A straight way to algebraic stacks

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These notes are an expansion of notes I had written for a pre&post-doc seminar on algebraic stacks, at the Kungliga Tekniska Högskolan (KTH) of Stockholm.

My original aim was to provide, roughly, the material filling the gap between Hartshorne's Algebraic Geometry and texts such as Vistoli's Appendix or Laumon and Moret-Bailly's book.

I included the case of descent by *faithfully flat universally open* morphisms (naturally denoted fpuo). Quite curiously fpuo descent is usually absent from the classical treatments. Proofs for fpuo descent are essentially the same as the proofs for fpqc descent, but the situations covered are different.

I assumed knowledge of the notions of flat, smooth, étale morphisms. In particular are recalled without proof some fundamental properties of flat morphisms, like openness or the preservation of properties of morphisms.

Apart from that, the proofs are as much as possible self-contained. I tried to make arguments concise and clear.

Descent theory

Introduction

Let X be a topological space and $\{U_i\}_{i\in I}$ an open cover. For each i let E_i be a given fibre space over U_i (of the type you prefer: vector bundle, principal G-bundle, projective bundle, topological covering space...). Do all the E_i glue to give a fibre space over X ? We know how to handle this problem, because the definition of the objects involved (bundles) is precisely in terms of the open covers (the topology) on X. Namely in order to glue we must have isomorphisms

$$\tau_{ij} : E_i|_{U_{ij}} \xrightarrow{\sim} E_j|_{U_{ij}} \tag{1}$$

(with $\tau_{ii} = \mathrm{id}$ and $\tau_{ji} = \tau_{ij}^{-1}$) such that on triple intersections U_{ijk} we have

$$\tau_{jk} \circ \tau_{ij} = \tau_{ik} \tag{2}$$

We can avoid indices by setting $U' = \coprod_{i\in I} U_i$ and view the datum of the E_i as a bundle $E' \to U'$; then obviously

$$U'' := U' \times_X U' = \coprod_{i,j\in I} U_{ij}$$

(this includes U_{ii}'s). The two projections to U' are given by : for $x \in U_{ij}$, $p_1(x) = x \in U_i$ and $p_2(x) = x \in U_j$. Then (1) is equivalent to an isomorphism

$$\tau : p_1^* E' \xrightarrow{\sim} p_2^* E' \tag{3}$$

and (2) is equivalent to

$$\pi_{23}^*\tau \circ \pi_{12}^*\tau = \pi_{13}^*\tau \tag{4}$$

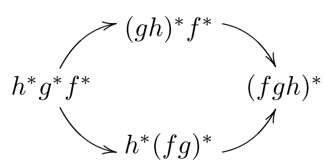
on $U''' := U' \times_X U' \times_X U' = \coprod_{i,j,k\in I} U_{ijk}$.

If we want to formalize these tautologies in the terms of the vocabulary of descent, then we will call an isomorphism τ satisfying (4) a *descent datum* for E' with respect to $U' \to X$, and what we said is just that it is the same thing to have a bundle over X, or a bundle over U' together with a descent datum. Roughly said, the discovery of Grothendieck is that, in Algebraic Geometry, the particular properties of flat morphisms allow to treat them as "coverings" (thus replacing the sums of open immersions) and get formally the same result. In particular the members of these new "coverings" will no longer be subsets of X ; the rule will be "replace intersections by fibre products".

The vocabulary of descent

Let's develop a little the formalism of descent. For simplicity we will only speak about categories fibred over the category of schemes $\mathcal{S}ch$. So let $\mathcal{C}$ be a category and $p : \mathcal{C} \to \mathcal{S}ch$ a functor. Any object $S \in \mathcal{S}ch$ defines a point category (the only object is S, the only map is id_S). The fibre category $\mathcal{C}(S) = p^{-1}(S)$ is the category whose objects are those above S, and whose maps are those above id_S.

Definition 1 We say that $\mathcal{C}$ is a *fibred category* over $\mathcal{S}ch$ if there are pullback functors $f^* : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ for any $f : S_2 \to S_1$, together with isomorphisms $c_{f,g} : g^* f^* \simeq (fg)^*$ whenever the composition fg makes sense, satisfying the cocycle relation expressing the expected associativity :



We often denote the canonical isomorphisms $c_{f,g}$ by simple equalities.

We could work with a more general definition ; on the contrary we could also state the descent theorems avoiding completely any general definition. This is the attitude adopted in the appendix of the book [BFFGK], but I don't find this very enlightening. The intermediary choice of definition 1 is well-suited for the purpose of speaking of algebraic stacks later on.

Definition 2 Let $\alpha : S' \to S$ be a morphism of schemes. Put $S'' = S' \times_S S'$ and $\gamma = \alpha p_1 = \alpha p_2$.

(i) We say that α is a *morphism of descent for* $\mathcal{C}$ if for all objects $\xi, \eta \in \mathcal{C}(S)$ we have that

$$\mathrm{Hom}_S(\xi, \eta) \longrightarrow \mathrm{Hom}_{S'}(\alpha^*\xi, \alpha^*\eta) \rightrightarrows \mathrm{Hom}_{S''}(\gamma^*\xi, \gamma^*\eta)$$

is an exact diagram.

(ii) Let $\xi' \in \mathcal{C}(S')$. We call *glueing datum for* ξ' *with respect to* α an isomorphism $\tau : p_1^*\xi' \simeq p_2^*\xi'$ between objects on S'' (p_1, p_2 are the two projections).

For example, if $\xi \in \mathcal{C}(S)$ then $\xi' = \alpha^*\xi$ has a canonical glueing datum given by $p_1^*\alpha^*\xi \simeq (\alpha p_1)^*\xi = (\alpha p_2)^*\xi \simeq p_2^*\alpha^*\xi$. There is an obvious notion of morphism between two pairs (ξ', τ) and (η', υ) : it is a morphism $\varphi : \xi' \to \eta'$ compatible with τ and υ i.e. such that the following square commutes :

$$\begin{array}{ccc} p_1^*\xi' & \xrightarrow{\tau} & p_2^*\xi' \\ {\scriptstyle p_1^*\varphi}\downarrow & & \downarrow{\scriptstyle p_2^*\varphi} \\ p_1^*\eta' & \xrightarrow{\upsilon} & p_2^*\eta' \end{array}$$

Definition 3 We say that a glueing datum for ξ' with respect to $\alpha : S' \to S$ is *effective* if ξ' together with its glueing datum is isomorphic to some $\alpha^*\xi$ with its canonical glueing datum.

In other words this means that ξ' "descends to S" or "comes from S". Of course in general there is no chance that all glueing data should be effective : a necessary condition of effectivity is to satisfy the cocycle condition on triple overlaps, i.e. on S'''.

Definition 4 We call *descent datum* for ξ' with respect to $\alpha : S' \to S$ a glueing datum $\tau : p_1^*\xi' \simeq p_2^*\xi'$ which satisfies $\pi_{23}^*\tau \circ \pi_{12}^*\tau = \pi_{13}^*\tau$.

Exercise 5 The case when α has a section ($i : S \to S'$ such that $\alpha i = \mathrm{id}$) is particularly important. Show that a given object ξ' can descend to at most one ξ ; that any glueing datum on ξ' is effective. So in particular all descent data for ξ' are effective and isomorphic.

With these definitions, we can give a simple statement of the result we aim at. Given $\alpha : S' \to S$, which we may think of as $\amalg U_i \to X$, consider, on one hand, the category $\mathcal{C}(S)$ of objects over S. On the other hand we have the category $\mathcal{C}(S')^{\mathrm{desc}}$ of objects over S' together with a descent datum. Then the "unglueing" functor $\alpha^* : \mathcal{C}(S) \to \mathcal{C}(S')^{\mathrm{desc}}$

- is fully faithful if and only if α is a morphism of descent for $\mathcal{C}$,
- is essentially surjective if and only if all descent data for objects on S' are effective.

If both properties are true we say that α is a *morphism of effective descent* for $\mathcal{C}$.

Faithfully flat descent

Now we have all in hand to state the theorems. There is a catalogue of results of descent for modules, schemes, schemes together with a locally free sheaf,... We won't try to be comprehensive, but rather give the most significant results and emphasize the ideas, so that reading of all of exposé VIII, [SGA1] should then become easy.

As a matter of notation, it is standard to denote faithfully flat and quasi-compact morphisms as *fpqc*. We will also use the (non-standard) acronym *fpuo* to denote faithfully flat and universally open morphisms. We prove basic descent theorems for fpqc and fpuo morphims ; then refinements exist with more assumptions. Roughly, the working principle of the proofs will be to reduce to a statement in commutative algebra (where the *fp* assumption will do the job) the geometric statement on schemes (thanks to either the *qc* or the *o* assumption).

Remark 6 A particular case of fpuo morphism is a *sum of open immersions*, by which we mean a morphism $\alpha : \amalg S_i \to S$ given by choosing an open covering $S = \cup_{i \in I} S_i$. The example of the introduction, transposed in the category of schemes, is just that sums of open immersions are morphisms of effective descent for the categories of sheaves, or morphisms,... This is just ordinary Zariski glueing.

For general faithfully flat descent, the first important result concerns descent of modules :

Theorem 7 *A morphism $\alpha : S' \to S$ which is either fpqc or fpo is a morphism of effective descent for the fibred category $\mathcal{Q}coh \to \mathcal{S}ch$ of quasi-coherent modules.*

Proof : First we reduce the fpo case to fpqc. Indeed if α is open, we can cover S' by open affines S'_i and S by the open images $S_i = \alpha(S'_i)$. Denote by $\alpha_i : S'_i \to S_i$ the restriction, then we have a diagram

$$\begin{array}{ccc} S'_i & \xhookrightarrow{m_i} & S' \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S_i & \xhookrightarrow{n_i} & S \end{array}$$

Consider the sums of open immersions $m = \amalg m_i$ and $n = \amalg n_i$. We have $\alpha \circ m = n \circ (\amalg \alpha_i)$, so $m^* \circ \alpha^* \simeq (\amalg \alpha_i^*) \circ n^*$. As m^* and n^* are equivalences of categories (remark 6) we are reduced to proving the theorem for α_i. This means we can assume that S' is affine, and in this case α is quasi-compact.

Second, we reduce to S and S' both affine. For this we cover S by open affines S_i and we put $S'_i = \alpha^{-1} S_i$. Then we have the same diagram as before, so we are reduced to proving the theorem for α_i, i.e. we can assume furthermore that S is affine. By quasi-compactness S' is covered by a finite number of open affine schemes. By the same trick again we can replace S' by the disjoint sum of these affine schemes, which is itself an affine scheme (this is not true for an *infinite* sum of affine schemes).

So finally $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(A')$. We note $A'' = A' \otimes_A A'$. To prove full faithfulness, given A-modules M, N (and $M' = M \otimes A'$, etc), we must show exactness of the following diagram :

$$\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{A'}(M', N') \rightrightarrows \operatorname{Hom}_{A''}(M'', N'')$$

By property of the tensor product $\cdot \otimes_A A'$, this diagram is none other than $\operatorname{Hom}_A(M, \cdot)$ of

$$\mathcal{D} : \; N \longrightarrow N' \rightrightarrows N''$$

Note that $\mathcal{D}$ can be rewritten as a diagram $0 \to N \to N' \to N''$ so, as $\operatorname{Hom}_A(M, \cdot)$ is left exact, the result will follow if we show that $\mathcal{D}$ is exact. But $\mathcal{D}$ is exact if and only if it is exact after tensor product by a faithfully flat algebra. If we choose to tensor by A' then $A \to A'$ acquires a section (namely $A' \otimes_A A' \to A'$, $x_1 \otimes x_2 \mapsto x_1 x_2$). Assume we reduced to this case, and note $s : A' \to A$ the section. Then injectivity of $N \to N'$ is clear. Now assume that an element of N', written $n' = \sum n_i \otimes x_i$, has its images that coincide in N'', meaning $\sum n_i \otimes 1 \otimes x_i = \sum n_i \otimes x_i \otimes 1$. Then applying the map $N'' \to N'$ which sends $n \otimes x \otimes y$ to $n \otimes s(x) y$, we get that $\sum n_i \otimes x_i = \sum n_i \otimes s(x_i) = \sum n_i s(x_i) \otimes 1$, as was to be shown.

To prove essential surjectivity, let N' be an A'-module together with an isomorphism $u : N''_1 \simeq N''_2$ between the extensions of N' via the two maps $A' \rightrightarrows A''$. We must show that (N', u) is isomorphic to an A-module N with its canonical descent datum. Consider the sub-A-module of N' of elements x such that $u(x \otimes 1_{A''}) = x \otimes 1_{A''}$ in N''. It remains to show that the obvious map $N \otimes_A A' \to N'$ is an isomorphism, which can again be checked after $\cdot \otimes_A A'$. But then we identify the problem with a problem of descent with respect to the morphism $\operatorname{Spec}(A' \otimes A') \to \operatorname{Spec}(A')$, which is a morphism of effective descent since it has a section (exercise 5). □

Remark 8 The "fully faithful" part of the theorem gives in particular, for any fpqc or fpo morphism $\alpha : S' \to S$, the exact diagram $\mathcal{O}_S \to \alpha_* \mathcal{O}_{S'} \rightrightarrows \gamma_* \mathcal{O}_{S''}$ (straightforward sheafification).

Let's look at descent of schemes. Here things are slightly different, because as we saw, the working principle of descent is to reduce to commutative algebra. But, locally on an open affine $U = \operatorname{Spec}(A)$ of S, an $\mathcal{O}_S$-module is a module over A, whereas an S-scheme can't be reduced to an A-algebra.

Of course this reduction can be done for schemes that are affine over S, i.e. $X = \operatorname{Spec}_{\mathcal{O}_S}(\mathcal{A})$. Furthermore as tensor product commutes with pullback, the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ does descends, applying theorem 7. So we get :

Theorem 9 *Any fpqc or fpo morphism is a morphism of effective descent for the fibred category $\mathcal{A}ff \to \mathcal{S}ch$ of relatively affine schemes.* □

It turns out that in general (i.e. for the whole category $\mathcal{S}ch$), an fpqc or fpuo morphism will at least be a morphism of descent. This rests on the following useful property of fpqc morphisms (SGA1, exposé VIII, cor. 4.3), which clearly holds also for fpo morphisms :

Proposition 10 *Let $\alpha : S' \to S$ be an fpqc morphism. Then a subset $Z \subset S$ is open (resp. closed) if and only if $\alpha^{-1}(Z)$ is open (resp. closed). In other words α makes S into a topological quotient of S'.* □

Theorem 11 *Any fpqc or fpuo morphism is a morphism of descent for the fibred category $\mathcal{S}ch \to \mathcal{S}ch$ of schemes.*

Proof : Let $\alpha : S' \to S$ be an fpqc (resp. fpuo) morphism ; we denote the usual pullbacks with the usual primes. We have to show that given S-schemes X and Y, we have an exact diagram

$$\mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'')$$

First we show that the left-hand map is injective. Let $f_1, f_2 : X \to Y$ be morphisms that coincide after pullback via α :

$$\begin{array}{ccccc} X' & \xrightarrow{f_1'=f_2'=f'} & Y' & \longrightarrow & S' \\ \downarrow v & & \downarrow w & & \downarrow \alpha \\ X & \xrightarrow{f_1,f_2} & Y & \longrightarrow & S \end{array}$$

Since v is fpqc (resp. fpo) , and the maps f_1 and f_2 are induced by wf' by the property of the topological quotient $X' \to X$ (proposition 10), they must agree as topological maps. If the comorphisms ${}^cf_i : \mathcal{O}_Y \to f_{i,*}\mathcal{O}_X$ also agree we will be done. But as v and w are fpqc (resp. fpo) we have $\mathcal{O}_X \hookrightarrow v_*\mathcal{O}_{X'}$ and $\mathcal{O}_Y \hookrightarrow w_*\mathcal{O}_{Y'}$ (use remark 8), so ${}^cf_1 = {}^cf_2$ being the restriction of ${}^cf'$.

We now consider $f' : X' \to Y'$ such that its pullbacks via $S'' \to S'$ coincide. So wf' is constant on the fibres of v, hence by proposition 10 it descends to a continuous map $f : X \to Y$ (by definition of a topological quotient). There is a map $\lambda : f^{-1}\mathcal{O}_Y \to v_*\mathcal{O}_{X'}$ obtained by adjunction from $\mathcal{O}_Y \to w_*\mathcal{O}_{Y'} \to w_*f'_*\mathcal{O}_{X'} = f_*v_*\mathcal{O}_X$. The assumption that $f_1'' = f_2''$ says that the compositions of λ with the maps $v_*\mathcal{O}_X \rightrightarrows z_*\mathcal{O}_{X''}$ coincide, but by remark 8 again the corresponding exact diagram for v gives a factorization of λ through $\mathcal{O}_X$ and we obtain a morphism of ringed spaces, which is what we wanted. □

The conclusion of our discussion is that for the descent of schemes, it is only effectivity of descent data that may fail. A lot of important criteria of effectivity are given in [SGA1], Exp. VIII, for different types of maps $\alpha : S' \to S$ with respect to different fibred categories of schemes $\mathcal{C} \to \mathcal{S}ch$. We will give below an instance of this. For the moment we must say that the technique of flat descent acquires great strength from the fact that an important number of properties of objects are equally tested after faithfully flat base extension. Precisely :

Theorem 12 *let $f : X \to Y$ be a morphism of S-schemes, let $\alpha : S' \to S$ be an fpqc or fpuo morphism, and let f' be the pullback by α. Let $\mathcal{P}$ be a property of morphisms among : injective, surjective, with finite fibres, bijective, radicial, (universally) open, (universally) closed, (universally) a homeomorphism, quasi-compact, (quasi-)separated, (locally) of finite type, (locally) of finite presentation, an open, closed or quasi-compact immersion, proper, an isomorphism, (quasi-)affine, (quasi-)finite, integral, flat, smooth, unramified, étale. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.*

Proof : For α fpqc this is EGA IV, 2.6.1, 2.6.2, 2.6.4, 2.7.1, 17.7.3. Now assume that α is fpuo. All properties are preserved by arbitrary base extension except $\mathcal{P}$ = open, closed and being a homeomorphism. But it is easy to check that these three are preserved by fpuo base extension. So there is only the "if" part left to check. Furthermore we can assume that $S = Y$ because $Y' \to Y$ is still fpuo. Finally all the properties $\mathcal{P}$ are local on the base, i.e. we can assume that Y is affine.

Then choose an open affine cover $Y' = \cup_{i\in I} Y_i'$. By the assumption that α is open, the images Y_i in Y form an open cover of Y. By quasi-compactness a finite number $Y_1, \dots, Y_n$ cover Y. Put $Y'' = \cup_{i=1}^n Y_i'$, so we have an open immersion $Y'' \hookrightarrow Y'$ such that $Y'' \to Y$ is fpqc. Consider f'', the pullback of f by $Y'' \to Y$. If f' has $\mathcal{P}$ then f'' still has $\mathcal{P}$, because all the properties $\mathcal{P}$ are preserved by base extension, except open, closed and being a homeomorphism, but these are at least preserved by restriction to an open set. Hence we are reduced to the fpqc case. □

We now prove effectivity of fpqc and fpuo morphisms for quasi-affine schemes :

Theorem 13

(i) *Any fpqc or fpo morphism is a morphism of effective descent for the fibred category* $(\mathcal{Q}c{+}\mathcal{A}mp) \to \mathcal{S}ch$ *of relatively quasi-compact schemes together with an ample sheaf.*

(ii) *Any fpqc or fpo morphism is a morphism of effective descent for the fibred category* $\mathcal{Q}\mathit{aff} \to \mathcal{S}ch$ *of relatively quasi-affine schemes.*

Proof : (sketch) Let $\alpha : S' \to S$ be fpqc (resp. fpo) , $f' : X' \to S'$ be quasi-compact, and $\mathcal{L}'$ an ample sheaf on X'. Look at the quasi-coherent sheaf of graded algebras $\mathcal{S}' := \oplus f'_* \mathcal{L}'^{\otimes n}$. By assumption X' is open in $P' = \mathrm{Proj}(\mathcal{S}')$. One uses th. 7 to descend $\mathcal{L}'$ and $\mathcal{S}'$ to $\mathcal{L}$ and $\mathcal{S}$, then X' descends to an open subscheme of $P = \mathrm{Proj}(\mathcal{S})$. Since a scheme is quasi-affine if and only if $\mathcal{O}_X$ is ample, making $\mathcal{L} = \mathcal{O}_X$ we get (ii). □

Remark 14 Contrary to fpqc, the notion of fpo morphism is not stable under base change. In view of the importance of base changes in the theory it would be preferable to work with *universally open* morphisms (hence we would look at *fpuo*). An important subclass of these morphisms is provided my the morphisms *locally of finite presentation* (ref). We note *fppf* for faithfully flat and locally of finite presentation. Be careful that for certain authors this stands for faithfully flat and of finite presentation, hence assuming quasi-compactness and quasi-separatedness.

Complements

There is a couple of important remarks to make before we close the topic. Namely we want to evoke the problems of non-effectivity and a few exotic (though very useful) situations of descent.

(rk1) The first remark is that when we work locally in the Zariski topology, usually we don't check the compatibility conditions on open set intersections, but rather content ourselves to claim that "it is clear that" they glue or that "being canonical", the construction globalizes. When we work locally in different "topologies" (this concept is introduced later) we often have the same somewhat lazy attitude, that is to say we often pass over the verification that a glueing datum is a descent datum, and sometimes we don't even provide a glueing datum. But one has to be careful in doing so that it needs a little practice and experience to feel exactly when the glueing will work, and when it won't.

(rk2) There is in *Néron Models* [BLR], § 6.1, th. 6, a refinement of theorem 11 that explains why some schemes may have non-effective descent data. One finds counter-examples to effective descent in [BGFFK] (App. on descent) and [BLR] (chap. 6).

(rk3) There are a few results of *non-flat* descent, i.e. where the "covering map" α is not flat and however α^* induces an equivalence of categories. For such examples see Laszlo-Beauville (*Un lemme de descente*, C.R.A.S. Paris, Sér. I 320, No.3, 335-340 (1995)) or Grothendieck (Bourbaki seminar, exposé 195, B. th.2, and exposé 190, A.4.e, and EGA IV_3 11.4 to 11.6).

(rk4) Lastly, we mention a situation in [BLR] (§ 6.2, prop. D.4) which can be considered also as non-flat descent. They study descent for $\alpha : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ where $R \subset R'$ is an extension of discrete valuation rings with same uniformizing element and same residue field (thus this descent of an arithmetic nature, like in the Galois example below). This includes an étale extension of discrete valuation rings with trivial residual extension, or R' = henselization or completion of R. Call K, K' the fraction fields. The scheme $S'_1 = \mathrm{Spec}(R')$ alone is a fpqc covering of $S = \mathrm{Spec}(R)$, but [BLR] considers together with it the scheme $S'_2 = \mathrm{Spec}(K)$ (also flat quasi-compact over S). Their result is that descent holds with a much lighter descent datum than usual, namely with a *glueing* datum on the *only* $S'_1 \cap S'_2$ (and not on S''_1, S''_2 as would seem natural).

Examples - counter-examples of flat maps

A sum of affine schemes which is not affine : $\amalg_{i=1}^{\infty} \mathrm{Spec}(k)$

An fpqc map which is not open : $\mathrm{Spec}(\mathbb{Z}) \amalg \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$

An fpo map which is not qc : $\amalg_{i=1}^{\infty} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ (trivial)

An fp map which is open, quasi-compact, but not universally open : take a field k, let $A = k[t]_{(t)}$ be the local ring of the origin in $\mathbb{A}^1_k$. Then the morphism $\mathrm{Spec}(\widehat{A} \otimes_A \widehat{A}) \to \mathrm{Spec}(\widehat{A})$ is not open.

An fp map which is not of effective descent for $\mathcal{Q}coh$: $\amalg_p \operatorname{Spec}(\mathbb{Z}_p) \to \operatorname{Spec}(\mathbb{Z})$. Indeed for this map we have

$$S' \times_S S' = \coprod_{p,q} \operatorname{Spec}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q)$$

An example of Galois descent

Descent theory is essential to do "relative" algebraic geometry, which studies objects *over a general base scheme* S rather than over an algebraically closed field k. The case of a base which is a non-algebraically closed field, although simple, can't usually be treated by classical methods and is a nontrivial application of descent. Let's work out an example in full detail.

Let C' be the conic in $\mathbb{P}^2_{\mathbb{C}}$ with equation

$$x^2 + 2ixy + z^2 = 0$$

It is actually defined over $\mathbb{Q}(i)$ (let $k_0 \subset k$ be an extension of fields, we say that a k-scheme X is defined over k_0 if there exists a k_0-scheme X_0 such that $X \simeq X_0 \otimes_{k_0} k$). This is descent already but it is trivial. From now on we see C' as a $\mathbb{Q}(i)$-curve.

A little less trivial fact is that actually it is defined over $\mathbb{Q}$: put $w = x + iy$ and you get the curve C_0 with equation

$$w^2 + y^2 + z^2 = 0$$

Note that a curve C_0 s.t. $C' \simeq C_0 \otimes_{\mathbb{Q}} \mathbb{Q}(i)$ is *not* unique. Indeed, if you put $u = x + iy$ and $v = y - iz$ you set up an isomorphism between C' and the projective line $\mathbb{P}^1_{\mathbb{Q}(i)}$ which of course is $\mathbb{P}^1_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(i)$. But you don't have $C_0 \simeq \mathbb{P}^1_{\mathbb{Q}}$ as $\mathbb{Q}$-schemes simply because C_0 has no rational point. (remark: the reason why C "comes from" $\mathbb{P}^1$ is that it has genus 0, like all conics !).

Clearly for less trivial examples it would be hard to see whether the curve is defined over a subfield; also in general there does not exist a minimal subfield.

In terms of descent here is the translation : put $S = \mathrm{Spec}(\mathbb{Q})$, $S' = \mathrm{Spec}(\mathbb{Q}(i))$, we have a curve C' over S' and we wonder if it descends to S. We have that S'' is the spectrum of $\mathbb{Q}(i) \otimes \mathbb{Q}(i) \simeq \mathbb{Q}(i) \times \mathbb{Q}(i)$ (an isomorphism is given by $x \otimes y \mapsto (xy, x\overline{y})$ where the bar is complex conjugation). Simplify notation to $* := \mathrm{Spec}(\mathbb{Q}(i))$, then

$$S' = *$$

$$S'' = \{*, *\}$$

$$S''' = \{*, *, *, *\} \quad \text{(careful !)}$$

We can compute all projection maps

$$p_1, p_2 : S'' \to S' \qquad \pi_{12}, \pi_{13}, \pi_{23} : S''' \to S'' \qquad \text{and} \qquad \pi_1, \pi_2, \pi_3 : S''' \to S'$$

$$\begin{array}{rcl} p_1^*(a) & = & (a, a) \\ p_2^*(a) & = & (a, \overline{a}) \end{array} \qquad \begin{array}{rcl} \pi_{12}^*(a, b) & = & (a, b, a, b) \\ \pi_{13}^*(a, b) & = & (a, a, b, b) \\ \pi_{23}^*(a, b) & = & (a, \overline{b}, b, \overline{a}) \end{array} \qquad \begin{array}{rcl} \pi_1^*(a) & = & (a, a, a, a) \\ \pi_2^*(a) & = & (a, \overline{a}, a, \overline{a}) \\ \pi_3^*(a) & = & (a, a, \overline{a}, \overline{a}) \end{array}$$

So p_1^*C is two copies of C, p_2^*C is a copy of C and a copy of $\overline{C}$, and so on. We see (...) that

- a glueing datum for C' with respect to $S' \to S$, that is to say an isomorphism $\tau : p_1^*C \simeq p_2^*C$, is just given by an isomorphism $\sigma : C \simeq \overline{C}$
- a descent datum (for C' with respect to $S' \to S$) is a glueing datum such that $\overline{\sigma} \circ \sigma = \mathrm{id}$

(we must check $\pi_{23}^*\tau \circ \pi_{12}^*\tau = \pi_{13}^*\tau$, but we have $\tau = (\mathrm{id}, \sigma)$ so $\pi_{12}^*\tau = (\mathrm{id}, \sigma, \mathrm{id}, \sigma)$ and so on...) Note that $\overline{C}$ has equation $x^2 - 2ixy + z^2 = 0$. We can give plenty of glueing data :

- $\sigma(x : y : z) = (-x : y : z)$
- $\sigma(x : y : z) = (x : -y : z)$
- $\sigma(x : y : z) = (-ix : -iy - x : z)$
- $\sigma(x : y : z) = (-ix + 2y : -x - iy : z)$

Exercise: the first three are descent data, the last is not (if I made no mistake). So it is not enough to give an isomorphism between C and $\overline{C}$...

In any case this shows that C' indeed descends to $S = \mathrm{Spec}(\mathbb{Q})$. The fact that the descended curve is not unique corresponds to the fact that we can choose different descent data (we can glue on S'' in different ways). But if we *fix* a descent datum then the curve descends uniquely.

Remarks 15

(i) Exercise : show that the elliptic curve with equation $y^2 + xy = x^3 + i$ does not descend to $\mathbb{Q}$.

(ii) Culture : let $k_0 \subset k$ be a Galois extension, then for an elliptic curve over k, any glueing datum w.r.t the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ is effective. It is not so easy to find curves over k with glueing data that don't descend to k_0, but there are (cf Dèbes & Douai, *Algebraic covers: Field of moduli versus field of definition*, Ann. Sc. ENS 1997).

(iii) The general arithmetic descent treated by Weil said that a k-variety descends to k_0 if and only if there exist isomorphisms $\varphi_g : C \simeq {}^gC$ for all $g \in \mathrm{Gal}(k/k_0)$ such that for all g, h we have $\varphi_{gh} = g\varphi_h \circ \varphi_g$.

Inductive limits

Here I give a few definitions and remarks that may be known already from the reader, but are rarely presented this way. It is also a pretext to introduce groupoids. This paragraph would perhaps better fit in the place of an appendix, but it is given here for the sake of logical order of reading.

Definitions

Briefly said, three major kinds of inductive limits will be of interest to us : the (amalgamated) sums, the quotients, and the filtered inductive limits [1].

Let I and $\mathcal{C}$ be categories. Any object $X \in \mathcal{C}$ defines a point category, which is a final object in $\mathcal{C}at$, so there is a unique functor $c_X : I \to X \to \mathcal{C}$, covariant in X.

Definition 16 Let I, I' and $\mathcal{C}$ be categories.

(i) The *inductive limit* of a functor $F : I \to \mathcal{C}$ is the functor $\mathcal{L}_F : \mathcal{C} \to \mathcal{E}ns$ defined by

$$\mathcal{L}_F(X) = \mathrm{Hom}_{\mathcal{F}unct}(F, c_X)$$

When it is representable we write $\varinjlim F$ the representing object, well-defined up to unique isomorphism.

(ii) A functor $\varphi : I' \to I$ is *cofinal* if the morphism $\mathcal{L}_F \to \mathcal{L}_{F\circ\varphi}$ is an isomorphism for all $F : I \to \mathcal{C}$.

In more down-to-earth terms a functor $\psi : F \to c_X$ is given by maps $A_i \to X$ for all $i \in I$, where $A_i = F(i)$, with commutativity of $A_i \to A_j \to X$ whenever $i \to j$ is a morphism in I (warning : different morphisms from i to j give rise to different morphisms form A_i to A_j).

Remarks 17 (i) As is clear already from the case of arbitrary direct sums (see 18(i) below), in general when the class of objects of I is not a set the inductive limit of a functor won't be representable. So in the sequel we will stick to *small* categories in the sense that $\mathrm{Ob}(I)$ is a set.

(ii) Another assumption can be made harmlessly. If we factor F through $I \to F(I) \to \mathcal{C}$, it is clear that $I \to F(I)$ is cofinal for F (in an obvious meaning), so we will always assume that I is a *subcategory* of $\mathcal{C}$ (via F).

(iii) If an object of I has one and only one morphism from it, then erasing this object together with its morphism gives a cofinal subcategory of I.

Examples 18 The following examples are essential. The first three describe the I's with low number of non-identity maps :

(i) no map : the inductive limit is by definition the *direct sum* of the objects $A_i = F(i)$, denoted $\coprod_{i\in I} A_i$.

(ii) one map : clearly $\{2\} \to \{1 \to 2\}$ is cofinal, bringing us back to (i).

(iii) two maps :
- when $I = \{1 \rightrightarrows 2\}$, then F is equally given by a diagram $A \rightrightarrows B$. The colimit is by definition the *cokernel* denoted $\mathrm{coker}\,(A \rightrightarrows B)$.
- when $I = \{1 \to 2, 1 \to 3\}$, then F is equally given by two morphisms $A \to B$ and $A \to C$. The colimit is by definition the *amalgamated sum* denoted $B \amalg_A C$.
- the last cases with two maps are $I = \{1 \to 3, 2 \to 3\}$ and $I = \{1 \to 2, 3 \to 4\}$, they are trivial as they have cofinal subcategories $\{3\}$ and $\{2, 4\}$ respectively.

(iv) another highly interesting example is the case of *quotients* which we will introduce later.

(v) examples of amalgamated sums are amalgams in topology and tensor product of A-algebras.

Case (i) is the "disconnected" situation ; it turns out that any case of (iii) is enough to encapture the "connected" situations and give all colimits :

Proposition 19 *Consider an inductive limit of a functor $F : I \to \mathcal{C}$ from a small category. If the category $\mathcal{C}$ has direct sums, and either cokernels or amalgamated sums, then it has all arbitrary small inductive limits.*

[1] A point of vocabulary : the following terms are used : colimits, inductive limits, (direct, disjoint, amalgamated) sums, cokernels, quotients. Dually we have limits, projective (or inverse) limits, (direct, fibred) products, fibred products, kernels.

Proof : We just give a sketch of the construction, and work out the recipe on examples rather than immerse into the verifications. First, assuming $\mathcal{C}$ has direct sums, the existence of cokernels or amalgamated sums are equivalent because

$$\mathrm{coker}\,(\,A \rightrightarrows B\,) = B \coprod_{A \amalg B} B \qquad \text{and} \qquad A \coprod_{C} B = \mathrm{coker}\,(\,C \rightrightarrows A \amalg B\,)$$

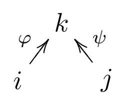
Now let $F : I \to \mathcal{C}$ be a functor. Consider the coproduct of all domains of maps of $F(I)$, and the coproduct of all objects of $F(I)$:

$$C_1 = \coprod_{i \to j \in \mathrm{Hom}(I)} F(i) \qquad \text{and} \qquad C_2 = \coprod_{i \in I} F(i)$$

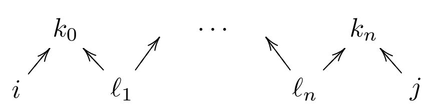
By definition of C_1, a map $C_1 \to C_2$ is given by maps $F(i) \to C_2$ for all $\varphi : i \to j$. Let's consider the two particular maps given (1) by the maps $F(i) \to F(i)$ and (2) by the maps $F(\varphi) : F(i) \to F(j)$. Then we have $\varinjlim F = \mathrm{coker}\,(\,C_1 \rightrightarrows C_2\,)$. □

Limits in $\mathcal{S}et$

We'll introduce filtered limits after we see them arise naturally in the computation of inductive limits in the particular case of $\mathcal{C} = \mathcal{S}et$. The procedure in the proof of the proposition exhibits an inductive limit of sets as the quotient of the direct sum $\coprod_{i \in I} A_i$ by an equivalence relation. Let $\smile$ be the reflexive symmetric relation defined by $a_i \smile a_j$ iff there exist k, φ, ψ as follows



such that $\varphi(x_i) = \psi(x_j)$. The transitive closure of $\smile$ is defined by $a_i \sim a_j$ iff there is a finite chain



with relations $a_i \smile b_1 \smile \cdots \smile b_n \smile a_j$. Then the inductive limit is $\varinjlim F = \coprod_{i \in I} A_i / \sim$. In the case of a cokernel $A \overset{\varphi,\psi}{\rightrightarrows} B$ it is easy to see that all amounts to a relation on B defined by $b \sim b'$ iff there is a chain of elements in B and A as below, and $\mathrm{coker}\,(\,A \rightrightarrows B\,) = B/\!\sim$.

The main trouble with this transitive closure process appears with the search for a commutation with products. Assume given two functors $F : I \to \mathcal{S}et$ and $G : J \to \mathcal{S}et$. Then there is an obvious product $F \times G : I \times J \to \mathcal{S}et$, but we don't have in general $\varinjlim F \times G = \varinjlim F \times \varinjlim G$ for lack of a possibility to "refine chains". Here is a counterexample : look at the two maps $\mathbb{R} \to \mathbb{R}$ given by the identity and $x \mapsto x+1$. Then $\mathrm{coker}\,(\,\mathbb{R} \rightrightarrows \mathbb{R}\,) = \mathbb{R}/\mathbb{Z}$. But, $\mathrm{coker}\,(\,\mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R} \times \mathbb{R}\,) = \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ because the product maps are the identity and $(x, y) \mapsto (x + 1, y + 1)$.

Remark 20 If F and G are direct sums then commutation holds : $\coprod_{i,j \in I \times J} A_i \times B_j = \coprod_{i \in I} A_i \times \coprod_{j \in J} B_j$. Indeed in this case there are no chains, so nothing needs refinement. This is why I looked for a counterexample among cokernels.

Filtered limits

The important case of filtered colimits is one where the pre-equivalence relation $\smile$ is already an equivalence relation, and all chains can be refined. In this case commutation with product holds, and even commutation with any *finite limit* (i.e. a limit such that the category of indices I is finite i.e. has finitely many objects and morphisms). We remain very concise here ; a good place where to find these definitions and results gathered is SGA4, exposé I, § 2.

Definition 21 A small category I is said to be *filtered* if

(i) for any $i, j \in I$ there exists k and maps $i \to k$, $j \to k$.

(ii) any two maps $h \to i$, $h \to j$ can be completed into a commutative square thanks to $i \to k$, $j \to k$.

Examples of filtered colimits are the local ring of a point in a ringed space, the sheafification process, the algebraic closure or henselization. Note that when I is a preordered set (that is to say there is at most one map between any two objects), condition (i) alone is the usual definition of filtered preordered sets, and (ii) is automatic.

Proposition 22 *In the categories $\mathcal{S}et$, A-$\mathcal{M}od$, A-$\mathcal{A}lg$ (for a commutative unitary ring A), $\mathcal{M}od(X)$ (for a topological space X), filtered colimits commute with finite limits.* □

To see this, one first shows that we can restrict to commutation of filtered colimits with fibred products (easy). Let there be given a filtered category I (or a filtered set for simplicity), three inductive systems (A_i, α_{ij}), (B_i, β_{ij}), (C_i, γ_{ij}), and morphisms of inductive systems $\{u_i\} : \{A_i\} \to \{C_i\}$ and $\{v_i\} : \{B_i\} \to \{C_i\}$. This means that for all $j \geqslant i$ we have $\gamma_{ij} u_i = u_j \alpha_{ij}$ and $\gamma_{ij} v_i = v_j \beta_{ij}$. Then there are induced arrows $\delta_{ij} : A_i \times_{C_i} B_i \to A_j \times_{C_j} B_j$ making the $D_i = A_i \times_{C_i} B_i$ into an inductive system. Furthermore let $A = \varinjlim A_i$, etc. Then there are morphisms $D \to A \to C$, $D \to B \to C$, inducing $D \to A \times_C B$, and what is to be checked is that this is an isomorphism. We won't prove this.

Groupoids in a category

First we introduce *quotients* that are particular inductive limits. Assume that I has a *quasi-terminal* object, meaning an object $* \in I$ such that any $i \in I$ has at least one map to $*$ (this is not standard terminology, so use it with precaution). Then we will say that the inductive limit functor $\mathcal{L}_F$ of a functor $F : I \to \mathcal{C}$ is a *quotient*. If the inductive limit exists, then the map $A_* \to \varinjlim F$ is an epimorphism and we say it is a quotient of A_*. Here are now examples in $\mathcal{S}et$.

(ex1) The most basic example is the cokernel $A \overset{\varphi,\psi}{\rightrightarrows} B$. As we said before, its inductive limit is computed by looking at the equivalence relation associated to the relation $\smile$ whose graph is the image of the map $(\varphi, \psi) : A \to B \times B$.

(ex2) An equivalence relation $R \rightrightarrows X$ is the particular case of cokernel when, in the notations above, (φ, ψ) is injective and $\smile$ is already... an equivalence relation.

(ex3) A group action of G on a set X yields a cokernel $G \times X \rightrightarrows X$ where the two maps are the action and the projection onto X. This is an equivalence relation exactly when the action is free.

(ex4) Let $\mathcal{C}$ be a category whose class of objects X_0 is a set. Then the class X_1 of morphisms is a set also, and there are maps s, t from X_1 to X_0 giving the source and target of a morphism. There is also a map $X_0 \to X_1$ giving the identity morphism of each object. Finally, we have the set X_2 of composable morphisms which is none other than the fibre product $X_2 = X_1 \times_{t, X_0, s} X_1$. There are three maps from X_2 to X_1, namely the two projections and the map giving composition in the category $\mathcal{C}$. As a conclusion, any small category gives rise to a quotient inductive limit functor $X_2 \Rrightarrow X_1 \rightrightarrows X_0$. Conversely it is clear that this diagram together with some commutative diagrams (expressing associativity of the composition in $\mathcal{C}$, compatibility with identities) allow to recover the category structure.

(ex5) A (small) groupoid is a (small) category $\mathcal{G}$ in which any morphism is invertible. In other words, in addition to the structure of a small category, there exists a map $X_1 \to X_1$ giving the inverse of any morphism. Note that groupoids include equivalence relations as well as group actions. A set X gives rise to a groupoid without morphisms ; a group G gives rise to a groupoid $B_0 G$ with one single object whose automorphism group is G, i.e. as the action of G on a point (necessarily trivial). Most often, as X_2 is determined by the rest of the data, groupoids are simply denoted $\mathcal{G} = (\ X_1 \rightrightarrows X_0\ )$.

Now let us indicate the way in which one can extend these concepts (and examples of other algebraic structures, usually defined on an underlying set, such as the structures of group, module, ...) to any category other than $\mathcal{S}et$, thanks to the Yoneda embedding. So let $\mathcal{C}$ be a category and denote by $\mathcal{C}^\wedge = \mathrm{Hom}_{\mathcal{C}at}(\mathcal{C}^0, \mathcal{S}et)$ the category of presheaves of sets on $\mathcal{C}$. First of all let us recall the well-known :

Proposition 23 *(Yoneda's lemma)*

(i) *(weak form) the functor* $h : \mathcal{C} \to \mathcal{C}^\wedge$ *which maps* X *to* $\mathrm{Hom}(\cdot, X)$ *is fully faithful.*

(ii) *(strong form) for any presheaf* $F \in \mathcal{C}^\wedge$ *and* $X \in \mathcal{C}$*, the map* $F(X) \to \mathrm{Hom}(h_X, F)$ *is a bijection.* $\square$

The proof is an exercise. Point (i) says that for any X, Y, the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ is a bijection. As $\mathrm{Hom}(X, Y) = h_Y(X)$, point (ii) says that (i) extends to presheaves F which are not necessarily of the form h_Y.

Now let $\mathcal{A}$ be a category of "algebraic" objects. By this we mean, quite generally, that an object $A \in \mathcal{A}$ is given by a structure supported by several sets (finitely many), related by several maps between them (finitely many), together with several diagrammatic conditions satisfied between these maps. For an example, a group structure is supported by a set, or more precisely by two sets (since it involves a unit element) and the multiplication and inverse maps. Other examples are rings, modules, equivalence relations or small categories (see above). If the structure in $\mathcal{A}$ involves only one set, there is a forgetful map $\omega : \mathcal{A} \to \mathcal{S}et$. If it involves two sets there is a forgetful map $\omega : \mathcal{A} \to \mathcal{S}et \times \mathcal{S}et$, and so on.

Definition 24 If the structure in $\mathcal{A}$ involves only one set, we say that an object $X \in \mathcal{C}$ is an $\mathcal{A}$*-object* if the functor $h_X : \mathcal{C}^0 \to \mathcal{S}et$ factors through $\mathcal{A}$, i.e. if the sets $\mathrm{Hom}(X, Y)$ are all objects of $\mathcal{A}$ functorially in Y. If the structure in $\mathcal{A}$ involves two sets, we say that $(X, R) \in \mathcal{C} \times \mathcal{C}$ is an $\mathcal{A}$*-object* if the functor $h_X \times h_R : \mathcal{C}^0 \times \mathcal{C}^0 \to \mathcal{S}et \times \mathcal{S}et$ factors through $\mathcal{A}$, and so on.

For example, a topological group is a group object in the category of topological spaces, while a group scheme is a group object in the category of schemes (observe that according to instances, the emphasis in the choices of terminology is put once on the group structure, once on the "variety" structure). Other well-known examples are Lie groups, ordered fields, G-sets...

By the weak form of Yoneda's lemma the maps giving the structure in $\mathcal{A}$ on the sets $\mathrm{Hom}(X, Y)$ come from maps on X. So, for example, a groupoid in a category $\mathcal{C}$ is an object $X_1 \rightrightarrows X_0$ such that $X_1(S) \rightrightarrows X_0(S)$ is a groupoid in $\mathcal{S}et$, functorially in $S \in \mathcal{C}$.

Sheaves and Grothendieck topologies

The second step we are led to consists, quite naturally, in trying to axiomize the definitions and properties of the more general "coverings" arising from faithfully flat descent, and try to see if we can carry on the study of sheaves, cohomology and so on, as it is developped in the framework of topological spaces (or schemes with the Zariski topology). If we don't make cohomology we don't actually need to go far into the theory.

Basic definitions

First of all, observe that by Yoneda's lemma 23, any object of $\mathcal{C}$ may be seen as a presheaf. When $\mathcal{C}$ is the category of schemes, the maps $T \to X$ are called the *points of X with values in T*. For example the maps $\mathrm{Spec}(k) \to X$ are ordinary points (or geometric points), the maps $\mathrm{Spec}(k[\epsilon]) \to X$ are the tangent vectors... Yoneda's lemma says that a scheme is determined by its points.

Remark 25 This raises natural questions : it is clear that a scheme is not determined by its points with values in a field (the naive points), but is it determined by its points with values in arbitrary artinian rings (i.e. all infinitesimal neighbourhoods of points) ? or complete local rings (formal neighbourhoods) ? when is a presheaf $X \in \mathcal{S}ch^\wedge$ a scheme ?...

We now address the question of giving a good axiomatic framework in order to pass from presheaves to sheaves. To this aim, we introduce here the concept of *pretopology*, which, for stack theory, is enough and is the one used in practice. (Note that the general concept of topology, with sieves, can't be avoided in topos theory, and also that different pretopologies may give the same topology, this has to be kept in mind). However for simplicity we use the word "topology" instead of "pretopology".

Definition 26 A *topology* $\mathcal{T}$ on a category with fibred products $\mathcal{C}$ is given by the data of sets $\mathrm{Cov}(U)$, for each object $U \in \mathcal{C}$, whose elements are families $\{U_i \to U\}$ of maps to U called *coverings* of U, all assumed to satisfy

(i) any isomorphism $U' \to U$ is a covering,

(ii) if $\{U_i \to U\}$ is covering and $V \to U$ is a map, then $\{U_i \times_U V \to V\}$ is covering,

(iii) if $\{U_i \to U\}$ is covering and for each i we are given a covering $\{V_{i,j} \to U_i\}$, then $\{V_{i,j} \to U\}$ is a covering.

Example 27 On $\mathcal{C} = \mathcal{S}ch$ we define the fpqc topology (resp. fppf, resp. étale) by choosing as coverings of a scheme U the families $\{U_i \to U\}$ where the images of the U_i cover U and each $U_i \to U$ is flat and quasi-compact (resp. flat and of finite presentation, or étale).

Definition 28 A category $\mathcal{C}$ endowed with a topology is called a *site*. Let $(\mathcal{C}, \mathcal{T})$ be a site, then a *sheaf* on $\mathcal{C}$ is a presheaf $F \in \mathcal{C}^\wedge$ such that for any $X \in \mathcal{C}$ and any covering $\{X_i \to X\}$, the following diagram is exact :

$$F(X) \longrightarrow \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j)$$

The category of sheaves on $(\mathcal{C}, \mathcal{T})$ is usually denoted $\mathcal{C}^\sim$.

Theorem 29 *Any scheme is a sheaf in the fpqc topology.*

Proof : This is theorem 11. □

In view of the theorem, when $\mathcal{C} = \mathcal{S}ch$ is endowed with the fpqc, fppf or étale topology, for the corresponding category of sheaves we have $\mathcal{C} \subset \mathcal{C}^\sim \subset \mathcal{C}^\wedge$. In general, given a site $(\mathcal{C}, \mathcal{T})$, it is not necessarily the case that $\mathcal{C} \subset \mathcal{C}^\sim$; actually there is a finest topology such that all representable presheaves are sheaves, called the *canonical topology.* Thus, continuing the questions in remark 25, *a necessary condition for a functor to be representable by a scheme is to be a sheaf for the fpqc topology.*

In the sequel we will deal with objects such as sheaves, algebraic spaces, and ultimately stacks, that won't be schemes any more - schemes will form a faitful subcategory of the previsous categories. Of course it will be nice, if it happens, to recognize when such an object is a scheme, and in the relative situation it will be as important to sort out those morphisms "whose fibres are schemes" :

Definition 30 A map of presheaves $u : F \to G$ is said to be *representable* if for all maps $X \to G$ from a scheme to G, the fibre product $F \times_G X$ is representable. For such a map, it makes sense to take a property of maps of schemes **P** and to say that, by definition, "u has **P**" if and only if $F \times_G X$ has **P**, for all $X \to G$.

If F is any presheaf, there is a way to sheafify it, using the following construction :

$$F^+(X) := \varinjlim_{\{X_i \to X\} \in \mathrm{Cov}(X)} \ker \Big(\prod F(X_i) \rightrightarrows \prod F(X_i \times_X X_j) \Big)$$

There is a canonical map $F \to F^+$. Here are the properties of the + operation (we only give the result, since the proof is neither appetizing nor really instructive for our needs) :

Theorem 31 *Let $(\mathcal{C}, \mathcal{T})$ be a site, and denote by $\mathcal{C}^\sim$ the category of sheaves on this site. Let F be a presheaf on $\mathcal{C}$, then*

(i) *the presheaf F^+ is separated (i.e. the injectivity part of the exact diagram is verified).*

(ii) *the presheaf F^{++} is a sheaf.*

(iii) *the functor $F \mapsto F^{++}$ is a left adjoint to the inclusion $\mathcal{C}^\sim \subset \mathcal{C}^\wedge$.* □

Main examples of topologies

Mention here fpqc-fppf-étale-Zariski. Describe the coverings (comparison lemma ?). Explain why fpqc causes trouble for sheafification (inverse limits don't exist). Mention the sequence of sheaves

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{x \mapsto x^p} \mathbb{G}_a \longrightarrow 0$$

that is exact only in the flat topology.

An example : the étale topology

This is an extremely important example because étale morphisms have very special properties, and in some sense étale localization for schemes plays the role of localization (for the usual complex topology) for complex varieties. Here are a few important situations.

Example 32 *Exact sequences of sheaves.* Some usual sequences of abelian sheaves that fail to be exact in the Zariski topology are exact in the étale topology. The standard examples are

- the Kummer sequence for schemes over $\mathbb{Z}[1/n]$: $0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \longrightarrow 0$

- the Artin-Schreier for schemes over $\mathbb{F}_p$: $0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{x \mapsto x^p - x} \mathbb{G}_a \longrightarrow 0$

In both cases, surjectivity fails in the Zariski topology.

Example 33 *Inverse mapping theorem.* Here is the algebraic form of the Inverse mapping theorem : *any smooth morphism $f : X \to S$ acquires a section after an étale extension $S' \to S$*, see EGA IV, 17.16.3.

Example 34 *G-bundles.* Let G be a finite group. Let X be a variety on which G acts freely, and let $Y = X/G$. Then the morphism $\pi : X \to Y$ has the properties of a G-bundle, in the sense that its geoemtric fibres are isomorphic to G, and however, locally in the Zariski topology, X is not isomorphic to G. The most trivial example of this we already saw : let $k_0 \subset k$ be a Galois extension of group G, then $\pi : \mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ is clearly not trivial Zariski-locally on $\mathrm{Spec}(k_0)$. However, after the étale extension $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$, it becomes trivial.

We are led to the definition of a *torsor*. The phenomenon above being similar for more general groups we include this here :

Definition 35 Let $G \to S$ be a flat, affine group scheme of finite presentation. A *G-torsor* is an S-scheme $X \to S$ with an action of G such that there exists an étale extension $S' \to S$ such that $X \times_S S'$ is isomorphic to the trivial torsor, that is to say $G \times_S S'$ with action by left multiplication.

Example 36 *Brauer-Severi schemes.* A proper flat S-scheme of finite presentation whose geometric fibres are isomorphic to some projective space $\mathbb{P}^r$, is not necessarily isomorphic to $\mathbb{P}^r \times U$ Zariski-locally. Such a scheme is called a Brauer-Severi scheme ; Grothendieck showed that they are exactly those isomorphic to projective S'-space $\mathbb{P}^r_{S'}$ after an étale extension. For an example, when we look at families of hyperelliptic curves (meaning a projective, smooth morphism $f : C \to S$ together with an involution $\sigma : C \to C$, such that the geometric fibres are hyperelliptic curves C_s with hyperelliptic involution σ_s), the quotient $D = C/\sigma$ naturally arises as a "curve of genus 0" over S (i.e. it is projective smooth over S, with fibres of genus 0). However it is not true that $D \simeq \mathbb{P}^1_S$, not even locally on S (a stupid counterexample is in these notes). This is true only after an *étale* extension $S' \to S$. Let's check this. The tool is to apply the Inverse mapping theorem to the map $D \to S$. Then the image $\Delta \subset D_{S'}$ of this section is a (relative) Cartier divisor, it induces an invertible sheaf $L = \mathcal{O}(\Delta)$ with degree 1 on all fibres over S. So $R^1 f_* L = 0$ where f is the structure map to S'. It follows that $V = f_* L$ is locally free and its formation commutes with base change (EGA III, 7.7 and 7.8 ; better see Mumford's *Geometric Invariant Theory*, 0.5). The natural map $f^* V \to L$ is surjective as is checked on the fibres. There is an induced morphism $D_{S'} \to \mathbb{P}(V)$, a little more work shows that this is an isomorphism. Localizing again (but for the Zariski topology this time) this trivializes. □

Finally let's give a definition that will be useful later on.

Definition 37 We can extend definition 24 of equivalence relations (ER's for short) in a category $\mathcal{C}$, to ER's in a site $(\mathcal{C}, \mathcal{T})$. Namely such an object is by definition an ER in $\mathcal{C}$, denoted $R \rightrightarrows U$, such that the two maps $R \to U$ are covering maps of $\mathcal{T}$. Exercise : an ER in a site $(\mathcal{C}, \mathcal{T})$ gives an ER in the category of sheaves of sets on $\mathcal{C}$.

Algebraic spaces

Introduction

Let X be a proper connected scheme over a field k. Then by quasi-compactness it can be covered by a finite number of affine schemes U_i, so their disjoint sum U is affine again. Then X is completely determined by U and the glueing data i.e. the open subsets $U_i \cap U_j = U_i \times_X U_j$ of X, whose disjoint union is $R = U \times_X U$, an affine scheme, by the assumption of separation. There is a canonical injection $R \to U \times U$ making R an equivalence relation (of affine schemes) on U. Considering the two projections $R \to U$, the covering map $U \to X$ makes X a quotient in the diagram $R \rightrightarrows U \to X$. The departure point of geometry is that all kinds of data on X are determined on U modulo glueing data on R.

Let us assume that we start from an equivalence relation $R \rightrightarrows U$, and we wish to do geometry on the "quotient" $\pi : U \to X$, provided it exists, by working on U. Descent theory tells us that this is possible, provided π happens to be a covering map for a topology such as fpqc, fppf, or étale. If it is so, then localizing thanks to π we have

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \pi \\ U & \xrightarrow{\pi} & X \end{array}$$

so we see that a crucial property is to have $R = U \times_X U$ (we say that the relation is *effective*) because then we can relate the properties of the quotient to those of the maps $R \to U$ we started from.

Hence we must be very careful that the quotient depends in general on the category in which it is considered, and so does effectivity. For instance if X is the proper scheme of the example above, we saw that $X = \operatorname{coker}(R \rightrightarrows U)$ in the category of schemes, and $R = U \times_X U$. But if we view this as a relation in the category of *affine* schemes, of course X can't be a quotient, and in fact it is easy to see that the quotient is $\operatorname{Spec}(k)$ so that $R = U \times_k U$ fails. We could look at the quotient in the category of locally ringed spaces, but the best-suited is to look at a quotient in a category of sheaves. This has the advantage of ensuring effectivity of equivalence relations (see below).

Finally a choice has to be made for the topology. An fpqc map allows descent of most objects on U to X, but is not enough in order to grasp, for instance, the dimension of X or local properties such as smoothness. Hence we will use the étale topology. Note that as long as we don't want to do cohomology, very few of sheaf theory is needed.

Epimorphisms and equivalence relations

Let $\mathcal{C}$ be a category.

Definition 38 Let $u : T \to S$ be a morphism in $\mathcal{C}$.

(i) We say u is an epimorphism if for all $X \in \mathcal{C}$, the map $X(S) \to X(T)$ is injective.

(ii) We say u is a universal epi. if it is an epimorphism after any base change $S' \to S$.

(iii) We say u is an effective epi. if it is an epi. and the diagram $T \times_S T \rightrightarrows T \to S$ is a cokernel.

Property (iii) means that "u is the quotient of the equivalence relation it defines", i.e. the relation on T given by equality of the images in S. For example, in the category of sets, epimorphisms are just surjective maps. They are all effective and universal (in general we will write UEE for "universal effective epimorphism"). This is not the case in a general category : the morphism $\operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Z})$ is an epimorphism in the category of affine schemes, but if p is a prime, then after the base change $\operatorname{Spec}(\mathbb{F}_p) \to \operatorname{Spec}(\mathbb{Z})$ it is no longer an epimorphism (the fibre product is empty), and it is not effective either because $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$. Another counter-example is of course the one given in the introduction : X proper connected over a field k gives rise to an equivalence relation of affine schemes $R \rightrightarrows U$ whose quotient in the latter category is $\operatorname{Spec}(k)$, hence effectivity fails.

Definition 39 Let $R \rightrightarrows U$ be an equivalence relation in $\mathcal{C}$ (definition 24). We say that it is (universally) *effective* if a quotient $X = \operatorname{coker}(R \rightrightarrows U)$ exists (universally), and if $R = U \times_X U$ (universally).

In this case, the quotient map $\pi : U \to X$ is an (universal) effective epimorphism, and conversely any (universal) effective epi. defines an (universally) effective equivalence relation.

In the category of sets all equivalence relations are universally effective, so this property clearly extends to presheaves of sets. It is not too hard to see that it remains valid in categories of sheaves because the "associated sheaf" functor is left exact (see SGA4, Exp. II, 4 or SGA3, Exp. IV, 4.4.3, 4.4.9) :

Proposition 40 *Let $(\mathcal{C}, \mathcal{T})$ be a site, and denote by $\mathcal{C}^\sim$ the category of sheaves on this site. Then all equivalence relations in $\mathcal{C}^\sim$ are universally effective.* □

Quotients of étale ER's of schemes as étale sheaves

Here we study *étale equivalence relations* (definition 37), i.e. ER's $R \rightrightarrows U$ in the category of schemes, such that both maps $R \to U$ are étale. We show that they satisfy two crucial properties (see discussion in the introduction) which will serve as a definition for algebraic spaces.

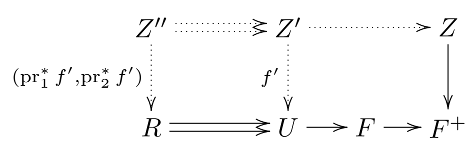
First of all let us give some examples of étale equivalence relations.

Basta for the examples. Given an étale ER denoted $R \rightrightarrows U$, we saw that it gives an ER in the category of étale sheaves (see def. 37 again). One essential point for the sequel is to understand the following description of the quotient. Assume that the quotient *presheaf* U/R fails to be a sheaf, i.e. given an étale covering $Z' \to Z$, some $f' \in U(Z')/R(Z')$ with a descent datum on Z'', does not descend to Z. Then we get the quotient *sheaf* by formally adding all such f' :

Lemma 41 *Let $R \rightrightarrows U$ be an equivalence relation in the category of étale presheaves $\mathcal{C} = \mathcal{S}ch^\wedge$. Let $F = U/R$ be the quotient presheaf. Then its associated sheaf can be described as follows :*

$$F^+(Z) = \left\{ \begin{array}{c} (Z'/Z, f') \text{ with } Z' \to Z \text{ an étale covering and } f' \in U(Z') \\ \text{such that } (\mathrm{pr}_1^* f', \mathrm{pr}_2^* f') \in R(Z'') \subset U(Z'') \times U(Z'') \end{array} \right\}$$

So using Yoneda's lemma (prop. 23), an element in $F^+(Z)$ is a filling into a commutative diagram

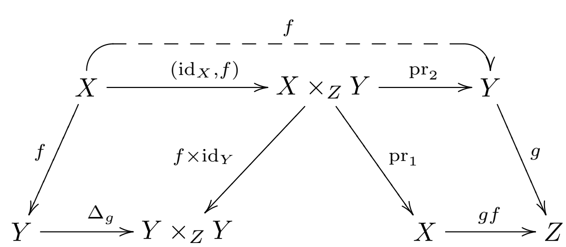


Proof : It is straightforward to define a map $F \to F^+$ and check the universal property. □

After the following easy preparatory lemma, we give the announced properties of étale ER's.

Lemma 42 *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes with gf and Δ_g both locally of finite type. Then f is also locally of finite type.*

Proof : The two parallelograms of the following diagram are cartesian :



□

Proposition 43 *Let $R \rightrightarrows U$ be an étale equivalence relation of schemes, with $\delta : R \to U \times U$ quasi-compact. Let X be the quotient étale sheaf. Then*

(AS1) *The diagonal $X \to X \times X$ is representable and quasi-compact.*

(AS2) *The map $\pi : U \to X$ is representable, étale and surjective (definition 30).*

Proof : First we will show that δ is quasi-affine. Like any monomorphism it is separated and has finite fibres. Recall that quasi-finite + separated implies quasi-affine (EGA IV 18.12.12) so by the assumption of quasi-compactness it is enough to show that δ is locally of finite type. For this we may replace the source of δ by an open affine scheme $\mathrm{Spec}(A)$ and the target by a product $\mathrm{Spec}(B) \times U$. Then the result follows from lemma 42 applied to $f = \delta$ with g equal to the projection $\mathrm{Spec}(B) \times U \to U$.

We now show (AS1). Clearly it is equivalent to show that for any schemes Y, Z, any fibre product $Y \times_X Z$ is representable by a scheme. The result will then follow by using the description in lemma 41 to reduce to the case where the maps $Y, Z \to X$ factor through U. Indeed let us show that we can reduce to $Z = U$. By lemma 41 there exists an étale map $Z' \to Z$ and an object $f' \in U(Z')$ with $\tau = (\mathrm{pr}_1^* f', \mathrm{pr}_2^* f') \in R(Z'')$. So we have a diagram

$$\begin{array}{ccccc} Z'' & \rightrightarrows & Z' & \longrightarrow & Z \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle f'} & & \downarrow \\ R & \rightrightarrows & U & \longrightarrow & X \end{array}$$

Clearly if we assume that $Y \times_X U$ is representable then $Y \times_X Z'$ is also, since it is $(Y \times_X U) \times_U Z'$. Furthermore I claim $Y \times_X Z' \to Y \times Z'$ is quasi-affine : to see this use lemma 41 again, that says there is an étale Y'/Y mapping to U, so after pulling back by $Y' \to Y$ the map we are interested in is a pullback of δ, then conclude with prop. 12. Finally note that τ is a glueing datum on the quasi-affine scheme $Y \times_X Z'$ with respect to the covering map $Y \times Z' \to Y \times Z$. So by effective descent (theorem 13(ii)) we get a quasi-affine $Y \times Z$-scheme which is none other than $Y \times_X Z$, q.e.d. Using this trick once again we now reduce to $Y = U$; and then $U \times_X U = R$ by effectivity of ER's of sheaves, prop. 40. We proved (AS1).

In particular $\pi : U \to X$ is representable. To show that it is étale and surjective we must prove it is so for any $U \times_X Z \to Z$. For this we reduce to $Z = U$ and then the result is just the assumption that $R \to U$ is étale (and surjective). We proved (AS2). □

Algebraic spaces

Property (AS2) is crucially nice for the following reason. We know that for a morphism of schemes $U \to X$, smoothness of X is equivalent to smoothness of U. Similarly, a lot of properties of X are preserved by taking an étale covering, such as reducedness, normality, having given dimension... So for any property which is local on the base for the étale topology, it will make sense to define "X has **P**" if and only if U has **P**.

Hence we showed that quotients of étale ER's have properties that make them workable for a geometer, according to our introductory considerations. These two properties give rise to fairly reasonable abstract objects.

Definition 44 An *algebraic space* over a scheme S is an étale sheaf $X \to S$ such that there exists a scheme U an a morphism $\pi : U \to X$ with properties (AS1) and (AS2) above. We call the map π an *atlas* or a *presentation* of X.

Quite obviously, an algebraic space X is the quotient of the equivalence relation $U \times_X U \rightrightarrows U$ for any chosen atlas U. So we didn't add a new concept.

There are strong results that indicate that we introduced a good, non-trivial notion. First, the quotient of an étale equivalence relation of algebraic spaces is again an algebraic space. This is "not too hard". But there is a much deeper result which was proved by Artin thanks to his algebraization theorems, namely that even the quotient of an *fppf* equivalence relation of algebraic spaces is an algebraic space (see Laumon & Moret-Bailly, cor. 10.4) :

Proposition 45 *Let $R \rightrightarrows U$ be an fppf equivalence relation of schemes, with $\delta : R \to U \times U$ quasi-compact. Then the quotient fppf sheaf is an algebraic space.* □

Algebraic stacks

¿From now on, fibred categories (as in the chapter on descent) are denoted by $\mathcal{M}$ instead of $\mathcal{C}$ (this is psychological).

Definition 46 Let $\mathcal{M}$ be a *fibred category* over the category $\mathcal{S}ch/S$, as above. We say that $\mathcal{M}$ is *fibred in groupoids* if the fibres $\mathcal{M}(T)$ are groupoids. We say that $\mathcal{M}$ is a *stack over S* if

(i) for any $x, y \in \mathcal{C}(T)$ the functors $T' \mapsto \mathrm{Hom}_{T'}(x_{T'}, y_{T'})$ are sheaves.

(ii) for any $x \in \mathcal{C}(T)$ and any covering $T' \to T$, every descent datum on x with respect to $T' \to T$ is effective.

Categories fibred in groupoids are sometimes abbreviated *CFG*'s, or simply *groupoids over S*. (Note that we could also call a CFG a *presheaf of groupoids* on $\mathcal{S}ch$, as described after definition 24, i.e. it is a groupoid object in the category $\mathcal{S}ch^\wedge$.) The structure that gathers them all is slightly more sophisticated than a simple category : it is called a *2-category*. Briefly and imprecisely, this is, by definition, a structure given by a class of objects (which we may see as 0-morphisms), for each pair of objects, a category of 1-morphisms, whose morphisms are called 2-morphisms, and functors of "composition" between the categories of 1-morphisms.

The reason for this is that some functors between groupoids, which we would like to be isomorphisms, may only be equivalences. In other words there may well be $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$ with $F \circ G$ and $G \circ F$ not equal to the identities, but only up to a *2-isomorphism.* In conclusion, categories, groupoids, or stacks, are the objects (or 0-morphisms) of

Examples 47 (i) Let X be an S-scheme. Then $\mathcal{S}ch/X$ is a groupoid over S, which we still write X by abuse of notation. Furthermore it is a stack.

(ii) Let G be a group scheme over S. Then the category whose fibre over T is the groupoid $B_0(G(T))$ is a groupoid over S, which we still denote B_0G.

Recall: a groupoid is a category where all morphisms are isomorphisms. Examples: any set X gives rise to a groupoid where objects are elements of X and morphisms are identities. Any group G gives rise to a groupoid where there is only one object and the elements of G are viewed as automorphisms of this object. in some sense these are the two extreme examples of groupoids.

Why ask only groupoids ? It's because the destiny of the morphisms in the CFG's we have in mind is to become isomorphisms between objects of a moduli problem ; in particular this will include automorphisms of objects.

Lemma 48 *Let $\mathcal{M}$ be a groupoid over S, then there exists an associated stack.* □

Example 49 Let G be a group scheme over S. Then B_0G is not a stack in general. However, if $G \to S$ is flat, affine and of finite presentation then the groupoid of G-torsors (see definition 35), denoted BG, is a stack. We can identify $\widetilde{B_0G}$ with BG, in the following way. We define a functor

$$B_0G \to BG$$

by mapping the point $* \in (B_0G)(T)$ to the trivial G-torsor, and mapping an element $g \in G(T)$ (as a morphism $* \to *$) to multiplication by g (as a map of torsors). This functor is fully faithful, and it is "locally essentially surjective" by definition of a torsor.

Now the definition of the notion of *algebraicity* for a stack mimicks that of algebraicity for a sheaf (giving rise to the notion of algebraic space) :

Definition 50 Let $\mathcal{M}$ be a stack over S. We say it is *algebraic* if

(i) the diagonal $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable, separated and quasi-compact.

(ii) there is an algebraic space U and a morphism $U \to \mathcal{M}$, called a *presentation* or an *atlas*, that is smooth and surjective.

An algebraic stack is a *Deligne-Mumford stack* if the atlas can be chosen étale. □

Example 51 The stack of stable curves of genus g is a Deligne-Mumford stack.

References

Here is a list of references for the topics we saw. Those I would recommend for a first reading have triple bullets ◎ , and for a second reading (more advanced landmarks) double bullets ◉ .

Descent theory

- ◎ Grothendieck's Bourbaki talk, exposé 190 of Bourbaki 1959/60.
- ◎ SGA1, chapter VIII.
  (the only place where you find all the descent results)
- Bosch, Lütkebohmert, Raynaud, *Néron Models*, chapter 6.
  (quite technical, but complete)
- [BFFGK] Book in preparation by Behrend, Fantechi, Fulton, Goettsche, Kresch.
  (see on Kresch's homepage www.math.upenn.edu/kresch/teaching/stacks.html)
- SGA 4 1/2 (Deligne et al.), LNM 569, Springer-Verlag (1977)
  (a summary of étale cohomology in the first chapter)

Grothendieck Topologies, Topoi

- SGA4, tome I.
- Mac Lane, Moerdijk, *Sheaves in geometry and logic: a first introduction to topos theory*, Springer-Verlag (1992).
  (funny but very logic-oriented)
- notes by Berndt Schwerdtfeger.
  (go to http://home.t-online.de/home/berndt.schwerdtfeger/articles.html)
  (they are nice for the exposition of topologies)

Algebraic spaces

- ◉ Artin, *The implicit function theorem in Algebraic Geometry*, Alg. Geom., Bombay Colloq. 1968, 13-34 (1969).
- Artin, *Algebraic Spaces*, Yale Mathematical Monographs 3, Yale University Press (1971).
  (these last two are fine introductory texts to algebraic spaces)
- Artin, *Algebraization of formal moduli I*, Global Analysis, Papers in Honor of K. Kodaira 21-71 (1969).
  (this one is tougher; it's the beginning of the application of his algebraization theorems)
- ◉ Knutson, *Algebraic Spaces*, LNM 203, Springer-Verlag (1971).
  (the most complete reference for algebraic spaces)

Stacks

- Deligne, Mumford, *The irreducibility of the space of curves of given genus*, IHÉS 36, 75-109 (1969).
  (the ancestor text on algebraic stacks; no proof at all)
- ◎ Appendix of Vistoli's article *Intersection theory on algebraic stacks and their moduli spaces*, Invent. Math. 97, no. 3, p. 613-670 (1989).
  (exposition of the definitions; perhaps the most readable of the kind)
- Artin, *Versal deformations and algebraic stacks*, Invent. Math. 27, 165-189 (1974).
  (hard ; these big theorems are perhaps more readable now in Laumon & Moret-Bailly, chapter 10)
- ◉ Laumon, Moret-Bailly, *Champs algébriques*, Springer (2000).
- lectures given by Vistoli at ICTP school on "Intersection theory and moduli".
  (go to www.ictp.trieste.it/ then click on "scientific calendar: <u>2002</u>"
  then go to "Intersection theory and moduli", sept. 2002)
  (very complete, above all on the "prerequisites")
- lectures of Kresch on his homepage (sketchy).
  (Kresch's homepage www.math.upenn.edu/kresch/teaching/stacks.html)