# Connected and irreducible components in families

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## 1 Motivation

Let $g \geq 2$ and $G$ be a finite group. Let $\mathscr{M}_g$ be the algebraic stack over $\mathbb{C}$ of smooth curves of genus $g$ and $\mathscr{M}_g(G)$ the (closed) locus of curves whose automorphism group contains a copy of $G$. This is reducible and any irreducible component $W$ has a *field of definition* which is a finite extension of $\mathbb{Q}$.

*Question (Pierre Lochak) :* in fact $\mathscr{M}_g$ and $\mathscr{M}_g(G)$ are defined over $\mathbb{Z}$. Does $W$ have a *ring of definition* which is a finite extension of $\mathbb{Z}$ ?

*Answer :* the stack $\mathscr{X} = \mathscr{M}_g(G)$ has a functor of relative irreducible components $\mathrm{Irr}(\mathscr{X}/S)$, and if we restrict to the base $S = \mathrm{Spec}(\mathbb{Z}[1/30|G|])$ then this functor is represented by a finite étale algebraic space. The connected component of the point of $\mathrm{Irr}(\mathscr{X}/S)$ corresponding to $W$ is finite étale over $S$, its function ring deserves the name of a ring of definition.

If the residue char. $p$ is a factor of $|G|$, the fibre of $\mathscr{M}_g(G)$ is not reduced and this causes some trouble (we'll see why). The reason for the factor $30 = 2.3.5$ is that for some (finitely many and known) groups, generic curves in $\mathscr{M}_g(G)$ have $[\mathrm{Aut}(C) : G] > 1$ with prime factors in $\{2, 3, 5\}$.

The topic of the talk is to describe functors of connected and irreducible components and to explain the "answer" above.

## 2 Open components

**Definitions.** $\mathscr{X} :=$ a fixed algebraic stack of finite presentation (f. p.) over $S$.

Wish that a relative connected component $\mathscr{C} \subset \mathscr{X}$ be determined by its support $\Rightarrow$ require $\mathscr{C} \subset \mathscr{X}$ to be open (N.B. the fibres $\mathscr{C}_s$ are open and closed but it is really too strong to require this for $\mathscr{C}$). Leads to :
1) Definition : an OCC = open (relative) conn. comp. is an open substack $\mathscr{C}$ flat finitely presented over $S$ such $\mathscr{C}_s$ is a conn. comp. of $\mathscr{X}_s$, for all geometric points $s \in S$. The functor $T/S \mapsto \{\text{OCC's of } \mathscr{X}_T/T\}$ is denoted $\pi_0(\mathscr{X}/S)$.
2) Definition : if $k = \overline{k}$ alg. closed field then an *open irreducible component* is the interior of some irred. comp. $W$, or the complement of all irred. comp.'s $W' \neq W$.
3) Definition : OIC's and the functor $\mathrm{Irr}(\mathscr{X}/S)$.

$\pi_0$ and Irr are fppf sheaves, quasi-compact over $S$, with open diagonal.
*By construction they are étale over $S$.*

**Representability.** Easy : $\pi_0$ and Irr are representable by alg. spaces over an open subscheme $U \subset S$ containing generic points. E.g. it is ok if $S$ is the spectrum of a field.

For general $S$ (one reduces easily to $S$ affine of finite type over $\mathbb{Z}$) we use Artin's criteria asserting that a locally finitely presented functor $F$ is an alg. space iff it has an obstruction theory $\mathcal{O}$ and satisfies the following conditions :

1) $F$ is an fppf sheaf,
2) The diagonal of $F$ is representable and of finite type,
3) The deformation theory $D$ satisfies Schlessinger's conditions S1 and S2,
4) For a complete local ring $R$ with residue field of finite type over $S$ the map $F(R) \to \varprojlim R(R/m^n)$ is injective with dense image,
5) $D$ and $\mathcal{O}$ satisfy various compatibilities.

For us the only issue is point 4) of *approximation of formal elements* : boils down to proving that $F(R) \to F(k)$ is bijective, where $k = R/m$. We must have a procedure for constructing components. This works if $\mathscr{X}/S$ has geometrically reduced fibres. In the following we let $\mathscr{U} \subset \mathscr{X}$ denote the *unicomponent locus* i.e. the locus of points that belong to a single irred. comp.

**Proposition.** *$\mathscr{X}$ f. p. with geometrically reduced fibres, $g : S \to \mathscr{X}$ a section.*
*If $\mathscr{X}$ is flat then the union of the conn. comp.'s containing $g(s)$ is open in $|\mathscr{X}|$.*
*If $g(S) \subset \mathscr{U}$ then the union of the irred. comp.'s containing $g(s)$ is open in $|\mathscr{X}|$.*

**Theorem.** *$\mathscr{X}$ flat f. p. with geometrically reduced fibres.*
*- The functor $\pi_0(\mathscr{X}/S)$ is representable by a quasi-compact étale algebraic space and there is a morphism $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ that induces an isomorphism $\mathscr{X}/\mathscr{R} \simeq \pi_0(\mathscr{X}/S)$.*
*- The functor $\mathrm{Irr}(\mathscr{X}/S)$ is representable by a quasi-compact étale algebraic space and there is a morphism $\mathscr{U} \to \mathrm{Irr}(\mathscr{X}/S)$ that induces an isomorphism $\mathscr{X}/\mathscr{S} \simeq \mathrm{Irr}(\mathscr{X}/S)$.*

**Corollary.** *Functoriality : an $S$-rational map $\mathscr{X} \dashrightarrow \mathscr{Y}$ induces $\pi_0(\mathscr{X}/S) \to \pi_0(\mathscr{Y}/S)$.*

# 3 Closed components

When $\mathscr{X}/S$ has reduced fibres, approximation of formal elements does not work.

Example : $X = \mathrm{Spec}(R[x]/(x^2 - \pi x))$ over a dvr $R$.

But when $\mathscr{X}/S$ is proper it is natural to look at *closed* substacks $\mathscr{C} \subset \mathscr{X}$ and $\mathscr{I} \subset \mathscr{X}$. The definitions of CCC's and CIC's are the same as the open versions with just open replaced by closed. We also consider one more notion : a RCC = reduced (closed) connected component is a CCC whose geometric fibres are reduced. Accordiingly we have functors $\pi_0(\mathscr{X}/S)^c$, $\pi_0(\mathscr{X}/S)^r$, $\mathrm{Irr}_(\mathscr{X}/S)^c$.

Example : $X_0$ proper geometrically connected over a field $k$ and $X = X_0[\epsilon]$. Then any closed subscheme $Z = V(I) \subset X_0$ gives rise to a CCC, namely $Z' = V(\epsilon I) \subset X$. This gives an isomorphism of functors $\mathrm{Hilb}(X_0) \simeq \pi_0(X/k)^c$. So in this case $\pi_0(X/k)^c$ is huge !

**Theorem.** *$\mathscr{X}$ proper f. p.*
*- $\pi_0(\mathscr{X}/S)^c$ is representable by a formal algebraic space locally f.p. and separated.*
*- $\pi_0(\mathscr{X}/S)^r$ is representable by a finite, separated formal scheme.*

The proof relies on the Hilbert space of $\mathscr{X}/S$. Inside the Hilbert space, the connected components are the closed substacks $W \subset \mathscr{X}$ whose support is open (open condition), that are geometrically connected (closed condition). So $\pi_0(\mathscr{X}/S)$ is the completion, in the former open subspace, along the latter closed subspace.

**Theorem.** *$\mathscr{X}$ proper flat, f.p. with geom. reduced fibres. Let $\mathscr{X} \to \mathrm{St}(\mathscr{X}/S) \to S$ be the Stein factorization. Then :*

$$\mathrm{St}(\mathscr{X}/S) \simeq \pi_0(\mathscr{X}/S)^c = \pi_0(\mathscr{X}/S)^r = \pi_0(\mathscr{X}/S)$$

*and $\mathrm{Irr}(\mathscr{X}/S)^c$ is open in $\mathrm{Irr}(\mathscr{X}/S)$ and is an étale separated scheme.*

Counter-example. Let $X$ be the universal plane conic over the moduli space of plane conics which is $S := \mathbb{P}_k^5$ with $char(k) \neq 2$. One can show that $\mathrm{Irr}(X/S)^c$ is not representable by a formal algebraic space. Roughly, the problem is that you can do successive specializations $s_0 \rightsquigarrow s_1 \rightsquigarrow s_2$ in $S$ with $X_0$ irreducible, $X_1$ reducible and $X_2$ irreducible again. This phenomenon is an obstacle to representability.

# 4 Application to $\mathscr{M}_g(G)$

**Proposition.**
*1) The locus $\mathscr{M}_g(G) \subset \mathscr{M}_g$ is closed, and the corresponding reduced substack over $\mathbb{Z}[1/|G|]$ is flat, f.p. with geometrically reduced fibres.*
*2) Over $\mathbb{Z}[1/30|G|]$, the scheme $\mathrm{Irr}(\mathscr{M}_g(G))$ is finite étale.*

Introduce the stack $\mathscr{H}$ classifying pairs $(C, \rho)$ of a curve and a faithful action of $G$ on $C$. The schematic image of the obvious map $\mathscr{H} \to \mathscr{M}_g$ is $\mathscr{M}_g(G)$ and its formation commutes with base change. Since $\mathscr{H}$ is smooth over $\mathbb{Z}[1/|G|]$, point 1) follows.

The group $\mathrm{Aut}(G)$ acts on $\mathscr{H}$ by $\theta.(C, \rho) = (C, \rho \circ \theta^{-1})$. Set $\mathscr{N} := \mathscr{H} / \mathrm{Aut}(G)$. The map $\mathscr{H} \to \mathscr{M}_g(G)$ induces $\psi : \mathscr{N} \to \mathscr{M}_g(G)$.

Claim : the normalization of $\mathscr{M}_g(G)$ is a disjoint sum of stacks similar to $\mathscr{N}$. Indeed, to compute the normalization we may assume $\mathscr{M}_g(G)$ irreducible. Let $C_\eta$ be the generic curve. Replacing $G$ by $G' = \mathrm{Aut}(C_\eta)$ we may assume that $\mathrm{Aut}(C_\eta) = G$. Then $\psi$ is birational and is the normalization.

By normality and functoriality we have $\pi_0(\mathscr{N}/S) \simeq \mathrm{Irr}(\mathscr{N}/S) \simeq \mathrm{Irr}(\mathscr{M}_g(G))$. Moreover there exists a smooth $S$-compactification $\overline{\mathscr{N}}$. Then $\pi_0(\mathscr{N}/S) \simeq \pi_0(\overline{\mathscr{N}})$. By the theorem, the latter is finite étale hence point 2).