The fundamental theorem of alternating functions

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Let A be a commutative ring with unit and $n \geq 1$ be an integer. The symmetric group in n letters acts on the polynomial ring $A[X_1, X_2, \dots, X_n]$ by permutation of the variables. The most famous invariant polynomials are the elementary symmetric functions S_i determined by the expansion $(T - X_1)(T - X_2)\dots(T - X_n) = T^n - S_1 T^{n-1} + \dots + (-1)^n S_n$. The fundamental theorem of symmetric functions asserts that the invariant ring for this action is the subring $A[S_1, S_2, \dots, S_n]$, and that the polynomials S_i are algebraically independent.

If one looks at the restricted action of the alternating group, there are more invariants. For example the Vandermonde polynomial $V_n = \prod_{1 \leq i < j \leq n}(X_i - X_j)$ is multiplied by the signature $\varepsilon(\sigma)$ under action of a permutation σ. When $2 \in A^\times$, it is known that the invariant ring is generated by the S_i $(1 \leq i \leq n)$ together with V_n. At the other extreme if $2 = 0$ in A then V_n is symmetric, so the same result does not hold. In this note we provide the correct invariant ring for the action of the alternating group, for any A.

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Keeping notations as above, consider the following four polynomials :

$$\Theta_n = \prod_{1 \leq i < j \leq n} (X_i + X_j) \quad ; \quad \Sigma_n = (V_n)^2 \quad ; \quad \Delta_n = \frac{1}{4}(\Sigma_n - \Theta_n^2) \quad ; \quad W_n = \frac{1}{2}(V_n + \Theta_n)$$

The first three are symmetric, while an odd permutation maps W_n to $\Theta_n - W_n$. Substituting $2W_n - \Theta_n$ to V_n in the equation $\Sigma_n = (V_n)^2$, one finds the integrality equation $(W_n)^2 - \Theta_n W_n - \Delta_n = 0$. Finally, all four polynomials have coefficients in $\mathbb{Z}$: for W_n this is clear, and the above equation shows that this is true also for Δ_n. As we now prove, for general A, the correct substitute for V_n is W_n :

Theorem *Let $n \geq 2$. The ring of alternating polynomials in the variables $X_1, \dots, X_n$ is $A[S_1, \dots, S_n, W_n]$ with relation $(W_n)^2 - \Theta_n W_n - \Delta_n = 0$.*

Proof : Let $B = A[S_1, \dots, S_n]$, and denote by C the ring of alternating polynomials. We first prove that C is a free module over B with basis $\{1, W_n\}$. Let F be an alternating polynomial. It is clear that $F^* = F - \tau F$ is independent of the choice of an odd permutation τ. In particular for $\sigma = (ij)$ this says that $X_i - X_j$ divides F^*. In the sequel, we use repeatedly the fact that $X_i - X_j$ is a nonzerodivisor in $A[X_{i,j}]$, for all (i, j). We will now prove that V_n divides F^*. We choose

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the lexicographic order on the pairs (i,j) with $1 \leq i < j \leq n$. Starting from $F^* = (X_1 - X_2)Q_1$ we assume by induction on N that there exists Q_N such that

$$F^* = \left(\prod_{(i,j) \leq N} (X_i - X_j) \right) Q_N$$

Let (u,v) be the $N+1$-th pair. Then Q_N vanishes when we specialize to $X_u = X_v$, because $X_u - X_v$ divides F^*. Hence $X_u - X_v$ divides Q_N, by a direct computation. After $n(n-1)/2$ steps we have $F^* = V_n Q$. Clearly, Q is uniquely defined and invariant under the full symmetric group. Now we check that the polynomial $P = F - W_n Q$ is also symmetric. If σ is odd we have $F - \sigma F = F^* = V_n Q$ and $\sigma W_n = \Theta_n - W_n = W_n - V_n$. Hence,

$$\sigma P = \sigma F - \sigma W_n \cdot \sigma Q = \sigma F - (W_n - V_n) Q = F - W_n Q = P$$

This proves that 1 and W_n generate C as a B-module. Furthermore, if $P = W_n Q$ with $(P,Q) \in B^2$, then after we apply an odd permutation we get $P = (\Theta_n - W_n)Q = (W_n - V_n)Q$. From this and $P = W_n Q$ it follows that $V_n Q = 0$ hence $Q = 0$. This shows that C is a free B-module. Therefore, the map $B[T]/(T^2 - \Theta_n T - \Delta_n) \to C$ defined by $T \mapsto W_n$ is a surjective map between free modules of the same rank, so it is an isomorphism. □

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