Models of group schemes of roots of unity and perfectoid spaces

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`Advice for next presentation of this talk: do not spend more than 20 minutes on Part 1. This means going reasonably fast and not making digressions.`

This is a report on work in progress. I'll present some recent work of Peter Scholze that might shed some light on two different approaches to models of μ_{p^n} that we've been investigating in the last years. In the first part of the talk, I'll review briefly the main features of our work on models of μ_{p^n}, and in the second part I'll introduce Scholze's perfectoid spaces and explain their relation with our work.

Let K be a char. 0 discretely valued complete field, R its ring of integers, k its char. p perfect residue field, $\pi \in R$ a uniformizer, $e = v(p)$ the absolute ramification index of R. Thus, concretely, R is a finite free degree e extension of the ring of Witt vectors $W(k)$, and K is its fraction field.

1 Models of $\mu_{p^n,K}$

Definition. A *model of* $\mu_{p^n,K}$ is a finite flat R-group scheme G with an isomorphism $G \otimes K \simeq \mu_{p^n,K}$.

For $n = 1$ there are finitely many models (one finds them all by iterated dilatations in $\mu_{p,R}$). For $n \geqslant 2$ there are infinitely many in general and no complete description is known, except for $n = 2$ by the PhD work of D. Tossici.

1.1 Some explicit constructions

In joint work with A. Mézard and D. Tossici, we used a formalism conceived by Sekiguchi and Suwa in the 90's to construct some explicit models of μ_{p^n}. Let us give an idea of how it goes.

The case $n = 1$**.** It is known [1] that the models of $\mathbb{G}_{m,K}$ with smooth connected fibres are the R-group schemes $\mathscr{G}^\ell = \mathrm{Spec}(R[x, \frac{1}{1+\pi^\ell x}])$, for $\ell \geqslant 0$, defined by the exact sequence in $\mathrm{Ab}(\mathrm{Spec}(R)_{\mathrm{fppf}})$:

$$(\star) \qquad 0 \longrightarrow \mathscr{G}^\ell \longrightarrow \mathbb{G}_{m,R} \longrightarrow i_*\mathbb{G}_{m,R/\pi^\ell} \longrightarrow 0. \qquad x \longmapsto x \bmod \pi^\ell$$

Now we have $\mu_{p,K} \subset \mathbb{G}_{m,K}$ and it is proved easily that all models of μ_p are found in one of the $\mathscr{G}^\ell$; in fact we get one such model iff the Kummer isogeny $x \mapsto x^p$ on $\mathbb{G}_{m,K}$ extends to $\mathscr{G}^\ell$.

The case $n > 1$**.** Since (by flat closure) any model of μ_{p^n} is an iterated extension of models of μ_p, it is natural to consider $(\mathbb{G}_{m,K})^n$-models iterated extensions $\mathscr{E}_n$ of groups $\mathscr{G}^{\ell_1}, \dots, \mathscr{G}^{\ell_n}$ and to look for models of μ_{p^n} inside there. We get one model every time the Kummer isogeny on $(\mathbb{G}_{m,K})^n$ (there is a natural one) extends to $\mathscr{E}_n$. This shifts focus to $\mathrm{Ext}^1(\mathscr{E}_{n-1}, \mathscr{G}^{\ell_n})$. It turns out that the coboundary

$$\mathrm{Hom}(\mathscr{E}_{n-1}, i_*\mathbb{G}_{m,R/\pi^{\ell_n}}) \longrightarrow \mathrm{Ext}^1(\mathscr{E}_{n-1}, \mathscr{G}^{\ell_n})$$

of the sequence $(\star)$ is *surjective*. Thus Ext^1 is parameterized by left-hand Hom-group whose elements are certain explicit *deformed Artin-Hasse exponentials* (this is one of the crucial results of Sekiguchi

[1] Waterhouse and Weisfeiler, 1980.

and Suwa). For $n = 2$ they are simply truncated ordinary exponentials $E_a(t) = \sum_{i=0}^{p-1} a^i t^i / i!$, and the parameters of the construction are gathered in a matrix

$$A_{\text{SS}} = \begin{pmatrix} [\pi^{\ell_1}] & [a] \\ 0 & [\pi^{\ell_2}] \end{pmatrix}$$

whose entries belong to $W^f(R)$ i.e. are finite Witt vectors.

The general picture. For general n the deformed exponentials are messier to describe but the general pattern is the same: we have a construction depending on some integers ℓ_i and coefficients $a_{ij} \in W^f(R)$ fitting into an upper-triangular square matrix A of size n. To any A whose entries satisfies some congruences (so that the Kummer isogeny extends) is attached a model G. This gives a map $A_{\text{SS}} \mapsto G$ from a certain set of matrices to the set of isom. classes of models of μ_{p^n}. We do not know how to measure the failure of injectivity of this map. We conjecture that it is surjective and we proved it under some restrictions for $n \geqslant 3$.

1.2 Dieudonné-type classification

Work of Dieudonné, Fontaine, Breuil, Kisin culminated in a classification of finite flat (commutative p-power order) R-group schemes for arbitrary e. Here's the result in the special case of models of μ_{p^n}.

Let $E(u) \in W(k)[u]$ be the minimum polynomial of π. Let $W_n = W_n(k)$ and $\phi : W_n((u)) \to W_n((u))$ be the extension of the Frobenius of W_n such that $\phi(u) = u^p$. Define a *lattice* of $W_n((u))$ to be a finitely generated sub-$W_n[[u]]$-module $\mathfrak{M} \subset W_n((u))$ such that $\mathfrak{M}[1/u] = W_n((u))$.

Theorem. *There is an equivalence between the category of models of $\mu_{p^n,K}$ and the category of lattices $\mathfrak{M} \subset W_n((u))$ such that $E(u)\mathfrak{M} \subset \langle \phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$.*

If $\mathfrak{M}$ is a lattice corresponding to a model of μ_{p^n}, we have a filtration by p-power kernels:

$$\mathfrak{M} = \mathfrak{M}[p^n] \supset \mathfrak{M}[p^{n-1}] \supset \cdots \supset \mathfrak{M}[p] \supset \mathfrak{M}[1] = 0$$

which after inverting u is $W_n((u)) \supset pW_n((u)) \supset \cdots \supset p^{n-1}W_n((u)) \supset 0$. The quotient $\mathfrak{M}[p^{i+1}]/\mathfrak{M}[p^i]$ is a submodule of $p^{n-i-1}W_n((u))/p^{n-i}W_n((u)) \simeq k((u))$ hence it has a unique generator of the form $u^{\ell_{n-i}}$. By lifting this generator in $\mathfrak{M}$ and normalizing a little bit, we can produce a *unique* generating system $(e_1, \dots, e_n)$ for $\mathfrak{M}$ whose coefficients on $1, \dots, p^{n-1}$ form the columns of a matrix:

$$A_{\text{BK}} = \begin{pmatrix} u^{\ell_1} & \dots & a_{ij} \\ 0 & \ddots & \vdots \\ 0 & 0 & u^{\ell_n} \end{pmatrix}$$

where $a_{ij} \in k[u]$. The map $A_{\text{BK}} \mapsto \mathfrak{M} \mapsto G$ is bijective hence is much better than what we had with the explicit construction. On the other hand, given A_{BK} we have hardly any information on the corresponding group scheme, unlike in the explicit construction.

1.3 Discussion

In 1.1 we have matrices with entries in R (think of a finite Witt vector as a finite collection of elements of R). In constract, in 1.2 we have matrices with entries in $k[[u]]$. On the one hand we have a dvr with char. 0, and on the other hand we have a dvr with char. p. How to relate them?

2 Perfectoid spaces

Our idea is that there is a moduli space for models of μ_{p^n}, which is a formal scheme, and passing to the inverse limit over all n and taking the generic fibre gives a *perfectoid space.*

2.1 Perfectoid spaces in a nutshell

It was noted by Fontaine and Wintenberger around 1980 that there is an isomorphism

$$G_{\mathbb{Q}_p(p^{1/p^\infty})} \simeq G_{\mathbb{F}_p((t))}$$

where G_* denotes absolute Galois groups. Let's sketch the proof. This is best understood after replacing $\mathbb{F}_p((t))$ by its purely inseparable extension $\mathbb{F}_p((t))(t^{1/p^\infty})$ (this does not affect Galois groups). Also since the Galois group is insensitive to completion, we may as well focus on the fields $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $K^\flat = \mathbb{F}_p((t))(t^{1/p^\infty})^\wedge$. To construct an isom., we shall then attach to each finite extension $L/K^\flat$ a canonical finite extension $L^\sharp/K$. Let's single out 3 steps.

1) Write $(-)^\circ$ for the subrings of integral elements. Then $\alpha(p^{1/p^n}) = t^{1/p^n}$ defines an isom.

$$K^\circ/p = \mathbb{Z}_p[p^{1/p^\infty}]/p \overset{\alpha}{\simeq} \mathbb{F}_p[t^{1/p^\infty}]/t = K^{\flat\circ}/t.$$

2) There is a continuous multiplicative map $K^{\flat\circ} \to K^\circ$, $x \mapsto x^\sharp := \lim_{n\to\infty} y_n^{p^n}$ where $y_n \in K^\circ$ is any lift under α of x^{1/p^n} mod t. (Here we see that completion is crucial: the isom. is in fact deeply analytic.) This induces an isom. $K^{\flat\circ} \to \varprojlim_{\mathrm{Frob}} K^\circ/p$, $x \mapsto (x^\sharp, (x^{1/p})^\sharp, \dots)$ and hence an identification:

$$(\star) \qquad K^\flat = \mathrm{Frac}\Big(\varprojlim_{\mathrm{Frob}} K^\circ/p\Big).$$

3) Let $L/K^\flat$ be a separable field extension. If L is the splitting field of $X^d + a_{d-1}X^{d-1} + \cdots + a_0$ then it is also that of $X^d + a_{d-1}^{1/p^n} X^{d-1} + \cdots + a_0^{1/p^n}$ for all $n \geqslant 0$. Then one can see that the splitting field of $X^d + (a_{d-1}^{1/p^n})^\sharp X^{d-1} + \cdots + (a_0^{1/p^n})^\sharp$ stabilizes when $n \to \infty$. We call this $L^\sharp$.

Definition. A *perfectoid field* is a complete nonarchimedean field K of residue characteristic p whose associated rank-1 valuation is nondiscrete and such that Frobenius is surjective on K°/p. Its *tilt* is the field $K^\flat$ defined by formula $(\star)$; it is perfectoid.

A perfectoid field may have char. 0 or p. In the latter case, in the definition $(\star)$ of the tilt one must replace p by an element $\varpi \in K^\circ$ such that $|p| \leqslant |\varpi| < 1$.

Definition. A *perfectoid K-algebra* is a Banach K-algebra R such that the subring R° of power-bounded elements is open and bounded, and Frobenius is surjective on R°/p. It has a *tilt* $R^\flat$.

These things glue:

Definition. A *perfectoid space* is an adic space X (=analytic space in the sense of Huber) that is locally isomorphic to $\mathrm{Spa}(R, R^\circ)$. It has a *tilt* $X^\flat$.

Theorem (Scholze). *There exist an isom. of sites $X_{\text{ét}} \simeq X^\flat_{\text{ét}}$. Moreover there are canonical bijections $X(L) = X^\flat(L^\flat)$, for all perfectoid fields L.*

2.2 Expectations

Let us consider π-adic formal R-schemes locally of finite type. To such a formal scheme $\mathfrak{X}$, one can attach a *generic fibre* X which is a K-analytic space. There are various notions of analytic spaces and either one is ok here: a rigid space in the sense of Tate, an analytic space in the sense of Berkovich, or an adic space in the sense of Huber. For example the generic fibre of the affine formal scheme $\mathrm{Spf}(A)$ is the maximal/Gelfand/adic spectrum of $A \otimes_R K$. If L/K is a valued extension, we have $\mathfrak{X}(L^\circ) = X(L)$ canonically.

A *family of finite flat group schemes* $\mathfrak{G} \to \mathfrak{X}$ is a group object in the category of formal schemes which is finite and flat. Thus for each valued extension L/K and point $x \in X(L) = \mathfrak{X}(L^\circ)$, there is a fibre $\mathfrak{G}_x$ which is a finite flat (formal) L°-group scheme.

We expect the following to hold. There is a formal scheme $\mathfrak{M}_n$, which is a moduli space for models of μ_{p^n}. It has a generic fibre M_n, which we view as an adic space, i.e. a K-analytic space in the sense of Huber. The inverse limit $M := \varprojlim M_n$ makes sense as an adic space and should be perfectoid. For fixed L, the finite flat models of μ_{p^n} over L° are the L-points of M; they should correspond to parameters of the explicit Sekiguchi-Suwa-based construction. They are in canonical bijection with the $L^\flat$-points of the tilt $M^\flat$, which should correspond to the parameters of Breuil-Kisin modules.