Polynomial bounds of 3×3 matrices

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Abstract

We exhibit a class of 3×3 matrices with a polynomial bound different of the completely bounded norm.

1 Introduction

Let us consider an operator T on a Hilbert space $(H, \|\cdot\|)$. Its polynomial bound $K(T)$ is defined by

$$K(T) = \sup\{\|p(T)\|\,; p \text{ polynomial}, \ |p(z)| \leq 1, \ \forall z \in \mathbb{D}\},$$

$\mathbb{D}$ being the closed unit disk of $\mathbb{C}$. If u_T denotes the homomorphism from polynomials to operators mapping p to $p(T)$, then $K(T)$ is the bounded norm of u_T. The corresponding completely bounded norm $M(T)$ has been characterized by Paulsen [6]

$$M(T) = \inf\{\|S\|\,\|S^{-1}\|\,; \|S^{-1}TS\| \leq 1\},$$

the infimum being considered onto all invertible operators, (or equivalently onto all self-adjoint positive definite operators). We use the convention $K(T) = +\infty$ if no S realizes $\|S^{-1}TS\| \leq 1$. Clearly, $M(T) = 1$ iff T is a contraction and, from the von Neumann inequality [8], $K(T) = 1$ iff T is a contraction. Using this inequality together with the relation $p(T) = Sp(S^{-1}TS)S^{-1}$ we deduce $K(T) \leq M(T)$.

A long-standing problem in operator theory is how $K(T)$ differs from $M(T)$. In his famous article [3], Halmos included the following question

Is every polynomially bounded operator similar to a contraction?

The answer *no* has been given by Pisier [7], of course in infinite dimension. For finite dimension the answer is clearly *yes*, but then the relevant question is how $K(T)$ controls $M(T)$? For large dimension, the best result is due to Bourgain [1]

$$M(T) \leq C_0 K(T)^4 \log(\dim H),$$

where C_0 is some universal constant. The case of low dimension has been studied by Holbrook. In a first paper [4], he has showed that $K(T) = M(T)$ if $\dim H = 2$ and that $K(T) < M(T)$ occurs for some T if $\dim H = 12$. Eighteen years later [5], he has exhibited some class of T with $K(T) < M(T)$ and $\dim H = 4$.

This paper is concerned by the case $\dim H = 3$. We show that, for the following class of matrices

$$T_\varepsilon = \begin{pmatrix} 0 & 2 & 0 \\ \varepsilon & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \\ 0 & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \end{pmatrix}, \quad -1/2 \leq \varepsilon \leq 1/2,$$

there holds $M(T_\varepsilon) > 2$, if $\varepsilon \neq 0$, and $K(T_\varepsilon) = 2$, if ε is small enough. Note that, if $|\varepsilon| > 2$, then the spectral radius of T_ε is larger than 1, and thus $K(T_\varepsilon) = +\infty$.

After completion of this paper, John Holbrook mentioned to me that, with Frank Gilfeather, they have obtained existence's evidence of 3×3 matrices T with $K(T) < M(T)$ by numerical experiments; this was consigned in an HPCREC 2000-01 internal report.

2 The completely bounded norm is larger than 2

We first note that, for all ε, there holds $2 \leq \|T_\varepsilon\| \leq K(T_\varepsilon) \leq M(T_\varepsilon)$. If we introduce the matrix $S = \mathrm{diag}(2, 1, 1)$, then

$$S^{-1} T_0 S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix};$$

thus $\|S^{-1} T_0 S\| = 1$ and $M(T_0) \leq \|S\| \, \|S^{-1}\| = 2$. This shows that $M(T_0) = K(T_0) = 2$.

Let us now consider the case $\varepsilon \neq 0$. We want to show that $M(T_\varepsilon) > 2$; to this end we argue ab absurdo. Otherwise, it would hold $M(T_\varepsilon) = 2$. Then, it would exist a self-adjoint positive matrix S with $\|S\| = 2$, $\|S^{-1}\| = 1$ and $\|S^{-1} T_\varepsilon S\| \leq 1$. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{C}^3$; using the relation

$$2e_1 = T_\varepsilon e_2 = S S^{-1} T_\varepsilon S S^{-1} e_2 = Sy, \quad \text{with } y = S^{-1} T_\varepsilon S S^{-1} e_2,$$

we would get $\|y\| \leq \|S^{-1} T_\varepsilon S\| \, \|S^{-1}\| \, \|e_2\| \leq 1$, $2 = \|Sy\| \leq \|S\| \, \|y\| \leq 2$. This would yield $Sy = 2y = 2e_1$, whence $y = e_1$. Now, we use

$$1 = \|y\| = \|(S^{-1} T_\varepsilon S) S^{-1} e_2\| \leq \|S^{-1} e_2\| \leq \|S^{-1}\| \, \|e_2\| \leq 1,$$

which yields $S^{-1} e_2 = e_2$, whence $S e_2 = e_2$. Combined with $S e_1 = 2e_1$, S self-adjoint positive definite, $\|S\| = 2$, and $\|S^{-1}\| = 1$, this implies $S = \mathrm{diag}(2, 1, \lambda)$ with $1 \leq \lambda \leq 2$. Then, there holds

$$S^{-1} T_\varepsilon S = \begin{pmatrix} 0 & 1 & 0 \\ 2\varepsilon & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \\ 0 & 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \end{pmatrix}, \quad \text{whence}$$

$$\|S^{-1} T_\varepsilon S\| = \max\left(1, \left\| \begin{pmatrix} 2\varepsilon & \frac{1-\varepsilon^2}{\sqrt{2}} \lambda \\ 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \end{pmatrix} \right\| \right) \geq \max\left(1, \left\| \begin{pmatrix} 2\varepsilon & \frac{1-\varepsilon^2}{\sqrt{2}} \\ 0 & \frac{1-\varepsilon^2}{\sqrt{2}} \end{pmatrix} \right\| \right) = \max(1, \sqrt{\mu_\varepsilon}),$$

where μ_ε is the largest root of $P_\varepsilon(\mu) := \mu^2 - (1+\varepsilon^2)^2 \mu + 2\varepsilon^2 (1-\varepsilon^2)^2$. We note that $P(1) = -5\varepsilon^4 + 2\varepsilon^6 < 0$, whence $\mu_\varepsilon > 1$, which contradicts $\|S^{-1} T_\varepsilon S\| \leq 1$.

3 The polynomial bound is 2

From now on, we assume that $|\varepsilon| < 1/2$; then the eigenvalues of T_ε have modulus < 1. In the definition

$$K(T_\varepsilon) = \sup\{\|p(T_\varepsilon)\| \, ; p \text{ polynomial}, \ |p(z)| \leq 1, \ \forall z \in \mathbb{D}\},$$

we can replace p polynomial by p holomorphic in $\mathbb{D}$. Then, it is easily seen that $K(T_\varepsilon)$ is attained, and that it is attained by a Blaschke product with (at most) two factors, i.e.,

$$K(T_\varepsilon) = \max\{\|M(\zeta_1, \zeta_2, \varepsilon)\| \,;\, \zeta_1, \zeta_2 \in \mathbb{D}\},$$

with $M(\zeta_1, \zeta_2, \varepsilon)$ defined by

$$M(\zeta_1, \zeta_2, \varepsilon) := (T_\varepsilon - \zeta_1)(1 - \bar\zeta_1 T_\varepsilon)^{-1}(T_\varepsilon - \zeta_2)(1 - \bar\zeta_2 T_\varepsilon)^{-1}.$$

We now denote by $\mu_1(M) \ge \mu_2(M) \ge \mu_3(M)$ the singular values of the matrix M. We have the following lemma

Lemma 1. *a) For all $\zeta_1, \zeta_2 \in \mathbb{D}$, there holds $\mu_1(M(\zeta_1, \zeta_2, 0)) \le 2$. Furthermore, $\mu_1(M(\zeta_1, \zeta_2, 0)) = 2$ if and only if $\zeta_1\zeta_2 = 0$ and either $|\zeta_1 + \zeta_2| = 1$ or $\zeta_1 + \zeta_2 = 1/\sqrt{2}$.*

b) There exist $c < 2$ and $\varepsilon_0 > 0$ such that

$$\zeta_1, \zeta_2 \in \mathbb{D} \text{ and } |\varepsilon| \le \varepsilon_0 \quad \Longrightarrow \quad \mu_2(M(\zeta_1, \zeta_2, \varepsilon)) \le c.$$

Proof. In all the proof, we assume $\zeta_1, \zeta_2 \in \mathbb{D}$.

a) Since $K(T_0) = 2$, there holds $\mu_1(M(\zeta_1, \zeta_2, 0)) = \|M(\zeta_1, \zeta_2, 0)\| \le 2$. Now, if $\|M(\zeta_1, \zeta_2, 0)\| = 2$, then there exists v such that $\|v\| = 1$ and $\|M(\zeta_1, \zeta_2, 0)v\| = 2$. Recall that, with $S = \operatorname{diag}(2, 1, 1)$ and $M = M(\zeta_1, \zeta_2, 0)$, there holds $\|S^{-1}T_0 S\| \le 1$; thus $\|S^{-1}MS\| \le 1$. We set $y = (S^{-1}MS)S^{-1}v$; then $\|y\| \le 1$ and $2 = \|Mv\| = \|Sy\| \le \|S\|\,\|y\| \le 2$. This yields $y = e^{i\theta}e_1$ and $Mv = 2y$. Changing v in $e^{-i\theta}$ if needed, we may assume that $y = e_1$. From $1 = \|y\| \le \|S^{-1}v\| \le \|S^{-1}\|\,\|v\| \le 1$ we infer $v_1 = 0$. From $Mv = 2e_1$ we infer

$$(T_0^2 - (\zeta_1{+}\zeta_2)T_0 + \zeta_1\zeta_2)v = 2\,(1 - (\bar\zeta_1{+}\bar\zeta_2)T_0 + \overline{\zeta_1\zeta_2}T_0^2)e_1 = 2\,e_1,$$

i.e.,

$$\begin{aligned} -2(\zeta_1{+}\zeta_2)\,v_2 + \sqrt{2}\,v_3 &= 2, \\ \zeta_1\zeta_2\,v_2 - (\zeta_1{+}\zeta_2)\frac{v_3}{\sqrt{2}} + \frac{1}{2}v_3 &= 0, \\ \zeta_1\zeta_2\,v_3 - (\zeta_1{+}\zeta_2)\frac{v_3}{\sqrt{2}} + \frac{1}{2}v_3 &= 0. \end{aligned}$$

This implies $\zeta_1\zeta_2\,(v_2{-}v_3) = 0$; thus

- either $\zeta_1\zeta_2 = 0$, in which case $v_3(\zeta_1{+}\zeta_2 - 1/\sqrt{2}) = 0$; thus
  - either $v_3 = 0$, $|v_2| = 1$, $\zeta_1{+}\zeta_2 = -\bar v_2$,
  - or $\zeta_1{+}\zeta_2 = \frac{1}{\sqrt{2}}$, $v_2 = -\frac{1}{\sqrt{2}}$, $v_3 = \frac{1}{\sqrt{2}}$.
- or $v_2 = v_3$; thus $|v_2| = |v_3| = \frac{1}{\sqrt{2}}$, $\zeta_1\zeta_2 - (\zeta_1{+}\zeta_2)\frac{1}{\sqrt{2}} + \frac{1}{2} = 0$, i.e., $\zeta_1 = \frac{1}{\sqrt{2}}$ or $\zeta_2 = \frac{1}{\sqrt{2}}$. This yields $2 = (\sqrt{2} - 2(\zeta_1{+}\zeta_2))v_2 = -2\sqrt{2}\zeta_1\zeta_2 v_2$, which contradicts $|\zeta_1\zeta_2| \le \frac{1}{\sqrt{2}}$ and $|v_2| = \frac{1}{\sqrt{2}}$.

So, we have proved that $\|M\| < 2$ except maybe if $\zeta_1\zeta_2 = 0$ and either $\zeta_1{+}\zeta_2 = e^{i\theta}$ or $\zeta_1{+}\zeta_2 = 1/\sqrt{2}$. It is easily verified that, if $\zeta_1\zeta_2 = 0$ and $\zeta_1{+}\zeta_2 = e^{i\theta}$, then $M = -e^{i\theta}T_0$ and $\|M\| = 2$; in the other case, if $\zeta_1\zeta_2 = 0$ and $\zeta_1{+}\zeta_2 = 1/\sqrt{2}$, then $M = \begin{pmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\|M\| = 2$ still holds.

b) In the two previous cases, there respectively holds $\mu_2(M) = 1$ and $\mu_2(M) = 0$. For the other values of $\zeta_1, \zeta_2 \in \mathbb{D}$, there holds $\mu_2(M(\zeta_1,\zeta_2,0)) \le \mu_1(M(\zeta_1,\zeta_2,0)) < 2$. The lemma then follows from the continuity of the function $\mu_2(M(\zeta_1,\zeta_2,\varepsilon))$ on the compact set $\mathbb{D}\times\mathbb{D}\times[-1/3,1/3]$. □

Corollary 2. *We set* $\Phi(\zeta_1,\zeta_2,\varepsilon) := \det(4I - M(\zeta_1,\zeta_2,\varepsilon)M(\zeta_1,\zeta_2,\varepsilon)^*)$. *Then, if* $|\varepsilon| \le \varepsilon_0$, $K(T_\varepsilon) \le 2$ *is equivalent to, for all* $\zeta_1,\zeta_2 \in \mathbb{D}$, $\Phi(\zeta_1,\zeta_2,\varepsilon) \ge 0$.

Proof. Indeed, $K(T_\varepsilon) \le 2$ is equivalent to "the matrix $4I - M(\zeta_1,\zeta_2,\varepsilon)M(\zeta_1,\zeta_2,\varepsilon)^*$ is positive semi-definite for all $\zeta_1,\zeta_2 \in \mathbb{D}$". From the previous lemma b), we know that the two largest eigenvalues of this matrix are strictly positive, thus the third eigenvalue will be greater or equal to 0 iff the determinant will be greater or equal to 0. □

Remark. Clearly, there holds $\Phi(\zeta_1,\zeta_2,\varepsilon) = \Phi(\zeta_2,\zeta_1,\varepsilon)$.

We admit for a while the following technical results

Lemma 3. *We have the following estimates*

$$\Phi(\zeta, e^{i\theta}, \varepsilon) = |\zeta|^2(54 + O(|\zeta|) + O(|\varepsilon|)), \tag{1}$$

$$\Phi(0, re^{i\theta}, 0) = 96\,(1-r) + O((1-r)^2), \tag{2}$$

$$\Phi(\zeta_1, \tfrac{1}{\sqrt{2}} + \zeta_2, \varepsilon) = 24\,|\sqrt{2}\,\varepsilon+\zeta_1|^2 + 48\,|\zeta_1+\zeta_2|^2 + 48\,|\zeta_2|^2 + O(|\varepsilon|^3+|\zeta_1|^3+|\zeta_2|^3). \tag{3}$$

From (1), we deduce that there exists ε_1 satisfying $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$\text{“}\theta \in \mathbb{R}, |\zeta| \le \varepsilon_1, \text{ and } |\varepsilon| \le \varepsilon_1\text{”} \implies \Phi(\zeta, e^{i\theta}, \varepsilon) \ge 0.$$

From (2), we infer that $\frac{\partial}{\partial r}\Phi(0, re^{i\theta}, 0)|_{r=1} = -96$. Thus, since Φ is a smooth real-valued function, there exists ε_2 satisfying $0 < \varepsilon_2 \le \varepsilon_1$ such that

$$\text{“}\theta \in \mathbb{R}, |\zeta| \le \varepsilon_2, 0 \le 1-r \le \varepsilon_2, \text{ and } |\varepsilon| \le \varepsilon_2\text{”} \implies \frac{\partial}{\partial r}\Phi(\zeta, re^{i\theta}, \varepsilon) \le 0.$$

Then, we conclude from $\Phi(\zeta, re^{i\theta}, \varepsilon) = \Phi(\zeta, e^{i\theta}, \varepsilon) - \int_r^1 \frac{\partial}{\partial s}\Phi(\zeta, se^{i\theta}, \varepsilon)\,ds$ that

$$\text{“}|\zeta_1| \le \varepsilon_2,\ 0 \le 1-|\zeta_2| \le \varepsilon_2, \text{ and } |\varepsilon| \le \varepsilon_2\text{”} \implies \Phi(\zeta_1,\zeta_2,\varepsilon) \ge 0.$$

Now, we consider $F(t) = F(t,\zeta_1,\zeta_2,\varepsilon) := \Phi(t\zeta_1, 1/\sqrt{2}+t\zeta_2, t\varepsilon)$. We deduce from (3) that $F(0) = F'(0) = 0$ and

$$F''(0) = 48\,|\sqrt{2}\,\varepsilon+\zeta_1|^2 + 96\,|\zeta_1+\zeta_2|^2 + 96\,|\zeta_2|^2 \ge 24(|\varepsilon|^2+|\zeta_1|^2+|\zeta_2|^2).$$

From the continuity of $F''(t)$ (considered as a quadratic form in ζ_1, $\bar\zeta_1$, ζ_2, $\bar\zeta_2$, ε) we infer that there exists ε_3 satisfying $0 < \varepsilon_3 \le \varepsilon_0$ such that

$$\text{“}t \in [0,1], |\zeta_1| \le \varepsilon_3,\ |\zeta_2| \le \varepsilon_3, \text{ and } |\varepsilon| \le \varepsilon_3\text{”} \implies F''(t) \ge 0.$$

Then, we infer from $\Phi(\zeta_1, 1/\sqrt{2}+\zeta_2, \varepsilon) = \int_0^1 (1-t)F''(t)\,dt$ that

$$\text{“}|\zeta_1| \le \varepsilon_3,\ |1/\sqrt{2}-\zeta_2| \le \varepsilon_3, \text{ and } |\varepsilon| \le \varepsilon_3\text{”} \implies \Phi(\zeta_1,\zeta_2,\varepsilon) \ge 0.$$

We have shown that $\Phi(\zeta_1,\zeta_2,\varepsilon) \ge 0$ in a neighborhood of the sets $E_1 = \{(\zeta_1,\zeta_2,0)\,;\zeta_1\zeta_2 = 0$ and $|\zeta_1+\zeta_2| = 1\}$ and $E_2 = \{(\zeta_1,\zeta_2,0)\,;\zeta_1\zeta_2 = 0$ and $\zeta_1+\zeta_2 = 1/\sqrt{2}\}$. We also know from Lemma 1 a) that, outside of these sets, $\Phi(\zeta_1,\zeta_2,0) > 0$. Therefore, we can conclude from the uniform continuity of Φ that there exists ε_4 satisfying $0 < \varepsilon_4 \le \min(\varepsilon_2,\varepsilon_3)$ such that $\Phi(\zeta_1,\zeta_2,\varepsilon) \ge 0$ for all $\zeta_1,\zeta_2 \in \mathbb{D}$ and all ε with $|\varepsilon| \le \varepsilon_4$. Summarizing, the next theorem will follow from Lemma 3.

Theorem 4. *For all* ε *with* $|\varepsilon| \le \varepsilon_4$, *there holds* $K(T_\varepsilon) = 2$.

4 Proof of Lemma 3

4.1 Proof of (1)

We first note that $M(\zeta, e^{i\theta}, \varepsilon) = -e^{i\theta} N_\varepsilon$, with $N_\varepsilon := (T_\varepsilon - \zeta)(1 - \bar\zeta T_\varepsilon)^{-1}$. Therefore, $\Phi(\zeta, e^{i\theta}, \varepsilon) = \det(4 - N_\varepsilon N_\varepsilon^*)$. Using the expansion $N_\varepsilon(\zeta) = T_\varepsilon - \zeta + \bar\zeta T_\varepsilon^2 + \bar\zeta^2 T_\varepsilon^3 - |\zeta|^2 T_\varepsilon + O(|\zeta|^3)$, we get

$$N_\varepsilon(\zeta) = \begin{pmatrix} -\zeta + 2\varepsilon\bar\zeta & 2(1-|\zeta|^2) & n_{13} \\ \varepsilon & -\zeta + 2\varepsilon\bar\zeta & n_{23} \\ 0 & 0 & n_{33} \end{pmatrix} + (|\zeta| + |\varepsilon|) O(|\zeta|^2), \tag{4}$$

with

$$\begin{aligned} n_{13} &= \sqrt{2}(1-\varepsilon^2)\bar\zeta + (1-\varepsilon^2)^2\bar\zeta^2 = \sqrt{2}(1-\varepsilon^2)\bar\zeta + O(|\zeta|^2), \\ n_{23} &= \frac{1-\varepsilon^2}{\sqrt{2}}\big(1 - |\zeta|^2 + (\bar\zeta + 2\varepsilon\bar\zeta^2)\frac{1-\varepsilon^2}{\sqrt{2}} + \bar\zeta^2\frac{(1-\varepsilon^2)^2}{2}\big) = \frac{1-\varepsilon^2}{\sqrt{2}} + O(|\zeta|), \\ n_{33} &= -\zeta + \frac{1-\varepsilon^2}{\sqrt{2}}\big(1 - |\zeta|^2 + \bar\zeta\frac{1-\varepsilon^2}{\sqrt{2}} + \bar\zeta^2\frac{(1-\varepsilon^2)^2}{2}\big) = \frac{1-\varepsilon^2}{\sqrt{2}} + O(|\zeta|). \end{aligned}$$

This yields

$$4 - N_\varepsilon N_\varepsilon^* = \begin{pmatrix} 5|\zeta|^2 & \bar\zeta & -\bar\zeta \\ \zeta & 7/2 & -1/2 \\ -\zeta & -1/2 & 7/2 \end{pmatrix} + (|\zeta| + |\varepsilon|) \begin{pmatrix} O(|\zeta|^2) & O(\zeta) & O(\zeta) \\ O(\zeta) & O(1) & O(1) \\ O(\zeta) & O(1) & O(1) \end{pmatrix},$$

whence $\Phi(\zeta, e^{i\theta}, \varepsilon) = \det(4 - N_\varepsilon N_\varepsilon^*) = 54\,|\zeta|^2 + |\zeta|^2 O(|\zeta| + |\varepsilon|)$.

4.2 Proof of (2)

A simple calculation gives

$$M(0, re^{i\theta}, 0) = \begin{pmatrix} 0 & -2re^{\theta} & \frac{2-2r^2}{\sqrt{2}-re^{-i\theta}} \\ 0 & 0 & \frac{1}{\sqrt{2}}\frac{1-\sqrt{2}e^{i\tau}}{\sqrt{2}-re^{-i\theta}} \\ 0 & 0 & \frac{1}{\sqrt{2}}\frac{1-\sqrt{2}e^{i\tau}}{\sqrt{2}-re^{-i\theta}} \end{pmatrix},$$

whence

$$4 - M(0, re^{i\theta}, 0)M(0, re^{i\theta}, 0)^* = \begin{pmatrix} 4(1-r^2) & 0 & 0 \\ 0 & 7/2 & -1/2 \\ 0 & -1/2 & 7/2 \end{pmatrix} + \begin{pmatrix} O((1-r)^2) & O(1-r) & O(1-r) \\ O(1-r) & O(1-r) & O(1-r) \\ O(1-r) & O(1-r) & O(1-r) \end{pmatrix},$$

which yields $\Phi(0, re^{i\theta}, 0) = 96(1-r) + O((1-r)^2)$.

4.3 Proof of (3)

We write $M(\zeta_1,\frac{1}{\sqrt2}{+}\zeta_2,\varepsilon)=M_\varepsilon N_\varepsilon$, with $M_\varepsilon=(T_\varepsilon{-}\frac{1}{\sqrt2}{-}\zeta_2)(1{-}(\frac{1}{\sqrt2}{+}\bar\zeta_2)T_\varepsilon)^{-1}$ and $N_\varepsilon=N_\varepsilon(\zeta_1)$ described in (4). Tedious calculations give, with the notation $|\zeta|=|\zeta_1|{+}|\zeta_2|{+}|\varepsilon|$,

$$
\begin{aligned}
m_{11}&=m_{22}=-\frac{1}{\sqrt2}+\frac{\varepsilon}{\sqrt2}-\zeta_2+\frac{\varepsilon^2}{\sqrt2}-\varepsilon\,\zeta_2+O(|\zeta|^3),\\
m_{21}&=\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2}-\frac{\varepsilon}{\sqrt2}\,(\zeta_2{+}\bar\zeta_2)+O(|\zeta|^3),\\
m_{12}&=1+\varepsilon-\sqrt2\,(\zeta_2{+}\bar\zeta_2)+\varepsilon^2-2\,|\zeta_2|^2-\sqrt2\,\varepsilon\,(\zeta_2{-}\bar\zeta_2)+O(|\zeta|^3),\\
m_{13}&=1+\varepsilon-\sqrt2\,(\zeta_2{-}\bar\zeta_2)-\varepsilon^2-6\,|\zeta_2|^2-\sqrt2\,\varepsilon\,(\zeta_2{-}3\,\bar\zeta_2)+O(|\zeta|^3),\\
m_{23}&=\frac{1}{\sqrt2}+\frac{\varepsilon}{\sqrt2}-\zeta_2-\frac{\varepsilon^2}{\sqrt2}-\varepsilon\,(\zeta_2{-}\bar\zeta_2)-2\sqrt2\,|\zeta_2|^2+O(|\zeta|^3),\\
m_{33}&=-2\zeta_2-\sqrt2\,\varepsilon^2-2\sqrt2\,|\zeta_2|^2+O(|\zeta|^3),\\
m_{31}&=m_{32}=0.
\end{aligned}
$$

$$
M_\varepsilon N_\varepsilon=\begin{pmatrix}\frac{\zeta_1}{\sqrt2}+\varepsilon & mn_{12} & mn_{13}\\ -\frac{\varepsilon}{\sqrt2} & \varepsilon+\frac{\zeta_1}{\sqrt2} & \varepsilon-\sqrt2\zeta_2-\frac{\zeta_1}{\sqrt2}\\ 0 & 0 & -\sqrt2\zeta_2\end{pmatrix}+\begin{pmatrix}O(|\zeta|^2) & O(|\zeta|^3) & O(|\zeta|^3)\\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2)\\ 0 & 0 & O(|\zeta|^2)\end{pmatrix}.
$$

with

$$
\begin{aligned}
mn_{12}&=-\sqrt2{+}\sqrt2\,\varepsilon{-}\zeta_1{-}2\,\zeta_2{+}\sqrt2\,\varepsilon^2{-}\varepsilon\,\zeta_1{+}2\,\varepsilon\,\bar\zeta_1{-}2\,\varepsilon\,\zeta_2{+}\sqrt2\,(|\zeta_1|^2{+}\zeta_1\zeta_2{+}\zeta_1\bar\zeta_2),\\
mn_{13}&=\sqrt2{+}\sqrt2\,\varepsilon-\zeta_1{-}2\,\zeta_2{+}-\sqrt2\,\varepsilon^2{-}\varepsilon\,\zeta_1{+}2\,\varepsilon\,\bar\zeta_1{-}2\,\varepsilon\,\zeta_2+4\,\varepsilon\,\bar\zeta_2\\
&\qquad -\sqrt2\,(|\zeta_1|^2{+}4|\zeta_2|^2{-}\zeta_1\zeta_2{+}\zeta_1\bar\zeta_2+2\,\bar\zeta_1\zeta_2).
\end{aligned}
$$

$$
4-(M_\varepsilon N_\varepsilon)(M_\varepsilon N_\varepsilon)^*=\begin{pmatrix}x_{11} & 2(\bar\zeta_1{+}\bar\zeta_2) & 2\bar\zeta_2\\ 2(\zeta_1{+}\zeta_2) & 4 & 0\\ 2\zeta_2 & 0 & 4\end{pmatrix}+\begin{pmatrix}O(|\zeta|^3) & O(|\zeta|^2) & O(|\zeta|^2)\\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2)\\ O(|\zeta|^2) & O(|\zeta|^2) & O(|\zeta|^2)\end{pmatrix},
$$

where

$$
x_{11}=3\varepsilon^2+\tfrac{11}{2}|\zeta_1|^2+8|\zeta_2|^2+\tfrac{3}{\sqrt2}\varepsilon\,(\zeta_1+\bar\zeta_1)+4(\zeta_1\bar\zeta_2{+}\bar\zeta_1\zeta_2).
$$

This shows that

$$
\begin{aligned}
\Phi(\zeta_1,\tfrac{1}{\sqrt2}+\zeta_2,\varepsilon)&=48\varepsilon^2+72\,|\zeta_1|^2+96|\zeta_2|^2+24\sqrt2\,\varepsilon\,(\zeta_1+\bar\zeta_1)+48(\zeta_1\bar\zeta_2{+}\bar\zeta_1\zeta_2)+O(|\zeta|^3)\\
&=24\,|\sqrt2\,\varepsilon{+}\zeta_1|^2+48\,|\zeta_1{+}\zeta_2|^2+48\,|\zeta_2|^2+O(|\varepsilon|^3{+}|\zeta_1|^3{+}|\zeta_2|^3).
\end{aligned}
$$

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