Abstracts

Images of finite schemes inside functors of homomorphisms

Matthieu Romagny

Let us fix a discrete valuation ring R with fraction field K. We consider a scheme X of finite type over R together with the action of a finite group G. Most often, in arithmetic geometry, such a situation arrives when one studies the reduction of a K-variety X_K with group action : if the variety has a model X satisfying some unicity property, then the action of G extends to X. Typical examples are the stable model of a smooth curve, or the Néron model of an abelian variety. In these cases, however, the action of G on the special fibre may fail to be faithful. The aim of the following result is to give a justification to the following naive idea : instead of G, one may look at the scheme-theoretic image of the morphism $G_R \to \mathrm{Aut}_R(X)$, where G_R is the finite flat R-group scheme associated to G. The problem that arises when one wants to do this is that the functor of automorphisms $\mathrm{Aut}_R(X)$, which is a sheaf for the fppf topology, is in general not representable by a scheme, in the absence of projective assumptions on X. However, it is possible to define the scheme-theoretic image in the setting of fppf sheaves, and our main result is :

Theorem 1. *Let X be an R-scheme locally of finite type, separated, flat and pure, of type* (FA)*. Let G be a finite flat R-group scheme acting on X, faithfully on the generic fibre. Then the schematic image of G in $\mathrm{Aut}_R(X)$ is representable by a finite flat R-group scheme.*

In this statement, the most important notion is that of a *pure R-scheme.* (We leave aside the details concerning other undefined less important notions, such as the (FA) assumption.) If R is henselian, then roughly speaking, a scheme locally of finite type, flat and pure, of type (FA), is an R-scheme locally of finite that has a covering by open affine schemes whose function rings are free R-modules. Most schemes that one encounters in pratical situations have these properties. The result above is based on some nice properties enjoyed by pure schemes. Especially important is the fact that for such a scheme X, the family of closed subschemes finite flat over R is schematically dense universally over R, and furthermore, X is the amalgamated sum of its generic fibre X_K and the family of all its closed subschemes finite flat over R. In fact, we prove :

Theorem 2. *Assume that R is henselian. Let X be an R-scheme locally of finite type, separated, flat and pure, of type* (FA)*. Then, the family of all closed subschemes $Z_\lambda \subset X$ finite flat over R is R-universally schematically dense, and* *for all separated R-schemes Y and all diagrams in solid arrows*

$$
\begin{tikzcd}
\coprod Z_{\lambda,K} \arrow[r] \arrow[d] & X_K \arrow[d] \arrow[ddr] & \\
\coprod Z_{\lambda} \arrow[r] \arrow[drr] & X \arrow[dr, dotted] & \\
& & Y
\end{tikzcd}
$$

there exists a unique morphism $X \to Y$ making the full diagram commutative.

Using theorem 2, we prove the following. Let X, Y be R-schemes locally of finite type, with X flat and pure, of type (FA) and Y separated. Let $\varphi : \Gamma \times X \to Y$ be an "action" of a finite flat R-scheme Γ, faithful on the generic fibre in an obvious sense. Then, the schematic image of Γ in $\mathrm{Hom}_R(X, Y)$ is representable by a finite flat R-scheme. Theorem 1 is an easy corollary of this.