Néron models of abelian varieties

Matthieu Romagny

Summer School on SGA3, September 3, 2011

Abstract : We present a survey of the construction of Néron models of abelian varieties, as an application of Weil's theorem of extension of birational group laws.

Acknowledgements : I wish to thank Q. Liu and C. Pépin for various interesting comments on a preliminary version of this note.

1 Introduction

Let S be a Dedekind scheme (a noetherian, integral, normal scheme of dimension 1) with field of rational functions K, and let A_K be a K-abelian variety.

A *model* of A_K over S is a pair composed of an S-scheme A and a K-isomorphism $A \times_S \operatorname{Spec}(K) \simeq A_K$. Usually, one refers to such a model by the letter A alone. If A is an S-model of A_K, we often say that its generic fibre "is" A_K. The nicest possible model one can have is a proper smooth S-model, but unfortunately this does not exist in general. In the search for good models for abelian varieties, Néron's tremendous idea is to abandon the requirement of properness, insisting on smoothness and existence of a group structure. He was led to the following notion.

Definition. A *Néron model* of A_K over S is a smooth, separated model of finite type A that satisfies the *Néron mapping property* : each K-morphism $u_K : Z_K \to A_K$ from the generic fibre of a smooth S-scheme Z extends uniquely to an S-morphism $u : Z \to A$.

Our aim is to prove that a Néron model exists. Note that once existence is established, the universal property implies that the Néron model A is unique up to canonical isomorphism; it implies also that the law of multiplication extends, so that A is an R-group scheme. Therefore it could seem that for the construction of the Néron model, we may forget the group structure and recover it as a bonus. The truth is that things go the other way round: the Néron model is constructed first and foremost as a group scheme, and then one proves that it satisfies the Néron mapping property.

An important initial observation is that A_K extends to an abelian scheme over the complement in S of a finite number of closed points s, so one can reduce the construction of the Néron model in general to the construction in the local case by glueing this abelian scheme together with the finitely many local Néron models (i.e. over the spectra of the local rings $\mathcal{O}_{S,s}$). Therefore it will be enough for us to consider the case where S is the spectrum of a discrete valuation ring R with field of fractions K and residue field k. We fix a separable closure $k \to k^{\mathrm{s}}$ and a strict henselisation $R \to R^{\mathrm{sh}}$; we have an extension of fractions fields $K \to K^{\mathrm{sh}}$.

If A satisfies the extension property of the above definition only for Z étale, we say that it is a *weak Néron model*. Alternatively, it is equivalent to require that A satisfies the extension property for $Z = \mathrm{Spec}(R^{\mathrm{sh}})$, as one can see using the fact that R^{sh} is the inductive limit of 'all' the discrete valuation rings R' that are étale over R. In contrast with Néron models, weak Néron models are not unique since their special fibre contains in general plenty of extraneous components, as we shall see. The Néron model will be obtained as the rightmost scheme in the following chain (hooked arrows denote open immersions):

$$\begin{tikzcd}[column sep=large]
A_0 & A_1 \arrow[l, "\substack{\text{Blowing-up}\\ \text{finitely}\\ \text{many times}}"'] & A_2 \arrow[l, hook', "\substack{\text{Taking}\\ \text{smooth}\\ \text{locus}}"'] & A_3 \arrow[l, hook', "\substack{\text{Removing}\\ \text{non-minimal}\\ \text{components}}"'] \arrow[r, hook, "\substack{\text{Applying}\\ \text{Weil's}\\ \text{theorem}}"] & A_4 \\
\substack{\text{Any flat}\\ \text{proper model}} & \substack{\text{Smoothening}\\ \text{of } A_0} & \substack{\text{Weak}\\ \text{Néron model}} & \substack{\text{Birational}\\ \text{group chunk}} & \text{Néron model}
\end{tikzcd}$$

2 Néron's smoothening process

2.1 Proper flat models and smoothenings

One way to start is to use Nagata's embedding theorem (see e.g. [Co]) in order to find an open immersion of A_K, viewed as an R-scheme, into a proper R-scheme B. Then the schematic closure of A_K inside B is a proper flat R-model A_0. In fact, it follows from the projectivity of abelian varieties (a classical consequence of the theorem of the square) that one may choose B to be some projective R-space, but this will not be useful in the sequel. Then, the valuative criterion of properness implies that the canonical map $A_0(R^{\mathrm{sh}}) \to A_K(K^{\mathrm{sh}})$ is surjective. Thus if A_0 happened to be smooth, it would be a weak Néron model of A_K; however, the special fibre $A_0 \otimes k$ may be singular, even nonreduced (note that it is proper and geometrically connected by [EGA] IV.15.5.9). In order to recover smoothness at least at integral points, in subsection 2.4 we will produce a *smoothening* of A_0 as defined in the following.

Definition 2.1.1 Let A be a flat R-scheme of finite type with smooth generic fibre. A *smoothening* of A is a proper morphism $A' \to A$ which is an isomorphism on the generic fibres and such that the canonical map $A'_{\mathrm{sm}}(R^{\mathrm{sh}}) \to A(R^{\mathrm{sh}})$ is surjective, where A'_{sm} is the smooth locus of A'.

In order to construct a smoothening, we will repeatedly blow up A along geometrically reduced closed subschemes of the special fibre containing the specializations of the points of $A(R^{\mathrm{sh}})$ that are "maximally singular", in a sense that we shall define soon. This leads to consider that the natural object to start with is a pair (A, E) where E is a given subset of $A(R^{\mathrm{sh}})$. Note that for any proper morphism $A' \to A$ which is an isomorphism on the generic fibres, the set E lifts uniquely to $A'(R^{\mathrm{sh}})$ and we will identify it with its image. The sense in which the singularity is maximal is measured by two invariants $\delta(A, E)$ and $t(A, E)$ which we now introduce.

2.2 Néron's measure for the defect of smoothness

Definition 2.2.1 Let A be a flat R-scheme of finite type with smooth generic fibre and let E be a subset of $A(R^{\mathrm{sh}})$. For each $a : \mathrm{Spec}(R^{\mathrm{sh}}) \to A$ in E, we set

$$\delta(a) = \text{the length of the torsion submodule of } a^*\Omega^1_{A/R}.$$

The integer $\delta(A, E) = \max\{\delta(a), a \in E\} \geqslant 0$ is called *Néron's measure for the defect of smoothness.*

It is easy to see that $\delta(a)$ remains bounded for $a \in E$, so that $\delta(A, E)$ is finite (see [BLR] 3.3/3). Moreover, this invariant does indeed measure the failure of smoothness:

Lemma 2.2.2 *We have $\delta(A, E) = 0$ if and only if $E \subset A_{\mathrm{sm}}(R^{\mathrm{sh}})$.*

Proof : Let $a \in E$ and let $d_K = \dim_{a_K}(A_K)$ and $d_k = \dim_{a_k}(A_k)$ be the local dimensions of the fibres of A. By the Chevalley semi-continuity theorem, we have $d_K \leqslant d_k$. If $\delta(a) = 0$ then $a^*\Omega^1_{A/R}$ is free generated by d_K elements. Then, at the point a_k, $\Omega^1_{A_k/k}$ can be generated by d_K elements, hence also by d_k elements, so that A_k is smooth according to [EGA] IV_4.17.15.5. Being R-flat, the scheme A is smooth at a_k and $a \in A_{\mathrm{sm}}(R^{\mathrm{sh}})$. Conversely, if $a \in A_{\mathrm{sm}}(R^{\mathrm{sh}})$ then $\Omega^1_{A/R}$ is locally free in a neighbourhood of a_k and hence $\delta(a) = 0$. $\square$

2.3 The length of the canonical partition

Starting from a pair (A, E) as above, we define geometrically reduced k-subschemes $Y^1, U^1, \ldots, Y^t, U^t$ of A_k and the *canonical partition*

$$E = E^1 \sqcup E^2 \sqcup \cdots \sqcup E^t$$

as follows:

(a) Y^1 is the Zariski closure in A_k of the specializations of the points of E,

(b) U^1 is the largest k-smooth open subscheme of Y^1 where $\Omega^1_{A/R}|_{Y^1}$ is locally free,

(c) E^1 is the set of points $a \in E$ whose specialization is in U^1.

Note that Y^1 is geometrically reduced because it contains a schematically dense subset of k^{s}-points (see [EGA] IV_3.11.10.7) and U^1 is dense by generic flatness. For $i \geqslant 1$, we remove $E^1 \sqcup \cdots \sqcup E^i$ from E and we iterate this construction. In this way we define Y^{i+1} as the Zariski closure in A_k of the specialization of the points of $E \setminus (E^1 \sqcup \cdots \sqcup E^i)$, U^{i+1} as the largest smooth open subscheme of Y^{i+1} where $\Omega^1_{A/R}$ is locally free, and E^{i+1} as the set of points $a \in E$ with specialization in U^{i+1}. Since A_k is noetherian, there is an integer $t \geqslant 0$ such that $Y^{t+1} = U^{t+1} = \emptyset$ and we end up with the canonical partition $E = E^1 \sqcup E^2 \sqcup \cdots \sqcup E^t$.

Definition 2.3.1 We write $t = t(A, E) \geqslant 1$ for the *length of the canonical partition.*

The crucial ingredient of the smoothening process is given by the following lemma, due to Néron and Raynaud.

Lemma 2.3.2 *Let $a \in E$ be such that a_k is a singular point of A_k. Assume that $a \in E_i$, let $A' \to A$ be the blow-up of Y_i, and let a' be the unique lifting of a to A'. Then $\delta(a') < \delta(a)$.*

Proof : This is an ingenious computation of commutative algebra, which we omit. We refer to [BLR] 3.3/5. $\square$

2.4 The smoothening process

For $E \subset A(R^{\mathrm{sh}})$, we denote by E_k the set of specializations of the points of E in the underlying topological space of A_k. We now make a definition that is tailor-made for an inductive proof of the theorem below.

Definition 2.4.1 Let A be a flat R-scheme of finite type with smooth generic fibre and let E be a subset of $A(R^{\mathrm{sh}})$. We say that a closed subscheme $Y \subset A_k$ is *E-permissible* if it is geometrically reduced and the set $F = Y \cap E_k$ satisfies:

(1) F lies in the smooth locus of Y,

(2) F lies in the largest open subscheme of Y where $\Omega^1_{A/R}|_Y$ is locally free,

(3) F is dense in Y.

We say that the blow-up $A' \to A$ with center Y is *E-permissible* if Y is E-permissible.

Theorem 2.4.2 *Let A be a flat R-scheme of finite type with smooth generic fibre and let E be a subset of $A(R^{\mathrm{sh}})$. Then there exists a morphism $A' \to A$, a finite sequence of E-permissible blow-ups, such that each point $a \in E$ lifts uniquely to a smooth point of A'.*

Proof : We proceed by induction on the integer $\delta(A,E) + t(A,E) \geqslant 1$. If $\delta(A,E) = 0$, then E lies in the smooth locus of A and no blow-up is needed at all; this covers the initial case of the induction. If $\delta(A,E) \geqslant 1$, we consider the canonical partition $E = E^1 \sqcup \cdots \sqcup E^t$. Let $A' \to A$ be the blow-up of the closed subscheme $Y^t \subset A_k$, which is E^t-permissible by construction. By lemma 2.3.2, we have $\delta(A', E^t) < \delta(A, E^t)$. By the inductive assumption, there exists a morphism $A'' \to A'$ which is a finite sequence of E^t-permissible blow-ups such that each point of E^t lifts uniquely to a point in the smooth locus of A''. If $t = 1$, we are done. Otherwise let $E'' \subset A''(R^{\mathrm{sh}})$ be obtained by looking at E as a subset of $A''(R^{\mathrm{sh}})$ and removing E^t, and for $1 \leqslant i \leqslant t-1$ let $(E'')^i$ be the set E^i viewed in E''. Since $A'' \to A$ is a sequence of E^t-permissible blow-ups, it does not affect $E^1 \sqcup \cdots \sqcup E^{t-1}$. In this way one sees that $E'' = (E'')^1 \sqcup \cdots \sqcup (E'')^{t-1}$ is the canonical partition of E'', therefore $t(A'', E'') < t(A, E)$. Applying the inductive assumption once again, we obtain a morphism $A''' \to A''$ which is a finite sequence of E''-permissible blow-ups such that points of E'' lift to smooth points of A'''. Then $A''' \to A$ is the morphism we are looking for. $\square$

3 Weak Néron models

3.1 Reminder on rational maps

Let X, Y be flat, finitely presented schemes over a scheme S. We recall that an open subscheme $U \subset X$ is *S-dense* if U_s is schematically dense in X_s for all points $s \in S$. An *S-rational map* $u : X \dashrightarrow Y$ is an equivalence class of morphisms $U \to Y$ with open S-dense domain, where $U \to Y$ and $V \to Y$ are equivalent if they agree on an S-dense sub-open subscheme $W \subset U \cap V$. An *S-birational map* is an S-rational map that can be represented by a morphism $U \to Y$ inducing an isomorphism with an S-dense open subscheme of Y. If $Y \to S$ is separated, there is a maximal S-dense open subscheme $U \subset X$ with a morphism $U \to Y$ representing u, called the *domain of definition* of u. Its reduced complement is called the *exceptional locus* of u.

3.2 The weak Néron mapping property

We started Section 2 with the schematic closure A_0 of our abelian variety A_K inside some proper R-scheme B. According to theorem 2.4.2 applied with $E = A_0(R^{\mathrm{sh}})$, there exists a proper morphism $A_1 \to A_0$ which is an isomorphism on the generic fibre, such that the smooth locus $A_2 = (A_1)_{\mathrm{sm}}$ is a weak Néron model. We now prove that weak Néron models satisfy a significant positive-dimensional reinforcement of their defining property.

Proposition 3.2.1 *Let A be a weak Néron model of A_K. Then A satisfies the* weak Néron mapping property *: each K-rational map $u_K : Z_K \dashrightarrow A_K$ from the generic fibre of a smooth R-scheme Z extends uniquely to an R-rational map $u : Z \dashrightarrow A$.*

Note that conversely, if the extension property of the proposition is satisfied for a smooth and separated model A of finite type, then one sees that A is a weak Néron model by taking $Z = \mathrm{Spec}(R')$ for varying étale extensions R'/R.

Proof : Since A is separated, we can first work on open subschemes of Z with irreducible special fibre and then glue. In this way, we reduce to the case where Z has irreducible special fibre. Then removing from Z the scheme-theoretic closure of the exceptional locus of u_K, we may assume that u_K is defined everywhere. Let $\Gamma_K \subset Z_K \times A_K$ be the graph of u_K, let $\Gamma \subset Z \times A$ be its scheme-theoretic closure, and let $p : \Gamma \to Z$ be the first projection. On the special fibre, the image of p_k contains all k^{s}-points $z_k \in Z_k$: indeed, since Z is smooth each such point lifts to an R^{sh}-point $z \in Z(R^{\mathrm{sh}})$ with generic fibre z_K, and since A is a weak Néron model the image $x_K = u_K(z_K)$ extends to a point $x \in A(R^{\mathrm{sh}})$, giving rise to a point $\gamma = (z, x) \in \Gamma$ such that $z_k = p_k(\gamma_k)$. Since the image of p_k is constructible, containing the dense set $Z_k(k^{\mathrm{s}})$, it contains an open set of Z_k.

In particular, the generic point η of Z_k is the image of a point $\xi \in \Gamma_k$. Since the local rings $\mathcal{O}_{Z,\eta}$ (a discrete valuation ring with the same uniformizer as R) and $\mathcal{O}_{\Gamma,\xi}$ are R-flat and $\mathcal{O}_{Z,\eta} \to \mathcal{O}_{\Gamma,\xi}$ is an isomorphism on the generic fibre, one sees that $\mathcal{O}_{\Gamma,\xi}$ is included in the fraction field of $\mathcal{O}_{Z,\eta}$. Given that $\mathcal{O}_{\Gamma,\xi}$ dominates $\mathcal{O}_{Z,\eta}$, it follows that $\mathcal{O}_{Z,\eta} \to \mathcal{O}_{\Gamma,\xi}$ is an isomorphism. The schemes Z and Γ being of finite presentation over R, the local isomorphism around ξ and η extends to an isomorphism $U \to V$ between open neighbourhoods $U \subset \Gamma$ and $V \subset Z$. By inverting this isomorphism and composing with the projection $\Gamma \to A$, one obtains an extension of u_K to V. □

4 The Néron model

In the final step of the construction, we make crucial use of the group structure of A_K and in particular of the existence of invariant volume forms.

4.1 Invariant differential forms and minimal components

Quite generally, if S is a scheme and G is a smooth S-group scheme of relative dimension d, it is known that the sheaf of differential forms of maximal degree $\Omega^d_{G/S} = \wedge^d \Omega^1_{G/S}$ is an invertible sheaf that may be generated locally by a left-invariant differential form (see [BLR] 4.2). This implies that on the Néron model of A_K, provided it exists, there should be a left-invariant global non-vanishing d-form, also called a *left-invariant volume form*, with $d = \dim(A_K)$. It is the search for such a form that motivates the following constructions.

We start by choosing a left-invariant volume form ω for A_K, uniquely determined up to a constant in K^*.

Lemma 4.1.1 *The left-invariant differential form ω is also right-invariant.*

Proof : We must prove that for each K-scheme T and each point $x \in A_K(T)$, we have $\rho_x^*\omega = \omega$ where $\rho_x : A_K \times T \to A_K \times T$ is the right-multiplication morphism. The argument will be functorial in T so for notational simplicity we may just as well restrict to $T = \mathrm{Spec}(K)$ and assume that x is a K-rational point. Consider the left-multiplication morphism $\lambda_x : A_K \to A_K$ and the conjugation $c_x = \lambda_x \circ \rho_{x^{-1}}$. Using the left-invariance of ω, we see that $c_x^*\omega = \rho_{x^{-1}}^*\omega$. Moreover since c_x is a group scheme automorphism, the form $c_x^*\omega$ is a left-invariant differential form, hence there exists a constant $\chi(x) \in \mathbb{G}_m(K)$ such that $c_x^*\omega = \chi(x)\omega$. Using the fact that $c : A_K \to \mathrm{Aut}(A_K)$ is a morphism of group schemes, one sees that in fact χ defines a character $A_K \to \mathbb{G}_m$. Since A_K is proper, this map must be trivial i.e. $\chi = 1$. Finally $\rho_{x^{-1}}^*\omega = c_x^*\omega = \omega$, as desired. $\square$

If A is a model of A_K which is smooth, separated and of finite type, then all its fibres have pure dimension d and the sheaf of differential d-forms $\Omega^d_{A/R} = \wedge^d\Omega^1_{A/R}$ is invertible. Moreover, if η is a generic point of the special fibre A_k, its local ring $\mathcal{O}_{A,\eta}$ is a discrete valuation ring with maximal ideal generated by a uniformizer π for R. Then the stalk of $\Omega^d_{A/R}$ at η is a free $\mathcal{O}_{A,\eta}$-module of rank one which may be generated by $\pi^{-r}\omega$ for a unique integer $r \in \mathbb{Z}$ called the *order of* ω *at* η and denoted $\mathrm{ord}_\eta(\omega)$. If W is the irreducible component with generic point η, this is also called the *order of* ω *along* W and denoted $\mathrm{ord}_W(\omega)$. Moreover, if ρ denotes the minimum of the orders $\mathrm{ord}_W(\omega)$ along the various components of A_k, then by changing ω into $\pi^{-\rho}\omega$ we may and will assume that $\rho = 0$. A component W with $\mathrm{ord}_W(\omega) = 0$ will be called *minimal.*

In the previous sections, we saw that blowing up in a clever way finitely many times in the special fibre of a model of A_K, and removing the non-smooth locus, we obtained a weak Néron model A_2. Now, we consider the open subscheme $A_3 \subset A_2$ obtained by removing all the non-minimal irreducible components of the special fibre.

Lemma 4.1.2 *The section ω extends to a global section of $\Omega^d_{A_2/R}$ and its restriction to A_3 is a global generator of $\Omega^d_{A_3/R}$.*

Proof : Since A_2 is normal and ω is defined in codimension $\leqslant 1$, it extends to a global section of $\Omega^d_{A_2/R}$. Now, recall that the zero locus of a nonzero section of a line bundle on an integral scheme has pure codimension 1. Thus since the restriction of ω to A_3 does not vanish in codimension $\leqslant 1$, it does not vanish at all and hence extends to a global generator of $\Omega^d_{A_3/R}$. $\square$

4.2 The Néron model

Now we denote by $m_K : A_K \times A_K \to A_K$ the multiplication of the abelian variety A_K.

Theorem 4.2.1 *The morphism $m_K : A_K \times A_K \to A_K$ extends to an R-rational map $m : A_3 \times A_3 \dashrightarrow A_3$ such that the R-rational maps $\Phi, \Psi : A_3 \times A_3 \dashrightarrow A_3 \times A_3$ defined by*

$$\Phi(x, y) = (x, xy)$$

$$\Psi(x, y) = (xy, y)$$

are R-birational. In other words, m is an R-birational group law on A_3.

Proof : Applying the weak Néron mapping property (proposition 3.2.1), we can extend m_K to an R-rational map $m : A_3 \times A_2 \dashrightarrow A_2$. We wish to prove that m induces a rational map $A_3 \times A_3 \dashrightarrow A_3$. Let $D \subset A_3 \times A_2$ be the domain of definition of m. We define a morphism $\varphi : D \to A_3 \times A_2$ by the formula $\varphi(x, y) = (x, xy)$ and we view it as a morphism of A_3-schemes in the obvious way. Denote by the same symbol ω' the pullback of ω via the projection $\mathrm{pr}_2 : A_3 \times A_2 \to A_2$ and its restriction to D. We claim that $\varphi^*\omega' = \omega'$: indeed, this holds on the generic fibre because φ is an A_3-morphism of left translation, so this holds everywhere by density. Now let $\xi = (\alpha, \beta)$ be a generic point of the special fibre of $A_3 \times A_3$ and $\eta = (\alpha, \gamma)$ its image under φ. Let $r = \mathrm{ord}_\gamma(\omega) = \mathrm{ord}_\eta(\omega') \geqslant 0$. Then ω' is a generator of Ω^d_{D/A_3} at ξ and $\pi^{-r}\omega'$ is a generator of $\Omega^d_{A_3 \times A_2/A_3}$ at η. It follows that $\varphi^*(\pi^{-r}\omega') = b\omega'$ for some germ of function b around ξ. Since $\varphi^*(\pi^{-r}\omega') = \pi^{-r}\omega'$, this implies that $r = 0$ hence $\eta \in A_3 \times A_3$. This shows that the set of irreducible components of the special fibre of $A_3 \times A_3$ is mapped into itself by φ. Setting $U = D \cap (A_3 \times A_3)$ we obtain morphisms $\varphi : U \to A_3 \times A_3$ and $m = \mathrm{pr}_2 \circ \varphi : U \to A_3$ that define the sought-for rational maps. Proceeding in the same way with the morphism $\psi : D \to A_3 \times A_2$ defined by $\psi(x, y) = (xy, y)$, we see that it also induces a morphism $\psi : U \to A_3 \times A_3$. In this way we obtain the R-rational maps m, Φ, Ψ of the theorem.

In order to prove that Φ induces an isomorphism of U onto an R-dense open subscheme, we show that $\varphi : U \to A_3 \times A_3$ is an open immersion. We saw above that the map

$$\varphi^*\Omega^d_{A_3 \times A_3/A_3} \to \Omega^d_{U/A_3}$$

takes the generator ω' to itself, so it is an isomorphism. This map is nothing else than the determinant of the morphism

$$\varphi^*\Omega^1_{A_3 \times A_3/A_3} \to \Omega^1_{U/A_3}$$

on the level of 1-forms which thus is also an isomorphism. It follows that $\mathrm{E}\varphi$ is étale, and in particular quasi-finite. Since it is an isomorphism on the generic fibre, it is an open immersion by Zariski's Main Theorem ([EGA] IV_3.8.12.10). One proves the required property for Ψ in a similar way. $\square$

If follows from Weil's extension theorem for birational group laws that there exists a smooth separated R-group scheme of finite type A_4 sharing with A_3 and R-dense open subscheme A'_3 and whose group law extends m. We refer to our previous lecture for an exposition of this theorem. Note that it can be shown easily that in fact one may choose $A'_3 = A_3$, that is, A_3 embeds as an R-dense open subscheme of A_4 (see [BLR] 5.1/5). The last thing we wish to do is to check that A_4 is the Néron model of A_K:

Proposition 4.2.2 *The group scheme A_4 is the Néron model of A_K, that is, each K-morphism $u_K : Z_K \to A_K$ from the generic fibre of a smooth R-scheme Z extends uniquely to an R-morphism $u : Z \to A_4$.*

Proof : Let us consider the K-morphism $\tau_K : Z_K \times A_K \to A_K$ defined by $\tau_K(z, x) = u_K(z)x$. Applying the weak Néron mapping property, this extends to an R-rational map $\tau_2 : Z \times A_2 \dashrightarrow A_2$. In a similar way as in the proof of 4.2.1, one proves that the induced R-rational map $Z \times A_2 \dashrightarrow Z \times A_2$ defined by $(z, x) \mapsto (z, \tau_2(z, x))$ restricts to an R-rational map $Z \times A_3 \dashrightarrow Z \times A_3$. Since A_3 is R-birational to A_4, the latter may be seen as an R-rational map $Z \times A_4 \dashrightarrow Z \times A_4$. Composing with the second projection, we obtain an R-rational map $\tau_4 : Z \times A_4 \dashrightarrow A_4$ extending the map τ_K. By Weil's theorem on the extension of rational maps from smooth R-schemes to smooth and separated R-group schemes ([BLR] 4.4/1), the latter is defined everywhere and extends to a morphism. Restricting τ_4 to the product of Z with the unit section of A_4, we obtain the sought-for extension of u_K. The fact that this extension is unique follows immediately from the separation of A_4. □

Remark 4.2.3 Raynaud proved that the Néron model A_4 is quasi-projective over R. In fact, one knows that there exists an ample invertible sheaf $\mathscr{L}_K$ on A_K. Raynaud proved that there exists an integer n such that the sheaf $(\mathscr{L}_K)^{\otimes n}$ extends to an R-ample invertible sheaf on A_4, see [Ra], theorem VIII.2.

References

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 21, Springer-Verlag, 1990.

[Co] B. Conrad, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. 22 (2007), 205–257.

[EGA] J. Dieudonné, A. Grothendieck, *Éléments de Géométrie Algébrique*, Publ. Math. IHÉS 4, 8, 11, 17, 20, 24, 28, 32.

[N] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Publ.Math. IHES no. 21, 1964.

[Ra] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Springer Lecture Notes in Math. 119, 1970.