/\sqrt{4-e^2}$ for an ellipse $\mathcal{E}$ of eccentricity e and $C_{cb}(\mathcal{P}) \le 2+2/\sqrt{3}$ for a parabola $\mathcal{P}$. The estimate $C_{cb}(S_0) \le 2+2/\sqrt{3}$ is also known for a strip, S_0 [5].

The only exact values known are for the half-plane case $C(\Pi) = C_{cb}(\Pi) = 1$ and for the disk case $C(D) = C_{cb}(D) = 2$, see [1]. The other bounds, and in particular the general bound $C_{cb}(\Omega) \le 11.08$, are very pessimistic.

Open problems.

- Is it true that $C(\Omega, d) = C_{cb}(\Omega, d)$ for any convex domain Ω ?

  (It is known from [10] that $C(\Omega, 2) = C_{cb}(\Omega, 2)$.)
- Is it true that $C_{cb}(S_0) \le 2$? (We especially mention this case, since then the constraint $W(A) \subset S_0$ is quite simple.)
- Is it possible to estimate $C_{cb}(S_\alpha)$ from the knowledge of $C_{cb}(S_0)$ and of $C_{cb}(S_{\pi/2}) = 1$? (By some interpolation trick à la Marcel Riesz.)
- Does the condition $C(\Omega, d) = 2$ imply Ω is a disk ? (This is the case if $d = 2$, see [3].)
- Is $C(\Omega)$ (resp. $C(\Omega, d)$) a continuous function of Ω (for instance with respect to the Hausdorff distance) ? (The lower semi-continuity is easily seen.) At least, is $C(\Omega)$ converging to 2 as Ω tends to the unit disk ?
- Is $\mathcal{Q} = \sup_\Omega\ C(\Omega)$ (resp. $\mathcal{Q}_{cb} = \sup_\Omega\ C_{cb}(\Omega)$) attained by some domain Ω ? (This is the case for $\mathcal{Q}(d)$ and $\mathcal{Q}_{cb}(d)$). Is it attained by a domain Ω which is symmetric with respect to the real axis ?
- Is $C(\Omega, d)$ (resp. $\sup_\Omega\ C_{cb}(\Omega, d)$) attained by some matrix A ? (This is the case for $C(\Omega, d)$, if Ω is bounded with an analytic boundary [3], and then attained by a Blaschke product r.)
- In the case where the boundary of Ω is a branch of hyperbola with angle 2α, is the equality $C(\Omega, d) = C(S_\alpha, d)$ valid ?
- In the case where Ω is symmetric with respect to the real axis, does the value of $C(\Omega, d)$ (resp. $C_{cb}(\Omega, d)$) change, if in the definition we restrict the matrices A to have real entries ? More generally is it possible to deduce some properties for some matrices A which realise $C(\Omega, d)$ from the symmetries of Ω ?

- Find a numerical method for the computation of $C(\Omega, d)$, Ω given, $d = 2, 3, \ldots$
  (I have only (partially) succeeded to do this for the strip S_0 and $d \leq 8$; it is known that $C(S_0, 2) = 1.5876598...$ and from my numerical experiments I have obtained the values $C(S_0, 4) = 1.6723401...$, $C(S_0, 6) = 1.72662...$, $C(S_0, 8) = 1.764577...$, but I cannot be sure that my optimisation algorithm has not converge to a local maximum.)
- My numerical experiments suggest that for the quarter plane $S_{\pi/4}$ we have $C(S_{\pi/4}, 4) = \sqrt{2}$. Is this true and is it true for all d ?

4 Some personal comments on the numerical range

We refer to [8] for a general exposure on the numerical range. This section is only devoted to few remarks.

The numerical range of a matrix is a compact and convex subset of the complex plane. Except in the 2×2 case (where it is an ellipse), its boundary is quite involved. From the convexity we know that it is the intersection of all tangent half-planes which contain it. More precisely, if we write $A = B + i\,C \in \mathbb{C}^{d,d}$, with B and C self-adjoint, if we set $P_A(u, v, w) := \det(uB + vC + wI)$, and if we denote by $w_m(u, v)$ the largest root of $P_A(u, v, .) = 0$ (all the roots are real since B and C are self-adjoint), then

$$W(A) = \{z = x+iy\,;\ x\cos\alpha + y\sin\alpha + w_m(\cos\alpha, \sin\alpha) \leq 0,\ \text{for all } \alpha \in [0, 2\pi]\}.$$

This provides an (exterior) approximation of $W(A)$ by computing a finite number of values of $w_m(\cdot, \cdot)$.

The tangential approach for the numerical range is simpler than the Cartesian one. From the previous formula we see that $W(A)$ is a part of the algebraic curve with tangential equation $P_A(u, v, w) = 0$. This curve is of class d, which means that the polynomial P_A is of degree d. The Cartesian equation of this curve is generically of degree $\frac{d(d-1)}{2}$, which is the maximal degree given by the Plücker relations.

An interesting characteristic of the numerical range is its good behaviour with respect to perturbations. If A and B denote two bounded operators on a Hilbert space, the Hausdorff distance $d_H(W(A), W(B))$ is bounded by $\|A - B\|$. The variational approach is a powerful tool for the analysis of P.D.E. problems. The assumptions are then generally imposed on the sesquilinear form $\langle Au, u\rangle$ (for instance in the Lax-Milgram Theorem) and can be translated in terms of localisation of the numerical range of a unbounded operator A. Furthermore, many numerical approximations (finite element methods, spectral methods, wavelets,...) use approximate sesquilinear forms $\langle A_h u_h, u_h\rangle$. The corresponding numerical range $W(A_h)$ then naturally inherits analogue properties to those of $W(A)$.

5 Supporting arguments for my conjecture

I have proposed the conjecture $\mathcal{Q} = 2$ in [3] (but I have been working on this for two years), I have tried to prove it, and also to find a counter-example, but up to now without success.

The main argument in favour of my conjecture is a symmetry reason. We have $\mathcal{Q} = \sup_\Omega C(\Omega)$, where Ω varies among the non empty bounded convex sets. The constant $C(\Omega)$ depending only on the shape of Ω, it is natural to think that the upper bound could be attained by a fully symmetric set, i.e., by a disk, but in this case $C(\Omega) = 2$. Another natural candidate for realising the upper bound is the very flat case where Ω is the strip S_0. For the strip it is known that $C(S_0, 2) = 1,58766...$ and I have obtained by numerical computations the values $C(S_0, 4) = 1,672...$, $C(S_0, 6) = 1,726...$, $C(S_0, 8) = 1,765...$, ... (by an empirical extrapolation this seems to confirm that $C(S_0) \leq 2$). But the complexity of computations drastically enlarges with the dimension d...

I have succeeded to show that $\mathcal{Q}(2) = 2$ [3]. I have made many numerical tests for 3×3 matrices and I am convinced that, if $\mathcal{Q}(3)$ were larger than 2, I would have succeeded to exhibit a 3×3 matrix with $\psi(A) > 2$. I have particularly explored the neighbourhood of matrices A such that $W(A)$ is a disk (which implies $\psi(A) \leq 2$) and $\psi(A) = 2$, and numerically verified that $\psi(A)$ then corresponds to a local maximum.

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Institut de Recherche Mathématique de Rennes, UMR CNRS n° 6625
Université de Rennes 1, Campus de Beaulieu, 35042 RENNES Cedex, France
michel.crouzeix@univ-rennes1.fr