

K -spectral sets and intersections of disks of the Riemann sphere

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Abstract

We prove that if two closed disks X_1 and X_2 of the Riemann sphere are spectral sets for a bounded linear operator A on a Hilbert space, then $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$ -spectral set for A . When the intersection $X_1 \cap X_2$ is an annulus, this result gives a positive answer to a question of A.L. Shields (1974).

1 Introduction and the statement of the main results.

Let X be a closed set in the complex plane and let $R(X)$ denote the algebra of bounded rational functions on X , viewed as a subalgebra of $C(\partial X)$ with the supremum norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} = \sup\{|f(x)| : x \in \partial X\}.$$

Here ∂X denotes the boundary of the set X .

1.1 Spectral and complete spectral sets.

Let $A \in \mathcal{L}(H)$ be a bounded linear operator acting on a complex Hilbert space H . For a fixed constant $K > 0$, the set X is said to be a K -spectral set for A if the spectrum $\sigma(A)$ of A is included in X and the inequality $\|f(A)\| \leq K\|f\|_X$ holds for every $f \in R(X)$. Notice that, for a rational function $f = p/q \in R(X)$, the poles of f are outside of X , and the operator $f(A)$ is naturally defined as $f(A) = p(A)q(A)^{-1}$ or, equivalently, by the Riesz holomorphic functional calculus. The set X is a spectral set for A if it is a K -spectral set with $K = 1$. Thus X is spectral for A if and only if $\|\rho\| \leq 1$, where $\rho : R(X) \mapsto \mathcal{L}(H)$ is the homomorphism given by $\rho(f) = f(A)$.

We let $M_n(R(X))$ denote the algebra of n by n matrices with entries from $R(X)$. If we let the n by n matrices have the operator norm that they inherit as linear transformations on the n -dimensional Hilbert space \mathbb{C}^n , then we can endow $M_n(R(X))$ with the norm

$$\|(f_{ij})\|_X = \sup\{\|(f_{ij}(x))\| : x \in X\} = \sup\{\|(f_{ij}(x))\| : x \in \partial X\}.$$

In a similar fashion we endow $M_n(\mathcal{L}(H))$ with the norm it inherits by regarding an element (A_{ij}) in $M_n(\mathcal{L}(H))$ as an operator acting on the direct sum of n copies of H . For a fixed constant $K > 0$, the set X is said to be a complete K -spectral set for A if $\sigma(A) \subset X$ and the inequality $\|(f_{ij}(A))\| \leq K\|(f_{ij})\|_X$ holds for every matrix $(f_{ij}) \in M_n(R(X))$ and every n . In terms of the complete bounded norm ([14]) of the homomorphism ρ , this means that $\|\rho\|_{cb} \leq K$. A complete spectral set is a complete K -spectral set with $K = 1$.

Spectral sets were introduced and studied by J. von Neumann [12] in 1951. In the same paper von Neumann proved that a closed disk $\{z \in \mathbb{C} : |z - \alpha| \leq r\}$ is a spectral set for A if and only if $\|A - \alpha I\| \leq r$. Also [12], the closed set $\{z \in \mathbb{C} : |z - \alpha| \geq r\}$ is spectral for $A \in \mathcal{L}(H)$ if and only if $\|(A - \alpha I)^{-1}\| \leq r^{-1}$. We refer to two books [4, 14] for a survey of known properties of spectral and complete spectral sets.

1.2 The annulus as a K -spectral set

Let r and R be two positive constants with $r < R$. Let $A \in \mathcal{L}(H)$ be an invertible operator such that $\|A\| \leq R$ and $\|A^{-1}\| \leq 1/r$. Then $X_1 = \{z \in \mathbb{C} : |z| \leq R\}$ and $X_2 = \{z \in \mathbb{C} : |z| \geq r\}$ are spectral sets for A . The annulus

$$X(r, R) = \{z \in \mathbb{C} : r \leq |z| \leq R\} = X_1 \cap X_2$$

is not necessarily spectral for a given invertible operator A . Examples can be found in [21, 11, 13]. Given an invertible operator A with $\|A\| \leq R$ and $\|A^{-1}\| \leq 1/r$, Shields proved in [17] that $X(r, R)$ is a K -spectral set for A with $K = 2 + ((R+r)/(R-r))^{1/2}$. The following questions were asked by Shields (see [17, Question 7]):

Question 1.1. Find the best constant $K(r, R)$, i.e., the smallest constant C such that $X(r, R)$ is a C -spectral set for all invertible $A \in \mathcal{L}(H)$ with $\|A\| \leq R$ and $\|A^{-1}\| \leq r^{-1}$.

Question 1.2. Fixing (for instance) R , is this best constant bounded (as a function of r) ?

In analogy with Question 1.1, we will denote by $K_{cb}(r, R)$ the smallest constant C such that $X(r, R)$ is a complete C -spectral set. The same proof of Shields (see also [7, 14]) shows that in fact $K_{cb}(r, R) \leq 2 + ((R+r)/(R-r))^{1/2}$.

1.3 Statement of the main results.

The aim of the present note is to study the intersection of two closed disks of the Riemann sphere which are spectral sets for a Hilbert space bounded linear operator. In the case of the annulus we give an estimate for $K(r, R)$ (a partial answer to Question 1.1) which allows to give a positive answer to Question 1.2.

We describe now the main results of this paper. By possibly multiplying the operator by a scalar, we see that $K(r, R) = K(\sqrt{r/R}, \sqrt{R/r})$. This allows to assume, without any loss of generality, that $r = R^{-1}$. We have the following result.

Theorem 1.3. *Let $R > 1$, $X = X(R^{-1}, R) = \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$, and denote by $K(R) = K(R^{-1}, R)$ (and $K_{cb}(R) = K_{cb}(R^{-1}, R)$, respectively), the smallest constant C such that X is a C -spectral set (and a complete C -spectral set, respectively) for any invertible $A \in \mathcal{L}(H)$ verifying $\|A\| \leq R$ and $\|A^{-1}\| \leq R$. Then*

$$\begin{aligned} \frac{2}{1+R^{-2}} &< K(R) \leq K_{cb}(R) \\ &\leq 2 + \min \left(\sqrt{\frac{R^2+2R+1}{R^2+R+1}}, \sqrt{\frac{R^2+1}{R^2-1}} \right) \leq 2 + \frac{2}{\sqrt{3}} < 3.2. \end{aligned}$$

In particular $K(R)$ and $K_{cb}(R)$ are bounded functions of R . We obtain the following consequence about normal dilations.

Corollary 1.4. *Let $R > 1$. Let $A \in \mathcal{L}(H)$ be an invertible operator verifying $\|A\| \leq R$ and $\|A^{-1}\| \leq R$. Let $X = \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$. Then there exist an invertible operator $L \in \mathcal{L}(H)$ with $\|L\| \cdot \|L^{-1}\| \leq 2 + 2/\sqrt{3}$, a larger Hilbert space $\mathcal{H} \supset H$ and an invertible normal operator $N \in \mathcal{L}(\mathcal{H})$ with $\sigma(N) \subset \partial X$ such that*

$$L^{-1}f(A)L = P_H f(N) |_H \quad (f \in R(X)).$$

Here P_H is the orthogonal projection of \mathcal{H} onto H .

Besides the annulus, (complete) K -spectral sets which are intersections of spectral disks of the complex plane have been considered in [19, 20, 10, 5, 3] ; we refer to [3] for a discussion of the best possible constant K . In the second part of our paper we consider the more general case of intersection of two closed disks X_1 and X_2 of the Riemann sphere. We prove the following result.

Theorem 1.5. *Let X_1 and X_2 be two closed disks of the Riemann sphere. If X_1 and X_2 are spectral sets for a bounded operator A in a Hilbert space, then $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$ -spectral set for A .*

This theorem extends previously known results concerning the intersection of two disks in \mathbb{C} to not necessarily convex or simply connected $X_1 \cap X_2$. Note that the case of finitely connected compact sets has been studied in [7, 14], however, without a uniform control on the constant K .

Note also that, if we consider two distinct bounded, convex and closed subsets X_1 and X_2 of the complex plane, and if we assume that X_1 and X_2 are spectral sets for A , then $X_1 \cap X_2$ is a complete 11.08-spectral set for A . Indeed, the fact that X_j is a spectral set for A implies that the numerical range $W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}$ is included in X_j , $j = 1, 2$, and according to [6] the closure of the numerical range $W(A)$ is a complete 11.08-spectral set for A . However, the result from [6] does not imply a solution of Shields' Question 1.2. We refer also to [15, 2, 6] for some normal dilation results for the numerical range, in the spirit of Corollary 1.4.

The remainder of the paper is organized as follows: we first show in §2 that Theorem 1.3 together with some results from [5, 3] implies Theorem 1.5. Our proof of Theorem 1.3 is based on a representation formula for $f(A)$ established in §3. Finally, the proofs of Theorem 1.3 and Corollary 1.4 are provided in §4.

2 Proof of Theorem 1.5 using Theorem 1.3

Let X_1 and X_2 be two closed disks of the Riemann sphere, which are spectral sets for a bounded linear operator A in a Hilbert space. Here six different situations have to be considered, see Figure 1.

Case 1: $X_1 \cap X_2 = \{\lambda\}$ is a singleton. Then we have $A = \lambda I$ and $X_1 \cap X_2$ clearly is a complete spectral set for A .

Case 2: $X_1 \cap X_2$ is a circle or a straight line. Then A is a normal operator with spectrum $\sigma(A)$ contained in $X_1 \cap X_2$. This yields that $X_1 \cap X_2$ is a complete spectral set for A .

Case 3: $X_1 \cap X_2$ is a convex sector or a strip of the complex plane. In this case, both X_1 and X_2 are half-planes, and a closed half-plane Π is a spectral set for A if and only if the numerical range $W(A)$ is a subset of Π . Thus $W(A) \subset X_1 \cap X_2$. It follows from [5] that $X_1 \cap X_2$ is a complete K -spectral set, with $K \leq 2 + 2/\sqrt{3}$.

Case 4: $\partial X_1 \cap \partial X_2 = \{\lambda_1, \lambda_2\}$ is a set consisting of two distinct points of \mathbb{C} . Here $X_1 \cap X_2$ is lens-shaped. If it is in addition convex, then from [3] we know that $X_1 \cap X_2$ is a complete K -spectral set, with $K \leq 2 + 2/\sqrt{3}$. The proof for not convex lenses is the same, we repeat here the main idea for the sake of completeness. Let us first assume that $\lambda_1 \notin \sigma(A)$ and set $B = \varphi(A)$ with $\varphi(z) = (\lambda_1 - z)^{-1}$ and $Y_j = \varphi(X_j)$, $j = 1, 2$. Then both Y_j are closed half-planes. The von Neumann inequality for disks shows that Y_j are spectral sets for B , see also [16, § 154, Lemma 2]. It follows from the previous case that $Y_1 \cap Y_2$ is a complete K -spectral set for B and thus $X_1 \cap X_2$ is a complete K -spectral set for A , with the same constant K . Finally, if $\lambda_1 \in \sigma(A)$, we can replace the disk X_1 of the Riemann sphere, of radius R_1 , by a concentric disk $X'_1 \supset X_1$, of radius $R_1 \pm \varepsilon$. Then, for $\varepsilon > 0$ small enough, $\partial X'_1 \cap \partial X_2 = \{\lambda'_1, \lambda'_2\}$ is still a set with two distinct points of \mathbb{C} , the set X'_1 is a spectral set for A and $\lambda'_1 \notin \sigma(A)$. We conclude that $X_1 \cap X_2$ is a complete K -spectral set for A by letting $\varepsilon \rightarrow 0$.

Case 5: $\partial X_1 \cap \partial X_2 = \emptyset$, but $X_1 \cap X_2$ is not a strip. For the special case $X_1 \cap X_2 = \{z \in \mathbb{C}; R^{-1} \leq |z| \leq R\}$, $R > 1$, Theorem 1.3 implies that $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$ -spectral set for A . In the general case, we may find $R > 1$ and a linear fractional transformation φ such that $\varphi(X_1) = \{z \in \mathbb{C}; |z| \leq R\}$ and $\varphi(X_2) = \{z \in \mathbb{C}; |z| \geq R^{-1}\}$. Then, setting $B = \varphi(A)$ and

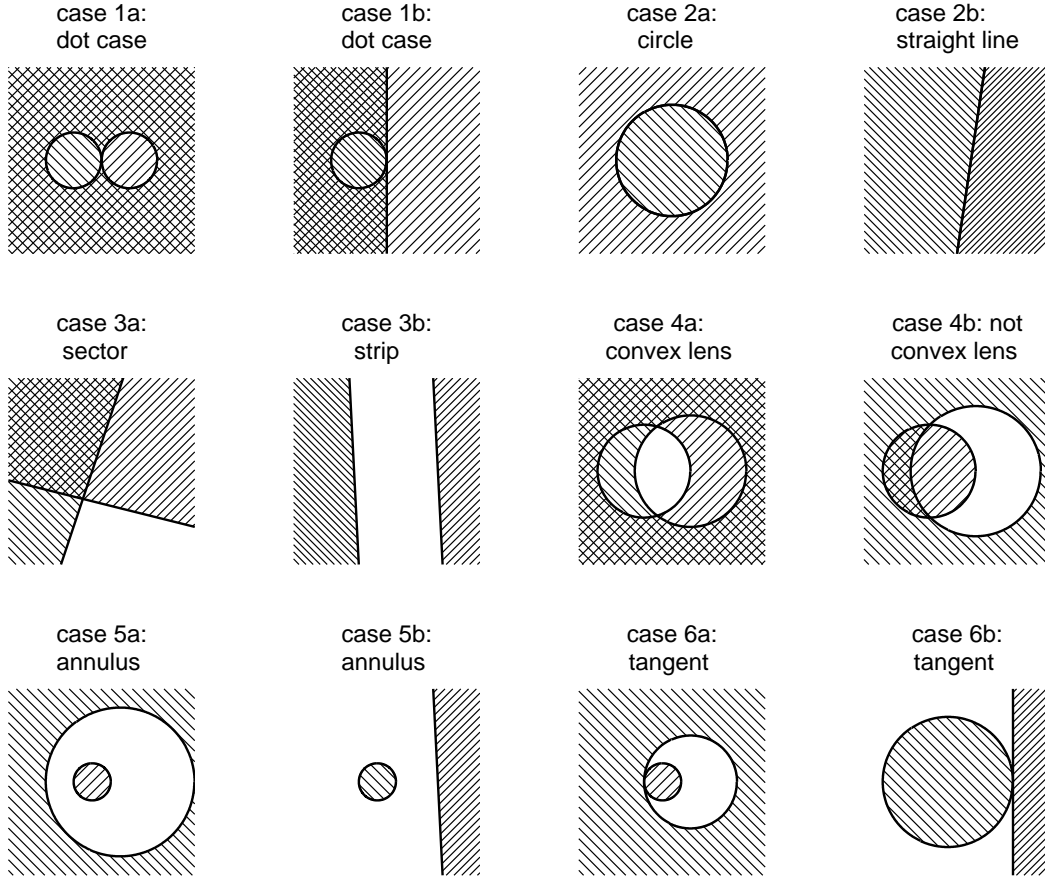


Figure 1: *The six different cases occurring by considering intersections of closed disks on the Riemann sphere.*

$Y_j = \varphi(X_j)$, $j = 1, 2$, we have that Y_j is a spectral set for B , see also [16, § 154, Lemma 2]. Thus $\{z \in \mathbb{C}; R^{-1} \leq |z| \leq R\} = \varphi(X_1 \cap X_2)$ is a complete $(2 + 2/\sqrt{3})$ -spectral set for B , which is equivalent to $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$ -spectral set for A .

Case 6: $\partial X_1 \cap \partial X_2 = \{\lambda\}$ is reduced to a single point, but $X_1 \cap X_2$ is neither a singleton, nor a sector nor a strip. In this case at least one of the sets X_j , $j = 1, 2$, is the interior or the exterior of a disk and the boundaries of the sets X_j are tangent in one point. We can replace the disk, say X_1 , of radius R_1 , by a concentric disk $X'_1 \supset X_1$, of radius $R_1 \pm \varepsilon$. Then, for $\varepsilon > 0$ small enough, $\partial X'_1 \cap \partial X_2 = \emptyset$, and we obtain from the previous case that $X_1 \cap X_2$ is a complete K -spectral set for A by letting $\varepsilon \rightarrow 0$.

3 A decomposition lemma for annuli

In order to give a proof of the upper bound of Theorem 1.3 we need the following representation formula for $f(A)$.

Lemma 3.1. *Let $A \in \mathcal{L}(H)$ be an operator satisfying $\|A\| < R$ and $\|A^{-1}\| < R$. We set $r = 1/R$ and denote by X the annulus $X = X(R^{-1}, R) = \{z \in \mathbb{C}; r \leq |z| \leq R\}$. For any bounded rational function f on X , we have the representation formula*

$$f(A) = \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) d\theta + \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) d\theta + \int_0^{2\pi} f(e^{i\theta}) M(\theta, A^*)^{-1} d\theta,$$

where

$$\begin{aligned}\mu(\theta, A) &= \frac{1}{4\pi} \left((1+e^{-i\theta}rA)(1-e^{-i\theta}rA)^{-1} + (1+e^{i\theta}rA^*)(1-e^{i\theta}rA^*)^{-1} \right), \quad \text{and} \\ M(\theta, A^*) &= \frac{2\pi}{R^2 - r^2} (R^2 + r^2 - (e^{i\theta}A^*)^{-1} - e^{i\theta}A^*).\end{aligned}$$

Proof. We get from the Cauchy formula

$$f(A) = \frac{1}{2\pi i} \int_{\partial X} f(\sigma) ((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}) + \frac{1}{2\pi i} \int_{\partial X} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} = F_1 + F_2.$$

Let us set $\Gamma_\rho = \{\rho e^{i\theta}; \theta \in [0, 2\pi]\}$. The part Γ_R of ∂X is counterclockwise oriented and, with $\sigma = Re^{i\theta}$, we have

$$\begin{aligned}\frac{1}{2\pi i} ((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}) &= \frac{1}{2\pi} ((Re^{i\theta} - A)^{-1} Re^{i\theta} + (Re^{-i\theta} - A^*)^{-1} Re^{-i\theta}) d\theta \\ &= \frac{1}{2\pi} ((1 - e^{-i\theta}rA)^{-1} + (1 - e^{i\theta}rA^*)^{-1}) d\theta \\ &= \frac{1}{2\pi} d\theta + \mu(\theta, A) d\theta.\end{aligned}$$

The other component Γ_r is clockwise oriented and, with $\sigma = re^{i\theta}$, we have

$$\begin{aligned}\frac{1}{2\pi i} ((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}) &= \frac{1}{2\pi} ((re^{i\theta} - A)^{-1} re^{i\theta} + (re^{-i\theta} - A^*)^{-1} re^{-i\theta}) d\theta \\ &= \frac{1}{2\pi} d\theta - \mu(-\theta, A^{-1}) d\theta.\end{aligned}$$

Noticing that $\int_0^{2\pi} f(Re^{i\theta}) d\theta = \int_0^{2\pi} f(re^{i\theta}) d\theta$, we obtain that

$$F_1 = \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) d\theta + \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) d\theta.$$

We consider now the second term F_2 . On the component Γ_R we have $\bar{\sigma} = R^2/\sigma$, and thus

$$\begin{aligned}\frac{1}{2\pi i} \int_{\Gamma_R} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} &= -\frac{1}{2\pi i} \int_{\Gamma_R} f(\sigma) (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma} d\sigma \\ &= -\frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma} d\sigma.\end{aligned}$$

Indeed, the last integrand is holomorphic in σ . Hence we can replace the integration path Γ_R by Γ_1 (counterclockwise oriented). We similarly have for the second component

$$\frac{1}{2\pi i} \int_{\Gamma_r} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} = \frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) (r^2 - \sigma A^*)^{-1} \frac{r^2}{\sigma} d\sigma$$

by taking into account the opposite orientation of Γ_r . Therefore

$$\begin{aligned}F_2 &= \frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) ((r^2 - \sigma A^*)^{-1} \frac{r^2}{\sigma} - (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma}) d\sigma \\ &= \int_0^{2\pi} f(e^{i\theta}) M(\theta, A^*)^{-1} d\theta,\end{aligned}$$

which completes the proof of the lemma. □

4 The complete bound in an annulus

We keep the notation from the previous section. The following lemma shows that $\operatorname{Re} M(\theta, A^*)$ is a positive operator.

Lemma 4.1. *Assume that $\|A\| < R$ and $\|A^{-1}\| < R$. Let $r = R^{-1}$. Then we have the lower bound*

$$\operatorname{Re} M(\theta, A^*) \geq N(\theta) := \frac{2\pi}{R^2 - r^2} \left((R^2 + r^2 - R - r) + \frac{R + r + 2}{4} (2 - e^{i\theta} U^* - e^{-i\theta} U) \right),$$

where U denotes the unitary operator such that $A = UG$, with G self-adjoint positive definite. Also, $N(\theta)$ is a positive invertible operator.

Proof. We have

$$\begin{aligned} \frac{R^2 - r^2}{2\pi} \operatorname{Re} M(\theta, A^*) &= R^2 + r^2 - \operatorname{Re}((e^{-i\theta} A)^{-1} + e^{i\theta} A^*) \\ &= R^2 + r^2 - \operatorname{Re}(e^{i\theta}(G^{-1} + G)U^*) \\ &= R^2 + r^2 - \frac{R+r+2}{2} \operatorname{Re}(e^{i\theta}U^*) - \operatorname{Re}(e^{i\theta}(G^{-1} + G - \frac{R+r+2}{2})U^*) \end{aligned}$$

We note that the assumptions $\|A\| \leq R$ and $\|A^{-1}\| \leq R$ are equivalent to $\|G\| \leq R$ and $\|G^{-1}\| \leq R$. Since G is self-adjoint, this means that $r \leq G \leq R$, and hence

$$\|G^{-1} + G - \frac{R+r+2}{2}\| \leq \sup_{r \leq x \leq R} |x^{-1} + x - \frac{R+r+2}{2}| = \frac{R+r-2}{2}.$$

It follows that

$$\begin{aligned} \frac{R^2 - r^2}{2\pi} \operatorname{Re} M(\theta, A^*) &\geq R^2 + r^2 - \frac{R+r+2}{2} \operatorname{Re}(e^{i\theta}U^*) - \frac{R+r-2}{2} \\ &= R^2 + r^2 - R - r + \frac{R+r+2}{2} \operatorname{Re}(1 - e^{i\theta}U^*), \end{aligned}$$

which completes the proof of the lemma. \square

Proof of the upper bound of Theorem 1.3. We can suppose that $\|A\| < R$ and $\|A^{-1}\| < R$. Using the notation of Lemma 3.1, it follows from the condition $\|A\| < R$ that $\mu(\theta, A) \geq 0$ for all $\theta \in \mathbb{R}$. Therefore we have

$$\left\| \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) d\theta \right\| \leq \left\| \int_0^{2\pi} \mu(\theta, A) d\theta \right\| \|f\|_X = \|f\|_X.$$

Here we have used that $\int_0^{2\pi} \mu(\theta, A) d\theta = 1$, which follows from the residue formula. Similarly we have $\mu(-\theta, A^{-1}) \geq 0$ and we get the estimate

$$\left\| \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) d\theta \right\| \leq \|f\|_X.$$

Using Lemma 3.1 and the positivity of $\operatorname{Re} M(\theta, A^*)$ for all $\theta \in \mathbb{R}$ (Lemma 4.1) we obtain the estimate

$$\|f(A)\| \leq K \|f\|_X, \quad \text{with} \quad K = 2 + \left\| \int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1} d\theta \right\|.$$

Let $\rho : R(X) \mapsto \mathcal{L}(H)$ be the homomorphism given by $\rho(f) = f(A)$. Therefore the norm of ρ is bounded by K . Furthermore, since we only have used arguments based on positivity of operators, it is easily seen that the complete bounded norm $\|\rho\|_{cb}$ is also bounded by K .

Taking into account the bound of Shields [17], for establishing the upper bound of Theorem 1.3 it suffices now to show that

$$\left\| \int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1} d\theta \right\| \leq \sqrt{\frac{R^2 + 2R + 1}{R^2 + R + 1}} \leq \frac{2}{\sqrt{3}}. \quad (1)$$

Consider the function

$$J(z) := \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \left((R^2 + r^2 - R - r) + \frac{R + r + 2}{4} (2 - e^{i\theta} z^{-1} - e^{-i\theta} z) \right)^{-1} d\theta.$$

Since U is a unitary operator, it follows from Lemma 4.1 that

$$\left\| \int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1} d\theta \right\| \leq \left\| \int_0^{2\pi} (N(\theta))^{-1} d\theta \right\| = \|J(U)\| = \sup \left\{ |J(e^{i\phi})| : e^{i\phi} \in \sigma(U) \right\}.$$

On the other hand, we have

$$\begin{aligned} J(e^{i\varphi}) &= \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{1}{(R^2 + r^2 - R - r) + \frac{R+r+2}{4}(2 - 2\cos(\theta - \varphi))} d\theta \\ &= \frac{R^2 - r^2}{2\pi} \int_{-\infty}^{\infty} \frac{2}{(R^2 + r^2 - R - r)(1 + s^2) + (R + r + 2)s^2} ds \\ &= \frac{R^2 - r^2}{2\pi} \int_{-\infty}^{\infty} \frac{2}{(R^2 + r^2 - R - r) + (R^2 + r^2 + 2)s^2} ds \\ &= \sqrt{\frac{R^2 + 2R + 1}{R^2 + R + 1}} = \sqrt{\frac{1}{1 - \frac{1}{(\sqrt{R+1}/\sqrt{R})^2}}} \leq \frac{2}{\sqrt{3}}, \end{aligned}$$

which implies (1). This gives a proof of the upper bound of Theorem 1.3 for $K_{cb}(R)$. \square

Proof of the lower bound of Theorem 1.3. For $t \in \mathbb{C}$, let $A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with inverse $A(t)^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ acting on the Hilbert space \mathbb{C}^2 . For $t_0 = R - R^{-1}$ we have $\|A(t_0)\| = \|A(t_0)^{-1}\| = R$ (compare with [14, p. 152]). We will make use of the following result from geometric function theory about the infinitesimal Carathéodory metric: it is shown by Simha in [18, Example (5.3)] that

$$\sup \left\{ \frac{|f'(1)|}{\|f\|_X} : f \text{ analytic in } X \text{ and } f(1) = 0 \right\} = \frac{2}{R} \prod_{n=1}^{\infty} \left(\frac{1 - R^{-8n}}{1 - R^{4-8n}} \right)^2,$$

with the supremum being attained for some function f_0 analytic in X , with $\|f_0\|_X = 1$ and $f_0(1) = 0$. Therefore

$$K(R) \geq \frac{1}{\|f_0\|_X} \|f_0(A(t_0))\| = \left\| \begin{pmatrix} f_0(1) & t_0 f_0'(1) \\ 0 & f_0(1) \end{pmatrix} \right\| = t_0 |f_0'(1)| = \gamma(R)$$

with

$$\begin{aligned} \gamma(R) &:= 2(1 - R^{-2}) \prod_{n=1}^{\infty} \left(\frac{1 - R^{-8n}}{1 - R^{4-8n}} \right)^2 = \frac{2}{1 + R^{-2}} \prod_{n=1}^{\infty} \frac{(R^{4n} - R^{-4n})^2}{(R^{4n} - R^{4-4n})(R^{4n} - R^{-4-4n})} \\ &= \frac{2}{1 + R^{-2}} \prod_{n=1}^{\infty} \left(1 - \frac{(R^2 - R^{-2})^2}{(R^{4n} - R^{-4n})^2} \right)^{-1}. \end{aligned}$$

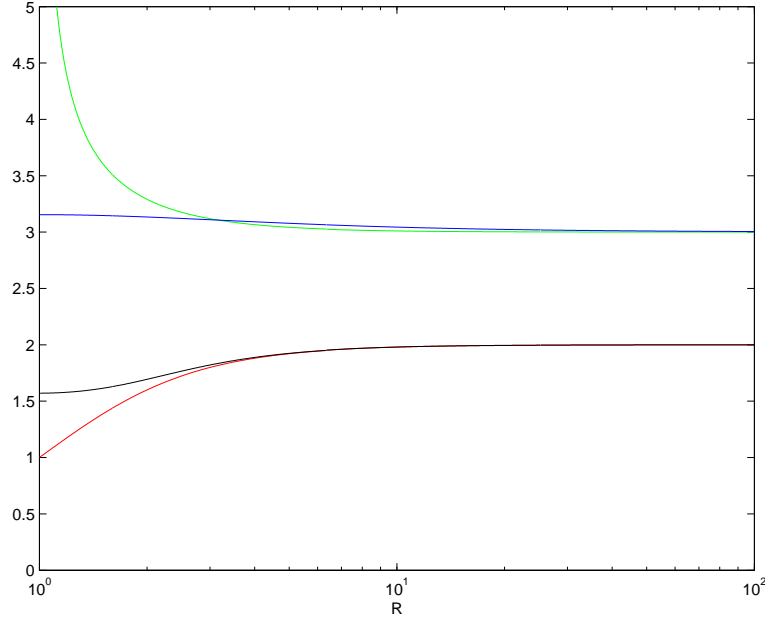


Figure 2: The two upper bounds and the lower bound for $K(R)$ from Theorem 1.3, and the lower bound $\gamma(R)$ from the proof of Theorem 1.3.

This yields the estimate

$$K(R) > \frac{2}{1 + R^{-2}}, \quad (2)$$

as claimed in Theorem 1.3. It remains to justify why we are allowed to take for a lower bound of $K(R)$ the function f_0 which is not a rational function. Indeed, by using instead of f_0 partial sums of the Laurent expansion of an extremal function for the infinitesimal Carathéodory metric on the annulus $1/R' < |z| < R'$ for some $R' > R$ we obtain the same conclusion after taking the limit $R' \rightarrow R$. \square

Remark 4.2. The final estimate (2) of the preceding proof is not very sharp for R close to one (see Figure 2), and $\gamma(R)$ is a sharper but less readable lower bound for $K(R)$. For instance, for $R \rightarrow 1$ the lower bound $2/(1 + R^{-2})$ of Theorem 1.3 tends to 1 but

$$\lim_{R \rightarrow 1} \gamma(R) = \lim_{R \rightarrow 1} \prod_{n=1}^{\infty} \left(1 - \frac{(R^2 - R^{-2})^2}{(R^{4n} - R^{-4n})^2}\right)^{-1} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)^{-1} = \frac{\pi}{2}.$$

In contrast, for our fixed matrix $A(t_0)$, it follows from [9, Theorem 1] and [18] that the function f_0 is extremal within the class of functions analytic in X .

Proof of Corollary 1.4. We use the terminology of Paulsen's book [14]. Let $\rho : R(X) \mapsto \mathcal{L}(H)$ be the homomorphism given by $\rho(f) = f(A)$. Theorem 1.3 implies that the complete bounded norm $\|\rho\|_{cb}$ of ρ is bounded by $2 + 2/\sqrt{3}$. Using a theorem of Paulsen [14, Theorem 9.1], there exists an invertible operator L with $\|L\| \cdot \|L^{-1}\| = \|\rho\|_{cb} \leq 2 + 2/\sqrt{3}$ such that $L^{-1}\rho(\cdot)L$ is a unital completely contractive homomorphism. Thus X is a complete spectral set for $L^{-1}AL$. Therefore, as a consequence of Arveson's extension theorem (see [14, Corollary 7.8]), $L^{-1}AL$ has a normal dilation with spectrum included in ∂X , as claimed in Corollary 1.4. \square

Remark 4.3. According to a deep result due to Agler [1], if X is a spectral set for A , then X is a complete spectral set for A , and thus A has a normal dilation with spectrum included in ∂X . The analogue of Agler's theorem is not true for triply connected domains (see [8]).

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