THE ANNULUS AS A K-SPECTRAL SET

Pour Michel Pierre, à l’occasion de son soixantième anniversaire

MICHEL CROUZEIX

Institut de Recherche Mathématique de Rennes, UMR no. 6625,
Université de Rennes 1, Campus de Beaulieu, 35042 RENNES Cedex, France

ABSTRACT. We consider the annulus $A_R$ of complex numbers with modulus and inverse
of modulus bounded by $R > 1$. We present some situations, in which this annulus is a
K-spectral set for an operator $A$, and some related estimates.

1. Introduction. Let us consider the annulus $A_R := \{ z \in \mathbb{C}; R^{-1} \leq |z| \leq R \}$ with
$R > 1$; $A_R$ is the intersection of two disks of the Riemann sphere $A_R = D_1 \cap D_2$, with
$D_1 := \{ z \in \mathbb{C}; |z| \leq R \}$ and $D_2 := \{ z \in \mathbb{C} \cup \{ \infty \}; |z|^{-1} \leq R \}$. Let $A \in B(H)$ be a
bounded operator acting on a complex Hilbert space $H$. The aim of this paper is to present
some assumptions on the pairs $(D_1, A)$ and $(D_2, A)$, ensuring that the annulus $A_R$ is a
(complete) $K$-spectral set for $A$.

Recall that, for a fix constant $K \geq 1$, a closed subset $X$ of the complex plane which
contains the spectrum $\sigma(A)$ is called a $K$-spectral set for $A$ if the inequality
\[
\| f(A) \| \leq K \| f \|_X,
\]
holds for all bounded rational functions $f$ (from $\mathbb{C}$ into $\mathbb{C}$) on $X$. Furthermore, if $K = 1$,
the set $X$ is said to be a spectral set for $A$, [5]. We also consider rational functions
$F = (f_{ij})$ on $X$ with values in the set $M_d(\mathbb{C})$ of complex $d \times d$ matrices; then
$F(A) = (f_{ij}(A))$ becomes a linear operator on $H^d$. The set $X$ is said to be a complete
$K$-spectral for $A$ if the inequality
\[
\| F(A) \| \leq K \| F \|_X,
\]
holds for all bounded rational functions $F$ on $X$ with values in $M_d(\mathbb{C})$, and for all values
of $d$. In the case $K = 1$, the set $X$ is said to be completely spectral for $A$.

There exists a best constant $C(R)$ (resp. $C_{cb}(R)$) such that each bounded rational function
$f$ on $A_R$, with values in $\mathbb{C}$ (resp. in $M_d(\mathbb{C})$), may be written as $f = f_1 + f_2$ (resp.
$F = F_1 + F_2$), with
\[
\| f_1 \|_{D_1} \leq C(R) \| f \|_{A_R} \quad \text{and} \quad \| f_2 \|_{D_2} \leq C(R) \| f \|_{A_R}
\]
(resp. $\| F_1 \|_{D_1} \leq C_{cb}(R) \| F \|_{A_R}$ and $\| F_2 \|_{D_2} \leq C_{cb}(R) \| F \|_{A_R}$).

It has been noticed, for instance in [4, 6, 7], that, if $D_1$ is a $K_1$-spectral set for $A$ and
if $D_2$ is a $K_2$-spectral for the same operator $A$, then $A_R$ is a $K$-spectral set for $A$, with
$K \leq C(R)(K_1 + K_2)$. Similarly, if $D_1$ is a complete $K_1$-spectral set for $A$ and if $D_2$
is a complete $K_2$-spectral set for $A$, then $A_R$ is a complete $K$-spectral for $A$, with $K \leq C_{cb}(R)(K_1 + K_2)$. In Section 2, we obtain some estimates of $C(R)$ and of $C_{cb}(R)$ that improve the ones given in [9] and in [8]. In particular we show that $C(R) = C_{cb}(R) = 1.5$ if $R \geq 2.3919$, and $\lim_{R \to 1} C(R) = \lim_{R \to 1} C_{cb}(R) = +\infty$. We do not know whether $C(R) = C_{cb}(R)$ for all $R > 1$.

The previous result is not fully satisfactory, in particular for $R$ closed to 1. Indeed, there exist situations in which the previous estimates may be strongly improved. For instance, it is shown in [2, Theorem 1.2] that, if $D_1$ is a spectral set for $A$ and $D_2$ is a spectral set for $A$ (or equivalently if $\|A\| \leq R$ and $\|A^{-1}\| \leq R$), then $A_R$ is a complete $K(R)$-spectral set for $A$, with $K(R) \leq 2 + \frac{R + 1}{\sqrt{R^2 + R + 1}}$. In particular we have $K(R) \leq 2 + 2\sqrt{3}$, for all $R$, while the previous estimate $K(R) \leq 2C_{cb}(R)$ blows up as $R \to 1$. In Section 3, we consider the assumptions “$w(A) \leq 2 R$ and $w(A^{-1}) \leq R$”, where $w(A) := \sup \{|\langle Av, v \rangle|; v \in H, \|v\| = 1\}$ is the numerical radius of $A$. We will say that $A_R$ is a numerical annulus for $A$ if these assumptions are satisfied. This situation infer that the sets $D_1$ and $D_2$ are completely 2-spectral for $A$ [1]; therefore, it follows from the previous part that the annulus $A_R$ is completely $K(R)$-spectral for $A$ with $K(R) \leq 4 C_{cb}(R)$. Using a method similar to [2], we show that $K(R) \leq 4 + \frac{R^2 - 1}{\sqrt{(R - 2)(R^2 - 1)^2}}$, for $R > 2$. More generally, if we add to the hypothesis “$A_R$ is a numerical annulus for $A$” the assumptions $\|A\| \leq \tau^2$ and $\|A^{-1}\| \leq \tau^2$, with $\sqrt{R} < \tau < R$, we show the estimate $K(R, \tau) \leq 4 + \frac{1}{\sqrt{1 - \gamma^2}}$, with $\gamma = \frac{R^2 - 1}{2 R^2}$. Note also that this estimate is still valid if $1 < \tau \leq \sqrt{R}$, but in this case the inequalities $\|A\| \leq R$ and $\|A^{-1}\| \leq R$ are satisfied, and then a better estimate $K(R) \leq 2 \sqrt{\frac{R + 1}{R^2 + R + 1}}$ holds.

From the well-known inequalities $w(A) \leq \|A\| \leq 2 w(A)$ and $w(A) w(A^{-1}) \geq 1$, we conclude that there exists a best (i.e. minimal) function $\varphi$ such that the inequality

$$\|A\| \leq w(A) \varphi (\sqrt{w(A) w(A^{-1})})$$

holds for all bounded operators $A$ with bounded inverses. The function $\varphi$ is defined on the interval $[1, +\infty)$ with values in $[1, 2]$. In [10], Stampfli has shown that the equality $w(A) w(A^{-1}) = 1$ holds, if and only if $A = \lambda U$, with $\lambda > 0$ and $U$ is a unitary operator; therefore $\varphi(1) = 1$. In Section 4, we prove the estimates

$$\max(1 + \sqrt{1 - x^2}, 2 - x^{-4}) \leq \varphi(x) \leq \min(1 + c_1 (x - 1)^{1/4}, 2 - c_2 x^{-4})$$

for some positive constants $c_1$ and $c_2$. In particular this shows that, if $w(A) \leq 1 + \varepsilon$ and $w(A^{-1}) \leq 1 + \varepsilon$, then there exists a unitary operator $U$ such that $\|A - U\| \leq c_3 \varepsilon^{1/4}$.

2. Decomposition of bounded rational functions in an annulus. Let $f$ be a bounded rational function in the annulus $A_R$. Then, $f$ may be written as $f = f_1 + f_2$, with rational functions $f_1$ bounded in $D_1$ and $f_2$ bounded in $D_2$. Note that, if $f = \varphi_1 + \varphi_2$ is another decomposition, with $\varphi_1$ and $\varphi_2$ holomorphic in the interior of $D_1$ and in the interior of $D_2$, respectively, $\varphi_2$ being furthermore assumed bounded at infinity, then the function $\varphi_1 - f_1 = f_2 - \varphi_2$ is holomorphic in the interior of $D_1$ and in the interior of $D_2$, thus in all the complex plane; furthermore the function $\varphi_1 - f_1$ is bounded in the unit disk while $f_2 - \varphi_2$ is bounded in the complementary of the unit disk. So, the function $\varphi_1 - f_1 = f_2 - \varphi_2$ is holomorphic and bounded in all the complex plane, therefore it is constant. This shows the uniqueness, up to an additive constant, of the decomposition $f = f_1 + f_2$. 


From now on, we use the notations
\[ \|f\|_{A_R} = \sup_{z \in A_R} |f(z)|, \quad \|f_1\|_{D_1} = \sup_{z \in D_1} |f_1(z)|, \quad \|f_2\|_{D_2} = \sup_{z \in D_2} |f_2(z)|. \]

**Lemma 2.1.** There exists a best constant \( C(R) \) such that all bounded rational functions in \( A_R \) may be written in the form \( f = f_1 + f_2 \), with \( \|f_1\|_{D_1} \leq C(R) \|f\|_{A_R} \) and \( \|f_2\|_{D_2} \leq C(R) \|f\|_{A_R} \).

Furthermore, the following estimates hold
\[
\begin{align*}
(a) & \quad C(R) \leq \max \left( 1.5, 1 + \sum_{n \geq 1} \frac{2}{R^n - 1} \right), \\
(b) & \quad C(R) \leq 1 + \frac{1}{2\pi} \int_0^\pi \left| \frac{R^2 + e^{i\theta}}{R^2 - e^{i\theta}} \right| d\theta, \\
(c) & \quad C(R) \geq 1.5, \\
(d) & \quad C(R) \geq \frac{1}{\pi} \log \frac{1}{R^{-1}}.
\end{align*}
\]

**Proof.** From the Cauchy formula, we may write \( f = f_1 + f_2 \) with
\[
\begin{align*}
f_1(z) &= \frac{1}{2\pi i} \int_{\partial D_1} f(\sigma) \left( \frac{1}{\sigma - z} - \frac{1}{\sigma} \right) d\sigma, \\
f_2(z) &= \frac{1}{2\pi i} \int_{\partial D_2} f(\sigma) \left( \frac{1}{\sigma - z} - \frac{1}{\sigma} \right) d\sigma,
\end{align*}
\]
by using a counterclockwise orientation for \( \partial D_1 \) and a clockwise for \( \partial D_2 \). The functions \( f_1 \) and \( f_2 \) are rational functions bounded in \( D_1 \) and in \( D_2 \), respectively.

a) We consider the Laurent series expansion, \( f(z) = \sum_{n \geq 0} a_n z^n \), then
\[
f_1(z) = \frac{1}{2} a_0 + \sum_{n \geq 1} a_n z^n \quad \text{and} \quad f_2(z) = \frac{1}{2} a_0 + \sum_{n \leq -1} a_n z^n.
\]

Without loss of generality, we may assume that \( \|f\|_{A_R} = 1 \) and \( a_0 \geq 0 \). We note that, for \( R^{-1} \leq r \leq R \),
\[
a_n r^n + \overline{a_n} r^{-n} = \frac{1}{2\pi} \int_0^{2\pi} \left( 1 - f(re^{i\theta}))e^{-ni\theta} \right) d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \left( 1 - f(re^{i\theta}))e^{-ni\theta} \right) d\theta = \frac{1}{\pi} \int_0^{2\pi} e^{-ni\theta} \text{Re} \left( 1 - f(re^{i\theta}) \right) d\theta.
\]
Using the fact that \( \text{Re} \left( 1 - f(re^{i\theta}) \right) \geq 0 \), which follows from \( \|f\|_{A_R} = 1 \), we get
\[
|a_n r^n + \overline{a_n} r^{-n}| \leq \frac{1}{\pi} \int_0^{2\pi} \text{Re} \left( 1 - f(re^{i\theta}) \right) d\theta = 2(1-a_0),
\]
and then, by taking \( r = R \) and \( r = R^{-1} \),
\[
|a_n R^n + \overline{a_n} R^{-n}| \leq 2(1-a_0), \quad |a_n R^{-n} + \overline{a_n} R^n| \leq 2(1-a_0);
\]
thus
\[
|a_n R^n| \leq 2(1-a_0) + |a_{-n}| R^{-n} \leq 2(1-a_0)(1 + R^{-2n}) + |a_n| R^{-3n},
\]
and
\[
|a_n| R^{-n} \leq \frac{2(1-a_0)}{R^{2n} - 1}.
\]

We note that, on the boundary \( \partial D_2 \),
\[
\|f_1\|_{L^\infty(\partial D_2)} \leq \frac{a_0}{2} + \sum_{n \geq 1} |a_n| R^{-n} \leq \frac{a_0}{2} + 2(1-a_0) \sum_{n \geq 1} \frac{1}{R^{2n} - 1};
\]
consequently, since $0 \leq \alpha_0 \leq 1$, we have $\|f_1\|_{L^\infty(\partial D_2)} \leq \max(\frac{1}{2}, \sum_{n \geq 1} \frac{2}{R^{2n} - 1})$. Then, using the maximum principle, we obtain

$$\|f_2\|_{D_2} = \|f_2\|_{L^\infty(\partial D_2)} = \|f - f_1\|_{L^\infty(\partial D_2)} \leq 1 + \max(\frac{1}{2}, \sum_{n \geq 1} \frac{2}{R^{2n} - 1}).$$

The same estimate for $\|f_1\|_{D_1}$ may be proved in a similar way; this infers the inequality (a).

b) For $z = R^{-1}e^{i\varphi} \in \partial D_2$, we have

$$f_1(z) = \frac{1}{2\pi i} \int_{\partial D_1} f(\sigma) \left(\frac{1}{\sigma - z} - \frac{1}{2\sigma}\right) d\sigma = \frac{1}{4\pi} \int_0^{2\pi} f(Re^{i\theta}) \left(Re^{i\theta} + R^{-1}e^{i\varphi}\right) d\theta,$$

It then follows that

$$\|f_1\|_{L^\infty(\partial D_2)} \leq \frac{1}{2\pi} \int_0^{\pi} \frac{|R^2 + e^{i\theta}|}{|R^2 - e^{i\theta}|} d\theta;$$

thus

$$\|f_2\|_{D_2} = \|f - f_1\|_{L^\infty(\partial D_2)} \leq 1 + \frac{1}{2\pi} \int_0^{\pi} \frac{|R^2 + e^{i\theta}|}{|R^2 - e^{i\theta}|} d\theta,$$

which shows the estimate (b).

c) We now consider the function $f = f_1 + f_2$, defined by

$$f_1(z) = \frac{1}{2} + \frac{z}{R - 1 + \varepsilon} \quad 0 < \varepsilon < 1, \quad f_2(z) = f_1(1/z).$$

The image of $D_1$ by $f_1$, as well as the image of $D_2$ by $f_2$, is the disk of radius $1$ centered in $1/2$. This infers

$$\min_{c \in \mathbb{C}} \left(\max\{\|f_1 - c\|_{D_1}, \|f_2 + c\|_{D_2}\}\right) = \|f_1\|_{D_1} = 1.5,$$

and then $1.5 \leq C(R) \|f\|_{A_R}$. Using the symmetry $f(z) = f(1/z)$, we note that

$$\|f\|_{A_R} = \max_{\theta} |f(Re^{i\theta})| \leq \|f_1 - \frac{1}{2}\|_{D_1} + \max_{\theta} |f_2(Re^{i\theta}) + \frac{1}{2}|$$

$$\leq 1 + \max_{\theta} \left|\frac{\varepsilon(1 + R^{-2}e^{-i\theta})}{1 - (1 - \varepsilon)R^{-2}e^{-i\theta}}\right|$$

$$\leq 1 + \frac{\varepsilon(1 + R^{-2})}{1 - (1 - \varepsilon)R^{-2}}.$$
This yields
\[ C(R) \geq \frac{1}{2\|f\|_{A_R}} \log\frac{2 + \varepsilon}{\varepsilon}. \]

From the maximum principle and the symmetries \( f(z) = -f(z^{-1}) \), \( f(\bar{z}) = \overline{f(z)} \), we have
\[\|f\|_{A_R} = \max_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| = \max_{0 \leq \theta \leq \pi} \left| \log \frac{g_1(\theta)}{g_2(-\theta)} \right|,\]
with \( g_1(\theta) = 1 + \varepsilon - e^{i\theta}, \ g_2(\theta) = 1 + \varepsilon - R^{-2}e^{i\theta}. \)

From one hand, for \( 0 \leq \theta \leq \pi \), we have the estimates \( -\frac{\pi}{2} \leq \arg g_1(\theta) \leq 0 \) and \( 0 \leq \arg g_2(-\theta) \leq \frac{\pi}{2} \); thus \( \left| \Im \left( \log \frac{g_1(\theta)}{g_2(-\theta)} \right) \right| \leq \pi \). From the other hand, the quantity
\[ \left| \frac{g_1(\theta)}{g_2(-\theta)} \right|^2 = \frac{(1+\varepsilon)^2 + 2(1+\varepsilon) \cos \theta}{(1+\varepsilon)^2 + R^{-4} - 2R^{-2}(1+\varepsilon) \cos \theta} \]
is an increasing function of \( \theta \) on \([0, \pi]\); this yields
\[ \left| \Re \left( \log \frac{g_1(\theta)}{g_2(-\theta)} \right) \right| \leq \max \left( \log \frac{1-R^{-2} \varepsilon}{\varepsilon}, \log \frac{2 \varepsilon}{1+\varepsilon} \right) = \log \frac{1-R^{-2} \varepsilon}{\varepsilon}. \]
Choosing \( \varepsilon = 1-R^{-2} \), we obtain \( \left| \Re \left( \log \frac{g_1(\theta)}{g_2(-\theta)} \right) \right| \leq \log 2; \) thus \( \|f\|_{A_R} \leq \sqrt{\pi^2 + \log^2 2} \leq 3.5 \), and finally
\[ C(R) \geq \frac{1}{4} \log \frac{3-R^{-2}}{1-R^{-1}} \geq \frac{1}{4} \log \frac{1}{R^{-1}}. \]

**Remark 2.2.** The rational functions \( f \) considered in this lemma take their values in \( C \). But the estimates would be exactly the same for functions with values in \( M_d(\mathbb{C}) \), independently of the value of \( d \). Therefore the bounds for \( C(R) \) given in this lemma are still valid for \( C_{cb}(R) \). It is clear that \( C(R) \leq C_{cb}(R) \), but we do not know whether \( C(R) = C_{cb}(R) \) for all \( R > 1 \).

**Remark 2.3.** In our choice, the functions \( f_1 \) and \( f_2 \) play symmetric roles with respect to the change of variables \( z \to 1/z \). This is not the case for the decomposition considered by Shields [9], which is slightly different. Translated in our context, his estimates would be
\[ C_{cb}(R) \leq 1 + \frac{1}{2} \left( \sqrt{\frac{R^2 + 1}{R^2 - 1}}. \right. \]

The estimate (a) is essentially a variant of one obtained by Paulsen and Singh [8, Theorem 4.2], it improves Shields’ estimate if \( R \geq 2.2227 \). The estimate (b) improves Shields’ estimate for all values of \( R \).

**Remark 2.4.** Choosing the best established estimate in each case, we obtain, with \( \varepsilon \approx 2.753 \times 10^{-5} \),
\[ C(R) = C_{cb}(R) = 1.5, \quad \text{if } R \geq 2.3919, \]
\[ 1.5 \leq C(R) \leq C_{cb}(R) \leq 1 + \frac{1}{2} \sum_{n \geq 1} \frac{2}{R^{2n} - 1}, \quad \text{if } 2.3634 \leq R \leq 2.3919, \]
\[ 1.5 \leq C(R) \leq C_{cb}(R) \leq 1 + \frac{1}{2\pi} \int_0^\pi \left| \frac{R^2 + \varepsilon e^{i\theta}}{R^2 - \varepsilon e^{i\theta}} \right| d\theta, \quad \text{if } 1 + \varepsilon < R \leq 2.3634, \]
\[ \frac{1}{\pi} \log \frac{1}{R-1} \leq C(R) \leq C_{cb}(R) \leq 1 + \frac{1}{2\pi} \int_0^\pi \left| \frac{R^2 + \varepsilon e^{i\theta}}{R^2 - \varepsilon e^{i\theta}} \right| d\theta, \quad \text{if } 1 < R \leq 1 + \varepsilon. \]
Remark 2.5. It is easily verified that
$$\sup\{|R^2 + e^{i\theta} - \frac{2}{R^2 - e^{i\theta}}|; R > 1, 0 \leq \theta \leq \pi\} < +\infty.$$ 

Therefore, in a neighborhood of $R = 1$,
$$1 + \frac{1}{2\pi} \int_{0}^{\pi} \left| \frac{R^2 + e^{i\theta}}{R^2 - e^{i\theta}} \right| d\theta = \frac{1}{2\pi} \int_{0}^{\pi} \left| \frac{2}{2(R - 1) - i\theta} \right| d\theta + O(1) = \frac{1}{\pi} \log \frac{1}{R - 1} + O(1).$$

This shows that the estimates (b) and (d) provide a good control of the behaviour of $C_{ch}(R)$ in this neighborhood.

3. Numerical annulus. In this section, we consider an operator $A$ which satisfies the assumptions $w(A) \leq R$, $w(A^{-1}) \leq R$, and $\max(\|A\|, \|A^{-1}\|) \leq \tau^2$, with $1 < \tau < R$. We will show the estimate
$$\|f(A)\| \leq \left(4 + \frac{1}{\sqrt{1 - \gamma^2}}\right) \|f\|_{A_R}, \quad \text{with} \quad \gamma = \frac{\tau - \tau^{-1}}{R - R^{-1}},$$

for all bounded rational functions $f$ in the annulus $A_R$. 

Proof of (1). It suffices to do it under the hypotheses $w(A) < R$ and $w(A^{-1}) < R$. Then we can write (using the appropriate orientations of $\partial D_1$ and of $\partial D_2$)
$$f(A) = \frac{1}{2\pi i} \int_{\partial D_1} f(\sigma)(\sigma - A)^{-1} d\sigma + \frac{1}{2\pi i} \int_{\partial D_2} f(\sigma)(\sigma - A)^{-1} d\sigma = F_1 + F_2 + F_3,$$

with
$$F_1 = \frac{1}{2\pi i} \int_{\partial D_1} f(\sigma)((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\bar{\sigma})$$
$$F_2 = \frac{1}{2\pi i} \int_{\partial D_2} f(\sigma)((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\bar{\sigma})$$
$$F_3 = \frac{1}{2\pi i} \int_{\partial D_1} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} + \frac{1}{2\pi i} \int_{\partial D_2} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}.$$

Setting $\sigma = Re^{i\theta}$, we note that
$$\frac{1}{2\pi} \int_{\partial D_1} ((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\bar{\sigma}) = \frac{R}{2\pi} \int_{\theta_0}^{\theta_1} \left( (R-e^{i\theta} A)^{-1} + (R-e^{-i\theta} A^*)^{-1} \right) d\theta.$$ 

The assumption $w(A) \leq R$ implies $(R-e^{i\theta} A)^{-1} + (R-e^{-i\theta} A^*)^{-1} \geq 0$. Therefore (see [2, Lemma 2.1])
$$\|F_1\| \leq \frac{1}{2\pi} \int_{\partial D_1} ((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\bar{\sigma}) \|f\|_{A_R} = 2 \|f\|_{A_R}.$$ 

Similarly, from $w(A^{-1}) \leq R$, we get $\|F_2\| \leq 2 \|f\|_{A_R}$. 

It remains to show that $\|F_3\| \leq (1 - \gamma^2)^{-1/2}$. For this, we note that $\sigma = R^2/\sigma$ on $\partial D_1$, while $\sigma = R^{-2}/\sigma$ on $\partial D_2$. Thus
$$F_3 = -\frac{1}{2\pi i} \int_{\partial D_1} f(\sigma) R^2 (R^2 - \sigma A^*)^{-1} \frac{d\sigma}{\sigma} + \frac{1}{2\pi i} \int_{\partial D_2} f(\sigma) R^{-2} (R^{-2} - \sigma A^*)^{-1} \frac{d\sigma}{\sigma}.$$ 

The integrands being holomorphic with respect to $\sigma$ in the annulus $A_R$, we can move the integration paths $\partial D_1$ and $\partial D_2$ into the unit circle. Taking into account the different
orientations of the paths, this gives
\[
F_3 = \frac{1}{2\pi i} \int_{|\sigma|=1} f(\sigma) (R^2 (R^2 - \sigma A^*)^{-1} - R^{-2} (R^{-2} - \sigma A^*)^{-1}) \frac{d\sigma}{\sigma}
\]
\[
= -\frac{R^2 - R^{-2}}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) (M(\theta, A^*)^{-1}) d\theta,
\]
with \( M(\theta, A^*) := R^2 + R^{-2} - e^{i\theta} A^* - (e^{i\theta} A^*)^{-1} \).

We now write \( A^* = UG \), with a unitary operator \( U \) and a positive self-adjoint operator \( G \). The assumptions \( \max(||A||, ||A^{-1}||) \leq \tau \) read \( \tau^{-1} \leq G \leq \tau \). Setting \( \rho = \frac{1}{2}(\tau + \tau^{-1}) \), we have
\[
\|G + G^{-1} - (\rho + 1)I\| \leq \max\{|x + x^{-1} - \rho| - 1; \tau^{-1} \leq x \leq \tau\} = \rho - 1.
\]
This yields, for the self-adjoint part of \( M(\theta, A^*) \),
\[
\Re M(\theta, A^*) = R^2 + R^{-2} - (\rho + 1) \Re(e^{i\theta} U) + \Re(e^{i\theta} U (G + G^{-1} - \rho - 1))
\geq R^2 + R^{-2} - (\rho + 1) \Re(e^{i\theta} U) - \rho + 1 \geq R^2 + R^{-2} - 2\rho > 0.
\]
We then have the estimate (see [2, Lemma 2.2])
\[
\|F_3\| \leq \frac{R^2 - R^{-2}}{2\pi} \int_0^{2\pi} (R^2 + R^{-2} - (\rho + 1) \Re(e^{i\theta} U) - \rho + 1)^{-1} d\theta = \|h(U)\|,
\]
where we have introduced the holomorphic function
\[
h(z) = \frac{R^2 - R^{-2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{R^2 + R^{-2} - \rho + 1 - (\rho + 1)(e^{i\theta} z + e^{-i\theta} z^{-1})/2}.
\]

Note that
\[
h(e^{i\varphi}) = \frac{R^2 - R^{-2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{R^2 + R^{-2} - \rho + 1 - (\rho + 1) \cos(\theta + \varphi)}
= \frac{R^2 - R^{-2}}{2\pi} \frac{2\pi}{(R + R^{-1})\sqrt{R^2 + R^{-2} - 2\rho}} = \frac{1}{\sqrt{1 - \gamma^2}} = h(1).
\]
This shows that \( h(U) = h(1) \) and gives the estimate
\[
\|F_3\| \leq h(1) = \frac{1}{\sqrt{1 - \gamma^2}}.
\]

Now, we only assume \( w(A) \leq R \) and \( w(A^{-1}) \leq R \). In the case \( R \geq 2 \), the inequality \( \max(||A||, ||A^{-1}||) \leq \tau^2 \) is automatically satisfied with \( \tau = \sqrt{2R} \), since \( ||A|| \leq 2w(A) \) and \( ||A^{-1}|| \leq 2w(A^{-1}) \). The inequality (1) provides the existence of the best constant \( K(R) \) such that
\[
\|f(A)\| \leq K(R) \|f\|_{A_R}, \quad \text{with} \quad K(R) \leq 4 + \frac{R^2 - 1}{\sqrt{(R - 2)(R^3 - 1/2)}},
\]
for all bounded rational functions \( f \) in the annulus \( A_R \) and for all operators \( A \) satisfying \( w(A) \leq R \) and \( w(A^{-1}) \leq R \).
**Remark 3.1.** We also have the estimate $K(R) \leq 4C(R)$, since $D_1$ and $D_2$ are 2-spectral sets for $A$. Choosing the best known estimate in each case, we obtain

\[
K(R) \leq 4 + \frac{R^2-1}{\sqrt{(R-2)(R^2-1)}}, \quad \text{if } R \geq 2.43618,
\]
\[
K(R) \leq 6, \quad \text{if } 2.3919 \leq R \leq 2.43618,
\]
\[
K(R) \leq 4 + \sum_{n \geq 1} \frac{8}{R^{2n} - 1}, \quad \text{if } 2.3634 \leq R \leq 2.3919,
\]
\[
K(R) \leq 4 + \frac{2}{\pi} \int_0^\pi \left| \frac{R^2 + e^{i\theta}}{R^2 - e^{i\theta}} \right| d\theta, \quad \text{if } 1 \leq R \leq 2.3634.
\]

**Remark 3.2.** These estimates blows up as $R \to 1$, but we do not know whether the best constant $K(R)$ is bounded as $R \to 1$.

**Remark 3.3.** In this section, we only have considered scalar functions, but all the estimates are still valid, with the same constants, in completely bounded form.

4. **Norm of operators and numerical radius.** From the classical inequalities $w(A) \leq \|A\| \leq 2w(A)$ and $w(A)w(A^{-1}) \geq 1$, it follows that there exists a minimal function $\varphi$ such that the inequality

\[
\|A\| \leq w(A)\varphi(\sqrt{w(A)w(A^{-1})})
\]

holds for all bounded operators $A$ on a Hilbert space $H$ with bounded inverses, and for all Hilbert spaces $H$. The function $\varphi$ is defined on the interval $[1, +\infty)$ with values in $[1, 2]$ and satisfies $\varphi(1) = 1$. In this section, we will show that $\varphi$ is an increasing function that satisfies the following estimates

\[
\varphi(x) \geq 1 + \sqrt{1-x^{-2}}, \quad \forall x \geq 1,
\]
\[
\varphi(x) \geq 2 - x^{-4}, \quad \forall x \geq 1,
\]
\[
\varphi(x) \leq 2 - c_2x^{-4}, \quad \forall x \geq 1, \quad \text{with a constant } c_2, \ 0 < c_2 < 1,
\]
\[
\varphi(x) \leq 1 + c_1(x-1)^{1/4}, \quad \forall x \geq 1, \quad \text{with a constant } c_1 > 0.
\]

**Proof that $\varphi$ is increasing.** Let $A \in B(H)$ be an invertible operator. We set $B = A \oplus \alpha$, with $\alpha = (t^2w(A^{-1}))^{-1}, \ t \geq 1$. Then, we have $0 < \alpha \leq \frac{1}{w(A^{-1})} \leq w(A) \leq \|A\|$; therefore $\|B\| = \|A\|$, $w(B) = w(A)$ and $w(B^{-1}) = t^2w(A^{-1})$. Replacing $A$ by $B$ in inequality (2), we obtain

\[
\|A\| \leq w(A)\varphi(t\sqrt{w(A)w(A^{-1})}), \quad \forall t \geq 1, \forall A \text{ and } A^{-1} \in B(H).
\]

From the minimality of $\varphi$, we deduce $\varphi(t\sqrt{w(A)w(A^{-1})}) \geq \varphi(\sqrt{w(A)w(A^{-1})})$ for all $t \geq 1$. This shows that $\varphi$ is increasing.

**Proof of the lower bound (3).** We use

\[
A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with } y = \sqrt{x^2 - 1}, \ x \geq 1.
\]

Then, we have $w(A) = w(A^{-1}) = x$ and $\|A\| = y + \sqrt{1+y^2} = x + \sqrt{x^2-1}$. We obtain (3) by using the matrix $A$ in (2). \qed
Proof of the lower bound (4). We will show a more precise inequality
\[ \varphi(x) \geq 2 - y, \quad \text{with} \quad y = \frac{4x^4 - x^2 + 1 - \sqrt{(4x^4 - x^2 + 1)^2 - 16x^4}}{4x^4}. \]

The lower bound (4) then follows by noticing that \( 0 < y \leq x^{-4} \). To this end, we take
\[ A = \begin{pmatrix} 0 & 0 & \sqrt{y} \\ 2 - y & 0 & 0 \\ 0 & \sqrt{y} & 0 \end{pmatrix}. \]

Using the formulae
\[ w \begin{pmatrix} 0 & 0 & b \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} = w \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{pmatrix} = a + \sqrt{a^2 + 8b^2} \frac{1}{4}, \]

it is easy to verify that \( \|A\| = 2 - y, \ w(A) = 1, \text{ and } w(A^{-1}) = x^2 \). The inequality \( \varphi(x) \geq 2 - y \) then follows by putting the matrix \( A \) in (2).

Proof of the upper bound (5). It suffices to show that if the operator \( A \) satisfies \( \|A\| = (2 - \varepsilon)w(A) \) with \( 0 < \varepsilon < 1 \), then it holds
\[ w(A)w(A^{-1}) \geq \frac{1}{6\sqrt{5\varepsilon}}. \]

For this, we can assume that \( w(A) = 1 \). Then, there exists a unit normed vector \( e_1 \) such that \( \|Ae_1\| \geq 2\sqrt{1 - \varepsilon} \). Replacing \( A \) by \( e^{i\theta}A \) if needed, we can assume that \( \alpha = \langle Ae_1, e_1 \rangle \geq 0 \). This allows to write \( Ae_1 = \alpha e_1 + \beta e_2, \ Ae_2 = \gamma e_1 + \delta e_2 + \omega e_3 \), with \( \beta \geq 0, \ \gamma \geq 0, \text{ and } e_1, e_2, e_3 \) being three orthonormal vectors in \( H \). We note that
\[ w(A^{-1}) \geq \frac{1}{2} \|A^{-1}\| \geq \frac{1}{2 \|Ae_2\|} = \frac{1}{2 \sqrt{\gamma^2 + \delta^2 + \omega^2}}. \]

Thus, it suffices to show that \( \gamma^2 + \delta^2 + \omega^2 \leq 45 \varepsilon \). Let us now consider the orthogonal projector \( P \) from \( H \) onto the subspace spanned by \( e_1, e_2 \) and \( e_3 \), and let us set \( A' = PA^*P \). Clearly \( 2 - \varepsilon \geq \|A'\| \geq \|A'e_1\| = \alpha^2 + \beta^2 \geq 2\sqrt{1 - \varepsilon} \) and \( w(A') \leq w(A) = 1 \). We identify \( A' \) with its corresponding matrix in the basis \( \{e_1, e_2, e_3\} \),
\[ A' = \begin{pmatrix} \alpha & \gamma & v \\ \beta & \delta & w \\ 0 & u & z \end{pmatrix} = B + C, \quad \text{with} \quad B = \text{Re}(A') = \frac{1}{2}(A' + A'^*), \quad C = \frac{1}{2}(A' - A'^*). \]

The condition \( w(A') \leq 1 \) also reads, for all \( \theta \in \mathbb{R}, \|\text{Re}(e^{i\theta}A')\| \leq 1 \), and, in particular, induces \( \|B\| \leq 1 \) and \( \|C\| \leq 1 \). It follows that
\[ \frac{1}{2} |\beta e^{i\theta} + \delta e^{-i\theta}| = |\langle \text{Re}(e^{i\theta}A')e_1, e_2 \rangle| \leq 1, \]

and then \( \beta + |\gamma| \leq 2 \), by a judicious choice of \( \theta \). We use
\[ 4 \text{Re}(Be_1, Ce_1) = 2\|A'e_1\|^2 - 2\|Be_1\|^2 - 2\|Ce_1\|^2 \geq 8(1 - \varepsilon) - 2 - 2, \]

that reads
\[ \beta^2 - |\gamma|^2 - |v|^2 \geq 4 - 8\varepsilon. \]

We also have
\[ \beta^2 + |\delta|^2 + |w|^2 = \|A'^*e_2\|^2 \leq (2 - \varepsilon)^2, \]

together with the previous inequality, this gives
\[ |\delta|^2 + |w|^2 + |\gamma|^2 + |v|^2 \leq 4\varepsilon + \varepsilon^2. \]
In particular, this shows $|w| \leq \sqrt{4\varepsilon + \varepsilon^2} \leq (2+\varepsilon)\sqrt{\varepsilon}$. Taking now the vectors $x_1^* = (1, 1, t)$ and $x_2^* = (1, -1, t)$, $t \in \mathbb{R}$, in the inequality

$$\text{Re} \left( \frac{1}{2}(A'x_1, x_1) - \frac{1}{2}(A'x_2, x_2) \right) \leq \frac{1}{2}((\|x_1\|^2 + \|x_2\|^2) = 2 + t^2,$$

we get

$$\beta + \text{Re} \gamma + t(u + \text{Re} w)| \leq 2 + t^2, \quad \forall t \in \mathbb{R};$$

thus, choosing $t = \frac{1}{2}(u+\text{Re} w)$ and using the inequalities $\beta+|\gamma| \leq 2$ and $\beta^2 - |\gamma|^2 \geq 4 - 8\varepsilon$,

$$\frac{|u + \text{Re} w|^2}{4} \leq 2 - \beta - \text{Re} \gamma \leq 2 - |\beta^2 - |\gamma|^2| \leq 4\varepsilon.$$

This yields $u \leq |w| + 4\sqrt{\varepsilon}$, and we finally obtain

$$|\gamma|^2 + |\delta|^2 + u^2 \leq |\delta|^2 + |w|^2 + |\gamma|^2 + |v|^2 + u^2 - |w|^2 \leq 4\varepsilon + \varepsilon^2 + u^2 - |w|^2 \leq 4\varepsilon + \varepsilon^2 + 8|w|\sqrt{\varepsilon} + 16 \varepsilon \leq 4\varepsilon + \varepsilon^2 + 16 \varepsilon + 8\varepsilon^2 + 16 \varepsilon \leq 36 \varepsilon + 9\varepsilon^2 \leq 45 \varepsilon.$$

\[\square\]

**Proof of the upper bound (6).** The work of Stampfli [10] has been an inspiration for this proof. We have to show that there exists a constant $c_1$ such that

$$\varphi(1+\varepsilon) \leq 1 + c_1 \varepsilon^{1/4}, \quad \forall \varepsilon > 0.$$

We shall obtain a constant $c_1 > 4$. Since $\varphi(1+\varepsilon) \leq 2$, the inequality will automatically be satisfied for $\varepsilon \geq \frac{1}{256}$. Thus, we only have to consider, from now on, the case $0 < \varepsilon < \frac{1}{256}$. Then, there exists an integer $n \geq 35$ such that

$$\frac{1}{\cos \frac{\pi}{n+1}} < 1 + \varepsilon \leq \frac{1}{\cos \frac{\pi}{n}}.$$

We set $t = \tan \frac{\pi}{n}$, and note that $t = \sqrt{2\varepsilon} + O(\varepsilon^{3/2})$ and $t \leq \frac{1}{17}$. In order to prove (6), it suffices to show that

$$\varphi(1+\varepsilon) \leq 1 + c\sqrt{t} + O(t) \quad \text{in a neighborhood of } t = 0.$$

To this end, we consider an operator $A$ satisfying $w(A) = w(A^{-1}) \leq 1 + \varepsilon$, and write it as $A = BU$, with $B$ self-adjoint positive and $U$ unitary. We introduce a partition of the unit circle in $n$ arcs

$$C_k = \{e^{i\theta}; \theta \in I_k\}, \quad I_k = [(2k-1)\pi/n, (2k+1)\pi/n), \quad k = 1, \ldots, n.$$

We consider the spectral decomposition of $U$ and the orthogonal projector $P_k$ onto the invariant subspace corresponding to the arc $C_k$:

$$U = \int_0^{2\pi} e^{it}dE(t), \quad P_k = E(I_k).$$

We admit, for the being, the following result

**Lemma 4.1.** Let $x \in P_k H$ be a unit element in the invariant subspace corresponding to $C_k$. Let us write $Bx = \lambda x + \beta t w$, with $\|x\| = \|w\| = 1$, $\langle x, w \rangle = 0$ and $\beta \geq 0$. Then, the following estimates hold

$$\frac{1}{1+3t^2} \leq \lambda \leq 1+8t^2, \quad 0 \leq \beta \leq 7.$$
For an arbitrary unit element \( x \in H, \|x\| = 1 \), we write
\[
x = \sum_{0 \leq k < n} \xi_k x_k \quad \text{with} \quad x_k \in P_k H, \quad \|x_k\| = 1, \quad \sum_k |\xi_k|^2 = 1.
\]
It follows from the lemma that \( Bx_k = \lambda_k x_k + \beta_k t w_k \), with \( \|w_k\| = 1, 0 < \lambda_k \leq 1 + 8 t^2 \) and \( 0 \leq \beta_k \leq 7 \). Thus,
\[
Bx = \sum_k \xi_k \lambda_k x_k + t \sum_k \xi_k \beta_k w_k.
\]
Using the orthonormality of the elements \( \{x_k\} \) and the Cauchy-Schwarz inequality, we get
\[
\|Bx\| \leq (\sum_k \lambda_k^2|\xi_k|^2)^{1/2} + t (\sum_k |\xi_k|^2)^{1/2} (\sum_k |\beta_k|^2)^{1/2} \leq 1 + 8 t^2 + 7 t \sqrt{n}.
\]
This shows that \( \|A\| = \|B\| \leq 1 + 7 \sqrt{\pi} \sqrt{t} + O(t) \), consequently
\[
\varphi(1+\varepsilon) \leq 1 + 7 \sqrt{\pi} \sqrt{t} + O(t),
\]
which infers the inequality (6).

\[\square\]

Proof of Lemma 4.1. Starting from \( x \in P_k H \), a unit element in the subspace corresponding to \( C_k \), we can write
\[
Ux = e^{i\psi} \cos \theta (x + \tan \theta y), \quad \text{with} \quad \|x\| = \|y\| = 1, \langle x, y \rangle = 0, \psi \in \mathbb{R}, \theta \in [0, \pi/2].
\]
As noticed by Donoghue [3], the complex number
\[
\cos \theta e^{i\psi} = \langle Ux, x \rangle = \int_{I_k} e^{itd} \|E(t)x\|^2
\]
belongs to the convex hull of \( C_k \). This infers that \( \cos \frac{\pi}{n} \leq \cos \theta \leq 1 \), i.e., \( 0 \leq \theta \leq \pi/n \); thus \( |\tan \theta| \leq t \). Recall that \( Bx = \lambda x + t \beta w \), with \( \|w\| = 1, \langle x, w \rangle = 0 \) and \( \beta \geq 0 \). Thus \( \lambda = \langle Bx, x \rangle \in \mathbb{R}^+ \). Using
\[
\langle Ax, x \rangle = \langle Ux, Bx \rangle = \cos \theta e^{i\psi} \langle x + \tan \theta y, \lambda x + t \beta w \rangle
\]
\[
= \cos \theta e^{i\psi} (\lambda + t \beta \tan \theta \langle y, w \rangle)
\]
together with the inequality \( w(A) \leq 1+\varepsilon \leq 1/\cos \frac{\pi}{n} \), we obtain
\[
|\lambda + t \beta \tan \theta \langle y, w \rangle| \leq \frac{1 + \varepsilon}{\cos \theta}; \quad \text{thus} \quad \lambda \leq 1 + t^2 + 2t^2 |\langle y, w \rangle|.
\]
In particular, there holds
\[
\lambda \leq 1 + (1+\beta)t^2. \quad (7)
\]
Starting now from the relation \( \lambda B^{-1}x = x - t \beta B^{-1}w \), we have
\[
\lambda \langle A^{-1}x, x \rangle = \langle \lambda B^{-1}x, Ux \rangle = \cos \theta e^{-i\psi} \langle x - t \beta B^{-1}w, x + t \tan \theta y \rangle
\]
\[
= \cos \theta e^{-i\psi} (1 + \frac{t^2 \beta^2}{\lambda}) \langle B^{-1}w, w \rangle - t \beta \tan \theta \langle B^{-1}w, y \rangle).
\]
We now use the assumption \( \lambda w(A^{-1}) \leq \lambda(1+\varepsilon) \), to get
\[
|1 + \frac{t^2 \beta^2}{\lambda} \langle B^{-1}w, w \rangle - t \beta \tan \theta \langle B^{-1}w, y \rangle| \leq \frac{1 + \varepsilon}{\cos \frac{\pi}{n}} \leq \lambda(1 + t^2).
\]
We also have
\[
\langle B^{-1}w, w \rangle \geq 1/\|B\| = 1/\|A\| \geq \frac{1}{2w(A)} \geq \frac{1}{2(1+\varepsilon)} \geq \frac{128}{257};
\]
\[
| \tan \theta \langle B^{-1}w, y \rangle | \leq \tan \frac{\pi}{n} \|B^{-1}\| = t \|A^{-1}\| \leq 2t \|w(A^{-1}) \| \leq \frac{257 t}{257}.
\]
this yields
\[ 1 + \frac{128 \beta^2 t^2}{257 \lambda} - \beta t^2 \frac{257}{128} \leq \lambda (1 + t^2), \]
or equivalently
\[ \beta^2 - a^2 \beta \lambda - a \left( \lambda^2 + \frac{\lambda^2 - \lambda}{t^2} \right) \leq 0, \quad \text{with } a = \frac{257}{128}. \quad (8) \]

The set of \((\lambda, \beta)\) satisfying (8) is the union of two convex parts delimited by a hyperbola \(H\), while the inequality (7) is corresponding to a half-plane. Recall that the inequalities \(\lambda > 0\) and \(\beta \geq 0\) also hold.

The hyperbola \(H\) is tangent to the axis \(\{\lambda = 0\}\) at the origin, and admits another vertical tangent at the point

\( (1 + t^2(1 + \beta_1), \beta_1) \) and \((1 + t^2(1 + \beta_2), \beta_2)\), with \(\beta_1 > 0\) and \(\beta_2 < 0\) being the roots of

\[ E_t(\beta) := \beta^2 - a \beta \frac{1 + a + 4t^2 + at^2 + 2t^4}{1 - at^2 - a^2t^2 - at^4} - \frac{(1 + t^3)(2 + t^3)}{1 - at^2 - a^2t^2 - at^4} = 0. \]

Recall that \(t < \frac{1}{11}\), and then \(E_t(7) \geq E_{1/11}(7) > 1.6308 > 0\). This shows the inequality \(\beta < 7\) and completes the proof of the lemma.

**Remark 4.2.** The estimates (4) and (5) give the fork

\[ 2 - x^{-4} \leq \varphi(x) \leq 2 - c_2 x^{-4}; \]

this gives a good control on the behaviour of \(\varphi\) for large \(x\), while the estimates (3) and (6) give a fork

\[ 1 + (1 - x^{-3})^{1/2} \leq \varphi(x) \leq 1 + c_1 (x-1)^{1/4}, \]

which gives a control in a neighborhood of \(x = 1\). We think that the exponent \(1/4\) in this estimate effectively corresponds to the behavior of \(\varphi\) for \(x\) close to 1. This intuition
is confirmed by numerical tests, that we have realized with the family of $n \times n$ matrices, $A = BD$, defined by, with $n = 4(2^k + 1)$,

$$B = I + \frac{1}{2n^{3/2}}E, \quad \text{with} \quad e_{ij} = 1 \text{ if } 3k + 2 \leq |i - j| \leq 5k + 3, \quad e_{ij} = 0 \text{ otherwise},$$

$$D = \text{diag}(e^{2i\pi/n}, \ldots, e^{2i\pi/n}, \ldots, e^{2n\pi/n}).$$

The points, with coordinates $$(\log \left( \frac{\|A\|_{w(A)} - 1}{\sqrt{w(A)w(A^{-1})}} \right), \log \left( \frac{\sqrt{w(A)w(A^{-1})}}{w(A)} \right))$$, computed for $k = 1, 2, \ldots, 12$, are close to a straight line with a slope 0.2506.

**Remark 4.3.** We think that the function $\varphi$ is continuous, but have not succeeded to prove it.

**REFERENCES**


E-mail address: michel.crouzeix@univ-rennes1.fr