Algebraic stacks

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These notes accompany my talk at the Séminaire RéGA (Paris) on Wed. 16th, November 2022. The reader is expected to be familiar with schemes and sheaf theory (for the fppf and/or étale topologies). Throughout, a base scheme B is fixed and most of the time absent from notation.

1 Moduli functors

1.0.1 Prehistory : classification problems. Algebraic stacks appeared with the needs of classification questions in algebraic geometry. Indeed, such questions were central from the very beginnings of the discipline. To quote Hartshorne's celebrated algebraic geometry textbook : '*In any branch of mathematics, there are usually guiding problems, which are so difficult that one never expects to solve them completely, yet which provide stimulus for a great amount of work, and which serve as yardsticks for measuring progress in the field. In algebraic geometry such a problem is the classification problem. In its strongest form, the problem is to classify all algebraic varieties up to isomorphism*' (Chapter I, § 8 in [Ha77]).

In algebraic geometry, such classification questions have a special flavour. In fact, in the earliest beginnings of the discipline, when the notion of variety emerged, it was realized that geometric objects of a given type could quite often be described by parameters (called 'moduli' by Riemann) varying in an algebraic variety (called a 'moduli space'). In some sense such moduli spaces had been considered for a long time : indeed, lines in a vector space are classified by a projective space. But for more sophisticated objects like abstract projective smooth connected curves of genus g, of whose elusive moduli space Riemann himself found the dimension $3g-3$, this was a truly fantastic discovery. As it turns out, this idea that 'nice' objects should have 'nice' moduli spaces pervaded algebraic geometry until the present day and helped shape the concept of moduli space : higher and derived algebraic geometry, and the principle of 'hidden smoothness', testify to this.

1.0.2 History : moduli functors and sheaves. The first genuine definition of moduli space was given by Grothendieck in the late 1950's with the advent of scheme theory. In his talks 'Techniques de construction et théorèmes d'existence en géométrie algébrique' at the Bourbaki seminar in the years 1959–1962 (known as 'Fondements de la géométrie algébrique'), he claimed that a classification problem should be defined by a functor of *flat families* of objects of a given type and that a moduli space should be a a scheme representing this functor – provided such a scheme exists. That is, there should exist a scheme M such that the functor F which to S associates the set of isomorphism classes of objects $C \to S$ is isomorphic to $\mathrm{Hom}(-, M)$. Here considering isomorphism classes is necessary in order for $F(S)$ to be a set ; it appears a natural thing to do at this stage.

Example 1. (*Genus g curves*) Throughout the notes we call *curve over S* a flat, proper, finitely presented morphism $C \to S$ whose fibres are smooth, projective, geometrically connected curves. Then the moduli functor of curves is defined by $F(S) :=$ the set of isomorphism classes of curves of genus g.

1.0.3 Why killing isomorphisms is a problem. At the same time, Grothendieck discovered that being representable imposes a strong condition on a functor : namely, it should be a sheaf for the fppf topology. Recall what this means : *the functor of points* $\mathrm{Hom}(-,M)$ *of a scheme* M *transforms sums into products, and for all faithfully flat, finitely presented morphisms* $S' \to S$, *the diagram*

$$\mathrm{Hom}(S,M) \longrightarrow \mathrm{Hom}(S',M) \ \rightrightarrows \mathrm{Hom}(S'\times_S S',M)$$

is exact. Shifting the focus to the objects classified by some moduli functor F, since isomorphisms provide glueing data for objects relative to fppf coverings $S' \to S$, it is not surprising that passing to isomorphism classes in defining the values $F(S)$ of a moduli functor tends to destroy the sheaf property.

Example 2. (*G-torsors*) Here is an example where the map $F(S) \to \ker(F(S') \rightrightarrows F(S'\times_S S'))$ is not injective. Let $G \to B$ be a flat, finitely presented group scheme. Write $G_S := G \times_B S$. Recall that a G_S-torsor is a flat, finitely presented scheme $E \to S$ with a free, transitive action of G_S, that is $G_S \times_S E \to E \times_S E$, $(g,x) \mapsto (x,gx)$ is an isomorphism. Let $F(S)$ be the set of isomorphism classes of G_S-torsors on S. For any nontrivial torsor $E \to S$, and $S' := E$, the torsor classes $[E],[G] \in F(S)$ have the same image in $F(S')$.

Example 3. (*the Picard functor*) Here is an example where the map $F(S) \to \ker(F(S') \rightrightarrows F(S'\times_S S'))$ is not surjective. Let X be a smooth projective curve of genus 0 over $\mathbb{R}$ with no real point (e.g. the smooth plane conic with equation $x^2+y^2+z^2=0$). Let $F = \mathrm{Pic}_{X/\mathbb{R}}$ be the (absolute) Picard functor of $X/\mathbb{R}$, defined by $F(S) = \mathrm{Pic}(X_S)$. Set $S = \mathrm{Spec}(\mathbb{R})$ and $S' = \mathrm{Spec}(\mathbb{C})$. Let $L' \in F(S') = \mathrm{Pic}(X_{\mathbb{C}})$ be a line bundle of degree 1. Since complex conjugation σ preserves the degree of line bundles, there exists an isomorphism $L' \simeq \sigma^*L'$. However, L' is not the pullback of a line bundle L over X. Indeed, for such an L we would have $\deg(L) = 1$, hence $h^0(S,L) = 2$ by Riemann-Roch, so $L \simeq \mathcal{O}(D)$ for some effective divisor D (e.g. the divisor of zeroes of a nonzero global section of L), a contradiction with the fact that X has no effective divisor of degree 1.

In special situations the sheafification of the functor of isomorphism classes is a better object (especially when 'being isomorphic' is a flat equivalence relation), but most of the time it is not.

Example 4. (*the sheaf of orbits*) Here is an example where the *sheaf* of isomorphism classes is not representable. Let G be a flat, finitely presented group scheme acting on a scheme X. Let

$$\mathrm{Orb}(S) = \big\{\ G_S\text{-equivariant maps } \omega : E \to X_S \text{ where } E \to S \text{ is a } G_S\text{-torsor}\ \big\}/\simeq .$$

Here an isomorphism $\omega \to \omega'$ is a G_S-equivariant morphism $u : E \to E'$ such that $\omega' \circ u = \omega$. The objects $\omega \in \mathrm{Orb}(S)$ are called 'orbits' because up to isomorphism ω is identified with its image sheaf in X, which is a subsheaf with transitive G-action.

The sheaf of orbits F is the sheafification of Orb. It is of fundamental importance because it gives the moduli interpretation of the quotient X/G in the category of sheaves. We sketch a proof that indeed F satisfies the universal property of the quotient sheaf. For each G-invariant map $f : X \to Y$ to a sheaf Y, define $F \to Y$ as follows. It is enough to define a map $\mathrm{Orb} \to Y$. Let $\omega \in \mathrm{Orb}(S)$, that is, $\omega : E \to X_S$ a G-equivariant morphism and $E \to S$ is a G_S-torsor. Then $x := f \circ \omega : E \to Y_S$ is G-invariant, i.e. the two pullbacks of $x \in Y_S(E)$ in $Y_S(G \times_S E)$ are equal. Given the torsor property $G \times_S E \simeq E \times_S E$, this means that $x \in \ker(Y_S(E) \rightrightarrows Y_S(E \times_S E))$. Since Y_S is a sheaf, x descends to a uniquely defined S-point $y \in Y(S)$. Then the map $F(S) \to Y(S)$, $\omega \mapsto y$ factors f.

As a general rule the sheaf $F = X/G$ is not representable by a scheme ([1]). Here is an example from Brochard [Br14], 3.10.2. Working over the field of complex numbers $k = \mathbb{C}$, let $X = \mathbb{A}^1$ be the affine line and $G = \mathbb{Z}/2\mathbb{Z}$ act by $x \mapsto -x$. We leave it as an exercise to verify that the quotient map in the category of schemes is $\pi : X \to \mathbb{A}^1$, $x \mapsto x^2$. On the other hand, the quotient presheaf P is described by $P(S) = \Gamma(S, \mathcal{O}_S)/\{\pm 1\}$ (quotient of multiplicative monoids). Assume that the quotient sheaf X/G is representable by a scheme. Then it satisfies the universal property of the quotient in the category of schemes, hence $X/G = \mathbb{A}^1$ canonically. Thus π factors as $X \to P \to X/G = \mathbb{A}^1$. We will derive a contradiction. Consider $T = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ and the points $x_0, x_1 \in X(T)$ defined by $x \mapsto 0$ and $x \mapsto \epsilon$. Since $\pi(x_0) = \pi(x_1)$, by construction of the associated sheaf the images of x_0 and x_1 in $P(T)$ have equal restrictions to some fppf cover $T' \to T$ which may be assumed affine. This implies that ϵ maps to 0 in $\Gamma(T', \mathcal{O}_{T'})/\{\pm 1\}$, hence already in $\Gamma(T', \mathcal{O}_{T'})$. This is impossible, because by faithful flatness $\Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$ is injective.

2 How not to kill isomorphisms : prestacks and stacks

2.1 Prestacks

As early as 1963, Grothendieck's ideas on descent and topologies prompted Mumford to introduce stacks, see [Mu63]. From a categorical point of view, the basic novelty is to allow moduli functors to take values in categories rather than sets.

2.1.1 Definition. A *prestack (in groupoids)* is given by three collections of data :

(i) categories $\mathscr{F}(S)$ where all morphisms are isomorphisms (a.k.a. groupoids), for all schemes S,

(ii) functors $f^* : \mathscr{F}(S) \to \mathscr{F}(S')$ for all morphisms $f : S' \to S$,

(iii) isomorphisms of functors $c_{f,g} : (fg)^* \to g^*f^*$ satisfying an 'obvious' condition of compatibility for triple compositions.

A *morphism of prestacks* is a collection of functors $\mathscr{F}(S) \to \mathscr{G}(S)$ compatible with the pullbacks f^*.

2.1.2 Remarks.

1. In practice the f^* will be obvious pullbacks and the $c_{f,g}$ will be obvious canonical isomorphisms.
2. Sometimes it is convenient to package $\{\mathscr{F}(S)\}$ as a single big category $\mathscr{F}$ whose objects are pairs (S, x) with $x \in \mathscr{F}(S)$, and whose morphisms $(S', x') \to (S, x)$ are morphisms $x' \to f^*x$ in $\mathscr{F}(S')$. In this case there is a functor $p_{\mathscr{F}} : \mathscr{F} \to$ (Schemes), $(S, x) \mapsto S$ and one calls $\mathscr{F}(S)$ the *fibre category of* $\mathscr{F}$ *at* S. A synonym for *prestack in groupoids* is *category fibred in groupoids*. In this guise, a morphism of prestacks is a functor $u : \mathscr{F} \to \mathscr{G}$ such that $p_{\mathscr{F}} = p_{\mathscr{G}} \circ f$.
3. Since any set can be viewed as a category all whose morphisms are identities, any fonctor F : (Schemes) $\to$ (Sets) can be viewed as a prestack.
4. There is a 2-Yoneda lemma which is just as useful and flexible as the classical one : for all schemes S and prestacks $\mathscr{F}$, the natural functor is an equivalence of categories $\mathrm{Hom}(S, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(S)$.

1. Or even by an algebraic space, if the reader knows what this is. In this talk we will stay silent about algebraic spaces. For readers wo don't know about this notion, it will be only a mild distortion of reality to consider that they are the same thing as schemes.

2.2 Stacks

In order to define stacks, the categorically enhanced analogues of sheaves, we need to add a condition meaning that objects *and* morphisms glue. For this we introduce :

- for objects $x, y \in \mathscr{F}(S)$, the *presheaf of (iso)morphisms* $\underline{\mathrm{Isom}}_{\mathscr{F}}(x,y)$,
- for an fppf covering map $S' \to S$, the category of *S'-objects with descent data* $\mathscr{F}^{\mathrm{dd}}(S' \to S)$.

The former is the functor $S' \mapsto \mathrm{Isom}_{\mathscr{F}(S')}(x_{S'}, y_{S'})$. The latter is the category of pairs (x', α) where $x' \in \mathscr{F}(S')$ and $\alpha : p_1^* x' \xrightarrow{\sim} p_2^* x'$ is an isomorphism such that $p_{13}^*\alpha = p_{23}^*\alpha \circ p_{12}^*\alpha$. Here $p_i : S' \times_S S' \to S'$, $p_{ij} : S' \times_S S' \times_S S' \to S' \times_S S'$ are the projections.

2.2.1 Definition. $\mathscr{F}$ is a *stack* if it transforms sums into products and the following conditions hold :

(i) for each $x, y \in \mathscr{F}(S)$ the presheaf $\underline{\mathrm{Isom}}_{\mathscr{F}}(x,y)$ is a sheaf ;

(ii) for each fppf covering $S' \to S$ the natural functor $\mathscr{F}(S) \to \mathscr{F}^{\mathrm{dd}}(S' \to S)$ is an equivalence.

A *morphism of stacks*, or *1-morphism of stacks*, is a morphisms of prestacks. An *isomorphism of stacks* is a morphism which is an equivalence of categories. (One can show that this is the same thing as a functor which is fully faithful and locally essentially surjective.)

Warning : we see that the notion of isomorphism of stacks which is in use is *not* the categorical notion.

2.2.2 Examples. All the examples of Section 1 have enhanced stack versions :

1. The *stack of curves of genus* $g \geqslant 2$ is the stack $\mathscr{M}_g$ with objects the curves $C \to S$ of genus g.
2. The *classifying stack* of G is the stack BG with objects the G-torsors $E \to S$.
3. The *Picard stack* of X/B is the stack $\mathscr{P}ic_{X/B}$ with objects the line bundles $L \to X_S$.
4. The *quotient stack of X by G* is the stack $[X/G]$ with objects the pairs (E, ω) where $E \to S$ is a G_S-torsor and $\omega : E \to X_S$ is a G_S-equivariant morphism.

In each case, the isomorphisms in the fibre categories are the obvious ones.

2.2.3 Two useful notions.

1. There is a *stackification* process analogous to sheafification.
2. A stack $\mathscr{F}$ has a *sheaf of isomorphism classes* $F = \mathscr{F}^{/\simeq}$ and this construction is left adjoint to the inclusion of sheaves into stacks. A stack 'is' a sheaf iff objects have trivial automorphism groups, iff the map $\mathscr{F} \to \mathscr{F}^{/\simeq}$ is an isomorphism.

2.2.4 The 2-category of stacks. A crucial point is that prestacks and stacks do not form a category but rather a 2-category. This is simply because 'equality' of objects is not a natural notion and should be replaced by the datum of an isomorphism. That this is natural and necessary for the theory to work well can be seen in a zillion ways (and is in fact the reason why we left the world of functors of isomorphism classes). Here is one. Consider the projective line $X = \mathbb{P}^1_k = U_0 \cup U_1$ and its standard covering by affine lines. Then glueing of sheaves should result in an equivalence of categories

$$\mathscr{P}ic_{X/k} \xrightarrow{\sim} \mathscr{P}ic_{U_0/k} \times_{\mathscr{P}ic_{U_0 \cap U_1/k}} \mathscr{P}ic_{U_1/k}.$$

Of course this is possible only if the indicated fibred product is the category of triples (L_0, L_1, α) with an isomorphism $\alpha : L_{0|U_0 \cap U_1} \xrightarrow{\sim} L_{1|U_0 \cap U_1}$ rather than an equality (what would 'equality' mean anyway ?).

We collect in one single definition two concepts of 2-category theory relevant to stacks.

2.2.5 Definition.

1. Let $u, v : \mathscr{F} \to \mathscr{G}$ be 1-morphisms between prestacks. A *2-morphism* $\alpha : u \to v$ is a natural transformation of functors. (Since all morphisms in $\mathscr{G}$ are isomorphisms, such a 2-morphism is necessarily an isomorphism.)
2. Let $u : \mathscr{F} \to \mathscr{H}$, $v : \mathscr{F} \to \mathscr{H}$ be 1-morphisms of prestacks. The (2-)fibred product $\mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ is the prestack whose S-points are triples (x, y, α) with $x \in \mathscr{F}(S)$, $y \in \mathscr{G}(S)$ and $\alpha : u(x) \xrightarrow{\sim} v(y)$ an isomorphism in $\mathscr{H}(S)$. Note that if $\mathscr{F}, \mathscr{G}$ are stacks then so is $\mathscr{F} \times_{\mathscr{H}} \mathscr{G}$.

3 How to do geometry : algebraic stacks

3.1 Algebraic stacks

The algebraic structure on a scheme is given by the existence of coverings in the Zariski topology by affine schemes. A stack will be declared *algebraic* if it has such a covering in the étale or the smooth topology. In order to make sense of this we need the following notion.

3.1.1 Definition. A morphism of stacks $u : \mathscr{F} \to \mathscr{G}$ is called *representable* (by schemes, resp. by algebraic spaces) if whenever $v : S \to \mathscr{G}$ is a morphism from a scheme S (viewed as a stack), the fibred product $\mathscr{F} \times_{\mathscr{G}} S$ is representable (by a scheme, resp. by an algebraic space).

3.1.2 Examples.

1. $\mathscr{M}_{g,n} \to \mathscr{M}_g$. Let $\mathscr{M}_{g,n}$ be the stack of n-pointed curves, i.e. tuples $(C, \sigma_1, \dots, \sigma_n)$ composed of a curve $C \to S$ together with fibrewise disjoint sections $\sigma_i : S \to C$. There is an obvious forgetful map $u : \mathscr{M}_{g,n} \to \mathscr{M}_g$ which is representable. Indeed, given a morphism $S \to \mathscr{M}_g$, which by Yoneda is a curve $C \to S$, the fibred product (the 'fibre of u at C') is representable by the scheme $C^n \setminus \mathbf{\Delta}$ where $\mathbf{\Delta}$ is the fat diagonal, where at least two components are equal.
2. Let $f : G \to H$ be a morphism of flat group schemes of finite presentation. Then to any G_S-torsor $E \to S$ we can attach the H_S-torsor $E \times^{G_S} H_S$, the *contracted product*, which is the quotient of $E \times H_S$ by the G-action $g \cdot (e, h) = (f(g)h, g^{-1}e)$. This gives rise to a morphism of stacks $BG \to BH$ which is representable only when f is a monomorphism (exercise left to the reader).
3. $n : \mathscr{P}ic_{X/k} \to \mathscr{P}ic_{X/k}$. For each $n \in \mathbb{Z}$ there is an n-power map $L \mapsto L^{\otimes n}$ on the Picard stack. This map is *not* representable. Indeed, let $S \to \mathscr{P}ic_{X/k}$ be a map, i.e. a line bundle L on X_S. The fibred product $n^{-1}(L) := \mathscr{P}ic_{X/k} \times_{n, \mathscr{P}ic_{X/k}, L} S$ is the category whose T-points are pairs (M, α) where $M \in \mathscr{P}ic(X_T)$ and $\alpha : M^{\otimes n} \xrightarrow{\sim} L_T$ is an isomorphism. An automorphism of (M, α) is an automorphism of line bundles $\varphi : M \to M$ such that $u \circ \varphi^{\otimes n} = u$. Since automorphisms of line bundles are just sections of $\mathbb{G}_{m, X_T}$ we find that φ should be a root of unity. Finally $\mathrm{Aut}(M, \alpha) = f_{T,*}\mu_n$ where $f : X \to \mathrm{Spec}(k)$ is the structure morphism. In particular, the stack $n^{-1}(L)$ has fibre categories which are *not* equivalent to sets.
4. $[X/G] \to BG$. The forgetful map which given a G_S-equivariant morphism $\omega : E \to X_S$ with source $E \to S$ remembers only $E \to S$ and forgets the map to X, is representable. Indeed, the fibre at some torsor $E \to S$ is representable by the functor $\mathrm{Hom}_G(E, X)$, which is a scheme (or rather an algebraic space in general).

3.1.3 Lemma. *Let $\mathscr{F}$ be a stack and $\Delta : \mathscr{F} \to \mathscr{F} \times \mathscr{F}$ its diagonal mapping. The following are equivalent.*

1. *The morphism Δ is representable.*
2. *Each morphism $U \to \mathscr{F}$ from a scheme is representable.*
3. *For any two morphisms $U \to \mathscr{F}$, $V \to \mathscr{F}$ from schemes, the fibred product $U \times_{\mathscr{F}} V$ is representable.*
4. *For any two objects $x, y \in \mathscr{F}(U)$, the sheaf of isomorphisms $\underline{\mathrm{Isom}}_{\mathscr{F}}(x,y)$ is representable.*

The main point of the proof is to observe that when the objects in 4. are seen as morphisms $x, y : U \to \mathscr{F}$ then the stack $U \times_{\mathscr{F}} U$ is isomorphic to the sheaf $\underline{\mathrm{Isom}}_{\mathscr{F}}(x,y)$. We omit the proof, which is playing with the definitions. The reason why the lemma is important comes from the following meaningful definition.

3.1.4 Definition. Let $u : \mathscr{F} \to \mathscr{G}$ be a representable map of stacks. Let P be a property of morphisms of schemes which is stable by base change. We say that u *has property* P if for all morphisms $U \to \mathscr{G}$ from a scheme, the morphism $\mathscr{F} \times_{\mathscr{G}} U \to U$ has property P. (Note that $\mathscr{F} \times_{\mathscr{G}} U$ is a scheme!)

$$\begin{array}{ccc} \mathscr{F} \times_{\mathscr{G}} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathscr{F} & \xrightarrow{\ u\ } & \mathscr{G} \end{array}$$

We are in position to define algebraicity.

3.1.5 Definition. We say that a stack $\mathscr{F}$ is *algebraic* if the following conditions are satisfied :

(i) The diagonal $\Delta : \mathscr{F} \to \mathscr{F} \times \mathscr{F}$ is representable.

(ii) There exists a scheme U and a smooth, surjective morphism $U \to \mathscr{F}$.

The map $U \to \mathscr{F}$ is called a *smooth atlas* or a *smooth presentation* for $\mathscr{F}$.

3.1.6 Examples. Under appropriate conditions, our four preferred examples are algebraic stacks. We review them in an order adapted to our arguments.

4. Let G be a flat, finitely presented group scheme acting on a scheme X. It is easy to see that the fibre of the map $X \to [X/G]$ at a point $S \to [X/G]$ incarnated by $\omega : E \to X_S$ is isomorphic to E. From this follows that if G is smooth over the base (so each G_S-torsor $E \to S$ is smooth also), then $X \to [X/G]$ is smooth thus we can take $U = X$ as a smooth atlas. (If G si only assumed flat, then similarly $X \to [X/G]$ is flat ; and a theorem of Artin shows that if a stack has an fppf atlas, then it is algebraic.)
2. The stack BG is $[X/G]$ with $X = B$, the base scheme : a special case of the previous point.
1. Let $g \geqslant 2$. Then for any genus g curve $f : C \to S$ the cube $\omega_{C/S}^{\otimes 3}$ of the canonical sheaf is very ample. Hence the canonical map $C \to \mathbb{P}(f_*\omega_{C/S}^{\otimes 3})$ is a closed embedding. Locally over S there exist isomorphisms $\mathbb{P}(f_*\omega_{C/S}^{\otimes 3}) \simeq \mathbb{P}^{5g-6}$ and any two such isomorphisms differ by an element of PGL_{5g-5}. Now building on the Hilbert scheme, there is a scheme H_g which classifies curves inside $\mathbb{P}^{5g-6}$. Assembling these remarks, we have an isomorphism $[H_g/\mathrm{PGL}_{5g-5}] \xrightarrow{\sim} \mathscr{M}_g$. Hence $\mathscr{M}_g$ is a quotient stack by a smooth group, which is algebraic.
3. Assume that X is flat, projective, of finite presentation over the base B. Then a theorem of Brochard [Br09], building on deep results of Artin giving criteria for algebraicity, states that $\mathscr{P}ic_{X/B}$ is algebraic.

3.2 Line bundles on algebraic stacks

In this section $\mathscr{F}$ is an algebraic stack. We present a simplified version of the sheaf theory on $\mathscr{F}$ which is enough for our needs. For a more complete account we refer to Olsson [Ol16].

3.2.1 Definition. A *quasi-coherent sheaf* $M = \{M_U\}$ *on* $\mathscr{F}$ is given by the following data :

(i) a collection of quasi-coherent sheaves M_U on U, for each smooth morphism $U \to \mathscr{F}$,

(ii) a collection of isomorphisms $\rho_{V,U} : f^*M_U \xrightarrow{\sim} M_V$, for all morphisms $f : V \to U$, satisfying the transitivity condition $\rho_{W,V} \circ g^*\rho_{V,U} = \rho_{W,U}$ for compositions $W \xrightarrow{g} V \xrightarrow{f} U$.

We say that M is *locally free of rank* r (resp. an *invertible sheaf*, or *line bundle*) if each M_U is.

There is a Picard group $(\mathrm{Pic}(\mathscr{F}), \otimes)$ and also a Picard stack $\mathscr{P}ic_{\mathscr{F}/B}$ (see [Br09]).

3.3 Gerbes

Finally we introduce a kind of stack which is somehow the opposite example of a sheaf : this is the notion of *gerbe*. Loosely speaking, a gerbe is a kind of twisted form of a classifying stack BG.

3.3.1 Definition. A stack $\mathscr{F}$ is called a *gerbe* if the following two conditions hold :

(i) $\mathscr{F}$ is locally nonempty, that is, for any scheme S there exists a cover in the fppf topology $S' \to S$ such that $\mathscr{F}(S') \neq \varnothing$. (Clearly it suffices to find some $S' = B'$ for the base scheme $S = B$.)

(ii) $\mathscr{F}$ is locally connected, that is, for any scheme S, any two objects $x, y \in \mathscr{F}(S)$ are locally (over S) isomorphic.

A morphism of stacks $u : \mathscr{F} \to \mathscr{G}$ is called a *gerbe* if it is representable by gerbes, that is, if for all morphisms $S \to \mathscr{G}$ from a scheme, the fibred product $\mathscr{F} \times_{\mathscr{G}} S$ is a gerbe over S. Equivalently, the fibres of u are locally nonempty and locally connected.

There is a related notion of G-*gerbe* where a group scheme is part of the data. We give two, nonequivalent variants which appear in the literature.

3.3.2 Definition. Let G be a flat, finitely presented group scheme.

1. ([EHKV01], § 3) A G-*gerbe* is a gerbe $\mathscr{F}$ such that there exists a faithfull flat locally finitely presented map $S' \to S$ and an isomorphism $\mathscr{F} \times_S S' \xrightarrow{\sim} BG \times_S S'$.
2. (see [Ol16], § 12.2) A G-*gerbe* is a gerbe $\mathscr{F}$ together with isomorphisms $\iota_x : G_S \xrightarrow{\sim} \mathrm{Aut}_S(x)$, for all S and $x \in \mathscr{F}(S)$, such that $c_f \circ \iota_x = \iota_y$ for all morphisms $f : x \to y$ (here $c_f : \mathrm{Aut}_S(x) \to \mathrm{Aut}_S(y)$ is conjugation by f).

The prototypical example of a gerbe is BG, because any two torsors are locally isomorphic (to the trivial torsor). Or rather, to make the local nonemptiness condition (i) nontrivial, the prototypical example of a gerbe is a twisted form of BG as in the first example below.

3.3.3 Examples.

1. (Gerbes of models of a torsor) Let k^*/k be a field extension; for each k-scheme X write $X^* := X_{k^*}$. Let G be a finitely presented k-group scheme and $P \to \mathrm{Spec}(k^*)$ a G^*-torsor. Let $\mathscr{F}(S)$ be the category of pairs (E, u) where $E \to S$ is a G_S-torsor and $u : E_{S^*} \xrightarrow{\sim} P_{S^*}$ is an isomorphism of torsors (note that E_{S^*} is a base change from S while P_{S^*} is a base change from k^*). Then $\mathscr{F}$ is a gerbe. For a more complete treatment see Giraud [Gi71], V.3.1.6.

2. (Root stacks) Let $\mathscr{P}ic = \mathscr{P}ic_{k/k}$ be the Picard stack of the point, a particular case of the Picard stack $\mathscr{P}ic_{X/k}$. Then we saw that given a line bundle $L \in \mathscr{P}ic(S)$, i.e. a point $S \to \mathscr{P}ic$, the fibre of the morphism $n : \mathscr{P}ic \to \mathscr{P}ic$ at L is the 'stack of n-th roots of L' composed of pairs (M, α) where $M \in \mathscr{P}ic(T)$ and $\alpha : M^{\otimes n} \xrightarrow{\sim} L_T$. This is a μ_n-gerbe.
3. (Elliptic curves) Let $\mathscr{E}ll/k$ be the Deligne-Mumford compactification of the moduli stack of elliptic curves, over a field k of characteristic prime to 6. Then one can show that $\mathscr{E}ll$ is isomorphic to the stacky projective line $\mathbb{P}^1(4,6)$, the stack quotient $[\mathbb{A}^2 \setminus \{0\}]/\mathbb{G}_m$ where $\mathbb{G}_m$ acts with weights 4 and 6 on the coordinates. Note that $\gcd(4,6) = 2$, which is related to existence of the elliptic involution. The natural morphism $\mathscr{E}ll = \mathbb{P}^1(4,6) \to \mathbb{P}^1(2,3)$ makes $\mathscr{E}ll$ a $\mathbb{Z}/2\mathbb{Z}$-gerbe over $\mathbb{P}^1(2,3)$. For more details see e.g. [Be13], 4.1.3.
4. More examples can be found on the MathOverflow post `https://mathoverflow.net/questions/263832/phenomena-of-gerbes`.
5. Even more examples in Siddharth Mathur's talk to come! (*Indeed my talk was followed by a talk by Siddharth Mathur.*)

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