INTERSECTIONS OF SEVERAL DISKS OF THE RIEMANN SPHERE AS $K$-SPECTRAL SETS

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Pour Philippe Ciarlet, à l’occasion de son soixante-dixième anniversaire

ABSTRACT. We prove that if $n$ closed disks $D_1, D_2, \ldots, D_n$, of the Riemann sphere are spectral sets for a bounded linear operator $A$ on a Hilbert space, then their intersection $D_1 \cap D_2 \cap \cdots \cap D_n$ is a complete $K$-spectral set for $A$, with $K \leq n + n(n-1)/\sqrt{3}$. When $n = 2$ and the intersection $X_1 \cap X_2$ is an annulus, this result gives a positive answer to a question of A. L. Shields (1974).

1. Introduction and the statement of the main results. Let $X$ be a closed set in the complex plane and let $\mathcal{R}(X)$ denote the algebra of bounded rational functions on $X$, viewed as a subalgebra of $C(\partial X)$ with the supremum norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} = \sup\{|f(x)| : x \in \partial X\}.$$

Here $\partial X$ denotes the boundary of the set $X$.

Spectral and complete spectral sets. Let $A \in \mathcal{L}(H)$ be a bounded linear operator acting on a complex Hilbert space $H$. For a fixed constant $K > 0$, the set $X$ is said to be a $K$-spectral set for $A$ if the spectrum $\sigma(A)$ of $A$ is included in $X$ and the inequality $\|f(A)\| \leq K\|f\|_X$ holds for every $f \in \mathcal{R}(X)$. The poles of a rational function $f = p/q \in \mathcal{R}(X)$ are outside of $X$, and the operator $f(A)$ is naturally defined as $f(A) = p(A)q(A)^{-1}$ or, equivalently, by the Riesz holomorphic functional calculus. The set $X$ is a spectral set for $A$ if it is a $K$-spectral set with $K = 1$. Thus $X$ is spectral for $A$ if and only if $\|\rho_A\| \leq 1$, where $\rho_A : \mathcal{R}(X) \rightarrow \mathcal{L}(H)$ is the homomorphism given by $\rho_A(f) = f(A)$.

We denote by $M_n(\mathcal{R}(X))$ the algebra of $n$ by $n$ matrices with entries from $\mathcal{R}(X)$. If we let the $n$ by $n$ matrices have the operator norm that they inherit as linear transformations on the $n$-dimensional Hilbert space $\mathbb{C}^n$, then we can endow $M_n(\mathcal{R}(X))$ with the norm $\|f_{ij}\|_X = \sup\{|(f_{ij}(x))| : x \in X\} = \sup\{|(f_{ij}(x))| : x \in \partial X\}$.

In a similar fashion we endow $M_n(\mathcal{L}(H))$ with the norm it inherits by regarding an element $(A_{ij})$ in $M_n(\mathcal{L}(H))$ as an operator acting on the direct sum of $n$ copies of $H$. For a fixed constant $K > 0$, the set $X$ is said to be a complete $K$-spectral set for $A$ if $\sigma(A) \subset X$ and the inequality $\|(f_{ij}(A))\| \leq K\|(f_{ij})\|_X$ holds for every matrix $(f_{ij}) \in M_n(\mathcal{R}(X))$ and every $n$. In terms of the complete bounded norm (161) of the homomorphism $\rho_A$, this

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means that $\|\rho_A\|_{cb} \leq K$. A complete spectral set is a complete $K$-spectral set with $K = 1$. Complete $K$-spectral sets are important in several problems of Operator Theory (see [16]).

Spectral sets were introduced and studied by J. von Neumann [14] in 1951. In the same paper von Neumann proved that a closed disk $\{z \in \mathbb{C} : |z - \alpha| \leq r\}$ is a spectral set for $A$ if and only if $\|A - \alpha I\| \leq r$. Also, the closed set $\{z \in \mathbb{C} : |z - \alpha| \geq r\}$ is spectral for $A \in \mathcal{L}(H)$ if and only if $\|(A - \alpha I)^{-1}\| \leq r^{-1}$, and the half-plane $\{\text{Re}(z) \geq 0\}$ is a spectral set if and only if $\text{Re}(Av, v) \geq 0$ for all $v \in H$. Therefore for any closed disk $D$ of the Riemann sphere (interior/exterior of a disk or a half-plane) it is easy to check whether $D$ is a spectral set.


**The problem.** Let $X$ be the intersection of $n$ disks of the Riemann sphere, each of them being a spectral set for a given operator $A \in \mathcal{L}(H)$. In the present paper we will be concerned with the question whether $X$ itself is a (complete) $K$-spectral set for $A$.

The intersection of two spectral sets is not necessarily a spectral set; counterexamples for the annulus are presented in [24, 13, 15]. However, the same question for $K$-spectral sets remains open. The counterexamples for spectral sets show that the same constant cannot be used for the intersection.

Some cases of the $K$-spectral set problem have been solved. If two $K$-spectral sets have disjoint boundaries, then by a result of Douglas and Paulsen [9] the intersection is a $K'$-spectral set for some $K'$. The case when the boundaries meet was considered by Stampfli [21, 22] and Lewis [12]. In particular, it is proved in [12] that the intersection of a (complete) $K$-spectral set for the bounded linear operator $A$ with the closure of any open set containing the spectrum of $A$ is a (complete) $K'$-spectral set for $A$.

**The main result.** In this paper we will show the following result, which gives a positive answer to a question raised by Michael A. Dritschel (personal communication).

**Theorem 1.1.** Let $A \in \mathcal{L}(H)$, and consider the intersection $X = D_1 \cap D_2 \cap \cdots \cap D_n$ of $n$ disks of the Riemann sphere $\mathbb{C}$, each of them being spectral for $A$. Then $X$ is a complete $K$-spectral set for $A$, with a constant $K \leq n + n(n-1)/\sqrt{3}$.

Theorem 1.1 extends previously known results concerning the intersection of two or more disks in $\mathbb{C}$ to not necessarily convex or even connected $X$. Note that the case of finitely connected compact sets has been studied in [9, 16], however, without a uniform control on the constant $K$.

If in Theorem 1.1 we add the requirement that the disks $D_j$ and thus $X$ are convex, then $X$ is a complete $11.08$-spectral set for $A$. Indeed, the fact that $D_j$ is a spectral set for $A$ implies that the numerical range $W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}$ is included in $D_j$, $j = 1, \ldots, n$, and, according to [8], the closure of the numerical range $W(A)$ is a complete $11.08$-spectral set for $A$.

Let us have a closer look at the case $n = 2$ of the intersection of two closed disks of the Riemann sphere. From Theorem 1.1 we may conclude that the intersection of two disks of the Riemann sphere is a $K$-spectral set, with $K \leq 2 + 2/\sqrt{3}$. This includes the special case of a sector/strip obtained by the intersection of two half-planes and discussed in [7], and the lens-shaped intersection of two disks considered already in [5]; see also [4, 6]. The case of the annulus is new and it permits to answer a question of A. Shields [20, Question 7] as described in the next paragraph.
Shields’ question for the annulus. Let $R > 1$. Given an invertible operator $A$ with \( \|A\| \leq R \) and \( \|A^{-1}\| \leq R \), Shields proved in [20] that the annulus $X = X(R) = \{ z \in \mathbb{C} : R^{-1} \leq |z| \leq R \}$ is a $K$-spectral set for $A$ with \( K = 2 + \sqrt{(R^2+1)/(R^2-1)} \). This bound is large if $R$ is close to 1. In this context, Shield raises the question of finding the smallest constant $K$ (as a function of $R$) such that the above annulus is $K$-spectral. In particular [20, Question 7], he asked whether this optimal constant $K$ remains bounded. With regard to his second question, we are able to give a positive answer.

**Theorem 1.2.** For $R > 1$, consider the annulus $X = X(R) = \{ z \in \mathbb{C} : R^{-1} \leq |z| \leq R \}$, and denote by $K(R)$ (and $K_\text{cb}(R)$, respectively), the smallest constant $C$ such that $X$ is a $C$-spectral set (and a complete $C$-spectral set, respectively) for any invertible $A \in \mathcal{L}(H)$ verifying $\|A\| \leq R$ and $\|A^{-1}\| \leq R$. Then

\[
\frac{4}{3} < K(R) \leq K_\text{cb}(R) \leq 2 + \frac{R+1}{\sqrt{R^2+R+1}} \leq 2 + \frac{2}{\sqrt{3}}.
\]

This statement has already been established in the unpublished manuscript [3], where one finds also a preliminary version of Theorem 1.1 for $n = 2$. One consequence of Theorem 1.2 is that the upper bound for $K$ of Theorem 1.1 is not sharp; furthermore, we expect this bound to be pessimistic for large $n$.

**Normal $\partial X$-dilations.** Based on results due to W. Arveson and V. Paulsen (see [16, Corollary 7.8, Theorem 9.1]), it is now a standard fact that if $X$ is a $K$-spectral set for $A$, then there is a normal $\partial X$-dilation for an operator $L^{-1}AL$ similar to $A$. Moreover, the best possible constant $K$ is equal to the infimum of all possible similarity constants (condition numbers) $\|L^{-1}\| \cdot \|L\|$. We say that $B \in \mathcal{L}(H)$ has a normal $\partial X$-dilation if there exists a Hilbert space $\mathcal{H}$ containing $H$ and a normal operator $N$ on $\mathcal{H}$ with $\sigma(N) \subset \partial X$ so that

\[
f(B) = P_\mathcal{H} f(N) \mid_{\mathcal{H}}
\]

for every rational function $f$ with poles off $X$. Here $P_\mathcal{H}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}$.

We obtain the following consequence:

**Corollary 1.3.** Let $A \in \mathcal{L}(H)$, and consider the intersection $X = D_1 \cap D_2 \cap \cdots \cap D_n$ of $n$ disks of the Riemann sphere $\overline{\mathbb{C}}$, each of them being spectral for $A$. Then there exists an invertible operator $L \in \mathcal{L}(H)$ with $\|L\| \cdot \|L^{-1}\| \leq n + n(n-1)/\sqrt{3}$, such that $L^{-1}AL$ has a normal $\partial X$-dilation.

If $\|A\| \leq R$ and $\|A^{-1}\| \leq R$, and $X = X(R)$ is an annulus, we can apply this corollary (with $n = 2$) and obtain a $\partial X(R)$-dilation for a similarity. We would also like to mention the following deep result due to Agler [1]: if an annulus $X$ is a spectral set for $A$, then it is a complete spectral set for $A$, that is, $A$ has a normal $\partial X$-dilation. However, the analogue of Agler’s result is not true for triply connected domains (see [2, 10, 18]).

**Outline of the proof and organization of the paper.** Let $A \in \mathcal{L}(H)$, and consider the intersection $X = D_1 \cap D_2 \cap \cdots \cap D_n$ of $n$ disks of the Riemann sphere $\overline{\mathbb{C}}$, each of them being spectral for $A$.

**Convention.** In what follows we will always suppose that for each spectral set $D_j$, the spectrum $\sigma(A)$ of $A$ is included in the interior of $D_j$. The general case will then follow by a limit argument by slightly enlarging the disks. Note that each superset of a spectral set is spectral.
Decomposition of the Cauchy kernel. Consider a disk \( D \) among \( D_1, \cdots, D_n \), which is centered in \( \omega \in \mathbb{C} \) (if \( D \) is a half-plane we take \( \omega = \infty \)). We chose an arclength parametrization \( s \mapsto \sigma = \sigma(s) \in \partial D \) of the boundary of \( D \) with orientation such that \( \frac{1}{i} \frac{d\sigma}{ds} \) is the outward normal to \( D \). Let \( A \in \mathcal{L}(H) \) be a bounded operator with \( \sigma(A) \subset \text{int}(\bar{D}) \). For \( \sigma \in \partial D \), we consider the following Poisson kernel
\[
\mu(\sigma, A, D) = \frac{1}{2\pi i} ((\sigma - A)^{-1} \frac{d\sigma}{ds} - (\sigma - A^*)^{-1} \frac{d\sigma}{ds} - \frac{1}{\omega - \sigma} \frac{d\sigma}{ds}).
\] Notice that in case \( \omega = \infty \) of a half-plane, the term involving \( \omega \) on the right-hand side of (1) vanishes.

The first important step in the proof is the decomposition of the Cauchy kernel
\[
\frac{1}{2\pi i} (\sigma - A)^{-1} d\sigma = \mu(\sigma, A, D) ds + \nu(\sigma, A, D) d\sigma
\] as the sum of the Poisson kernel and a residual kernel.

Decomposition of \( f(A) \). For a rational function \( f \in \mathcal{R}(X) \), the above decomposition of the Cauchy kernel leads to a decomposition of \( f(A) \) as
\[
f(A) = g_p(f) + g_r(f),
\] with
\[
g_p(f) = \sum_{j=1}^{n} \int_{X \cap \partial D_j} f(\sigma) \mu(\sigma, A, D_j) ds, \quad g_r(f) = \sum_{j=1}^{n} \int_{X \cap \partial D_j} f(\sigma) \nu(\sigma, A, D_j) d\sigma.
\] Here \( p \) stands for "Poisson" and \( r \) for "residual". This decomposition reduces to the one used before in [3, 7, 6] in the special case \( \omega = \infty \) of a half-plane. The proof will consist in showing that the complete bounded norm of the map \( f \mapsto g_p(f) \) is bounded by \( n \), while the complete bounded norm of the map \( f \mapsto g_r(f) \) can be estimated by \( n(n-1)/\sqrt{3} \).

Two basic lemmas on operator-valued integrals are proved in Section 2 and, as application, the Poisson term is estimated. In order to make clearer the basic ideas of our reasoning for the control of the residual term, we start with the annulus case. A proof for Theorem 1.2 (implying Theorem 1.1 for an annulus) is presented in Section 3. A proof of Theorem 1.1 for the case \( n = 2 \) is presented in Section 4. For the residual term, the decisive step is the invariance under Möbius maps (also called fractional linear transformations or homomorphic transformations) of our representation formula. We also use the fact that the variable \( \sigma \) which appears in \( g_r(f) \) can be expressed in terms of \( \sigma \) due to the particular form of \( \partial X \). Subsequently, a new path of integration is used in order to monitor the complete bounded norm of the residual term. The new path of integration will be the circle of radius 1 in case of the annulus \( \{ R^{-1} \leq |z| \leq R \} \), and the positive real line in case of the sector \( \{ |\arg(z)| \leq \theta \} \) for \( \theta \in (0, \pi/2) \). We call these median lines.

Our proof of Theorem 1.1 for \( n > 2 \), presented in Section 5, is obtained by constructing a Voronoi-like tessellation of the Riemann sphere based on the reciprocal of the infinitesimal Carathéodory pseudodistance (see [11]), sometimes also called infinitesimal Carathéodory metric (see, e.g., [23]). Here the new paths of integration (the median lines) are obtained using suitable edges of this tessellation, which requires some combinatorial considerations. Then the arguments used for the case of two disks will allow us to conclude.

2. Two basic lemmas. The following two basic lemmas, formulated in a slightly more general setting, are obtained using positivity in a crucial way.
Lemma 2.1. We consider a measurable subset $E$ of $\mathbb{C}$, a complex-valued measure $m$ on $E$, and a bounded and continuous function $M(.) \in C(E; \mathcal{L}(H))$ defined on $E$ such that the operator $M(\sigma)$ is self-adjoint and positive for each $\sigma \in E$. We also assume that there exist a positive bounded measure $\muN$ with $|\muN| \leq \muN$ on $E$. Then the map
\[
  r \mapsto g_1(r), \quad g_1(r) = \int_E r(\sigma) M(\sigma) d\muN(\sigma)
\]
is completely bounded from the algebra $\mathcal{R}(E)$ into the algebra $\mathcal{L}(H)$, with complete bound $\|g_1\|_{cb} \leq \|\int_E M(\sigma) \, d\muN(\sigma)\|$.

Lemma 2.2. We consider a measurable subset $E$ of $\mathbb{C}$, a complex-valued measure $m$ on $E$, and a bounded and continuous function $M(.) \in C(E; \mathcal{L}(H))$ defined on $E$ with bounded invertible operator values. We assume that there exist a continuous function $N(.) \in C(E; \mathcal{L}(H))$ defined on $E$ with bounded self-adjoint operator values, a positive number $\alpha$ and a positive bounded measure $\muN$ such that $\frac{1}{2}(M(\sigma) + M(\sigma)^*) \geq \muN(\sigma) \geq \alpha > 0$, for all $\sigma \in E$ and $|\muN| \leq \muN$ on $E$. Then the map
\[
  r \mapsto g_2(r), \quad g_2(r) = \int_E r(\sigma) (M(\sigma))^{-1} d\muN(\sigma)
\]
is well-defined and completely bounded from the algebra $\mathcal{R}(E)$ into the algebra $\mathcal{L}(H)$, with complete bound $\|g_2\|_{cb} \leq \|\int_E (N(\sigma))^{-1} d\muN(\sigma)\|$.

Proof. We only give the proof of Lemma 2.2, the other being easier and similar. We consider $R(\sigma) = (r_{ij}(\sigma)) \in M_n(\mathbb{C})$, where the components $r_{ij}(\cdot)$ are rational functions bounded on $E$ with values in $\mathbb{C}$. We associate to $R$, the operator $G_2(R)$ on $\mathcal{H}^n$ defined by $G_2(R)_{ij} = g_2(r_{ij})$. To say that $g_2(r)$ is completely bounded with complete bound $\|g_2\|_{cb}$ means that
\[
  \|G_2(R)\|_{\mathcal{H}^n \to \mathcal{H}^n} \leq \|g_2\|_{cb}\|R\|_E,
\]
for all such $R$ and all values of $n$.

Without loss of generality, we may assume that $\muN(\sigma) = \frac{1}{2}(M(\sigma) + M(\sigma)^*)$. Then we define $B(\sigma) = N(\sigma)^{-1/2}$, so that we can write $M(\sigma) = B(\sigma)^{-1}(I + iD(\sigma))B(\sigma)^{-1}$, with a self-adjoint operator $D(\sigma)$. Let us consider $u$ and $v \in \mathcal{H}^n$, with $\|u\| = \|v\| = 1$, then we have
\[
  \langle G_2(R)u, v \rangle = \int_E \langle (R(\sigma) \otimes M(\sigma)^{-1})u, v \rangle d\muN(\sigma)
  = \int_E \langle (R(\sigma) \otimes (I + iD(\sigma))^{-1}B(\sigma)u, B(\sigma)v \rangle d\muN(\sigma).
\]
We note that $\|((I + iD(\sigma))^{-1})\| \leq 1$, whence
\[
  \|R(\sigma) \otimes (I + iD(\sigma))^{-1}\|_{\mathcal{H}^n \to \mathcal{H}^n} \leq \|R\|_E.
\]
This yields
\[
  |\langle G_2(R)u, v \rangle| \leq \|R\|_E \int_E \|B(\sigma)u\| \|B(\sigma)v\| d\muN(\sigma) \leq \frac{1}{2} \|R\|_E \left( \int_E \|B(\sigma)u\|^2 d\muN(\sigma) + \int_E \|B(\sigma)v\|^2 d\muN(\sigma) \right).
\]
But we have
\[
  \int_E \|B(\sigma)u\|^2 d\muN(\sigma) = \langle \int_E B^2(\sigma) d\muN(\sigma)u, u \rangle \leq \|\int_E (N(\sigma))^{-1} d\muN(\sigma)\|,
\]
and the same inequality with \( v \) in place of \( u \). Finally we have obtained
\[
|\langle G_2(R)u, v \rangle| \leq \|R\|_E \left\| \int_E (N(\sigma))^{-1} \, d\mu(\sigma) \right\|
\]
for all \( u, v \), with \( \|u\| = \|v\| = 1 \), which completes the proof of the lemma.

As an application of Lemma 2.1, we will prove an estimate for the Poisson term defined in the introduction.

**Corollary 2.3.** (a) Let the closed disk \( D \) of the Riemann sphere be a spectral set for \( A \in \mathcal{L}(H) \), and let \( \Gamma \subset \partial D \). With the above notation, the map
\[
f \mapsto g(f) := \int_{\Gamma} f(\sigma) \mu(\sigma, A, D) \, ds
\]
is completely bounded from the algebra \( \mathcal{R}(D) \) into \( \mathcal{L}(H) \), with complete bound \( 1 \).

(b) Let \( A \in \mathcal{L}(H) \), and consider the intersection \( X = D_1 \cap D_2 \cap \cdots \cap D_n \) of \( n \) disks of the Riemann sphere \( \overline{\mathbb{C}} \), each of them being spectral for \( A \). With the above notation, the map
\[
f \mapsto g_p(f) = \sum_{j=1}^n \int_{X \cap \partial D_j} f(\sigma) \mu(\sigma, A, D_j) \, ds
\]
is completely bounded from the algebra \( \mathcal{R}(X) \) into \( \mathcal{L}(H) \), with complete bound \( \leq n \).

**Proof.** For (a) we want to apply Lemma 2.1 with \( E = \Gamma, M(\sigma) = \mu(\sigma, A, D) \), \( dm = ds \) and \( dn = ds \). Hence it is sufficient to verify that
\[
\mu(\sigma, A, D) \geq 0, \quad \text{for all } \sigma \in \partial D, \quad \mu(\sigma, A, D) \, ds = 1, \quad \text{identity on } H.
\]
For a proof of (3), consider for instance a disk of the form \( D = \{ |z - \omega| \geq r \} \) (proofs for compact disks and half-planes are similar, we omit here the details). Since \( D \) is a spectral set for \( A \), we have \((A - \omega)(A^* - \bar{\omega}) - r^2 \geq 0 \). Then with the parametrization \( \sigma = \omega + re^{-is/r} \) we get
\[
\mu(\sigma, A, D) = \frac{1}{2\pi} \left( r(r - e^{is/r}(A - \omega))^{-1} + r(r - e^{-is/r}(A^* - \bar{\omega}))^{-1} - 1 \right)
\]
\[
= \frac{1}{2\pi} (r - e^{is/r}(A - \omega))^{-1} \left( (A - \omega)(A^* - \bar{\omega}) - r^2 \right) (r - e^{-is/r}(A^* - \bar{\omega}))^{-1},
\]
which shows that \( \mu(\sigma, A, D) \geq 0 \). Finally, the relation (4) easily follows from the Cauchy formula. The proof of (b) follows by applying part (a) to each term of the sum. \qed

3. **The annulus case: proof of Theorem 1.2.** We start by establishing the upper bounds in Theorem 1.2, implying that Theorem 1.1 holds for \( n = 2 \) in the special case of an annulus.

**Proof of the upper bounds in Theorem 1.2.** We have by assumption that the disks
\[
D_1 = \{ z \in \mathbb{C} : |z| \leq R \}, \quad \text{and} \quad D_2 = \{ z \in \mathbb{C} : |z| \geq R^{-1} \} \cup \{ \infty \}
\]
are spectral for \( A \), and \( X = X(R) = D_1 \cap D_2 \).

Taking into account the complete bound \( n = 2 \) of the map \( f \mapsto g_p(f) \), it suffices to show that the map \( f \mapsto g_r(f) \) is completely bounded by \( (R+1)/\sqrt{R^2+R+1} \). Recall that
\[
g_r(f) = \int_{\partial D_1} f(\sigma) \nu(\sigma, A, D_1) \, d\sigma + \int_{\partial D_2} f(\sigma) \nu(\sigma, A, D_2) \, d\sigma,
\]
and that the orientations of $\partial D_1$ and $\partial D_2$ are opposite. We have on $\partial D_1$ the identity

$$\nu(\sigma, A, D_1) \, d\sigma = \frac{1}{2\pi i} (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} + \sigma^{-1} d\sigma$$

$$= \frac{1}{2\pi i} A^* (\sigma A^* - R^2)^{-1} d\sigma.$$  

Note that $(\sigma A^* - R^2)^{-1}$ is holomorphic in $\sigma$, for $|\sigma| \leq R$. Therefore we can deform the boundary $\partial D_1$ into the unit circle and get

$$\int_{\partial D_1} f(\sigma) \nu(\sigma, A, D_1) \, d\sigma = \int_{|\sigma| = 1} f(\sigma) \nu(\sigma, A, D_1) \, d\sigma.$$  

Similarly, with $\nu(\sigma, A, D_2) = \frac{1}{2\pi i} A^* (\sigma A^* - R^2)^{-1}$, and keeping for the unit circle the orientation of $\partial D_1$,

$$\int_{\partial D_2} f(\sigma) \nu(\sigma, A, D_2) \, d\sigma = - \int_{|\sigma| = 1} f(\sigma) \nu(\sigma, A, D_2) \, d\sigma$$

and

$$g_r(f) = \int_{|\sigma| = 1} f(\sigma) (\nu(\sigma, A, D_1) - \nu(\sigma, A, D_2)) \, d\sigma.$$  

Setting $\sigma(\theta) = e^{i\theta}$, we get

$$g_r(f) = - \frac{R^2 - R^{-2}}{2\pi} \int_0^{2\pi} f(e^{i\theta}) M(\theta, A^*)^{-1} d\theta,$$

where $M(\theta, A^*) := R^2 + R^{-2} - e^{i\theta} A^* - e^{-i\theta} (A^{-1})^*.$

We write $A^* = UG$, with a unitary operator $U$ and a positive operator $G$. Then the assumption “$D_1$ and $D_2$ spectral sets for $A$” reads $R^{-1} I \leq G \leq R I$. Setting $\rho = \frac{R + R^{-1}}{2}$, we have

$$\|G + G^{-1} - (\rho + 1) I\| \leq \max_{R^{-1} \leq x \leq R} |x + x^{-1} - \rho - 1| = \rho - 1.$$  

This yields, for the self-adjoint part of $M(\theta, A^*)$,

$$\text{Re } M(\theta, A^*) = R^2 + R^{-2} - (\rho+1) \text{Re } (e^{i\theta} U) + \text{Re } (e^{i\theta} U (G + G^{-1} - \rho - 1))$$

$$\geq R^2 + R^{-2} - (\rho+1) \text{Re } (e^{i\theta} U) - \rho + 1 \geq R^2 + R^{-2} - 2\rho > 0.$$  

We then apply Lemma 2.2 with

$$N(\theta) = R^2 + R^{-2} - (\rho+1) \text{Re } (e^{i\theta} U) - \rho + 1, \quad dm = dn = \frac{R^2 - R^{-2}}{2\pi} d\theta,$$

and get that $g_r$ is completely bounded with a complete bound

$$\|g_r\|_{cb} \leq \frac{R^2 - R^{-2}}{2\pi} \left\| \int_0^{2\pi} (R^2 + R^{-2} - (\rho+1) \text{Re } (e^{i\theta} U) - \rho + 1)^{-1} d\theta \right\|.$$  

Elementary calculations show that, if we set,

$$h(z) = \int_0^{2\pi} \frac{d\theta}{R^2 + R^{-2} - \rho + 1 - (\rho+1)(e^{i\theta} z + e^{-i\theta} z^{-1})/2},$$
then we have
\[ h(e^{i\varphi}) = \frac{2\pi}{\sqrt{R^2 + R^{-2} - R - R^{-1}} \sqrt{R^2 + R^{-2} + 2}} \]
\[ = \frac{2\pi}{R - R^{-1}} \frac{R^2 - R^{-2} \sqrt{R^2 + R^{-2} - R - R^{-1}}}{\sqrt{R^2 - R^{-2}} \sqrt{R + R^{-1} + 1}} = h(1). \]

This yields that \( h(U) = h(1) \) and thus
\[ \|g_1\|_{cb} \leq \frac{R^2 - R^{-2}}{2\pi} \|h(U)\| = \frac{\sqrt{R + \sqrt{R^{-1}}}}{\sqrt{R + R^{-1} + 1}} = \frac{R + 1}{\sqrt{R^2 + R + 1}}, \]
as required for the upper bound claimed in Theorem 1.2.

**Proof of the lower bound in Theorem 1.2.** For \( t \in \mathbb{R} \), let
\[ A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \]
acting on the Hilbert space \( \mathbb{C}^2 \). For \( t = R - R^{-1} \) we have \( \|A(t)\| = \|A(t)^{-1}\| = R \) (compare with [16, p. 152]). For a rational function \( f \) we have
\[ f(A(t)) = \begin{pmatrix} f(1) & tf'(1) \\ 0 & f(1) \end{pmatrix}, \]
which implies \( \|f(A(t))\| \geq t|f'(1)|. \) This shows that
\[ K(R) \geq (R-R^{-1}) \sup\{|f'(1)|/\|f\|_{X(R)} : f \in \mathcal{R}(X)\}. \]

It follows from the computation of the infinitesimal Carathéodory pseudodistance on the annulus by Simha [23, Example (5.3)] that
\[ \sup\{|f'(1)|/\|f\|_{X(R)} : f \in \mathcal{R}(X), f(1) = 0\} = \frac{2}{R} \prod_{n=1}^{\infty} \left( \frac{1 - R^{-8n}}{1 - R^{4-8n}} \right)^2. \]

Thus
\[ K(R) \geq \gamma(R) := 2(1-R^{-2}) \prod_{n=1}^{\infty} \left( \frac{1 - R^{-8n}}{1 - R^{4-8n}} \right)^2 = \lim_{k \to \infty} \gamma_k(R), \]
with
\[ \gamma_k(R) := \frac{2}{1 + R^{-2}} \prod_{n=1}^{k} \left( \frac{1 - R^{-8n}}{1 - R^{4-8n}} \frac{1 - R^{-8n}}{1 - R^{-4-8n}} \right). \]

For \( k = 1 \) we have
\[ \gamma_1(R) = \psi(R^2), \quad \text{with} \quad \psi(t) := \frac{2t(1+t^2)^2}{(1+t)(1+t^2+t^4)}. \]
It is easily seen that \( \psi'(t) > 0 \), which shows that \( \gamma_1(\cdot) \) is an increasing function. We have \( \gamma_1(1) = 4/3 \), and \( \gamma_1(\infty) = 2 \). We deduce that \( K(R) \geq \gamma_1(R) \geq \gamma_1(1) = 4/3 \). Note also that, for each \( R \) the sequence \( \gamma_k(R) \) is increasing with \( k \), with upper bound 2.  \( \square \)
Remark 3.1. We have
\[
\lim_{R \to 1} \gamma(R) = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}, \quad \lim_{R \to \infty} \gamma(R) = 2.
\]
Numerical computations show that \(\gamma(R)\) increases with \(R\), which leads to a better lower bound \(\gamma(R) \geq \gamma(1) = \pi/2\). For the sake of comparison, we have drawn in Figure 1 the different lower and upper bounds for \(K(R)\) as a function of \(R\). Returning to the second question of Shields mentioned in the introduction, we see that an optimal constant independent of \(R\) for our annulus \(X(R)\) has to lie between 2 and \(2 + 2/\sqrt{3}\).

Remark 3.2. We owe to a suggestion of Vern Paulsen an improvement of upper bounds of \(K(R)\) for large values of \(R\). More precisely, adapting the arguments of [17, Theorem 4.1], we obtain
\[
K(R) \leq \max(3, 2 + \psi(R)) \quad \text{with} \quad \psi(R) = \sum_{n \geq 1} \frac{4}{1 + R^{2n}}.
\]
This shows that \(K(R) \leq 3\), for \(R \geq 2.0952978\). This estimate is better than the upper bound \(2 + (R+1) / \sqrt{R^2 + R+1}\) of Theorem 1.2, if \(R \geq 1.9878813\), and better than the upper bound of Shields if \(R \geq 1.85443\).

Proof. Let \(f\) be a rational function bounded by 1 in the annulus \(X(R)\). We write
\[
f(z) = f_1(z) + f_2(z) \quad \text{with} \quad f_1(z) = \sum_{n \geq 0} a_n z^n, \quad f_2(z) = \sum_{n < 0} a_n z^n.
\]
It suffices to show that \( \|f_1\|_{D_1} + \|f_2\|_{D_2} \leq \max(3, 2 + \psi(R)) \). For that we may assume, without loss of generality that \( a_0 \geq 0 \). For each integer \( n > 0 \), each real \( r \in [1/R,R] \), and \( \varphi \in \mathbb{R} \), we have

\[
a_n r^n e^{i\varphi} = -\frac{1}{2\pi} \int_0^{2\pi} (1 - e^{i\varphi}f(re^{i\theta}))e^{-in\theta}d\theta,
\]

\[
\overline{a_n} r^{-n} e^{-i\varphi} = -\frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\varphi}f(re^{i\theta}))e^{-in\theta}d\theta,
\]

and

\[
a_n r^n e^{i\varphi} + \overline{a_n} r^{-n} e^{-i\varphi} = -\frac{1}{\pi} \int_0^{2\pi} \text{Re}(1 - e^{i\varphi}f(re^{i\theta}))e^{-in\theta}d\theta.
\]

We note that \( \text{Re}(1 - e^{i\varphi}f(re^{i\theta})) \geq 0 \), thus

\[
|a_n r^n e^{i\varphi} + \overline{a_n} r^{-n} e^{-i\varphi}| \leq \frac{1}{\pi} \int_0^{2\pi} \text{Re}(1 - e^{i\varphi}f(re^{i\theta}))d\theta = 2(1 - \text{Re}(e^{i\varphi}a_0)).
\]

With a judicious choice of \( \varphi \) we obtain

\[
|a_n|^n + |a_{-n}|r^{-n} \leq 2(1 - a_0), \quad \text{for all } r \in [1/R,R].
\]

Writing this inequality with the values \( r = R \) and \( r = 1/R \), and then adding, we deduce

\[
(|a_n| + |a_{-n}|)(R^n + R^{-n}) \leq 4(1 - a_0).
\]

We note that, from the maximum principle,

\[
\|f_1\|_{D_1} = \max_{|z|=R} (|f(z) - f_2(z)|) \leq 1 + \max_{|z|=R} (|f_2(z)|)
\]

\[
\leq 1 + \sum_{n \geq 1} |a_n| R^{-n}.
\]

Similarly we have

\[
\|f_2\|_{D_2} = \max_{|z|=1/R} (|f(z) - f_1(z)|) \leq 1 + a_0 + \sum_{n \geq 1} |a_n| R^{-n},
\]

and finally

\[
\|f_1\|_{D_1} + \|f_2\|_{D_2} \leq 2 + a_0 + \sum_{n \geq 1} (|a_n| + |a_{-n}|) R^{-n}
\]

\[
\leq 2 + a_0 + 4(1 - a_0) \sum_{n \geq 1} (1 + R^{2n})^{-1}.
\]

The result follows by noticing that \( a_0 \in [0,1] \).

4. The general case of two closed disks of the Riemann sphere. Recall our basic assumption that for each spectral set \( D \) for a given operator \( A \), the spectrum \( \sigma(A) \) is included in the interior of \( D \). Recall also that we have defined the residual kernel in a point \( \sigma \) on the boundary of a disk \( D \) centered in \( \omega \) by

\[
\nu(\sigma, A, D) d\sigma = \frac{1}{2\pi i} (\sigma - A)^{-1} d\sigma - \mu(\sigma, A, D) ds = \frac{1}{2\pi i} (\sigma - A^*)^{-1} d\sigma + \frac{d\sigma}{\sigma - \omega}.
\]
If \( \partial D = \{ z ; |z-\omega| = r \} \), then \( \bar{\sigma} - \bar{\omega} = r^2/(\sigma - \omega) \), thus
\[
\nu(\sigma, A, D) = \frac{1}{2\pi i} \left( \frac{1}{\sigma - \omega} - \frac{r^2}{\sigma - \omega} (r^2 - (\sigma - \omega)(\sigma - \omega)^{-1}) \right) = \frac{1}{2\pi i} (A^* - \bar{\omega})((\sigma - \omega)(A^* - \bar{\omega}) - r^2)^{-1}.
\]
Note that \( ((z-\omega)(A^* - \bar{\omega}) - r^2)^{-1} \) is also well-defined, and analytic in \( z \), if \( z \in D \), under the assumption \( \sigma(A) \subset \text{int}(D) \). This allows us to extend the previous definition of \( \nu \) by
\[
\nu(z, A, D) := \frac{1}{2\pi i} (A^* - \bar{\omega})((z-\omega)(A^* - \bar{\omega}) - r^2)^{-1}, \quad \text{for } z \in D.
\]
Similarly, if \( D \) is a half-plane \( D = \Pi = \{ z ; \Re(e^{-i\theta}(z-a)) \geq 0 \} \), we use from now on the definition
\[
\nu(z, A, \Pi) := (z-a+e^{2i\theta}(A^* - \bar{a}))^{-1}, \quad \text{for } z \in \Pi.
\]

An important point in our study is the homographic invariance of the differential form
\[
(\nu(z, A, D_j) - \nu(z, A, D_k)) \, dz.
\]

More precisely we have the following result

**Lemma 4.1.** Let \( D_j \) and \( D_k \) be two disks of \( \mathbb{C} \), and let \( A \) be an operator with spectrum \( \sigma(A) \) contained in the interior of \( D_j \cap D_k \). Then, for all Möbius map \( \varphi \) with poles off \( \sigma(A) \), we have
\[
(\nu(z, A, D_j) - \nu(z, A, D_k)) \, dz = (\nu(\zeta, B, \Delta_j) - \nu(\zeta, B, \Delta_k)) \, d\zeta,
\]
where \( \zeta = \varphi(z) \), \( B = \varphi(A) \), \( \Delta_j = \varphi(D_j) \), and \( d\zeta = \varphi'(z) \, dz \).

**Proof.** For the sake of the clarity we will assume in the proof that \( D_j, D_k, \Delta_j \) and \( \Delta_k \) are not half-planes. The case of half-planes could be obtained either by a limiting argument, by replacing half-planes by suitable supersets being unbounded disks, or otherwise directly by slightly changing the arguments below. We omit the (elementary but tedious) details.

Recall that a Möbius map of the Riemann sphere is an automorphism \( \varphi \) of \( \mathbb{C} \) which has the form
\[
\varphi(z) = \frac{az+b}{cz+d}, \quad \text{with} \quad ad \neq bc, \; a, b, c, d \in \mathbb{C}.
\]

These transformations form a group generated by the translations: \( \varphi(z) = z + b \), the dilations centered in \( 0 \): \( \varphi(z) = \lambda z, \; \lambda \neq 0 \), and the inversion-symmetry: \( \varphi(z) = 1/z \). The lemma is easily verified if \( \varphi \) is a translation or a dilation, therefore it suffices to consider the case \( \varphi(z) = 1/z \).

So, if \( \omega_j \) and \( r_j \) denote respectively the center and the radius of \( D_j \), after the transformation \( \varphi(z) = 1/z \), we get a disk \( \Delta_j = \varphi(D_j) \) with center and radius
\[
o_j = \frac{\bar{\omega}_j}{|\omega_j|^2 - r_j^2}, \quad \rho_j = \frac{r_j}{|\omega_j|^2 - r_j^2}.
\]
Setting \( \zeta = 1/z, \; B = A^{-1} \), we have
\[
B^* - o_j = \frac{1}{|\omega_j|^2 - r_j^2} \left( |\omega_j|^2 - r_j^2 - \omega_j A^* \right),
\]
and thus
\[(\zeta - o_j)(B^* - o_j) = \frac{(A^{-1})^*}{z((\omega_j)^2 - r_j^2)^2} \left((|\omega_j|^2 - r_j^2 - \omega_j A^*)(|\omega_j|^2 - r_j^2 - \bar{\omega}_j z) - z r_j^2 A^*)\right)\]
\[= \frac{(A^{-1})^*}{z((\omega_j)^2 - r_j^2)} \left(|\omega_j|^2 - r_j^2 - \omega_j A^* + z(A^* - \bar{\omega}_j)\right),\]

implying that
\[2\pi i \nu(\zeta, B, \Delta_j) = z(|\omega_j|^2 - r_j^2 - \omega_j A^*)(|\omega_j|^2 - r_j^2 - \omega_j A^* + z(A^* - \bar{\omega}_j))^{-1}\]
\[= z - z^2(A^* - \bar{\omega}_j)(|\omega_j|^2 - r_j^2 - \omega_j A^* + z(A^* - \bar{\omega}_j))^{-1}\]
\[= z - 2\pi i z^2 \nu(z, A, D_j).\]

Using that \(d\zeta = -z^{-2} dz\), we get
\[\left(\nu(\zeta, B, \Delta_j) - \nu(\zeta, B, \Delta_k)\right) d\zeta = (\nu(z, A, D_j) - \nu(z, A, D_k)) dz,\]
which completes the proof. \(\square\)

We are now prepared to present our proof of Theorem 1.1 for the case \(n = 2\) of the intersection \(X = D_1 \cap D_2\) of two arbitrary closed disks of the Riemann sphere.

We warn again the reader that each disk is a spectral set for the bounded operator \(A\), and that the spectrum of \(A\) is interior to \(X\). We have seen in Corollary 2.3 that the map \(f \mapsto g_p(f)\) is completely bounded, with constant 2, it remains to show that the map \(f \mapsto g_r(f)\) is completely bounded, with constant \(2/\sqrt{3}\). We have
\[g_r(f) = \int_{X \cap \partial D_1} f(\sigma) \nu(\sigma, A, D_1) d\sigma + \int_{X \cap \partial D_2} f(\sigma) \nu(\sigma, A, D_2) d\sigma.\] (6)

Three cases may occur.

**Case 1.** \(\partial D_1 \cap \partial D_2 = \emptyset\). Then there exist a Möbius map \(\varphi\) and a real \(R > 1\) such that
\[\Delta_1 = \varphi(D_1) = \{z \in \mathbb{C} : |z| \leq R\} \quad \text{and} \quad \Delta_2 = \varphi(D_2) = \{z \in \overline{\mathbb{C}} : |z|^{-1} \leq R\}.\]

Note that the pole of the map \(\varphi\), which is \(\varphi^{-1}(\infty)\), is not in \(D_1\) and thus does not belong to \(\sigma(A)\). We introduce a median circle, \(C_{12} := \{z \in \overline{\mathbb{C}} : |\varphi(z)| = 1\}\).

**Case 2.** \(\partial D_1 \cap \partial D_2 = \{\alpha, \beta\}, \alpha \neq \beta\). Then there exist a Möbius map \(\varphi\) and a real \(\theta \in (0, \pi/2)\) such that
\[\Delta_1 = \varphi(D_1) = \{z \in \mathbb{C} : \text{Re}(e^{i\theta} z) \geq 0\} \cup \{\infty\} \quad \text{and} \quad \Delta_2 = \varphi(D_2) = \{z \in \mathbb{C} : \text{Re}(e^{-i\theta} z) \geq 0\} \cup \{\infty\}.\]

The pole of the map \(\varphi\) now is \(\alpha\) or \(\beta\) and does not belong to \(\sigma(A)\). We introduce a median line, \(C_{12} := \{z \in \overline{\mathbb{C}} : \varphi(z) \in [0, \infty]\}\).

**Case 3.** \(\partial D_1 \cap \partial D_2 = \{\alpha\}\). Then there exists a Möbius map \(\varphi\) such that
\[\Delta_1 = \varphi(D_1) = \{z \in \mathbb{C} : \text{Im} z \leq 1\} \cup \{\infty\} \quad \text{and} \quad \Delta_2 = \varphi(D_2) = \{z \in \mathbb{C} : \text{Im} z \geq -1\} \cup \{\infty\}.\]

The pole of \(\varphi\) now is \(\alpha\) which does not belong to \(\sigma(A)\). We set \(C_{12} := \{z \in \overline{\mathbb{C}} : \varphi(z) \in \mathbb{R}\}\).

In each case we can continuously deform, staying in the set \(X, X \cap \partial D_1\) onto \(C_{12}\) as well as \(X \cap \partial D_2\) onto \(C_{12}\). We can choose \(C_{12}\) with the same orientation that \(\partial D_1\),
which is opposite to the orientation of $\partial D_2$. Since $f(.)\nu(.,A,D_j)$ is holomorphic in $X$, we deduce from (6)
\[ g_r(f) = \int_{C_{12}} f(z) (\nu(z,A,D_1) - \nu(z,A,D_2)) \, dz. \]

Theorem 1.1, in the case $n = 2$, then follows from the following lemma

**Lemma 4.2.** Consider two distinct disks $D_1$ and $D_2$ of $\mathbb{C}$, and a bounded operator $A \in \mathcal{L}(H)$ such that $D_1$ and $D_2$ are spectral for $A$. Then, for any subarc of the median line $\Gamma \subset C_{12}$, the map
\[ f \mapsto g(f) = \int_{\Gamma} f(z) \left( \nu(z,A,D_1) - \nu(z,A,D_2) \right) \, dz, \]
is completely bounded from the algebra $\mathcal{R}(D_1 \cap D_2)$ into $\mathcal{L}(H)$, with complete bound $\leq 2/\sqrt{3}$.

**Proof.** We use the Möbius map $\varphi$ previously described in the three possible cases and set $\zeta = \varphi(z)$, $h(\zeta) = f(z)$, $B = f(A)$. Note that, since $\varphi$ is a bijection, the disks $\varphi(D_1)$ and $\varphi(D_2)$ are spectral for $B$, and the spectrum of $B$ is included in the interior of these disks. We get from Lemma 4.1
\[ g(f) = \int_{\varphi(\Gamma)} f(\zeta) \left( \nu(\zeta,B,\Delta_1) - \nu(\zeta,B,\Delta_2) \right) \, d\zeta. \]

Therefore, it suffices to show the lemma in the three following cases:
- $D_1 \cap D_2$ is an annulus $X(R) = \{ z \in \mathbb{C} ; R^{-1} \leq |z| \leq R \}$,
- $D_1 \cap D_2$ is a sector $X = \{ z \in \mathbb{C} ; \text{Re}(e^{i\theta}z) \geq 0 \text{ and } \text{Re}(e^{-i\theta}z) \geq 0 \} \cup \{ \infty \}$,
- $D_1 \cap D_2$ is a strip $X = \{ z \in \mathbb{C} ; -1 \leq \text{Im } z \leq 1 \} \cup \{ \infty \}$.

The annulus case has been treated in Section 3, except that the set $\{ z ; |z| = 1 \}$ is now replaced by a subset $\Gamma$. The necessary modifications in the proof are rather minor, and so will be omitted here.

**The sector case.** We have $\Gamma \subset \mathbb{R}_+$ and
\[ g(f) = \int_{\Gamma} f(x) \left( \nu(x,A,D_1) - \nu(x,A,D_2) \right) \, dx \]
\[ = \frac{\sin 2\theta}{\pi} \int_{0}^{\infty} f(x) M(x)^{-1} 1_{\Gamma}(x) \, dx, \]
with $M(x) = A^* + 2x \cos 2\theta + x^2 (A^*)^{-1}$. The conditions $D_1$ and $D_2$ spectral sets for $A$ means that $\langle Av,v \rangle \in D_1 \cap D_2 \subset \{ z \in \mathbb{C} ; \text{Re}(z) \geq 0 \}$ for all $v \in H$. Hence the operator $B = \frac{1}{2}(A + A^*)$ is a positive operator, and we have $A = B^{1/2}(I+iC)B^{1/2}$, with a self-adjoint operator $C$ satisfying $||C|| \leq 1/\tan \theta$. Note that
\[ \text{Re}(I-iC)^{-1} \geq \inf_{\lambda \in \sigma(C)} \frac{1}{1-i\lambda} = \inf_{\lambda \in \sigma(C)} \frac{1}{1+\lambda^2} \geq \sin^2 \theta. \]

Hence we have $\text{Re}(A^*)^{-1} \geq \sin^2 \theta B^{-1}$ and $\text{Re} M(x) \geq x^2 \sin^2 \theta B^{-1} + 2x \cos 2\theta + B$. We then apply Lemma 2.2 with
\[ N(x) = x^2 \sin^2 \theta B^{-1} + 2x \cos 2\theta + B, \quad dn = 1_{\Gamma} \, dn, \quad dn = \frac{\sin 2\theta}{\pi} \, dx. \]
Setting
\[
    h(z, \theta) = \frac{\sin \theta}{\pi} \int_{\theta}^{\infty} \frac{dx}{x^2 \sin^2 \theta / z + 2x \cos \theta + z}
\]
for \( z \geq 0 \), we observe that \( h(z, \theta) = h(1, \theta) \) does not depend on \( z \), and Lemma 2.2 allows us to conclude that \( g \) is completely bounded with a complete bound
\[
    \|g\|_{cb} \leq \|h(B, \theta)\| = h(1, \theta) \leq \max_{\theta' \in [0, \pi/2]} h(1, \theta') = h(1, \pi/2) = 2/\sqrt{3}.
\]
We should notice that the estimate \( h(1, \theta) \leq h(1, \pi/2) \), though sufficient for the purpose of Theorem 1.1, is quite rough. Alternately, one may evaluate the integral expression for \( h(1, \theta) \). Also, our Corollary 2.3(b) may be improved for a sector. We refer the reader to [4] and [6, Chapter 4] for a discussion of optimized upper bounds as a function of \( \theta \).

The strip case. We now have \( \Gamma \subset \mathbb{R} \) and
\[
    g(f) = \int_{\Gamma} f(x) \left( (\nu(x, A, D_1) - \nu(x, A, D_2)) \right) dx
    = \frac{2}{\pi} \int_{-\infty}^{+\infty} f(x) M(x)^{-1} 1_\Gamma(x) dx,
\]
with \( M(x) = (x-A^*)^2 + 4 \). Since \( D_1 \) and \( D_2 \) are spectral sets for the operator \( A \), we have \( |\text{Im} \langle Av, v \rangle| \leq 1 \) for all \( v \in H \) with \( \|v\| = 1 \). Hence, if we denote \( B = \frac{1}{2}(A + A^*) \) and write \( A = B + ic \), the self-adjoint operator \( C \) satisfies \( \|C\| \leq 1 \) and
\[
    \text{Re} M(x) = (x-B)^2 + 4 - C^2 \geq N(x) := (x-B)^2 + 3.
\]
We then deduce from Lemma 2.2 that \( g \) is completely bounded with a complete bound
\[
    \|g\|_{cb} \leq \|h(B)\| = 2/\sqrt{3},
\]
where
\[
    h(\lambda) = \frac{2}{\pi} \int_{\mathbb{R}} \frac{dx}{(x-\lambda)^2 + 3} = \frac{2}{\sqrt{3}}.
\]

Remark 4.3. The key points in the proof of this lemma has been the homographic invariance of the differential forms, and the choice of the median lines as paths of integration. For our proof for more than two disks presented below, it will be useful to give a more abstract definition of median lines in terms of sets of equal "distance" to some parts of the boundary of \( X \): given a disk \( D \) of the Riemann sphere, we introduce the following notation
\[
    d(z, D) = \left\{ \frac{(r^2 - |z-\omega|^2)}{r}, \text{ if } D \text{ is a disk with center } \omega \text{ and radius } r, \right. \\
    \left. \text{Re}(e^{-i\theta}(z-a)), \text{ if } D \text{ is the half-plane } \{ z : \text{Re}(e^{-i\theta}(z-a)) \geq 0 \} \right\}.
\]
By means of elementary computations we have
\[
    \frac{1}{d(z, D)} = \frac{|\phi'(z)|}{1 - |\phi(z)|^2}, \quad \text{for } z \in D,
\]
with the Riemann conformal map \( \phi \) from \( D \) onto the unit disk \( \mathbb{D} \). As a consequence, \( 1/d(z, D) \) is the infinitesimal Carathéodory pseudodistance. In particular, it becomes obvious that
\[
    d(\varphi(z), \varphi(D)) = |\varphi'(z)| d(z, D),
\]
for any Möbius map \( \varphi \).
Note also that \( d(z, D) \to 0 \), if \( z \) tends to the boundary of \( D \).
Now, if \( D_j \) and \( D_k \) are two disks with non empty intersection, we introduce their median line \( C_{jk} \)
\[
C_{jk} := \{ z \in D_j \cap D_k : d(z, D_j) = d(z, D_k) \}.
\]
From (8) it follows that, if \( C_{jk} \) is the median line of \( D_j \) and \( D_k \), and if \( \varphi \) is a Möbius map, then \( \varphi(C_{jk}) \) is the median line of \( \varphi(D_j) \) and \( \varphi(D_k) \). Also, the circle \( \{ z ; |z| = 1 \} \) is the median line in the annulus case \( X(R) \), the positive real semi axis is the median line in the sector case, the real axis is the median line in the strip case. This shows that the present definition effectively coincides with the notion of median line used in the beginning of this section with the curve \( C_{12} \).

5. **Completing the proof of Theorem 1.1.** We can now present the proof of the general case of Theorem 1.1.

*Proof of Theorem 1.1 for the case \( n \geq 3 \).* We consider \( n \) closed disks \( D_{1}, D_{2}, \ldots, D_{n} \) of the Riemann sphere, with \( D_{j} \not= D_{k} \), for all \( j \neq k \), such that \( X = D_{1} \cap D_{2} \cap \ldots \cap D_{n} \) has a non-empty interior (these two assumptions can be added without loss of generality in the statement of Theorem 1.1). Given \( n \) points \( z_{1}, \ldots, z_{n} \) in the complex plane, the classical Voronoi tessellation of \( X \) consists in dividing up the set \( X \) in sets \( X_{j} \) of points \( \sigma \in X \) such that \( \sigma \) is at least as close to \( z_{j} \) as to any of the other points \( z_{k} \) for \( k \neq j \). In order to find the new integration paths in our representation formula for the residual term, we will construct a similar Voronoi tessellation, but use the distance to the boundaries \( \partial D_{j} \) instead of the distance to points. In our case, this distance will be measured in terms of \( d \), the reciprocal of the infinitesimal Carathéodory pseudodistance (see Remark 4.3). We recall that \( d(\sigma, D_{j}) \to 0 \) if \( \sigma \in X \) tends to \( \partial D_{j} \).

To be more precise, let
\[
X_{j} = \left\{ \sigma \in X : d(\sigma, D_{j}) = \min_{k=1,\ldots,n} d(\sigma, D_{k}) \right\}.
\]
By construction, the union \( X_{1} \cup \ldots \cup X_{n} \) equals our set \( X \). We notice a difference between the present case and the \( n = 2 \) case: the intersection \( X_{j} \cap X_{k} \) for \( j \neq k \) is a subset of the median line \( C_{jk} \) of the disks \( D_{j} \) and \( D_{k} \), which, in general, is a proper subset, since the median line \( C_{jk} \) does not need to be a subset of \( X_{j} \), see Figures 2–4. This situation does not occur in the case \( n = 2 \) of two disks.

In order to make the situation clearer in the general case \( n \geq 3 \), recall that the boundary of \( X \) may be written as a union of the sets \( \partial D_{j} \cap X \) for \( j = 1, \ldots, n \), where any two of these sets have at most 2 points in common (otherwise two of the disks \( D_{j} \) would have identical boundary, in contradiction to the assumptions on the disks mentioned above). From the asymptotic behavior of \( d(\sigma, D_{j}) \) for \( \sigma \in X \) approaching \( \partial D_{j} \) it follows that \( \partial D_{j} \cap X = \partial X \cap X_{j} \) for \( j = 1, \ldots, n \), that is, \( X_{j} \) can be considered as a kind of closed neighborhood in \( X \) of \( \partial D_{j} \cap X \).

It is now important to observe that, by construction, the boundary of \( X_{j} \) may be written as the union of \( \partial D_{j} \cap X \) and of \( X_{j} \cap X_{k} \) for \( k \neq j \) or, in other words, of the parts of the median lines \( C_{jk} \) for \( k \neq j \) being a subset of \( X_{j} \). Again, any two of these pieces of \( \partial X_{j} \) being all circular arcs have at most two points in common, since a non-trivial part of \( \partial D_{j} \) cannot be a (part of a) median line for disks \( D_{j} \) and \( D_{k} \) for \( k \neq j \), and two medians \( C_{jk} \) and \( C_{jl} \) only have more than two points in common for \( k, \ell \neq j \) if \( D_{k} = D_{\ell} \).

Therefore, we have shown that, for a function analytic in \( X_{j} \subset X \), the path of integration \( X \cap \partial D_{j} \) can be replaced by the union of paths formed by the parts of the median lines
\( C_{jk} \) for \( k = 1, \ldots, n, k \neq j \), which have a non-empty intersection with \( X_j \) (and thus also with \( X_k \)). Since the orientation of these median lines is changed if we permute \( j \) and \( k \), we
may conclude that, for any $f \in R(X)$,

$$g_r(f) = f(A) - \sum_{j=1}^{n} \int_{X \cap \partial D_j} f(\sigma)\mu(\sigma, A, D_j) ds$$

$$= \sum_{j,k=1, j \neq k}^{n} \int_{C_{jk} \cap X_j \cap X_k} f(\sigma) \left( \nu(\sigma, A, D_j) - \nu(\sigma, A, D_k) \right) d\sigma.$$

Thus Theorem 1.1 follows from Corollary 2.3 and Lemma 4.2.

REFERENCES

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