

Our purpose of this exposé (is to give the equivalence):

$$P_{G_0}(Gr, K) \cong \text{Rep } G_{HK} \quad (\text{given by } F \cong H^* \text{ the global cohomology functor})$$

where  $P_{G_0}(Gr, K)$  stands for the cat. of  $G_0$ -equiv. presheaves with  $K$ -coefficients on  $Gr$ .

Notations:  $G$ : red. gp /  $k$ ,  $K = \mathbb{C}(\!(t)\!)^*$ ,  $\mathcal{O} = \mathbb{C}[[t]]$ ,  $Gr = G(K) / G(\mathcal{O})$  = an ind-scheme,  $G_k \leftrightarrow G(k)$  - ind-scheme,  $G_{\mathcal{O}} \leftrightarrow G(\mathcal{O})$  - scheme,  $G_{HK}$  the dual gp of  $G/k$ ,  $K$ : Noeth. commutative unitary ring.

Steps:

1. Decomposition of  $F = F = \bigoplus F_{\lambda}$
2. Construct a convolution product "\*" on  $P_{G_0}(Gr, K)$  s.t.  $F$  is a tensor functor.
3. Construct  $\tilde{G}$  s.t.  $P_{G_0}(Gr, K) \cong \text{Rep } \tilde{G}$
4. Identify  $\tilde{G}$  to  $G_{HK}$

§1.

Fix  $T$  a max. torus  $T \subset B$  a Borel.

Consider the left action  $G_0 \curvearrowright Gr$ , whose orbits can be para by  $X_{\lambda}(T)$ ,  $\forall \lambda \in X_*(T)$ ,  $G_{\lambda}$  denotes the  $G_0$ -orbit  $\leftrightarrow \lambda$ .

Let  $S$  denote the stratification induced by  $G_0$ -orbits, we have:

$$P_{G_0}(Gr, K) \cong P_S(Gr, K)$$

$\mathcal{D}_S = P_S(Gr, K)$  a full subcat of  $\mathcal{D}_G(Gr, K)$ : the bounded  $S$ -constructible derived cat of  $K$ -sheaves.  
 - why this equiv useful? =  $\exists$  cohomological functor.

$$PH^0: \mathcal{D}_G(X, K) \rightarrow P_S(X, K)$$

Whence  $I_{\lambda}(\lambda, M) := PH^0 \circ j_{!}(M[\dim Gr]^{ob}) \in P_{G_0}(X, K)$  (same as  $I_{\lambda}(\lambda, M)$ ,  $I_{\lambda}(\lambda, M)$ )  
 $M$  a fin. gen.  $K$ -module. It is useful to construct  $\tilde{G}$

Notations:  $N$ : unip. radical of  $B$ .  $N_k$ : ind-sch  $\leftrightarrow N(K) = (N_k^-)$ . The  $N_k$ -orb para by  $X_{\lambda}(T) \in S_{\lambda}$  denote the  $N_k$ -orb  $\leftrightarrow \lambda \in X_*(T)$ , where  $S_{\lambda} = \bigcup_{\mu} X_{\mu}(T)$ : ind-var. infinite dim and infinite codim.

Thm 1:  $\forall A \in P_{G_0}(Gr, K)$   $\exists$  can. iso

$$H_c^k(S_{\lambda}, A) \cong H_{T_0}^k(Gr, A)$$

both sides vanish for  $k \neq 2p(\lambda)$  (where  $p = \frac{1}{2}$  sum of positive roots with resp to  $B$ ).  
 Let  $F_{\lambda}$  denote  $H_c^k(S_{\lambda}, -)$ ,  $\lambda \in X_*(T)$ , it is a

We can have morph. of functors:

$$H_{T_0}^{2p(\lambda)}(Gr, -) \rightarrow H_c^{2p(\lambda)}(-) \rightarrow H_c^{2p(\lambda)}(S_{\lambda}, -)$$

s.t. the comp. is the can. equiv. in [Thm 1]. Consider  $Gr = U \cap V = U \cap S_0$ , the two filtrations of  $Gr$  by  $F_0$ 's and  $S_0$ 's, both indexed by  $X_*(T)$ . The equiv. above  $\Rightarrow$  The two filtrations of  $H^*$  split each other.

[Thm 2]  $\exists$  a natural equiv. of functors:

$$H^* \cong F = \bigoplus_{\nu \in X_*(T)} H_c^{2p(\nu)}(S_0, -) = P_{G_0}(Gr, k) \rightarrow \text{Mod}_k$$

$F$  and this equiv. are indep. of the choice of the pair  $(T, B)$

(I will introduce a technique in the proof of Thm 1 dimension estimate, which is a crucial fact given by Vilonen & Mirzakhani)

"pf of Thm 1":

Upper bound of  $H_c^{2p(\nu)}(S_0, A)$  — (1)  
 Lower bound of  $H_{G_0}^{2p(\nu)}(Gr, A)$

$H_c^{2p(\nu)}(S_0, A) \cong H_{G_0}^{2p(\nu)}(Gr, A)$  (equivalence induced from [Braden]). — (2)

(1):  $\forall$  dominant  $\eta \in X_*(T)$ :

$f_* A(G_r^\eta) \in \mathcal{D}^{-2p(\eta)}(Gr, k)$ , i.e. in degrees  $\leq -\dim(G_r^\eta)$  — (3)

$H_c^k(S_0 \cap G_r^\eta, k) = 0$ , for  $k > \dim(S_0 \cap G_r^\eta)$  — (4)

(Thm 3.2 in M.V)  $\hookrightarrow = 2p(\nu) + \eta$

$H_c^*(S_0 \cap G_r^\eta, A) = H_c^*(S_0 \cap G_r^\eta) \times_{\mathbb{Z}}^{G_r^\eta} A \otimes k$

(By Künneth formula +  $k$  constant on  $S_0 \cap G_r^\eta$ )  $= H_c^*(S_0 \cap G_r^\eta, k) \otimes H_c^*(G_r^\eta, A)$

By (4):

$H_c^k(S_0 \cap G_r^\eta, A) = 0$  for  $k > 2p(\nu) = 2p(\eta) + \eta + (-2p(\eta))$ .

+ Spectral sequence of stratification space  $\Rightarrow H_c^k(S_0, A) = 0$  for  $k > 2p(\nu)$  (Strat:  $G_r^\eta$   $\eta$  dominant)

(2): Consider the graph:

$$\begin{array}{ccc} X_*(T) & \xrightarrow{f^+} & U \cap S_0 \xrightarrow{g^+} Gr \\ & \xrightarrow{f^-} & U \cap V \xrightarrow{g^-} Gr \end{array} \xrightarrow{[B_r]} \begin{array}{ccc} A^{i+} & & A^i \\ \vdots & & \vdots \\ (f^+)^* & (g^+)^* A \cong & (f^-)^* (g^-)^* A \end{array}$$

Fact

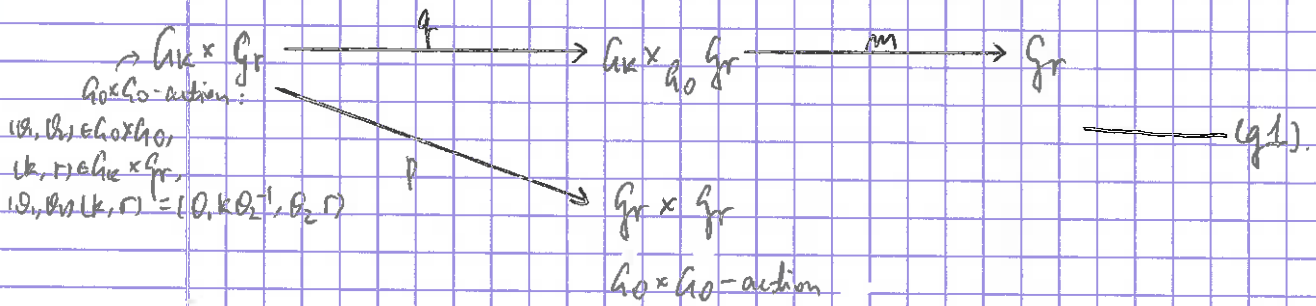
$A_v^{i+} = H_c^{2i}(S_0, A)$   $A_v^{i-} = H_{G_0}^{2i}(Gr, A) \Rightarrow$  The equiv.

§2. Convolution product and commutative constraint

Purpose = make  $P_{G_0}(Gr, k)$  a tensor cat. and  $F$  to be a tensor functor

2.1. Convolution product:

Consider the diagram: (don't clean the graph!!!)



Consider the gp immersions:

$$\begin{array}{l}
 \text{I: } G_0 \hookrightarrow G_0 \times G_0, \quad \vartheta \mapsto (\vartheta, \vartheta), \quad \text{Im(I)} = G_0^{\text{I}} \\
 \text{II: } G_0 \hookrightarrow G_0 \times G_0, \quad \vartheta \mapsto (\vartheta, \vartheta), \quad \text{Im(II)} = G_0^{\text{II}} \Rightarrow G_0^{\text{II}} \cong G_0 \times G_0 / G_0^{\text{I}}
 \end{array}$$

$G_0^{\text{II}}$ -action on  $G_0 \times G_0$  is free,  $G_0 \times_{G_0} G_0$  the quotient of  $G_0 \times G_0$  by  $G_0^{\text{II}}$ -action  
 ( $\hookrightarrow$  By  $h_0 = ak$ , the  $h_0$  term is unique)

$p, q$ : proj and  $G_0 \times G_0$ -equiv. ( $G_0 \times G_0$ -action on  $G_0 \times_{G_0} G_0$  is given by  $h_0 \times h_0 \rightarrow h_0 \times h_0 / G_0^{\text{I}} \cong G_0^{\text{II}}$ -action)

$$m := (k, r) \mapsto k \cdot r, \quad m: G_0^{\text{I}}\text{-equiv}$$

By [Bernstein-Lunts]:  $\exists! \tilde{A} \in P_{G_0^{\text{I}}}(G_0 \times_{G_0} G_0, k)$  s.t.

$$q^* \tilde{A} = p^* (PH^0(A_1, A_2)), \quad \text{define } A_1 * A_2 := Rm_* \tilde{A} \quad (3)$$

$m$  is a stratified semi-small map with resp. to the  $G_0^{\text{I}}$ -orbits strat on  $G_0 \times_{G_0} G_0$  and the  $G_0$ -orb strat on  $G_0 \Rightarrow$

**Proposition 3:**  $A_1 * A_2 \in P_{G_0}(G_0, k)$

By analogue (3) can define  $A_1 * A_2 * A_3$  and a can iso:  $(A_1 * A_2) * A_3 \cong A_1 * (A_2 * A_3) \cong A_1 * A_2 * A_3$ , which gives the associative constraint of this product (More details, see: Zhu [1.3] [1.2-1F])

### 2.2 Commutative constraint:

(Explain without writing notes on blackboard:

Look back to (g1). Let  $\mathcal{D}: G_0 \hookrightarrow G_0 \times G_0$  be the diagonal immersion, let  $G_0^{\mathcal{D}}$  denote its image, the natural commutative constraint  $\tau$  of  $PH^0(\mathcal{D})$  on  $P_{G_0}(G_0, k) \times P_{G_0}(G_0, k)$  is an  $G_0^{\mathcal{D}}$ -equiv. iso, whence  $p^* \tau$  on  $G_0^{\mathcal{D}}$ -equiv. iso on  $P_{G_0^{\text{I}}}(G_0 \times_{G_0} G_0, k)$ , but  $G_0 \times_{G_0} G_0$  is the quotient of  $G_0^{\text{II}}$ -action.

$\Rightarrow$  Couldn't be induced on  $(P_{G_0}(G_0, k), *)$

### Beilinson-Drinfeld Grassmannians:

To motivate this Grassmannians, we see an interpretation of affine Grass:

Why need  
 $\times K$  Grassmannian  
 $\hookrightarrow$

$\forall R$  a  $G$ -alg  $\forall x \in |X|$  a smooth class pt, let  $\text{Gr}_x$  denote the presheaf:

$$\text{Gr}_x(R) = \{ (F, \nu) \mid F \text{ a } G\text{-torsor on } X_R, \nu: F|_{X_R} \cong F^0|_{X_R} \text{ a trivialization} \}$$

where  $F^0$  the trivial torsor.  $X_R^x := X_R - \{x\}$ .

In fact,  $\text{Gr}$  is a presheaf defined as:

$$\text{Gr}(R) = \{ (F, \nu) \mid F \text{ a } G\text{-torsor on } X_R, \nu: F|_{X_R} \cong F^0|_{X_R} \text{ a trivialization} \}$$

where  $X_R^x := X^x \times_{\text{Spec}(R)} \text{Spec}(R)$ ,  $X^x := \text{Spec}(C((t)))$ ,  $X := \text{Spec}(C[[t]])$ .

Thm of Beilinson-Logvinov:  $\text{Gr}_x \cong \text{Gr}$ .

We globalize the definition of  $\text{Gr}_x$ :

-  $\text{Gr}_x$  denote the ind-scheme w.r.p the functor:

$$R \mapsto \{ (x_1, \dots, x_n) \in X^n(R), F \text{ a } G\text{-torsor on } X_R; \nu_{x_1, \dots, x_n}: F|_{X_R - \cup x_i} \cong F^0|_{X_R - \cup x_i} \text{ a triv.} \}$$

-  $\widetilde{\text{Gr}}_x \times \text{Gr}_x$  denotes the ind-scheme w.r.p the functor:

$$R \mapsto \{ (x_1, x_2) \in X^2(R); F_1, F_2 \text{ } G\text{-torsors on } X_R; \nu_i: \text{triv. of } F_i \text{ on } X_R - x_i, i=1,2; \mu_i: \text{a triv. of } F_i \text{ on } (X_R)_{x_i} \}$$

(where  $(X_R)_{x_i}$  the formal neighborhood of  $x_i$  in  $X_R$ .)

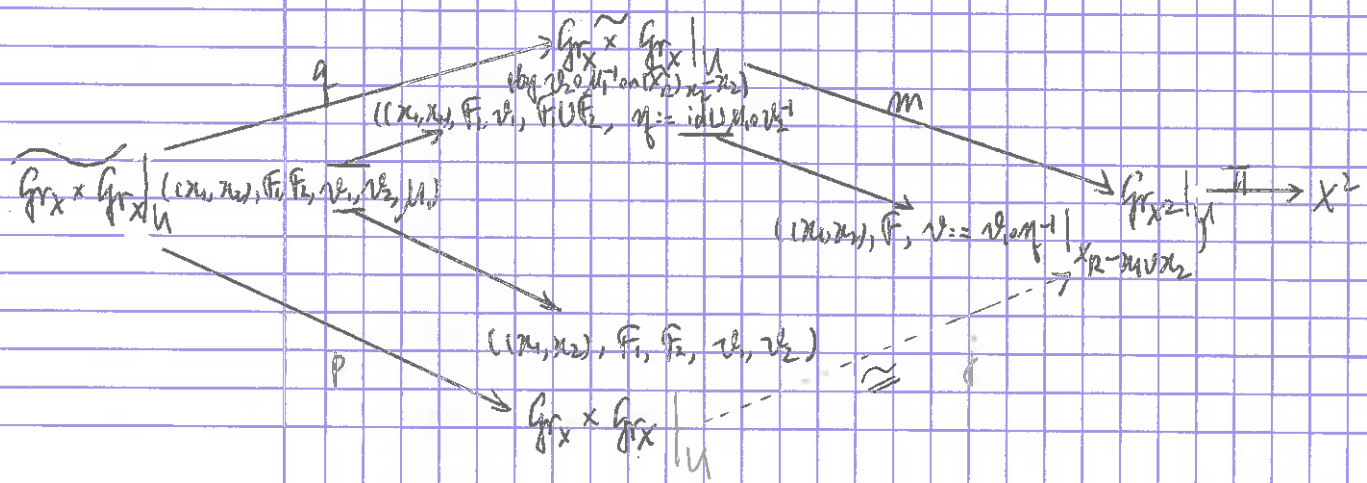
-  $\text{Gr}_x \times \widetilde{\text{Gr}}_x$  denotes the ind-sch ... :

$$R \mapsto \{ (x_1, x_2) \in X^2(R); F_1, F_2 \text{ } G\text{-torsors on } X_R; \nu_i \text{ a triv. of } F_i \text{ on } X_R - x_i; \eta: F_1|_{(X_R)_{x_1}} \cong F_2|_{(X_R)_{x_2}} \}$$

-  $G_{x_1, x_2}$  a gp-sch ... : (Attention the triv  $F^0|_{\dots} \rightarrow F^0|_{\dots}$ )

$$R \mapsto \{ (x_1, \dots, x_n) \in X^n(R), \mu_{(x_1, \dots, x_n)} \text{ a triv. of } F^0 \text{ on } (X_R)_{(x_1, \dots, x_n)} \}$$

We have now =



For  $B_1, B_2 \in \mathcal{P}_{G_{X,0}}(G_X, K)$ , define convolution product by:

$$B_1 * B_2 = R\pi_* \tilde{B}, \text{ where } q^* \tilde{B} = p^*(P^H^0(B_1 \boxtimes B_2))$$

Explain:

$G_{X,0} = \begin{cases} 1^{st} \text{ action } \leadsto \widetilde{G_X \times G_X} \text{ altering } U \\ 2^{nd} \text{ action } \leadsto \widetilde{G_X \times G_X} \dots \mu \text{ and } \nu \end{cases}$

Both actions are free  $\Rightarrow p, q$  as  $G_{X,0}$  torsors, whence  $\tilde{B}$  is well defined

$\exists$  projections  $G_X \xrightarrow{\pi} G_X$  (cause  $G_X \cong G_X \times^{X^*} X^*$  twisted product) in Zhu)

**Proposition:**  $\exists$  can. iso:  $\Delta: X \rightarrow X \times X = X^2$  diagonal

$$i: G_X \cong G_{X^2} \times_{X^2} X, \quad j: G_X|_{X^2=\Delta} = U \cong (G_X \times G_X)|_{X^2=\Delta} = U$$

explain without notes  $\rightarrow$

In addition,  $\exists S_2$ -action on  $G_{X^2}$  s.t.  $j$  is  $S_2$ -equiv., where  $S_2$  acts on  $G_X \times G_X$  by permuting the two factors. And  $S_2$  commutes with the  $G_{X,0}$ -action on both sides.

(Go back to the graph, explain the pencil part.)

$$G_X \xrightarrow{i} G_{X^2} \xleftarrow{j} G_X \times G_X|_U$$

$$\text{Consider: } \begin{cases} T^0 := T^*(A) = P_{H^0}(G_X, K) \rightarrow P_{G_{X,0}}(G_X, K) \\ i^0 := i^*(A) = P_{H^0}(G_{X^2}, K) \rightarrow P_{G_{X,0}}(G_X, K) \end{cases}$$

We have:

$$(a) T^0 A_1 * T^0 A_2 \cong j_* (P^H^0(T^0 A_1 \boxtimes T^0 A_2)|_U)$$

$$(b) T^0(A_1 * A_2) \cong i^0(T^0 A_1 \oplus_X T^0 A_2) \quad (\text{see Zhu Lemma 5.4.6 for details})$$

a functorial iso  
cause induced by  
functorial iso of

$$\Rightarrow T^0(A_1 * A_2) \cong T^0(A_1 * A_2) \quad (\text{The commutative constraint}).$$

**Fact:**  $R\pi_* (T^0 A_1 \oplus_X T^0 A_2)|_U = H^*(G_X, A_1) \oplus H^*(G_X, A_2)$  (as const. sheaf)

$$R\pi_* (\dots)|_\Delta = T^0(H^*(G_X, A_1 * A_2))$$

Specialize on one point =

Let  $i_x: \{x, x, x\} \rightarrow X$ , after composing  $i_x^*$  we have a functor from  $(P_{H^0}(G_X, K), *) \rightarrow (\text{Mod}_K, \otimes)$  which is a tensor functor  $\simeq H^*$

### §3. Construction of $\widehat{G}$

**Proposition:**  $Z \subset G$  a closed subset which is a finite union of  $G_0$ -orb.  $F_Z$  restrict to  $P_{G_0}(Z, K)$  is repr. by a proj. obj  $P_Z(\mathcal{O}_Z, K)$  of  $P_{G_0}(Z, K)$ .

$$P_Z(K) := \bigoplus_{\mathcal{O}_Z} P_Z(\mathcal{O}_Z, K) \text{ repr. } F \text{ on } P_{G_0}(Z, K)$$

**Proposition:**

(a) Let  $Y \subset Z$  be a closed subset consisting of  $G_0$ -orb. Then:

and  $\exists$  a can. surj.  $P_Y(K) = PL^0(P_Z(K) | Y)$

$$P_Z(K) \rightarrow P_Y(K)$$

(b)  $F(P_Z(K)) \cong H^0(P_Z(K))$  is free over  $K$

Rg:

(b) is given by the fact:  $P_Z(K)$  has a filter with associated graded:

$$Gr(P_Z(K)) = \bigoplus_{\mathcal{O}_Z} F(I_{\mathcal{O}_Z}(K))^* \oplus I_{\mathcal{O}_Z}(K)$$

This is the standard graded, i.e.  $F(P_Z(K)) = \bigoplus F(A \oplus B)$   
 $\downarrow$  free  $\downarrow$  free

**Proposition:**  $\exists$  a gp-scheme  $\widehat{G}_K$  over  $K$  s.t.  $(\text{Rep}(\widehat{G}_K), K) \oplus \cong P_{G_0}(G, K)$   
 Furthermore, the coordinate ring  $K[\widehat{G}_K]$  is free over  $K$  and  $\widehat{G}_K = \text{Spec}(K) \times_{\text{Spec}(K)} \widehat{G}$

Pl:

Denote  $A_Z(K)$  as  $F(P_Z(K))$ . The functor  $F: P_{G_0}(Z, K) \rightarrow \text{Mod}_K$  lifts to an equiv.

$$P_{G_0}(Z, K) \xrightarrow{\cong} \text{Mod}_{A_Z(K)} \text{ (as ab. cat.)}$$

(In fact  $P_Z(K)$  is the proj generator of  $P_{G_0}(Z, K)$  i.e. it's proj. and  $\text{Hom}(P_Z(K), -)$  a faithful functor. By thm of Morita, we see the result. Ref: [Generators in modules and comodule categories.]

Consider the  $K$ -dual of  $A_Z(K)$ , denote it as  $B_Z(K) \cong$  a coalg. by (b)

$$\Rightarrow P_{G_0}(Z, K) \cong \text{Comodule}_{B_Z(K)} \text{ (as ab. cat.)}$$

By using Verdier dual  $\dagger$  (a):  $Z \subset Z' \Rightarrow \exists B_Z(K) \rightarrow B_{Z'}(K)$ . Let

$$B(K) = \varinjlim B_Z(K) \Rightarrow P_{G_0}(G, K) \cong \text{Comodule}_{B(K)} \text{ (as ab. cat.)}$$

To define  $\widehat{G}_K = \text{Spec}(B(K))$

-  $\text{Spec}(B(k))$  makes sense. (1)

- ... a gp-scheme. (2)

For (1) = endow a commutative  $k$ -structure on  $B(k)$ , i.e. a morph:

$$B(k) \otimes_{k} B(k) \rightarrow B(k) \text{ (Easy, see [M.V.])}$$

Now  $\widehat{G}_k$  is well-defined and an affine monoid.

For (2): Consider the eqn:

$$\begin{array}{ccc} \text{Rep}_{\widehat{G}_k}^{\text{proj}} & \xrightarrow{\cong} & \text{Proj}_{G_0}^{\text{proj}}(S_r, k) \\ \downarrow \text{note} & & \downarrow \text{note} \\ (\Delta) & & (\square) \end{array}$$

In (□), we can endow an anti-involution given from  $i: G_k \rightarrow G_k$  which makes (Δ) a rigid tensor cat.  $\Rightarrow$  By "categories Tannakiennes"  
 $\Rightarrow \text{Proj}(S_r, k) \xrightarrow{\cong} \text{Rep } \widehat{G}_k$

§4 Not this time