Polynomial bounds of $3 \times 3$ matrices

Michel CROUZEIX

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Abstract

We exhibit a class of $3 \times 3$ matrices with a polynomial bound different of the completely bounded norm.

1 Introduction

Let us consider an operator $T$ on a Hilbert space $(H, \| \cdot \|)$. Its polynomial bound $K(T)$ is defined by

$$K(T) = \sup \{ \| p(T) \| ; p \text{ polynomial}, |p(z)| \leq 1, \forall z \in \mathbb{D} \},$$

$\mathbb{D}$ being the closed unit disk of $\mathbb{C}$. If $u_T$ denotes the homomorphism from polynomials to operators mapping $p$ to $p(T)$, then $K(T)$ is the bounded norm of $u_T$. The corresponding completely bounded norm $M(T)$ has been characterized by Paulsen [6]

$$M(T) = \inf \{ \| S \| \| S^{-1} \| ; \| S^{-1}TS \| \leq 1 \},$$

the infimum being considered onto all invertible operators, (or equivalently onto all self-adjoint positive definite operators). We use the convention $K(T) = +\infty$ if no $S$ realizes $\| S^{-1}TS \| \leq 1$.

Clearly, $M(T) = 1$ iff $T$ is a contraction and, from the von Neumann inequality [8], $K(T) = 1$ iff $T$ is a contraction. Using this inequality together with the relation $p(T) = Sp(S^{-1}TS)S^{-1}$ we deduce $K(T) \leq M(T)$.

A long-standing problem in operator theory is how $K(T)$ differs from $M(T)$. In his famous article [3], Halmos included the following question

*Is every polynomially bounded operator similar to a contraction?*

The answer *no* has been given by Pisier [7], of course in infinite dimension. For finite dimension the answer is clearly *yes*, but then the relevant question is how $K(T)$ controls $M(T)$? For large dimension, the best result is due to Bourgain [1]

$$M(T) \leq C_0K(T)^4\log(\text{dim } H),$$

where $C_0$ is some universal constant. The case of low dimension has been studied by Holbrook. In a first paper [4], he has showed that $K(T) = M(T)$ if $\text{dim } H = 2$ and that $K(T) < M(T)$ occurs for some $T$ if $\text{dim } H = 12$. Eighteen years later [5], he has exhibited some class of $T$ with $K(T) < M(T)$ and $\text{dim } H = 4$.

This paper is concerned by the case $\text{dim } H = 3$. We show that, for the following class of matrices

$$T_\epsilon = \begin{pmatrix} 0 & 2 & 0 \\ \epsilon & 0 & \frac{1-\epsilon^2}{\sqrt{2}} \\ 0 & \frac{1-\epsilon^2}{\sqrt{2}} & \sqrt{2} \end{pmatrix}, \quad -1/2 \leq \epsilon \leq 1/2,$$
there holds $M(T_\varepsilon) > 2$, if $\varepsilon \neq 0$, and $K(T_\varepsilon) = 2$, if $\varepsilon$ is small enough. Note that, if $|\varepsilon| > 2$, then the spectral radius of $T_\varepsilon$ is larger than 1, and thus $K(T_\varepsilon) = +\infty$.

After completion of this paper, John Holbrook mentioned to me that, with Frank Gilfeather, they have obtained existence’s evidence of $3 \times 3$ matrices $T$ with $K(T) < M(T)$ by numerical experiments; this was consigned in an HPCREC 2000-01 internal report.

2 The completely bounded norm is larger than 2

We first note that, for all $\varepsilon$, there holds $2 \leq \|T_\varepsilon\| \leq K(T_\varepsilon) \leq M(T_\varepsilon)$. If we introduce the matrix $S = \text{diag}(2, 1, 1)$, then

$$S^{-1}T_0S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix};$$

thus $\|S^{-1}T_0S\| = 1$ and $M(T_0) \leq \|S\| \|S^{-1}\| = 2$. This shows that $M(T_0) = K(T_0) = 2$.

Let us now consider the case $\varepsilon \neq 0$. We want to show that $M(T_\varepsilon) > 2$; to this end we argue ab absurdo. Otherwise, it would hold $M(T_\varepsilon) = 2$. Then, it would exist a self-adjoint positive matrix $S$ with $\|S\| = 2$, $\|S^{-1}\| = 1$ and $\|S^{-1}T_\varepsilon S\| \leq 1$. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{C}^3$; using the relation

$$2e_1 = T_\varepsilon e_2 = SS^{-1}T_\varepsilon SS^{-1}e_2 = Sy,$$

with $y = S^{-1}T_\varepsilon SS^{-1}e_2$,

we would get $\|y\| \leq \|S^{-1}T_\varepsilon S\| \|S^{-1}\| \|e_2\| \leq 1$, $2 = \|Sy\| \leq \|S\| \|y\| \leq 2$. This would yield $Sy = 2y = 2e_1$, whence $y = e_1$. Now, we use

$$1 = \|y\| = \|(S^{-1}T_\varepsilon S)S^{-1}e_2\| \leq \|S^{-1}e_2\| \leq \|S^{-1}\| \|e_2\| \leq 1,$$

which yields $S^{-1}e_2 = e_2$, whence $Se_2 = e_2$. Combined with $Se_1 = 2e_1$, $S$ self-adjoint positive definite, $\|S\| = 2$, and $\|S^{-1}\| = 1$, this implies $S = \text{diag}(2, 1, \lambda)$ with $1 \leq \lambda \leq 2$. Then, there holds

$$S^{-1}T_\varepsilon S = \begin{pmatrix} 0 & 1 & 0 \\ 2\varepsilon & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \\ 0 & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \end{pmatrix},$$

whence

$$\|S^{-1}T_\varepsilon S\| = \max \left(1, \left\| \begin{pmatrix} 2\varepsilon & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \\ 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \end{pmatrix} \right\| \right) \geq \max \left(1, \left\| \begin{pmatrix} 2\varepsilon & \frac{1-\varepsilon^2}{\sqrt{2}} \\ 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \end{pmatrix} \right\| \right) = \max(1, \sqrt{\mu_\varepsilon}),$$

where $\mu_\varepsilon$ is the largest root of $P_\varepsilon(\mu) := \mu^2 - (1+\varepsilon^2)^2\mu + 2\varepsilon^2(1-\varepsilon^2)^2$. We note that $P(1) = -5\varepsilon^4 + 2\varepsilon^6 < 0$, whence $\mu_\varepsilon > 1$, which contradicts $\|S^{-1}T_\varepsilon S\| \leq 1$.

3 The polynomial bound is 2

From now on, we assume that $|\varepsilon| < 1/2$; then the eigenvalues of $T_\varepsilon$ have modulus $< 1$. In the definition

$$K(T_\varepsilon) = \sup \{\|p(T_\varepsilon)\| : p \text{ polynomial, } |p(z)| \leq 1, \forall z \in \mathbb{D}\},$$

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we can replace \( p \) polynomial by \( p \) holomorphic in \( \mathbb{D} \). Then, it is easily seen that \( K(T_e) \) is attained, and that it is attained by a Blaschke product with (at most) two factors, i.e.,

\[
K(T_e) = \max\{ \| M(\zeta_1, \zeta_2, \varepsilon) \| ; \zeta_1, \zeta_2 \in \mathbb{D} \},
\]

with \( M(\zeta_1, \zeta_2, \varepsilon) \) defined by

\[
M(\zeta_1, \zeta_2, \varepsilon) := (T_\varepsilon - \zeta_1)(1 - \bar{\zeta_1}T_\varepsilon)^{-1}(T_\varepsilon - \zeta_2)(1 - \bar{\zeta_2}T_\varepsilon)^{-1}.
\]

We now denote by \( \mu_1(M) \geq \mu_2(M) \geq \mu_3(M) \) the singular values of the matrix \( M \). We have the following lemma

**Lemma 1.** a) For all \( \zeta_1, \zeta_2 \in \mathbb{D} \), there holds \( \mu_1(M(\zeta_1, \zeta_2, 0)) \leq 2 \). Furthermore, \( \mu_1(M(\zeta_1, \zeta_2, 0)) = 2 \) if and only if \( \zeta_1 \zeta_2 = 0 \) and either \( |\zeta_1 + \zeta_2| = 1 \) or \( \zeta_1 + \zeta_2 = 1/\sqrt{2} \).

b) There exist \( c < 2 \) and \( \varepsilon_0 > 0 \) such that

\[
\zeta_1, \zeta_2 \in \mathbb{D} \text{ and } |\varepsilon| \leq \varepsilon_0 \implies \mu_2(M(\zeta_1, \zeta_2, \varepsilon)) \leq c.
\]

**Proof.** In all the proof, we assume \( \zeta_1, \zeta_2 \in \mathbb{D} \).

a) Since \( K(T_0) = 2 \), there holds \( \mu_1(M(\zeta_1, \zeta_2, 0)) = \| M(\zeta_1, \zeta_2, 0) \| \leq 2 \). Now, if \( \| M(\zeta_1, \zeta_2, 0) \| = 2 \), then there exists \( v \) such that \( \| v \| = 1 \) and \( \| M(\zeta_1, \zeta_2, 0)v \| = 2 \). Recall that, with \( S = \text{diag}(2,1,1) \) and \( M = M(\zeta_1, \zeta_2, 0) \), there holds \( \| S^{-1}T_0S \| \leq 1 \); thus \( \| S^{-1}MS \| \leq 1 \). We set \( y = (S^{-1}MS)S^{-1}v \); then \( \| y \| \leq 1 \) and \( 2 = \| Mv \| = \| Sy \| \leq \| S\| \| y \| \leq 2 \). This yields \( y = e^{i\theta}e_1 \) and \( Mv = 2y \). Changing \( v \) in \( e^{-i\theta} \) if needed, we may assume that \( y = e_1 \). From \( 1 = \| y \| \leq \| S^{-1}v \| \leq \| S^{-1}\| \| v \| \leq 1 \) we infer \( v_1 = 0 \). From \( Mv = 2e_1 \) we infer

\[
(T_0^2 - (\zeta_1 + \zeta_2)T_0 + \zeta_1\zeta_2)v = 2(1 - (\zeta_1 + \zeta_2)T_0 + \zeta_1\zeta_2\bar{T_0})e_1 = 2e_1,
\]

i.e.,

\[
-2(\zeta_1 + \zeta_2)v_2 + \sqrt{2}v_3 = 2,
\]

\[
\zeta_1\zeta_2v_2 - (\zeta_1 + \zeta_2)\frac{v_3}{\sqrt{2}} + \frac{1}{2}v_3 = 0,
\]

\[
\zeta_1\zeta_2v_3 - (\zeta_1 + \zeta_2)\frac{v_3}{\sqrt{2}} + \frac{1}{2}v_3 = 0.
\]

This implies \( \zeta_1\zeta_2 (v_2 - v_3) = 0 \); thus

- either \( \zeta_1\zeta_2 = 0 \), in which case \( v_3(\zeta_1 + \zeta_2 - 1/\sqrt{2}) = 0 \); thus
  - either \( v_3 = 0 \), \( |v_2| = 1 \), \( \zeta_1 + \zeta_2 = -\bar{v}_2 \),
  - or \( \zeta_1 + \zeta_2 = \frac{1}{\sqrt{2}} \), \( v_2 = -\frac{1}{\sqrt{2}} \), \( v_3 = \frac{1}{\sqrt{2}} \),

- or \( v_2 = v_3 \); thus \( |v_2| = |v_3| = \frac{1}{\sqrt{2}} \), \( \zeta_1\zeta_2 - (\zeta_1 + \zeta_2)\frac{1}{\sqrt{2}} + \frac{1}{2} = 0 \), i.e., \( \zeta_1 = \frac{1}{\sqrt{2}} \) or \( \zeta_2 = \frac{1}{\sqrt{2}} \). This yields \( 2 = (\sqrt{2} - 2(\zeta_1 + \zeta_2))v_2 = -2\sqrt{2}\zeta_1\zeta_2v_2 \), which contradicts \( |\zeta_1\zeta_2| \leq \frac{1}{\sqrt{2}} \) and \( |v_2| = \frac{1}{\sqrt{2}} \).

So, we have proved that \( \| M \| < 2 \) except maybe if \( \zeta_1\zeta_2 = 0 \) and either \( \zeta_1 + \zeta_2 = e^{i\theta} \) or \( \zeta_1 + \zeta_2 = 1/\sqrt{2} \). It is easily verified that, if \( \zeta_1\zeta_2 = 0 \) and \( \zeta_1 + \zeta_2 = e^{i\theta} \), then \( M = -e^{i\theta}T_0 \) and \( \| M \| = 2 \); in the other case, if \( \zeta_1\zeta_2 = 0 \) and \( \zeta_1 + \zeta_2 = 1/\sqrt{2} \), then \( M = \begin{pmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), and \( \| M \| = 2 \) still holds.
b) In the two previous cases, there respectively holds \( \mu_2(M) = 1 \) and \( \mu_2(M) = 0 \). For the other values of \( \zeta_1, \zeta_2 \in \mathbb{D} \), there holds \( \mu_2(M(\zeta_1, \zeta_2, 0)) \leq \mu_1(M(\zeta_1, \zeta_2, 0)) < 2 \). The lemma then follows from the continuity of the function \( \mu_2(M(\zeta_1, \zeta_2, \varepsilon)) \) on the compact set \( \mathbb{D} \times \mathbb{D} \times [-1/3, 1/3] \).

**Corollary 2.** We set \( \Phi(\zeta_1, \zeta_2, \varepsilon) := \det(4I - M(\zeta_1, \zeta_2, \varepsilon)M(\zeta_1, \zeta_2, \varepsilon)^*) \). Then, if \( |\varepsilon| \leq \varepsilon_0 \), \( K(T_\varepsilon) \leq 2 \) is equivalent to, for all \( \zeta_1, \zeta_2 \in \mathbb{D} \), \( \Phi(\zeta_1, \zeta_2, \varepsilon) \geq 0 \).

**Proof.** Indeed, \( K(T_\varepsilon) \leq 2 \) is equivalent to “the matrix \( 4I - M(\zeta_1, \zeta_2, \varepsilon)M(\zeta_1, \zeta_2, \varepsilon)^* \) is positive semi-definite for all \( \zeta_1, \zeta_2 \in \mathbb{D} \).” From the previous lemma b), we know that the two largest eigenvalues of this matrix are strictly positive, thus the third eigenvalue will be greater or equal to 0 iff the determinant will be greater or equal to 0.

**Remark.** Clearly, there holds \( \Phi(\zeta_1, \zeta_2, \varepsilon) = \Phi(\zeta_2, \zeta_1, \varepsilon) \).

We admit for a while the following technical results

**Lemma 3.** We have the following estimates

\[
\Phi(\zeta, e^{i\theta}, \varepsilon) = |\zeta|^2 (54 + O(|\zeta|) + O(|\varepsilon|)), \quad (1)
\]

\[
\Phi(0, re^{i\theta}, 0) = 96 (1-r) + O((1-r)^2), \quad (2)
\]

\[
\Phi(\zeta_1, \frac{1}{\sqrt{2}} + \zeta_2, \varepsilon) = 24 |\sqrt{2} \varepsilon + |\zeta_1|^2 + 24 |\zeta_1 + \zeta_2|^2 + 24 |\zeta_2|^2 + O(|\varepsilon|^3 + |\zeta_1|^3 + |\zeta_2|^3). \quad (3)
\]

From (1), we deduce that there exists \( \varepsilon_1 \) satisfying \( 0 < \varepsilon_1 \leq \varepsilon_0 \) such that

\[
"\theta \in \mathbb{R}, |\zeta| \leq \varepsilon_1, \text{ and } |\varepsilon| \leq \varepsilon_1^0 \implies \Phi(\zeta, e^{i\theta}, \varepsilon) \geq 0.
\]

From (2), we infer that \( \frac{\partial}{\partial r} \Phi(0, re^{i\theta}, 0) |_{r = 1} = -96 \). Thus, since \( \Phi \) is a smooth real-valued function, there exists \( \varepsilon_2 \) satisfying \( 0 < \varepsilon_2 \leq \varepsilon_1 \) such that

\[
"\theta \in \mathbb{R}, |\zeta| \leq \varepsilon_2, 0 \leq 1-r \leq \varepsilon_2, \text{ and } |\varepsilon| \leq \varepsilon_2^0 \implies \frac{\partial}{\partial r} \Phi(\zeta, re^{i\theta}, \varepsilon) \leq 0.
\]

Then, we conclude from \( \Phi(\zeta, re^{i\theta}, \varepsilon) = \Phi(\zeta, e^{i\theta}, \varepsilon) - \int_r^1 \frac{\partial}{\partial s} \Phi(\zeta, se^{i\theta}, \varepsilon) ds \) that

\[
"|\zeta_1| \leq \varepsilon_2, 0 \leq 1-|\zeta_2| \leq \varepsilon_2, \text{ and } |\varepsilon| \leq \varepsilon_2^0 \implies \Phi(\zeta_1, \zeta_2, \varepsilon) \geq 0.
\]

Now, we consider \( F(t) = F(t, \zeta_1, \zeta_2, \varepsilon) := \Phi(t\zeta_1, 1/\sqrt{2} + t\zeta_2, t\varepsilon) \). We deduce from (3) that \( F(0) = F'(0) = 0 \) and

\[
F''(0) = 48 |\sqrt{2} \varepsilon + |\zeta_1|^2 + 96 |\bar{\zeta}_1 + \bar{\zeta}_2|^2 + 96 |\zeta_2|^2 \geq 24(|\varepsilon|^3 + |\zeta_1|^3 + |\zeta_2|^3).
\]

From the continuity of \( F''(t) \) (considered as a quadratic form in \( \zeta_1, \zeta_2, \zeta_2, \varepsilon \)) we infer that there exists \( \varepsilon_3 \) satisfying \( 0 < \varepsilon_3 \leq \varepsilon_0 \) such that

\[
"t \in [0, 1], |\zeta_1| \leq \varepsilon_3, |\zeta_2| \leq \varepsilon_3, \text{ and } |\varepsilon| \leq \varepsilon_3^0 \implies F''(t) \geq 0.
\]

Then, we infer from \( \Phi(\zeta_1, 1/\sqrt{2} + \zeta_2, \varepsilon) = \int_0^1 (1-t)F''(t) dt \) that

\[
"|\zeta_1| \leq \varepsilon_3, 1/\sqrt{2} - \zeta_2 | \leq \varepsilon_3, \text{ and } |\varepsilon| \leq \varepsilon_3^0 \implies \Phi(\zeta_1, \zeta_2, \varepsilon) \geq 0.
\]

We have shown that \( \Phi(\zeta_1, \zeta_2, \varepsilon) \geq 0 \) in a neighborhood of the sets \( E_1 = \{(\zeta_1, \zeta_2, 0); \zeta_1 \zeta_2 = 0 \text{ and } |\zeta_1 + \zeta_2| = 1\} \) and \( E_2 = \{(\zeta_1, \zeta_2, 0); \zeta_1 \zeta_2 = 0 \text{ and } \zeta_1 + \zeta_2 = 1/\sqrt{2}\}. \) We also know from Lemma 1 a) that, outside of these sets, \( \Phi(\zeta_1, \zeta_2, 0) > 0 \). Therefore, we can conclude from the uniform continuity of \( \Phi \) that there exists \( \varepsilon_4 \) satisfying \( 0 < \varepsilon_4 \leq \min(\varepsilon_2, \varepsilon_3) \) such that \( \Phi(\zeta_1, \zeta_2, \varepsilon) \geq 0 \) for all \( \zeta_1, \zeta_2 \in \mathbb{D} \) and all \( \varepsilon \) with \( |\varepsilon| \leq \varepsilon_4 \). Summarizing, the next theorem will follow from Lemma 3.

**Theorem 4.** For all \( \varepsilon \) with \( |\varepsilon| \leq \varepsilon_4 \), there holds \( K(T_\varepsilon) = 2 \).
4 Proof of Lemma 3

4.1 Proof of (1)

We first note that \( M(\zeta, e^{i\theta}, \varepsilon) = -e^{i\theta}N_\varepsilon \), with \( N_\varepsilon := (T_\varepsilon - \zeta)(1 - \zeta T_\varepsilon)^{-1} \). Therefore, \( \Phi(\zeta, e^{i\theta}, \varepsilon) = \det(4 - N_\varepsilon N_\varepsilon^*) \). Using the expansion \( N_\varepsilon(\zeta) = T_\varepsilon - \zeta + \zeta T_\varepsilon^* + \zeta^2 T_\varepsilon^3 - |\zeta|^2 T_\varepsilon + O(|\zeta|^3) \), we get

\[
N_\varepsilon(\zeta) = \begin{pmatrix}
-\zeta + 2\varepsilon \zeta & 2(1 - |\zeta|^2) n_{13} & 0 \\
\varepsilon & -\zeta + 2\varepsilon \zeta & n_{23} \\
0 & 0 & n_{33}
\end{pmatrix} + (|\zeta| + |\varepsilon|)O(|\zeta|^2),
\]

(4)

with

\[
n_{13} = \sqrt{2}(1-\varepsilon)^2 \zeta + (1-\varepsilon)^2 \zeta^2 = \sqrt{2}(1-\varepsilon)^2 \zeta + O(|\zeta|^2),
\]

\[
n_{23} = \frac{1-\varepsilon^2}{\sqrt{2}} (1 - |\zeta|^2 + (\zeta + 2\varepsilon \zeta)^2) \frac{1 - \varepsilon^2}{\sqrt{2}} + \frac{\zeta^2 (1-\varepsilon^2)^2}{2} = \frac{1-\varepsilon^2}{\sqrt{2}} + O(|\zeta|),
\]

\[
n_{33} = -\zeta + \frac{1-\varepsilon^2}{\sqrt{2}} (1 - |\zeta|^2 + (\zeta - 2\varepsilon \zeta)^2) \frac{1 - \varepsilon^2}{\sqrt{2}} + \frac{\zeta^2 (1-\varepsilon^2)^2}{2} = \frac{1-\varepsilon^2}{\sqrt{2}} + O(|\zeta|).
\]

This yields

\[
4 - N_\varepsilon N_\varepsilon^* = \begin{pmatrix}
5|\zeta|^2 & \bar{\zeta} & -\bar{\zeta} \\
\zeta & 7/2 & -1/2 \\
-\bar{\zeta} & -1/2 & 7/2
\end{pmatrix} + (|\zeta| + |\varepsilon|) \begin{pmatrix}
O(|\zeta|^2) & O(\zeta) & O(\zeta) \\
O(\zeta) & O(1) & O(1) \\
O(\zeta) & O(1) & O(1)
\end{pmatrix},
\]

whence \( \Phi(\zeta, e^{i\theta}, \varepsilon) = \det(4 - N_\varepsilon N_\varepsilon^*) = 54 |\zeta|^2 + |\zeta|^2 O(|\zeta| + |\varepsilon|) \).

4.2 Proof of (2)

A simple calculation gives

\[
M(0, re^{i\theta}, 0) = \begin{pmatrix}
0 & -2re^\theta & \frac{2-2^2}{\sqrt{2-2re^{-i\theta}}} \\
0 & 0 & \frac{1}{\sqrt{2}} \frac{1-\sqrt{2}re^{-i\theta}}{\sqrt{2-2re^{-i\theta}}} \\
0 & 0 & \frac{1}{\sqrt{2}} \frac{1-\sqrt{2}re^{-i\theta}}{\sqrt{2-2re^{-i\theta}}}
\end{pmatrix},
\]

whence

\[
4 - M(0, re^{i\theta}, 0)M(0, re^{i\theta}, 0)^* = \begin{pmatrix}
4(1-r^2) & 0 & 0 \\
0 & 7/2 & -1/2 \\
0 & -1/2 & 7/2
\end{pmatrix} + \begin{pmatrix}
O((1-r)^2) & O(1-r) & O(1-r) \\
O(1-r) & O(1-r) & O(1-r) \\
O(1-r) & O(1-r) & O(1-r)
\end{pmatrix},
\]

which yields \( \Phi(0, re^{i\theta}, 0) = 96(1-r) + O((1-r)^2) \).
4.3 Proof of (3)

We write $M(\zeta_1, \zeta_2, \varepsilon) = M_\varepsilon N_\varepsilon$, with $M_\varepsilon = (T_\varepsilon - \frac{1}{\sqrt{2}} - \zeta_2)(1 - (\frac{1}{\sqrt{2}} + \bar{\zeta}_2)T_\varepsilon)^{-1}$ and $N_\varepsilon = N_\varepsilon(\zeta_1)$ described in (4). Tedium calculations give, with the notation $|\zeta| = |\zeta_1| + |\zeta_2| + |\varepsilon|,$

$$m_{11} = m_{22} = -\frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} - \zeta_2 + \frac{\varepsilon^2}{\sqrt{2}} - \varepsilon \zeta_2 + O(|\zeta|^3),$$

$$m_{21} = \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} - \frac{\varepsilon}{\sqrt{2}}(\zeta + \bar{\zeta}_2) + O(|\zeta|^3),$$

$$m_{12} = 1 + \varepsilon - \sqrt{2}(\zeta_2 + \bar{\zeta}_2) + \varepsilon^2 - 2 |\zeta_2|^2 - \sqrt{2} \varepsilon (\zeta_2 - \bar{\zeta}_2) + O(|\zeta|^3),$$

$$m_{13} = 1 + \varepsilon - \sqrt{2}(\zeta_2 - \bar{\zeta}_2) - \varepsilon^2 - 6 |\zeta_2|^2 - \sqrt{2} \varepsilon (\zeta_2 - 3 \bar{\zeta}_2) + O(|\zeta|^3),$$

$$m_{23} = \frac{1}{\sqrt{2}} + \frac{\varepsilon}{2} - \zeta_2 - \frac{\varepsilon^2}{2} - \varepsilon (\zeta_2 - \bar{\zeta}_2) - 2 \sqrt{2} |\zeta_2|^2 + O(|\zeta|^3),$$

$$m_{33} = -2 \zeta_2 - \sqrt{2} \varepsilon^2 - 2 \sqrt{2} |\zeta_2|^2 + O(|\zeta|^3),$$

$$m_{31} = m_{32} = 0.$$  

$$M_\varepsilon N_\varepsilon = \begin{pmatrix} \frac{\zeta_1}{\sqrt{2}} + \varepsilon & mn_{12} & \frac{mn_{13}}{\sqrt{2}} \\ -\sqrt{2} \zeta_2 & \varepsilon & -\sqrt{2} \zeta_2 - \frac{\zeta_1}{\sqrt{2}} \\ 0 & 0 & -\sqrt{2} \zeta_2 \\ \end{pmatrix} + \begin{pmatrix} O(|\zeta|^2) & O(|\zeta|^3) & O(|\zeta|^3) \\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2) \\ 0 & 0 & O(|\zeta|^2) \end{pmatrix},$$

with

$$mn_{12} = -\sqrt{2} + \sqrt{2} \varepsilon - \zeta_1 - 2 \zeta_2 + \sqrt{2} \varepsilon^2 - \varepsilon \zeta_1 + 2 \varepsilon \zeta_2 + \sqrt{2} (|\zeta_1|^2 + \zeta_1 \zeta_2 + \zeta_1 \bar{\zeta}_2),$$

$$mn_{13} = \sqrt{2} + \sqrt{2} \varepsilon - \zeta_1 - 2 \zeta_2 + \sqrt{2} \varepsilon^2 - \varepsilon \zeta_1 + 2 \varepsilon \zeta_2 + 4 \varepsilon \bar{\zeta}_2 - \sqrt{2}(|\zeta_1|^2 + 4 |\zeta_2|^2 - \zeta_1 \zeta_2 + \zeta_1 \bar{\zeta}_2 + 2 \zeta_2).$$

$$4 - (M_\varepsilon N_\varepsilon)(M_\varepsilon N_\varepsilon)^* = \begin{pmatrix} x_{11} & 2(\zeta_1 + \bar{\zeta}_2) & 2 \bar{\zeta}_2 \\ 2(\zeta_1 + \bar{\zeta}_2) & 4 & 0 \\ 2 \bar{\zeta}_2 & 0 & 4 \end{pmatrix} + \begin{pmatrix} O(|\zeta|^3) & O(|\zeta|^2) & O(|\zeta|^2) \\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2) \\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2) \end{pmatrix},$$

where

$$x_{11} = 3 \varepsilon^2 + \frac{11}{2} |\zeta_1|^2 + 8 |\zeta_2|^2 + \frac{3}{\sqrt{2}} \varepsilon (\zeta_1 + \bar{\zeta}_1) + 4(\zeta_1 \zeta_2 + \bar{\zeta}_1 \zeta_2).$$

This shows that

$$\Phi(\zeta_1, \frac{1}{\sqrt{2}} + \zeta_2, \varepsilon) = 48 \varepsilon^2 + 72 |\zeta_1|^2 + 96 |\zeta_2|^2 + 24 \sqrt{2} \varepsilon (\zeta_1 + \bar{\zeta}_1) + 48(\zeta_1 \zeta_2 + \bar{\zeta}_1 \zeta_2) + O(|\zeta|^3)$$

$$= 24 |\sqrt{2} \varepsilon + \zeta_1|^2 + 48 |\zeta_1 + \zeta_2|^2 + 48 |\zeta_2|^2 + O(|\varepsilon|^3 + |\zeta_1|^3 + |\zeta_2|^3).$$

References


