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« Contributions à la géométrie algébrique imparfaite en caractéristique positive »

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Titre : *Contributions à la géométrie algébrique imparfaite en caractéristique positive*

Mots clés : schéma en groupes fini plat, torseur, modèle maximal, coperfection, groupoïde fondamental étale, espace de modules de revêtements.

Résumé : Ce travail de thèse, composé de quatre parties, est consacré à l'étude de la géométrie algébrique en caractéristiques mixte et positive.

Dans la première partie, motivés par une théorie conjecturale de la ramification pour les torseurs inséparables, nous étudions les modèles maximaux des torseurs sur un corps local, qui sont une généralisation des anneaux des entiers dans la théorie classique de la ramification. Nous prouvons la maximalité et la fonctorialité des modèles maximaux et nous les calculons explicitement pour les schémas en groupes finis plats d'ordre p.

La deuxième partie est un travail en commun avec Giulio Orecchia et Matthieu Romagny. Nous étudions la perfection des algèbres et la coperfection des espaces et champs algébriques. Nous prouvons que l'espace des composantes connexes fournit la coperfection d'un espace algébrique et il représente la colimite du système de Frobenius relatifs. Dans le cas des champs algébriques, nous construisons le pro-groupoïde fondamental étale, nous prouvons qu'il fournit la coperfection, et il représente la colimite du système de Frobenius relatifs dans le cas de Deligne-Mumford.

Dans la troisième partie, nous prouvons quelques résultats de platitude et de représentabilité des espaces de modules de torseurs sous certains schémas en groupes, qui découlent naturellement de l'espace de modules propre des p-revêtements galoisiens. Nous discutons également de la relation avec les jacobiennes généralisées des courbes ouvertes.

Dans la dernière partie, nous nous intéressons à un nouveau type de géométrie analytique non-archimédienne, avec des valuations à valeurs dans des monoïdes commutatifs totalement ordonnés. Nous étudions quelques exemples de schémas et d'espaces adiques.

Title : *Contributions to imperfect algebraic geometry in positive characteristic*

Keywords : finite flat group scheme, torsor, maximal model, coperfection, étale fundamental groupoid, moduli space of covers.

Abstract : This thesis work, consisting of four parts, is devoted to the study of algebraic geometry in mixed and positive characteristics.

In the first part, motivated by a conjectural ramification theory for inseparable torsors, we study the maximal model of a torsor over a local field, which is a generalization of integer rings in classical ramification theory. We prove the maximality and functoriality of maximal models, and calculate them explicitly for some finite flat group schemes of order p.

The second part is a joint work with Giulio Orecchia and Matthieu Romagny. We study perfection of algebras and coperfection of algebraic spaces and stacks. We prove that the space of connected components provides the coperfection of an algebraic space, and it represents the colimit of relative Frobenii. In the case of algebraic stacks, we construct the étale fundamental pro-groupoid, and prove that it provides the coperfection, and it represents the colimit of relative Frobenii in Deligne-Mumford case.

In the third part, we prove some results on flatness and representability of moduli spaces of torsors under certain group schemes, which naturally arise from the proper moduli space of Galois p-covers (stable p-torsors). We also discuss the relation with generalized Jacobians of open curves.

In the last part, we are interested in a new kind of nonarchimedean analytic geometry, with valuations on totally ordered commutative monoids. We study some examples from schemes and adic spaces.

Contributions à la géométrie algébrique imparfaite en caractéristique positive

Yuliang HUANG

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Introduction (en français)

Ce travail de thèse est consacré à l'étude de la géométrie algébrique en caractéristiques mixtes et positives, notamment de la géométrie algébrique sur une base imparfaite à caractéristique positive. Il se compose de quatre parties ayant des origines de motivation différentes mais également liées. Dans ce qui suit, nous donnons quelques introductions.

Modèles maximaux de torseurs sur un corps local

Dans le livre de Serre [Se79], la théorie classique de la ramification de corps locaux est bien documentée. L'hypothèse d'un corps résiduel parfait est cruciale dans cette théorie classique. Abbes et Saito (cf. [AS02], [AS03], [Sa12]) ont développé la théorie de la ramification pour les schémas de dimension supérieure (cf. [AS02], [Sa12]), utilisant des techniques de la géométrie analytique rigide. Plus tard, Saito [Sa19] a également donné une approche schématique. Cette théorie plus générale est adaptée dans les travaux ultérieurs (cf. [KS08], [KS13]) de Kato et Saito sur la théorie de la ramification de variétés de dimensions supérieures sur des corps parfaits et des corps locaux.

La ramification des extensions finies *séparables* de corps locaux avec un corps résiduel imparfait est éventuellement bien comprise, grâce à la théorie d'Abbes-Saito. Une question intéressante est: qu'en est-il des extensions finies de corps locaux *arbitraires*? Par exemple, si le corps local a une caractéristique positive, il existe de nombreuses extensions inséparables qui ne peuvent pas être reconnues dans le groupe de Galois absolu. Ont-elles une théorie raisonnable de ramification? Afin de réfléchir à cette question, nous devons clarifier certaines choses. Premièrement, qu'est-ce que signifie "Galois" pour les extensions inséparables? Dans le contexte classique, le groupe de Galois $G_{L/K}$ d'une extension galoisienne L/K de corps locaux agit sur L, ce qui rend le schéma $\mathrm{Spec}(L)$ un $G_{L/K}$-torseur sur $\mathrm{Spec}(K)$. Pour une extension purement inséparable K'/K, il n'y a pas de groupe abstrait nontrivial agissant sur K' fixant K. Au lieu de cela, il pourrait y avoir un schéma en groupes infinitésimal (ou appelé local) agissant sur K' et transformant $\mathrm{Spec}(K')$ en torseur sur $\mathrm{Spec}(K)$. Une différence importante est qu'il peut y avoir plus d'un groupe agissant sur K, et en particulier, il est raisonnable de s'attendre à ce qu'ils aient des comportements de ramification différents.

De nombreuses tentatives ont été étudiées, en particulier, la notion d'une action *modérée* a été considérablement développée par diverses personnes, cf. [CEPT96]. Dans [Za16], Zalamansky a prouvé une formule analogue de Riemann–Hurwitz pour des revêtements inséparables sous des schémas en groupes diagonalisables infinitésimaux. Pour des cas plus généraux, il faut encore de nouvelles idées.

Notre cadre de base est comme suivant. Soit K un corps local, $\mathcal{O}_K$ l'anneau d'entiers et k le corps résiduel. Soit G un groupe sur $\mathcal{O}_K$, puis nous commençons avec un G_K-torseur P_K sur $\mathrm{Spec}(K)$. Le premier problème auquel nous sommes confrontés est de trouver un modèle entier P de P_K avec une G-action étendue. Dans le cas classique, ceci est standard, à savoir, nous prenons simplement la clôture intégrale de $\mathcal{O}_K$ dans le corps d'extension, avec l'action naturelle du groupe de Galois. Dans notre contexte, un bon candidat est la notion de "modèles minimaux" de torseurs sur K, à la Lewin-Ménégaux. La construction trouve son origine dans l'article de Raynaud, [Ra67], page 82-83, dans lequel il construisit des schémas abéliens et entendait prétendre qu'il était *minimal* au sens habituel:

"c) Soient A un anneau de valuation discrète, K son corps de fractions, G une K-variété

> *abélienne qui possède une bonne réduction sur A, de sorte que G se prolonge en un schéma abélien $\mathscr{G}$ sur $S = \mathrm{Spec}\ A$ et soit X un K-espace principal homogène sous G. Montrons que X se prolonge en un S-schéma* projectif *et* régulier $\mathscr{X}$ *(qui sera d'ailleurs un modèle minimal de X dans la terminologie de [2]). ...”*

où [2] est [Ne67] dans le même volume. Plus tard, l'idée a été développée par Lewin-Ménégaux dans son article [LM83]. Mais malheureusement, son article est sommaire et laisse le terme “minimal” inexpliqué. Il s'avère que de façon surprenante, un “modèle minimal” n'est pas minimal, mais en fait *maximal*, parmi tous les modèles entiers.

0.0.0.1 Proposition.[Proposition 2.2.0.2] *Soit $X_K \to \mathrm{Spec}(K)$ un G_K-torseur, nous supposons que son modèle maximal $X \to S$ existe. Alors X est maximal parmi tous les modèles intégraux de $X_K \to \mathrm{Spec}(K)$, à savoir, si $\mathcal{X}$ est un autre modèle intégral, alors il existe un unique morphisme de modèles unique $X \to \mathcal{X}$.*

Comme dans la proposition ci-dessus, nous changeons la terminologie en *modèle maximal* qui est plus raisonnable. Le modèle maximal a quelques propriétés agréables. Par exemple, il est unique à isomorphisme près s'il existe, [1] et il recouvre la situation classique, à savoir, si L/K est une extension galoisienne avec le groupe de Galois $G_{L/K}$, alors le modèle maximal du $G_{L/K}$-torseur $\mathrm{Spec}(L)$ est exactement $\mathrm{Spec}(\mathcal{O}_L)$, où $\mathcal{O}_L$ est l'anneau d'entiers de L. Les modèles maximaux sont fonctoriels par rapport aux schémas en groupes. Plus important, les modèles maximaux sont compatibles avec les inductions, et en particulier, avec les quotients. Nous prouvons les résultats suivants:

0.0.0.2 Proposition.[Proposition 2.2.0.4] *Soit $\varphi : G \to H$ un homomorphisme de S-schémas en groupes plats, X_K un G_K-torseur. Supposons que le modèle maximal X de X_K existe. Si le H_K-torseur induit Y_K a un modèle maximal Y, alors $\mathrm{Ind}_G^H X = X \times^G H$ est représentable par Y.*

Cette propriété est nécessaire et importante pour tenter de construire une théorie de ramification générale.

Overview. Dans son article [LM83], Lewin-Ménégaux a prouvé l'unicité et la fonctorialité du modèle maximal, l'existence dans les cas de schémas abéliens et de schémas en groupes finis plats et commutatifs, ainsi que la propriété de la différente. Cependant, son article est très sommaire, nous donnons donc des détails et représentons ses résultats dans les sections 2.3, 2.5 et la partie de la fonctorialité dans la section 2.2. En outre, nous prouvons la maximalité et la compatibilité avec les inductions dans la section 2.2. Dans la section 2.4, nous prouvons un résultat sur l'existence d'un modèle maximal de torseurs sous un produit semi-direct des schémas en groupes. Dans la section 2.6, nous classifions les modèles maximaux de torseurs sous quelques schémas en groupes finis plats d'ordre p.

Coperfection et le pro-groupoïde fondamental étale en caractéristique positive

Au chapitre 3 (travail en commun avec Giulio Orecchia et Matthieu Romagny), nous étudions des espaces et des champs algébriques, plats de présentation finie en caractéristique $p > 0$, ainsi que les applications à des espaces et champs parfaits (ceux dont le morphisme de Frobenius est un isomorphisme). Notre intérêt est la perfection des algèbres et la coperfection des espaces et des champs algébriques. Nos résultats sont meilleurs pour les morphismes à fibres géométriquement

[1] L'unicité n'est pas simplement une conséquence de la maximalité, c'est vrai pour les modèles maximaux sur des bases plats arbitraires, à condition qu'ils existent. Voir Proposition 2.1.0.4. D'autre part, nous n'avons que la maximalité sur S.

réduites, également appelés *séparables*. Laissez-nous maintenant les décrire.

Pour chaque espace algébrique S et chaque champ algébrique plats de présentation finie $\mathscr{X} \to S$, nous construisons son ***pro-groupoïde fondamental étale*** $\Pi_1(\mathscr{X}/S)$. C'est un 2-pro-objet de la 2-catégorie des champs algébriques étales. Si $\mathscr{X}/S$ est séparable, alors leur espace de module grossier est l'espace des composantes connexes $\pi_0(X/S)$ (voir [Rom11]), considéré comme un 2-pro-objet constant. Lorsque S est le spectre d'un corps κ et que $\mathscr{X}$ est géométriquement connexe, le pro-groupoïde fondamental étale $\Pi_1(\mathscr{X}/S)$ est représentable dans la 2-catégories des champs par la gerbe fondamentale étale $\Pi^{\text{ét}}_{\mathscr{X}/\kappa}$ de Borne et Vistoli [BV15, § 8]. Si S est de caractéristique p, soit

$$\mathrm{F}_i : \mathscr{X}^{p^i/S} \longrightarrow \mathscr{X}^{p^{i+1}/S}$$

le morphisme de Frobenius relatif de $\mathscr{X}^{p^i/S}$, le i-ème twist de Frobenius de $\mathscr{X}/S$.

Théorème A. (3.5.1.1, 3.5.3.5) *Soit S un espace algébrique noetherien de caractéristic p.*

(i) *Soit $X \to S$ un morphisme plat de présentation fini, séparable d'espaces algébriques. Le système inductif des Frobenii relatifs*

$$X \xrightarrow{\mathrm{F}_0} X^{p/S} \xrightarrow{\mathrm{F}_1} X^{p^2/S} \longrightarrow \dots$$

admet une colimite dans la catégorie des espaces algébriques sur S. Cette colimite est l'espace algébrique des composantes connexes $\pi_0(X/S)$; il est une coperfection de $X \to S$.

(ii) *Soit $\mathscr{X} \to S$ un champ algébrique plat de présentation fini, séparable. Le système inductif des Frobenii relatives*

$$\mathscr{X} \xrightarrow{\mathrm{F}_0} \mathscr{X}^{p/S} \xrightarrow{\mathrm{F}_1} \mathscr{X}^{p^2/S} \longrightarrow \dots$$

admet une colimite dans la 2-catégorie des champs de Deligne–Mumford quasi-séparés sur S. Cette colimite est le pro-groupoïde fondamental étale $\Pi_1(\mathscr{X}/S)$, et il est une coperfection de $\mathscr{X}/S$.

Notez que le point (ii) inclut le point (i) en tant que cas particulier, parce que $\Pi_1(\mathscr{X}/S)$ a pour espace de module grossier $\pi_0(\mathscr{X}/S)$. Nous incluons (i) pour souligner et aussi parce que la preuve procède en déduisant (ii) de (i).

Dans la catégorie des algèbres, la situation est en quelque sorte plus subtile. Étant donné un anneau R en caractéristique p et une algèbre $R \to A$, soit

$$\mathrm{F}_i : A^{p^{i+1}/R} \to A^{p^i/R}$$

le morphisme de Frobenius relatif de $A^{p^i/R}$, le i-ème twist de Frobenius de A. Définissons la *préperfection*:

$$A^{p^\infty/R} = \lim \left(\cdots\ A^{p^2/R} \xrightarrow{\mathrm{F}_1} A^{p/R} \xrightarrow{\mathrm{F}_0} A\right).$$

Le nom est expliqué par un fait surprenant: l'algèbre $A^{p^\infty/R}$ n'est pas parfaite en général, même si $R \to A$ est plat, de présentation fini et séparable. Nous en donnons un exemple avec R égal à l'anneau local d'une singularité de courbe nodale (voir 3.4.5.2). Dans notre exemple, la double préperfection est parfaite, mais nous ne savons pas si les préperfections itérées devraient converger vers une algèbre parfaite en général. Dans le cas affine $S = \operatorname{Spec}(R)$ et $X = \operatorname{Spec}(A)$, nous écrivons $\pi_0(A/R)$ au lieu de $\pi_0(X/S)$. Ce que le Théorème A implique dans ce cas est qu'il existe un isomorphisme de R-algèbres:

$$\mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Ici $\mathcal{O}(-)$ est le foncteur des fonctions globales. Compte tenu des mauvaises propriétés des anneaux considérés, on ne pouvait vraiment pas anticiper cela: en effet, généralement $\mathcal{O}(\pi_0(A/R))$ n'est pas étale et $A^{p^\infty/R}$ n'est pas parfait. De même que ci-dessus, la structure de la preuve consiste à établir d'abord cet isomorphisme d'algèbres (voir 3.4.3.2), puis à en déduire l'énoncé géométrique pour des espaces et des champs (Théorème A ci-dessus).

Cela appelle une étude plus approfondie de la perfection des algèbres. Nous nous attendons généralement à ce que, pour les algèbres de type fini, il existe une plus grande sous-algèbre étale qui soit (au moins proche de) la perfection de $R \to A$. En essayant de matérialiser cette image, nous étudions plus en détail les enveloppes étales. Nous reprenons le travail récent de Ferrand [Fe19] et prouvons le résultat suivant qui n'est pas particulier à la caractéristique p.

Théorème B. (3.3.1.8) *Soit $f : X \to S$ un morphisme fidèlement plat de présentation fini d'espaces algébriques. Supposons que S est noethérien, géométriquement unibranche et sans points immergés. Alors la catégorie $\mathsf{E}^{\mathrm{aff},\mathrm{dom}}(X/S)$ est un ensemble réticulé, c'est-à-dire, deux objets quelconques ont un suprémum et un infimum (pour la relation évidente de domination). De plus, elle a un élément le plus grand $\pi^a(X/S)$.*

L'élément le plus grand $\pi^a(X/S)$ est le spectre relatif d'un faisceau de $\mathcal{O}_S$-algèbres, qui est la plus grande sous-algèbre étale de $f_*\mathcal{O}_X$. Quand S est artinien ou $X \to S$ est séparable, de sorte que $\pi_0(X/S)$ est un espace algébrique étale, nous avons des morphismes:

$$X \longrightarrow \pi_0(X/S) \longrightarrow \pi^a(X/S).$$

Quand $S = \mathrm{Spec}(R)$ et $X = \mathrm{Spec}(A)$, la plus grande sous-algèbre étale est notée $A^{\text{ét}/R} \subset A$, et $\pi^a(A/R) = \mathrm{Spec}(A^{\text{ét}/R})$. Nous obtenons alors les résultats positifs suivants sur la perfection.

Théorème C. (3.4.2.1, 3.4.4.1) *Soit $R \to A$ un morphisme plat de type fini d'anneaux noetheriens en caractéristique p. Supposons l'une des conditions suivantes:*

(1) *R est artinien,*

(2) *R est géométriquement $\mathbb{Q}$-factoriel (e.g. régulier) et $R \to A$ est séparable,*

(2) *R est 1-dimensionnel, intégral, géométriquement unibranche, et $R \to A$ est séparable.*

Alors les applications naturelles donnent lieu à des isomorphismes:

$$A^{\text{ét}/R} \xrightarrow{\sim} \mathcal{O}(\pi(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Overview. Dans la section 3.2, nous commençons par des informations de base sur la coperfection. Dans la section 3.3 qui ne fait aucune hypothèse sur la caractéristique, nous donnons des compléments sur le foncteur π_0. Nous focalisons d'abord sur la propriété de "enveloppe étale", à savoir que π_0 est adjoint à gauche de l'inclusion de la catégorie des espaces étale de présentation fini dans la catégorie des espaces plats de présentation fini; nous prouvons le Théorème 3.3.1.8 qui donne l'existence d'une enveloppe étale *affine*. Nous prouvons ensuite quelques résultats liés à la définition de π_0 comme un espace de modules pour les composantes connexes; ceci inclut deux résultats cruciaux permettant de présenter $\pi_0(X/S)$ comme recollé à partir de pièces plus simples (les pièces les plus simples étant soit π_0 d'un atlas, Prop. 3.3.4.3, ou une complétion d'une fibre spéciale, Prop. 3.3.5.2). Dans la section 3.4, nous étudions l'algèbre commutative de perfection, avec les applications $A^{\text{ét}/R} \to \mathcal{O}(\pi(A/R)) \to A^{p^\infty/R}$ en tant que personnages principaux. Les principaux résultats sont les Théorèmes 3.4.2.1 et 3.4.3.2. Enfin dans la section 3.5, nous dérivons le calcul de la coperfection d'espaces algébriques ou de champs, tout d'abord dans la catégorie des espaces algébriques (Théorème 3.5.1.1), puis dans la 2-catégorie des (pro-)champs Deligne–Mumford quasi-séparés (Théorème 3.5.3.5). Les techniques préparatoires pour la construction du pro-groupoïde fondamental étale fondamental peuvent être trouvés dans les sous-sections 3.5.2 et 3.5.4.

Les espaces de modules de p-torseurs stables

Les courbes modulaires, qui classifient les courbes elliptiques équipées de structures de niveau divers, sont des objets géométriques classiques qui jouent un rôle important en arithmétique et en géométrie algébrique. Elles ont fait l'objet d'études approfondies au cours des dernières décennies et sont bien comprises à présent. En particulier, l'étude de la structure entière de courbes modulaires sur l'anneau $\mathbb{Z}$ et de leurs réductions modulo des nombres premiers mauvais, est réalisée grâce aux travaux d'Igusa, Deligne–Rapoport [DR73], Katz–Mazur [KM85] et finalement Conrad [Con07]. La réduction modulo un nombre premier mauvais d'une courbe modulaire s'avère être l'union de ses composants irréductibles avec des croisements à des points supersinguliers.

Il y a deux généralisations naturelles à ce problème. L'une s'oriente vers les dimensions supérieures, à savoir l'étude du modèle entier de l'espace de modules des variétés abéliennes à structure de niveau et de sa mauvaise réduction. Une autre consiste à considérer les courbes de genre supérieur. Nous nous intéressons à la deuxième généralisation. Par exemple, la courbe modulaire $\mathcal{X}_1(p)$ classifie les courbes elliptiques généralisées équipées d'une $\Gamma_1(p)$-structure, qui sont des paires $(E/S, \varphi)$ où E/S est une courbe elliptique et φ représente un sous-schéma en groupes cyclique G d'ordre p de E/S avec un générateur au sens de Katz–Mazur [KM85]. Alternativement, une telle paire est aussi un p-revêtement galoisien $E \to E/G$ entre des courbes elliptiques avec un générateur de son groupe de Galois G. La question vise donc à étudier le modèle entier de l'espace de modules de revêtements galoisiens entre des courbes propres, et sa mauvaise réduction.

Dans l'article [AR12] d'Abramovich et Romagny, utilisant de l'idée de courbes tordues apparue dans les œuvres pionnières d'Abramovich–Vistoli [AV02] sur la compactification de champs de modules d'applications stables de Kontsevich, et d'Abramovich–Olsson–Vistoli [AOV11] de sa généralisation aux caractéristiques positives, les auteurs construisent un champ de modules propre $\mathbf{ST}_{p,g,h,n}$ sur $\mathbb{Z}$ qui contient un sous-champ ouvert classifiant les p-revêtements galoisiens n-marqués $Y \to X$ entre courbes semi-stables du genres g et h respectivement. Rappelons brièvement les principaux résultats de [AR12].

L'idée principale de *op.cit.* est d'étendre la notion classique de *p-revêtements galoisiens*, en une nouvelle notion *p-torseurs stables*, après quoi on obtient un champ de modules propre qui compactifie le champ de modules classique de p-revêtements galoisiens admissibles modérés en caractéristique 0. Abramovich–Vistoli [AV02] observe que, dans le champ de modules de Kontsevich, quand on remplace la variété cible par un champ de Deligne–Mumford, il n'est pas propre en général. Un ingrédient clé est qu'il faut envisager des applications stables à partir de courbes semi-stables *tordues* plutôt que de courbes semi-stables. Plus tard, Abramovich–Olsson–Vistoli généralise ce travail aux caractéristiques positives [AOV11], en utilisant la notion de *champ d'Artin modéré* [AOV08].

Dans le cas classique, un *p-revêtement galoisien admissible* en caractéristique 0 est un morphisme $Y \to X$ entre deux courbes semi-stables, de sorte qu'il existe une action génériquement libre par $\mathbb{Z}/p\mathbb{Z}$ sur Y et on identifie $X \simeq Y/(\mathbb{Z}/p\mathbb{Z})$, les caractères des branches en un point nodal sont inverses l'un de l'autre. Pour les p-torseurs stables en caractéristique arbitraire, le groupe n'est plus constant $\mathbb{Z}/p\mathbb{Z}$, mais un schéma en groupe cyclique rigidifié au sens de Katz–Mazur [KM85]:

0.0.0.3 Definition. Un *schéma en groupes cyclique rigidifié* $G \to S$ d'ordre p est un schéma en groupes localement libre, avec un générateur $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$.

Pour la notion de générateur d'un schéma en groupes fini localement libre, nous vous référons à *op.cit.* Il existe également une notion utile appelée schéma en groupes cocyclique d'ordre p:

0.0.0.4 Definition. Un *schéma en groupes cocyclique rigidifié* $G \to S$ d'ordre p est un schéma en groupes fini localement libre, avec un *cogénérateur* $\kappa : G \to \mu_{p,S}$, dont le dual de Cartier $\kappa^D : (\mathbb{Z}/p\mathbb{Z})_S \to G^D$ est un générateur.

Le schéma en groupes cocyclique sera un objet central dans le chapitre 4.

0.0.0.5 Definition. Soit S la base. Un *p-torseur stable n-marqué* de genres (h, g) sur S est un triple $\{(\mathscr{X}, \{\Sigma_i\}_{i=1}^n), (\mathscr{G}, \gamma), (Y, \{P_i\}_{i=1}^n)\}$, où

(1) $(\mathscr{X}, \{\Sigma_i\}_{i=1}^n)/S$ est une courbe tordue n-marquée de genre h;

(2) $(Y, \{P_i\}_{i=1}^n)$ est une courbe stable de genre g avec les diviseurs étales marqués P_i/S;

(3) $\mathscr{G} \to \mathscr{X}$ est un schéma en groupes cyclique rigidifié d'ordre p, avec un générateur $\gamma : (\mathbb{Z}/p\mathbb{Z})_{\mathscr{X}} \to \mathscr{G}$;

(4) $Y \to \mathscr{X}$ est un $\mathscr{G}$-torseur et $P_i = \Sigma_i \times_{\mathscr{X}} Y$ pour chaque i.

Pour la définition des courbes tordues, les références standard sont [O07] (pour les courbes tordues de Deligne–Mumford) et l'appendice A dans [AOV11] (pour les courbes tordues générales).

Le théorème principal de [AR12] est le suivant:

0.0.0.6 Theorem.[Theorem 1.6 in *op.cit.*] *L'espace de modules $\mathbf{ST}_{p,g,h,n}/\mathbb{Z}$ est un champ de Deligne–Mumford propre de type fini sur $\mathbb{Z}$.*

Après avoir obtenu un champ de modules propre contenant un sous-champ ouvert classifiant les p-revêtements galoisiens, la question prochaine est de trouver la compactification de ce sous-champ ouvert. En d'autres termes, il s'agit d'identifier la clôture schématique de $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$ dans $\mathbf{ST}_{p,g,h,n}$, qui est également un modèle propre plat de $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$. Dans la section 4.2, nous remarquons que le champ de Deligne–Mumford $\mathbf{ST}_{p,g,h,n}$ n'est pas connexe.

Dans l'appendice A de *op.cit.*, les auteurs introduisent la notion de cogénérateur et prouvent une suite exacte de type de Kummer et la compatibilité avec la suite classique de Kummer:

0.0.0.7 Theorem.[Theorem A.2 in *op.cit.*] *Soit G/S un schéma en groupes cocyclique fini localement libre d'ordre p, avec un cogénérateur $\kappa : G \to \mu_{p,S}$. Alors κ peut être prolongé canoniquement dans un diagramme commutatif avec des lignes exactes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & \mathcal{G}^{\lambda} & \xrightarrow{\varphi^{\kappa}} & \mathcal{G}^{\lambda^p} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{p,S} & \longrightarrow & \mathbb{G}_{m,S} & \xrightarrow{(-)^p} & \mathbb{G}_{m,S} & \longrightarrow & 0
\end{array}$$

où φ^{κ} est une isogénie entre les S-schémas en groupes lisses affines 1-dimensionnels $\mathcal{G}^{\lambda}$ avec $\lambda \in S$.

Ce résultat est très utile pour étudier les torseurs sous les schémas en groupes cocycliques d'ordre p. Expliquons également l'importance des schémas en groupes cocycliques dans le problème de la recherche d'un modèle plat de $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$. En caractéristique différente de p, un schéma en groupe cyclique G/S trouvé dans $\mathbf{ST}_{p,g,h,n}(S)$ n'admet pas toujours un cogénérateur. Par exemple, prenons $S = \mathrm{Spec}(\mathbb{Q})$ et $G = (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}}$. Alors le seul homomorphisme $G \to \mu_{p,\mathbb{Q}}$ est l'homomorphisme trivial, ainsi que l'homomorphisme dual. En les caractéristiques autres que p, l'homomorphisme trivial $(\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}} \to \mu_{p,\mathbb{Q}}$ n'est pas un cogénérateur. Soit $\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n}/\mathbb{Z}$ le champ de modules classifiant les p-torseurs stables dont le schéma en groupe cyclique est également cocyclique. Ceci est un sous-champ propre de $\mathbf{ST}_{p,g,h,n}$, mais après tensorisation avec $\mathbb{Q}(\zeta_p)$, où ζ_p est une racine p-ème de l'unité, alors nous obtenons la même chose:

$$\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p) = \mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$$

par conséquent, la fibre spéciale du modèle plat de $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$ coïncide avec la fibre spéciale du modèle plat de $\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$.

Mais c'est très difficile de trouver le modèle plat de $\mathbf{ST}_{p,g,h,n}$ et d'étudier sa réduction mod p. Dans le chapitre 4, nous souhaitons éclairer ce problème, par l'étude sur des espaces de modules des torseurs découlant de $\mathbf{ST}_{p,g,h,n}$. En particulier, les champs de modules des torseurs sous les groupes G et $\mathcal{G}^\lambda$ sont des objets centraux de ce chapitre.

Overview. Le chapitre 4 est consacré à l'étude des torseurs sur les courbes sous les schémas en groupes lisses et affines 1-dimensionnels $\mathcal{G}^\lambda$ et les schémas en groupes cocycliques d'ordre p. Dans la section 4.2, nous construisons explicitement le relèvement local tordu des revêtements d'Artin–Schreier et le Frobenius de la droite projective, et faisons une remarque sur l'existence de composantes supplémentaires du champ de modules des p-torseurs stables en caractéristique p. Dans la section 4.3, nous étudions les espaces de modules des torseurs sur les courbes propres sous les schémas en groupes $\mathcal{G}^\lambda$ et les schémas en groupes cocycliques G. En particulier, nous prouvons quelques résultats de représentabilité et de platitude de ces espaces de modules dans certains cas:

0.0.0.8 Theorem.[Proposition 4.3.4.2] *Supposons que λ est régulière dans les fibres. Alors le champ de modules $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ est représentable et lisse de dimension $g + d - 1$ sur S, où $d = \deg D$ et g est le genre de X.*

0.0.0.9 Theorem.[Theorem 4.3.4.3] *Supposons que λ est régulière dans les fibres. Alors $\mathbf{TORS}_X(G)$ est représentable par un schéma en groupes fini plat, d'ordre p^{2g+d-1}.*

0.0.0.10 Theorem.[Theorem 4.3.5.3] *Soit S un anneau de valuation discrète, X/S une courbe propre génériquement irréductible, $\mathscr{L}$ un faisceau inversible sur X, et $\lambda \in H^0(X, \mathscr{L}^{-1})$ satisfaisant la condition* (*) *(voir la section 4.3.5). Alors le champ de modules $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ est représentable par un S-schéma en groupes lisse de dimension $g + d - 1$.*

Dans la section 4.4, nous discutons la relation entre les espaces de modules des $\mathcal{G}^\lambda$-torseurs sur les courbes propres et les jacobiennes généralisées de courbes ouvertes. Dans la section 4.5, nous prouvons une classification catégorique de $\mathcal{G}^\lambda$-torseurs qui généralise légèrement un résultat de Andreatta–Gasbarri [AG07].

Sur les espaces monadiques

Dans le dernier chapitre, nous nous intéressons à un nouveau type de géométrie analytique non-archimédienne, en généralisant les valuations à valeurs dans des groupes aux valuations à valeurs dans des monoïdes commutatifs totalement ordonnés. Nous étudions quelques exemples de schémas, d'espaces analytiques rigides et d'espaces adiques.

Contents

1
Introduction

The thesis work is devoted to the study of algebraic geometry in mixed and positive characteristics, especially about algebraic geometry over an imperfect base in positive characteristic. It consists of four parts with different but also related origins of motivation.

1.1 Maximal models of torsors over a local field

In the book of Serre [Se79], the classical ramification theory of local fields is well-documented. The assumption of residue field being perfect is crucial in this classical theory. In the need of pursuing ramification theory for higher dimensional schemes, Abbes and Saito (cf. [AS02], [AS03], [Sa12]) have developed the ramification theory for local fields with possibly imperfect residue field, using techniques of rigid analytic geometry. Later on, Saito [Sa19] also gave a schematic approach. This more general theory is adapted in the later works (cf. [KS08], [KS13]) of Kato and Saito on ramification theory of higher dimensional varieties over perfect fields and local fields.

Ramification of finite ***separable*** extensions of local fields with possibly imperfect residue field is well-understood by now, thanks to the theory of Abbes–Saito. An interesting question is, what about ***arbitrary*** finite extensions of local fields? For example, if the local field has positive characteristic, there are many inseparable extensions which cannot be recognized in the absolute Galois group. Do they have a reasonable theory of ramification? In order to think about this question, we have to make certain things clear. First, what does it mean by "Galois" for inseparable extensions? In the classical setting, the Galois group $G_{L/K}$ of a Galois extension L/K of local fields acting on L, and it makes the scheme $\mathrm{Spec}(L)$ a $G_{L/K}$-torsor over $\mathrm{Spec}(K)$. For a purely inseparable extension K'/K, there is no nontrivial abstract group acting on K' fixing K. Instead, there could be some infinitesimal (or called local) group scheme acting on K', and making $\mathrm{Spec}(K')$ a torsor over $\mathrm{Spec}(K)$. A significant difference is that, there might be more than one group scheme acting on K', and in particular, it is reasonable to expect that they could have different behaviors of ramification.

Many attempts have been studied, especially, the notion of *tame action* by group schemes has been greatly developed by various people, cf. [CEPT96]. In [Za16], Zalamansky proved an analogous Riemann–Hurwitz formula for inseparable covers under infinitesimal diagonalizable group schemes. For more general case, there still needs new ideas.

Our basic setting is as follows. Let K be a local field, $\mathcal{O}_K$ the ring of integers, and k the residue field. Let G be a group scheme over $\mathcal{O}_K$, and then we start with a G_K-torsor P_K over $\mathrm{Spec}(K)$. The first problem we are facing is to find an integral model P of P_K with extended G-action. In the classical case, this is standard, namely, we simply take the integral closure of $\mathcal{O}_K$ in the extension field, with the natural action by Galois group. In our setting, a good candidate is Lewin-Ménégaux's notion of "minimal models" of torsors over K. The construction originated in Raynaud's paper [Ra67], page 82–83, where he did the construction for abelian schemes, and intended to claim that it was *minimal* in the usual sense:

> *"c) Soient A un anneau de valuation discrète, K son corps de fractions, G une K-variété abélienne qui possède une bonne réduction sur A, de sorte que G se prolonge en un schéma abélien $\mathscr{G}$ sur $S = \mathrm{Spec}\ A$ et soit X un K-espace principal homogène sous G. Montrons que X se prolonge en un S-schéma* projectif *et* régulier *$\mathscr{X}$ (qui sera d'ailleurs un modèle minimal de X dans la terminologie de [2]). ..."*

where [2] is [Ne67] in the same volume. Later, the idea was further developed by Lewin-Ménégaux in her paper [LM83]. But unfortunately, her paper is sketchy and left "minimal" unexplained. It turns out surprisingly, a "minimal model" is not minimal but in fact *maximal*, among all the integral models.

1.1.0.1 Proposition.[Proposition 2.2.0.2] *Let $X_K \to \mathrm{Spec}(K)$ be a G_K-torsor, we assume that the maximal model $X \to S$ exists. Then X is maximal among all the integral models of $X_K \to \mathrm{Spec}(K)$, namely, if $\mathcal{X}$ is another integral model, then there is a unique model morphism $X \to \mathcal{X}$.*

As in the above proposition, we reasonably change the terminology to *maximal model.* Maximal model has some pleasant features. For example, it is unique up to isomorphisms if exists,[1] and it recovers the classical situation, namely, if L/K is a Galois extension with the Galois group $G_{L/K}$, then the maximal model of the $G_{L/K}$-torsor $\mathrm{Spec}(L)$ is exactly $\mathrm{Spec}(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of L. Maximal models are functorial with respect to group schemes. More importantly, maximal models are compatible with inductions, and in particular they are compatible with quotients. We prove the following results

1.1.0.2 Proposition.[Proposition 2.2.0.4] *Let $\varphi : G \to H$ be a homomorphism of flat S-group schemes, X_K is a G_K-torsor. Assume that the maximal model X of X_K exists. If the induced H_K-torsor Y_K has a maximal model Y, then $\mathrm{Ind}_G^H X = X \times^G H$ is representable by Y.*

This property is very necessary and important for trying to build a general ramification theory.

Overview. In Lewin-Ménégaux's paper [LM83], she has proved uniqueness and functoriality of maximal model, the existence for the case of abelian schemes and finite flat commutative group schemes, and property of the different. However, her paper is highly sketchy, so we give details and represent her results in Section 2.3, 2.5, and the part of functoriality in Section 2.2. Besides, we prove the maximality, and the compatibility with inductions in Section 2.2. In Section 2.4, we prove a result on the existence of maximal model of torsors under a semi-direct product of

[1] The uniqueness is not merely a consequence of maximality, it is true for maximal models over arbitrary flat S-schemes, provided existence. See Proposition 2.1.0.4. On the other hand, we only have maximality over S.

group schemes. In Section 2.6, we classify maximal models of torsors under some finite flat group schemes of order p.

1.2 Coperfection and the étale fundamental pro-groupoid in positive characteristic

In Chapter 3 (joint work with Giulio Orecchia and Matthieu Romagny), we study flat, finitely presented algebraic spaces and stacks in characteristic $p > 0$, and their maps to perfect ones (those whose relative Frobenius morphism is isomorphic). Our interest is in perfection of algebras and coperfection of algebraic spaces and stacks. Our results are better for morphisms with geometrically reduced fibres, also called *separable*. Let us now describe them.

For each algebraic space S and flat, finitely presented algebraic stack $\mathscr{X} \to S$, we construct its *étale fundamental pro-groupoid* $\Pi_1(\mathscr{X}/S)$. This is a 2-pro-object of the 2-category of étale algebraic stacks. If moreover $\mathscr{X}/S$ is separable, then its coarse moduli space is the space of connected components $\pi_0(X/S)$ (see [Rom11]), seen as a constant 2-pro-object. When S is the spectrum of a field κ and $\mathscr{X}$ is geometrically connected, the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ is representable in the 2-category of stacks by the étale fundamental gerbe $\Pi^{\text{ét}}_{\mathscr{X}/\kappa}$ of Borne and Vistoli [BV15, § 8]. If S has characteristic p, we let

$$\mathrm{F}_i : \mathscr{X}^{p^i/S} \longrightarrow \mathscr{X}^{p^{i+1}/S}$$

denote the relative Frobenius of $\mathscr{X}^{p^i/S}$, the i-th Frobenius twist of $\mathscr{X}/S$.

Theorem A. (3.5.1.1, 3.5.3.5) *Let S be a noetherian algebraic space of characteristic p.*

(i) *Let $X \to S$ be a flat, finitely presented, separable morphism of algebraic spaces. The inductive system of relative Frobenii*

$$X \xrightarrow{\mathrm{F}_0} X^{p/S} \xrightarrow{\mathrm{F}_1} X^{p^2/S} \longrightarrow \dots$$

admits a colimit in the category of algebraic spaces over S. This colimit is the algebraic space of connected components $\pi_0(X/S)$; it is a coperfection of $X \to S$.

(ii) *Let $\mathscr{X} \to S$ be a flat, finitely presented, separable algebraic stack. The inductive system of relative Frobenii*

$$\mathscr{X} \xrightarrow{F_{\mathscr{X}/S}} \mathscr{X}^{p/S} \xrightarrow{F_{\mathscr{X}^{p}/S}} \mathscr{X}^{p^2/S} \longrightarrow \dots$$

admits a colimit in the 2-category of quasi-separated Deligne–Mumford stacks over S. This colimit is the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ and it is a coperfection of $\mathscr{X}/S$.

Note that point (ii) includes point (i) as a special case, because $\Pi_1(\mathscr{X}/S)$ has coarse moduli space $\pi_0(\mathscr{X}/S)$. We include (i) for emphasis and also because the proof actually proceeds by deducing (ii) from (i).

Within the category of algebras, the situation is somehow more subtle. Given a characteristic p ring R and an algebra $R \to A$, let

$$\mathrm{F}_i : A^{p^{i+1}/R} \to A^{p^i/R}$$

denote the relative Frobenius of $A^{p^i/R}$, the i-th Frobenius twist of A. Define the ***preperfection***:

$$A^{p^\infty/R} = \lim\big(\cdots A^{p^2/R} \xrightarrow{\mathrm{F}_1} A^{p/R} \xrightarrow{\mathrm{F}_0} A\big).$$

The name is explained by a surprising fact: the algebra $A^{p^\infty/R}$ is not perfect in general, even if $R \to A$ is flat, finitely presented and separable. We give an example of this with R equal to the local ring of a nodal curve singularity (see 3.4.5.2). In our example the double preperfection is perfect but we do not know if iterated preperfections should converge to a perfect algebra in general. In the affine case $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, we write $\pi_0(A/R)$ instead of $\pi_0(X/S)$. What Theorem A implies in this case is that there is an isomorphism of R-algebras:

$$\mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Here $\mathcal{O}(-)$ is the functor of global functions. Given the bad properties of the rings under consideration, this could not really be anticipated: indeed, in general $\mathcal{O}(\pi_0(A/R))$ is not étale and $A^{p^\infty/R}$ is not perfect. Similarly as above, the structure of the proof is actually to first establish this isomorphism of algebras (see 3.4.3.2) and then deduce the geometric statement for spaces and stacks (Theorem A above).

This begs for a further study of perfection of algebras. Our general expectation is that for algebras of finite type, there should exist a largest étale subalgebra and this should be (at least close to) the perfection of $R \to A$. In striving to materialize this picture, we study étale hulls in more detail. We take up recent work of Ferrand [Fe19] and prove the following result which is not special to characteristic p.

Theorem B. (3.3.1.8) *Let $f : X \to S$ be a faithfully flat, finitely presented morphism of algebraic spaces. Assume that S is noetherian, geometrically unibranch, without embedded points. Then the category $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$ is a lattice, that is, any two objects have a supremum and an infimum. Moreover $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$ has a largest element.*

The largest element $\pi^a(X/S)$ is the relative spectrum of a sheaf of $\mathcal{O}_S$-algebras which is the largest étale subalgebra of $f_*\mathcal{O}_X$. When S is artinian or $X \to S$ is separable, so that $\pi_0(X/S)$ is an étale algebraic space, we have morphisms:

$$X \longrightarrow \pi_0(X/S) \longrightarrow \pi^a(X/S).$$

When $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, the largest étale subalgebra is written $A^{\text{ét}/R} \subset A$, that is $\pi^a(A/R) = \mathrm{Spec}(A^{\text{ét}/R})$. We then obtain the following positive results on perfection.

Theorem C. (3.4.2.1, 3.4.4.1) *Let $R \to A$ be a flat, finite type morphism of noetherian rings of characteristic p. Assume that one of the following holds:*

(1) *R is artinian,*

(2) *R is geometrically $\mathbb{Q}$-factorial (e.g. regular) and $R \to A$ is separable,*

(2) *R is one-dimensional, integral, geometrically unibranch, and $R \to A$ is separable.*

Then the natural maps give rise to isomorphisms:

$$A^{\text{ét}/R} \xrightarrow{\sim} \mathcal{O}(\pi(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Overview. In Section 3.2 we start with basic facts on coperfection. In Section 3.3 which makes no characteristic assumption, we give complements on the functor π_0. We first focus the “étale hull" property, namely that π_0 is left adjoint to the inclusion of the category of étale finitely presented spaces into the category of flat, finitely presented spaces; we prove Theorem 3.3.1.8 which gives existence of an *affine* étale hull. We then prove some results related to the definition of π_0 as a moduli space for connected components; this includes two crucial pushout results that allow to view $\pi_0(X/S)$ as glued from simpler pieces, (the simpler pieces being either π_0 of an atlas, Prop. 3.3.4.3, or a completion from a special fibre, Prop. 3.3.5.2). In Section 3.4 we study the commutative algebra of perfection, with the maps $A^{\text{ét}/R} \to \mathcal{O}(\pi(A/R)) \to A^{p^\infty/R}$ as main characters. The main results are Theorems 3.4.2.1and 3.4.3.2. Finally in Section 3.5 we derive the computation of the coperfection of algebraic spaces or stacks, first in the category of algebraic spaces (Theorem 3.5.1.1) and then in the 2-category of (pro-)quasi-separated Deligne–Mumford stacks (Theorem 3.5.3.5). The construction of the étale fundamental pro-groupoid necessitates technical preparations to be found in Subsections 3.5.2 and 3.5.4.

1.3 Moduli spaces of stable p-torsors

Modular curves, which classify elliptic curves equipped with various level structures, are classical geometric objects which play significant roles in arithmetics and algebraic geometry. They are intensively studied over the last few decades, and are well-understood by now. Especially, the study of integral structure of modular curves over the integer $\mathbb{Z}$, and their reductions modulo bad primes, is accomplished thanks to the works of Igusa, Deligne–Rapoport [DR73], Katz–Mazur [KM85], and eventually Conrad [Con07]. The reduction modulo a bad prime of a modular curve turns out to be the union of its irreducible components with crossings at supersingular points.

There are two natural generalizations of this problem. One is towards higher dimensions, namely, to study integral model of moduli space of abelian varieties with level structures and its bad reduction. Another is to consider curves of higher genus. We are interested in the second generalization. For example, the modular curve $\mathcal{X}_1(p)$ classifies generalized elliptic curves equipped with a $\Gamma_1(p)$-structure, which are pairs $(E/S, \varphi)$ where E/S is an elliptic curve and φ stands for a cyclic subgroup scheme G of E/S of order p together with a generator in the sense of Katz–Mazur [?]. Alternatively, such a pair is also a Galois p-cover $E \to E/G$ between elliptic curves together with a generator of its Galois group G. Thus the question aims to study integral model of moduli space of Galois covers between proper curves, and its bad reduction.

In the paper [AR12] by Abramovich and Romagny, using the idea of twisted curves appeared in the pioneer works of Abramovich–Vistoli [AV02] on the compactification of Kontsevich's moduli stacks of stable maps, and Abramovich–Olsson–Vistoli [AOV11] its generalization to positive characteristics, they construct a proper moduli stack $\mathbf{ST}_{p,g,h,n}$ over $\mathbb{Z}$ which contains an open substack classifying n-marked Galois p-covers $Y \to X$ between semistable curves of genus g and h respectively. Let us briefly recall the main results of [AR12].

The main idea of *op.cit.* is to extend the classical notion of *Galois p-covers*, to a new notion named *stable p-torsors*, after which one obtains a proper moduli stack that compactifies the classical moduli stack of tame admissible Galois p-covers in characteristic 0. It is observed by Abramovich–Vistoli [AV02] that, in Kontsevich's moduli stack of stable maps, when one replaces the target variety by a Deligne–Mumford stack, then it will not be proper in general. A key ingredient is that one should consider stable maps from *twisted* semistable curves rather than merely semistable curves. Later, Abramovich–Olsson–Vistoli generalize this work to positive characteristics [AOV11], by using the notion of *tame Artin stacks* [AOV08].

In the classical case, an *admissible Galois p-cover* in characteristic 0 is a morphism $Y \to X$ between two semistable curves, such that there is a generically free action of $\mathbb{Z}/p\mathbb{Z}$ on Y and one identifies $X \simeq Y/(\mathbb{Z}/p\mathbb{Z})$, and each pair of characters at two components of a node are mutually inverse. As for stable p-torsors in arbitrary characteristic, the group is not longer constant $\mathbb{Z}/p\mathbb{Z}$, but a rigidified cyclic group scheme in the sense of Katz–Mazur [KM85]:

1.3.0.1 Definition. A *rigidified cyclic group scheme* $G \to S$ of order p is a finite locally free group scheme, together with a generator $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$.

For the notion of generators of a finite locally free group scheme, we refer to *op.cit.* There is also a useful notion named cocyclic group scheme of order p:

1.3.0.2 Definition. A *rigidified cocyclic group scheme* $G \to S$ of order p is a finite locally free group scheme, together with a *cogenerator* $\kappa : G \to \mu_{p,S}$, which is to require that the Cartier dual $\kappa^D : (\mathbb{Z}/p\mathbb{Z})_S \to G^D$ is a generator.

Cocyclic group scheme will be an important object in Chapter 4.

1.3.0.3 Definition. Fix a base scheme S. A *stable n-marked p-torsor* of genus (h, g) over S is a triple $\{(\mathscr{X}, \{\Sigma_i\}_{i=1}^n), (\mathscr{G}, \gamma), (Y, \{P_i\}_{i=1}^n)\}$ where

(1) $(\mathscr{X}, \{\Sigma_i\}_{i=1}^n)/S$ is a n-marked twisted curve of genus h;

(2) $(Y, \{P_i\}_{i=1}^n)$ is a stable curve of genus g with étale marking divisors P_i/S;

(3) $\mathscr{G} \to \mathscr{X}$ is a rigidified cyclic group scheme of order p, with a generator $\gamma : (\mathbb{Z}/p\mathbb{Z})_{\mathscr{X}} \to \mathscr{G}$;

(4) $Y \to \mathscr{X}$ is a $\mathscr{G}$-torsor and $P_i = \Sigma_i \times_{\mathscr{X}} Y$ for all i.

For the definition of twisted curves, standard references are [O07] (for Deligne–Mumford twisted curves) and the Appendix A of [AOV11] (for general twisted curves).

The main theorem of [AR12] is the following:

1.3.0.4 Theorem.[Theorem 1.6 in *op.cit.*] *The moduli space $\mathbf{ST}_{p,g,h,n}/\mathbb{Z}$ is a proper Deligne–Mumford stack of finite type over $\mathbb{Z}$.*

After having a proper moduli stack containing an open substack classifying Galois p-covers, the next question is to find the compactification of this open substack. In other words, it aims to identify the schematic closure of $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$ in $\mathbf{ST}_{p,g,h,n}$, which is also a proper flat model of $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$. In Section 4.2, we remark that the DM stack $\mathbf{ST}_{p,g,h,n}$ is not connected.

In Appendix A of *op.cit.*, they introduce the notion of cogenerator, and proved a Kummer-type short exact sequence and the compatibility with the classical Kummer sequence:

1.3.0.5 Theorem.[Theorem A.2 in *op.cit.*] *Let G/S be a finite locally free cocyclic group scheme of order p, with a cogenerator $\kappa : G \to \mu_{p,S}$. Then κ can be canonically embedded into a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & \mathcal{G}^{\lambda} & \xrightarrow{\varphi^{\kappa}} & \mathcal{G}^{\lambda^p} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\kappa} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{p,S} & \longrightarrow & \mathbb{G}_{m,S} & \xrightarrow{(-)^p} & \mathbb{G}_{m,S} & \longrightarrow & 0
\end{array}$$

where φ^κ is an isogeny between affine smooth one-dimensional S-group schemes $\mathcal{G}^\lambda$ for $\lambda \in S$.

This result is a very useful tool for studying torsors under cocyclic group schemes of order p. Let us also explain the importance of cocyclic group schemes in the problem of finding flat model of $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}$. In characteristic not equal to p, a cyclic group scheme G/S appearing in $\mathbf{ST}_{p,g,h,n}(S)$ does not always admit a cogenerator. For example, let $S = \mathrm{Spec}(\mathbb{Q})$, and $G = (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}}$. Then the only homomorphism $G \to \mu_{p,\mathbb{Q}}$ is the trivial homomorphism, so is the dual homomorphism. While in characteristics other than p, the trivial homomorphism $(\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}} \to \mu_{p,\mathbb{Q}}$ is not a cogenerator. Let $\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n}/\mathbb{Z}$ be the moduli stack classifying stable p-torsors whose cyclic group scheme is also cocyclic. This is a proper substack of $\mathbf{ST}_{p,g,h,n}$, but after tensoring $\mathbb{Q}(\zeta_p)$, where ζ_p is a p-th root of unity, then we get the same:

$$\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p) = \mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$$

hence the special fiber of the flat model of $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$ coincides with the special fiber of the flat model of $\mathbf{ST}^{\mathrm{bal}}_{p,g,h,n} \otimes \mathbb{Q}(\zeta_p)$.

However, the problems of finding the flat model of $\mathbf{ST}_{p,g,h,n}$ and the study of its reduction are very difficult. In Chapter 4, we wish to shed some light, by investigating relevant moduli spaces of torsors arising from $\mathbf{ST}_{p,g,h,n}$. Especially, the moduli stacks of torsors under groups $G, \mathcal{G}^\lambda$ are central objects of this chapter. Let us give an overview.

Overview. Chapter 4 devotes to the study of torsors over curves under affine smooth one-dimensional group schemes $\mathcal{G}^\lambda$ and cocyclic group schemes of order p, where those group schemes naturally arise from moduli stack of stable p-torsors. In Section 4.2, we construct explicitly the local twisted lifting of Artin–Schreier covers and Frobenius of the projective line, and make a remark on the existence of extra components of moduli stack of stable p-torsors in characteristic p. In Section 4.3, we study moduli spaces of torsors over proper curves under group schemes $\mathcal{G}^\lambda$ and cocyclic group schemes G. In particular, we prove some representability and flatness results of these moduli spaces in certain cases:

1.3.0.6 Theorem.[Proposition 4.3.4.2] *Assume that λ is fiberwise regular. Then the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable and smooth of dimension $g + d - 1$ over S, where $d = \deg D$ and g is the genus of X.*

1.3.0.7 Theorem.[Theorem 4.3.4.3] *Assume that λ is fiberwise regular. Then $\mathbf{TORS}_X(G)$ is representable by a finite flat group scheme, of degree p^{2g+d-1}.*

1.3.0.8 Theorem.[Theorem 4.3.5.3] *Let S be a discrete valuation ring, X/S a proper, geometrically connected and generically irreducible curve, $\mathscr{L}$ an invertible sheaf over X, and $\lambda \in H^0(X, \mathscr{L}^{-1})$ satisfying the condition (*). Then the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable by a smooth S-group scheme of dimension $g + d - 1$.*

In Section 4.4, we discuss the relation between moduli spaces of $\mathcal{G}^\lambda$-torsors over proper curves and the generalized Jacobians of open curves. In Section 4.5, we prove a categorical classification of $\mathcal{G}^\lambda$-torsors which slightly generalizes a result of Andreatta–Gasbarri [AG07].

1.4 Monoid-valued adic spaces

In the final chapter, we are interested in a new kind of nonarchimedean analytic geometry, by generalizing the valuation on groups to the valuation on totally ordered commutative monoids. We study some examples from schemes, rigid analytic spaces, and adic spaces.

2

Maximal models of torsors over a local field

Throughout this chapter, we work with the category of *separated* schemes. When we work over a DVR base $S = \mathrm{Spec}(R)$, we always assume that R is a Japanese ring. Equivalently, it means that for any finite extension K' of the fraction field K of R, the normalization R' of R is a finite R-module.

2.1 Basic definitions (general setting)

Let R be a discrete valuation ring, with residue field k and fraction field K. Let $S = \mathrm{Spec}(R)$. If Y is a S-scheme, then we always denote its generic and special fiber by Y_K and Y_k respectively.

Let T be a S-scheme. Let G be a flat commutative S-group scheme, and $X_K \to T_K$ a G_K-torsor. First let us define the notion of *integral model* of such a torsor.

2.1.0.1 Definition. Let $X_K \to T_K$ be a G_K-torsor. An *integral model* of $X_K \to T_K$ is a faithfully flat separated morphism $\mathcal{X} \to T$ whose generic fiber is $X_K \to T_K$, together with an extended G-action on $\mathcal{X}$. A *model morphism* between integral models of $X_K \to T_K$ is a G-equivariant morphism whose generic fiber is an isomorphism of G_K-torsors.

Among all the integral models, we are interested in finding a convenient one. The following notion of maximal model serves as a candidate.

2.1.0.2 Definition. A *maximal S-model* (or *maximal R-model*) of the G_K-torsor X_K is an integral model $X \to T$ of $X_K \to T_K$ satisfying the following condition: there exists a finite extention K'/K and a finite flat G-equivariant morphism $f : G \times_S T_{S'} \to X$, where S' denotes the normalization of S in K'.

2.1.0.3 Remark.

(1) Notice that the original notion in Lewin-Ménégaux [LM83] is called “minimal model”, the idea goes back to Raynaud [Ra67]. We have found it awkward because later we will see that such a “minimal” model is actually maximal. To avoid further ambiguity, we decided to correct the terminology.

(2) From the last condition, we have the diagram

$$\begin{tikzcd}
G_{T_{S'}} \arrow[dr, dashed, "g"] \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& X_{T_{S'}} \arrow[r] \arrow[d] & X \arrow[d] \\
& T_{S'} \arrow[r] & T
\end{tikzcd}$$

where f factors through g, which is a T'-morphism. In particular, the morphism $g_{K'}$ on the generic fiber is an isomorphism of trivial T'-torsors, since K' trivializes $X_K \to T_K$.

(3) Note that it is meaningless to speak of maximal model of a G_K-torsor $X_K \to T_K$ which cannot be trivialized by any finite extension K'/K. If the torsor can be trivialized by a finite extension K'/K, and it naturally extends to a G-torsor $X \to T$ which is trivialized by the normalization S'/S, then this G-torsor is automatically a maximal model of $X_K \to T_K$. Indeed, according to the previous diagram

$$\begin{tikzcd}
G_{T_{S'}} \arrow[dr, dashed, "\simeq"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& X_{T_{S'}} \arrow[r] \arrow[d] & X \arrow[d] \\
& T_{S'} \arrow[r] & T
\end{tikzcd}$$

the morphism f is isomorphic to the projection $X_{T_{S'}} \to X$, which is flat since S'/S is flat.

2.1.0.4 Proposition. *A maximal model X of X_K, if it exists, is unique up to T-isomorphisms.*

Proof : Let X_1 and X_2 be maximal S-models of X_K. Indeed, we can choose the same finite extension $K_1' = K_2' = K'/K$ which trivializes the torsor X_K. Let $T' = T \times_S S'$ where S' is the normalization of S in K', and denote the morphisms of two models from $G_{T'}$ by

$$f_i : \; G_{T'} := G \times_S T' \;\longrightarrow\; X_i, \qquad i = 1, 2.$$

The finite flat T-morphism f_1 (resp. f_2) realizes X_1 (resp. X_2) as the cokernel of the finite flat groupoids

$$R_1 := G_{T'} \times_{X_1} G_{T'} \;\rightrightarrows\; G_{T'},$$

(resp. R_2). It is clear that $R_1|_{T_K'} \cong R_2|_{T_K'}$. The morphism $f_2 : G_{T'} \to X_2$ is both R_1- and R_2-invariant, since it is invariant on the generic fiber, which implies the invariance by flatness of the groupoid. Therefore it induces a T-morphism $\alpha : X_1 \to X_2$ such that $f_2 \circ \alpha = f_1$. Similarly, there is a T-morphism $\beta : X_2 \to X_1$ with $f_1 \circ \beta = f_2$, and we have

$$\begin{cases} f_1 = f_2 \circ \alpha = f_1 \circ (\beta\alpha) \\ f_2 = f_1 \circ \beta = f_2 \circ (\alpha\beta) \end{cases}$$

which indicates that $\beta\alpha = \mathrm{id}_{X_1}$ and $\alpha\beta = \mathrm{id}_{X_2}$, because f_1, f_2 are epimorphisms. □

2.1.0.5 Example. Let G be a constant finite group, and K'/K a finite Galois extension with Galois group G. Then $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is a G_K-torsor, and it is trivialized by the field extension K'/K. Let $R' \subset K'$ be the normalization of R in K', then $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a maximal R-model. Indeed, the finite flat G-equivariant morphism is given by projection. In particular, the maximal model $\mathrm{Spec}(R')$ remains to be a G-torsor if and only if the extension K'/K of local fields is unramified.

2.1.0.6 Example. Let $K = k(\!(t)\!)$, where k is a field of characteristic $p > 0$. Consider the purely inseparable extension $L = k(\!(t^{1/p})\!)$ as an $\alpha_{p,K}$-torsor over K

$$\mathrm{Spec}(L) := \mathrm{Spec}(k(\!(t^{1/p})\!)) \longrightarrow \mathrm{Spec}(K),$$

where the $\alpha_{p,K}$-action is given by [1]

$$t^{1/p} \longmapsto \frac{t^{1/p}}{1 + at^{1/p}},$$

here a is the coordinate of $\alpha_{p,K}$. The action naturally extends to the integer rings $\mathcal{O}_L$ of L, and the $\mathcal{O}_K$-scheme $\mathrm{Spec}(\mathcal{O}_L)$ is the maximal model of $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$. Indeed, the $\alpha_{p,K}$-torsor is tautologically trivialized by L/K, and we have the diagram

$$\begin{tikzcd}
\alpha_{p,\mathcal{O}_L} \arrow[dr, dashed, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& \mathrm{Spec}(\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_L) \arrow[d] \\
& \mathrm{Spec}(\mathcal{O}_L) \arrow[r] & \mathrm{Spec}(\mathcal{O}_K)
\end{tikzcd}$$

where f is clearly finite flat. Here the maximal model is not a torsor, hence it is "ramified".

2.2 Maximality and functoriality of maximal models of torsors

From now on, we restrict ourselves to maximal models of torsors over a discrete valuation ring $S = \mathrm{Spec}(R)$.

The maximality in the name of maximal model agrees with the usual sense that for any S-model, there is a dominant morphism from the maximal S-model. Let us consider the following example.

2.2.0.1 Example. Let $K = k(\!(t)\!)$ be a local field of equicharacteristic where k has characteristic $p > 0$, its ring of integers is $R = k[\![t]\!]$. Consider the constant group $G = \mathbb{Z}/p\mathbb{Z}$ which we view

[1]It is the Frobenius of $\mathbb{P}^1_k$ at ∞.

as a constant group scheme over R, and $P_K \to \mathrm{Spec}(K)$ the trivial G_K-torsor. Obviously its maximal model is the trivial G-torsor $P = \mathbb{Z}/p\mathbb{Z} \to \mathrm{Spec}(R)$. Yet we have another integral model $\mathcal{P} = \mathrm{Spec}(R[a]/(a^p - t^{p-1}a))$, where the action of $G = \mathrm{Spec}(R[e]/(e^p - e))$ is given by

$$a \longmapsto a + et.$$

It is straightforward that this is an integral model of the original G_K-torsor, but it is not a maximal S-model. There is a natural morphism from the maximal model $\mathbb{Z}/p\mathbb{Z}$ to $\mathcal{P}$

$$\begin{array}{ccc} R[a]/(a^p - t^{p-1}a) & \longrightarrow & R[e]/(e^p - e) \\ a & \longmapsto & et \end{array}$$

which is a *full set of sections* in the sense of Katz–Mazur [KM85].

2.2.0.2 Proposition. *Let $X_K \to \mathrm{Spec}(K)$ be a G_K-torsor, we assume that the maximal model $X \to S$ exists. Then X is maximal among all the integral models of $X_K \to \mathrm{Spec}(K)$, namely, if $\mathcal{X}$ is another integral model, then there is a unique model morphism $X \to \mathcal{X}$.*

Proof : First, we claim that there exists a section $y \in \mathcal{X}(S')$ for some finite flat extension S'/S, where S' is the normalization of some finite extension of fraction field K. Indeed, since $\mathcal{X}/S$ is surjective, we choose a closed point x_0 of the special fiber $\mathcal{X}_k$. Because its local ring $\mathscr{O}_{\mathcal{X},x_0}$ is flat over S, there is a generization x_1 of x_0. The schematic closure $\overline{\{x_1\}}$ is irreducible and faithfully flat over S, we choose a closed point x_0' of its special fiber. By Proposition 10.1.36 in [Liu02], there exists a closed point x_1' of $\overline{\{x_1\}} \otimes K$, such that x_0' is a specialization of x_1'. The residue field K' of x_1' is a finite extension of K, let S' be the normalization of S in K', then the schematic closure of x_1' gives a section in $\mathcal{X}(S')$.

Let $S' \to S$ be a finite flat base change via some extension of fraction field K'/K, which trivializes the G_K-torsor X_K and gives rise to a finite flat morphism $G_{S'} \to X$. This is guaranteed by the definition. Moreover by our previous claim, we may assume that there exists a section $y \in \mathcal{X}(S')$. Consider the finite flat groupoid

$$\Gamma_X := G_{S'} \times_X G_{S'} \rightrightarrows G_{S'} \to X,$$

and note that the morphism $G_{S'} \to X$ is effectively epimorphic. The integral point y also gives rise to a groupoid

$$\Gamma_{\mathcal{X}} := G_{S'} \times_{\mathcal{X}} G_{S'} \rightrightarrows G_{S'} \to \mathcal{X}$$

where the map $G_{S'} \to \mathcal{X}$ is contructed via

$$G_{S'} \simeq G_{S'} \times_{S'} S' \xrightarrow{\mathrm{id} \times y} G_{S'} \times_{S'} \mathcal{X}_{S'} \longrightarrow \mathcal{X}_{S'} \longrightarrow \mathcal{X}$$

The generic fibers of above two groupoids are isomorphic. Notice that Γ_X and $\Gamma_{\mathcal{X}}$ are contained in the same background scheme $G_{S'} \times_S G_{S'}$. Since Γ_X is flat over S, it is the flat closure of the generic fiber, hence we have a closed immersion

$$\Gamma(X) \simeq \text{schematic closure of } \Gamma(\mathcal{X}) \otimes K \hookrightarrow \Gamma(\mathcal{X}).$$

Now the two compositions

$$\Gamma_X \rightrightarrows G_{S'} \to \mathcal{X}$$

coincide with the restriction of the compositions $\Gamma_{\mathcal{X}} \rightrightarrows G_{S'} \to \mathcal{X}$ to Γ_X, hence they agree on Γ_X. By the fact that $G_{S'} \to X$ is an effective epimorphism, it induces a unique morphism from X to $\mathcal{X}$

$$\begin{tikzcd} \Gamma_X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & G_{S'} \arrow[r] \arrow[d, equal] & X \arrow[d, dashed] \\ \Gamma_{\mathcal{X}} \arrow[r, shift left] \arrow[r, shift right] & G_{S'} \arrow[r] & \mathcal{X} \end{tikzcd}$$

which is an isomorphism on the generic fibers. □

Now let us study the functorial behavior of maximal models.

2.2.0.3 Lemma. *Let $\varphi : G \to H$ be a homomorphism of S-group schemes, and $f_K : X_K \to Y_K$ a G_K-equivariant morphism from a G_K-torsor X_K to a H_K-torsor Y_K. Suppose that the maximal models of X_K and Y_K exist, and denote them by X, Y respectively. Then f_K extends to a G-equivariant morphism f which fits into the diagram*

$$\begin{tikzcd} G_{S'} \arrow[r, "\varphi_{S'}"] \arrow[d, "\pi_X"'] & H_{S'} \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where S'/S trivializes the torsors X_K, Y_K, and π_X, π_Y are the finite flat morphisms as in the definition of maximal model. Moreover, if φ is faithfully flat, then so is f.

Proof : Indeed, we have the following G_K-equivariant morphism on the generic fiber

$$(\varphi_{S' \otimes K}, \varphi_{S' \otimes K}) : (G_{S'} \times_X G_{S'})_K \longrightarrow (H_{S'} \times_Y H_{S'})_K$$

it induces a G-equivariant morphism of their schematic closures

$$(\varphi_{S'}, \varphi_{S'}) : G_{S'} \times_X G_{S'} \longrightarrow H_{S'} \times_Y H_{S'}$$

which gives a morphism of groupoids

$$\begin{tikzcd} G_{S'} \times_X G_{S'} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & G_{S'} \arrow[d] \\ H_{S'} \times_Y H_{S'} \arrow[r, shift left] \arrow[r, shift right] & H_{S'} \end{tikzcd}$$

hence it induces a G-equivariant morphism $f : X \to Y$ which extends f_K.

If φ is faithfully flat, we apply the fiberwise criterion for flatness. By restricting $f : X \to Y$ to fibers, we obtain fibers of homomorphisms $\varphi : G \to H$, which are all flat. Since $\varphi_{S'}, \pi_X, \pi_Y$ are all faithful, it implies that f is also faithful. □

Another nice property of maximal model is the compatibility with inductions. Let $\varphi : G \to H$ be a homomorphism of flat S-group schemes, and X_K is a G_K-torsor over $\mathrm{Spec}(K)$ whose maximal model exists and is denoted by X. Let Y_K be the induced H_K-torsor

$$Y_K := \mathrm{Ind}_{G_K}^{H_K} X_K = X_K \times^{G_K} H_K,$$

and moreover we assume that its maximal model Y exists. In case that X is an actual G-torsor, we know that $Y \simeq X \times^G H$ is the induced H-torsor. In general, this isomorphism remains true.

2.2.0.4 Proposition. *Let $\varphi : G \to H$ be a homomorphism of flat S-group schemes, X_K is a G_K-torsor. Assume that the maximal model X of X_K exists. If the induced H_K-torsor Y_K has a maximal model Y, then $\mathrm{Ind}_G^H X = X \times^G H$ is representable by Y.*

Proof : We only need to verify the universal property of induction for Y. Let Z be an S-scheme acting by H, with an G-equivariant morphism from X

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Z \\ & Y \arrow[ur, dashed] & \end{tikzcd}$$

We want to construct the dashed arrow as an H-equivariant morphism from Y, and to show that it is unique. The dashed arrow is already uniquely defined on the generic fiber, namely, we have a unique factorization by an H_K-equivariant morphism from Y_K

$$\begin{tikzcd} X_K \arrow[rr] \arrow[dr] & & Z_K \\ & Y_K \arrow[ur] & \end{tikzcd}$$

In particular, if there exist two H-equivariant morphisms $Y \to Z$ which extend $Y_K \to Z_K$, then they must coincide.

Let Γ_K be the graph of the morphism $Y_K \to Z_K$, and let Γ be the schematic closure of Γ_K in $Y \times Z$. The H_K-action on Γ_K naturally extends to an H-action on Γ. Since Γ_K is isomorphic to Y_K, the schematic closure Γ is an integral model of Y_K. By maximality of Y, the model morphism $\Gamma \to Y$ is necessarily an isomorphism, and hence we obtain the unique factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Z \\ & Y \simeq \Gamma \arrow[ur] & \end{tikzcd}$$

□

2.2.0.5 Corollary. *Let $\varphi : G \to H$ be a faithfully flat homomorphism of flat S-group schemes with $N = \ker(\varphi)$. Let X_K be a G_K-torsor and assume that its maximal model X exists, and let Y_K be the induced H_K-torsor $\mathrm{Ind}_{G_K}^{H_K} X_K$. If the maximal model Y of Y_K exists, then the induction $\mathrm{Ind}_G^H X$ and the quotient X/N are representable by Y.*

Proof : Representability of $\mathrm{Ind}_G^H X$ is by Proposition 2.2.0.4. We claim that X/N is representably by $\mathrm{Ind}_G^H X$. To show this, We verify the universal property of quotient for $\mathrm{Ind}_G^H X$. Let $F : X \to Z$ be an N-invariant map. Consider the map $\tilde{F} : G \times X \to Z$ by sending (g, x) to $f(gx)$. With the N-action on $G \times X$ via left multiplication on G, and by the N-invariance of F, the map $\tilde{F}$ factors through $H \times X$

$$\begin{tikzcd} G \times X \arrow[rr, "\tilde{F}"] \arrow[dr, "/N"'] & & Z \\ & H \times X \arrow[ur, "\tilde{F}'"'] & \end{tikzcd}$$

Since $\tilde{F}'$ is G-invariant, it induces a map

$$\mathrm{Ind}_G^H X = (H \times X)/G \longrightarrow Z$$

and we obtain a factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Z \\ & \mathrm{Ind}_G^H X \arrow[ur] & \end{tikzcd}$$

The uniqueness of this factorization follows from the uniqueness on the generic fiber. Therefore $\mathrm{Ind}_G^H X = Y$ represents the quotient X/N. □

2.3 Existence of maximal models

In this section, we study the existence of the maximal model of a torsor over $\mathrm{Spec}(K)$ under various group schemes. The answers for smooth proper group schemes and finite flat commutative group schemes are positive, while in general the answer is unknown.

2.3.1 The case of proper smooth group schemes

Let G be a proper smooth S-group scheme. Suppose that we have a G_K-torsor X_K over K, trivialized by a finite extension K'/K of fields, i.e., we have the diagram

$$\begin{tikzcd} X_{K'} \arrow[r, "\sim"] \arrow[d] & G_{K'} \arrow[dl, "v"] \\ X_K & \end{tikzcd}$$

one obtains a finite flat K-morphism v, which is G_K-equivariant.

Let S' be the normalization of S in K'. Consider the graph $Z = G_{K'} \times_{X_K} G_{K'} \subset G_{K'} \times_K G_{K'}$ of the equivalence relation defined by v, and let $\overline{Z}$ be the schematic closure of Z in $G_{S'} \times_S G_{S'}$.

2.3.1.1 Lemma. *The projection $p_1 : \overline{Z} \to G_{S'}$ is finite flat.*

Proof : Since $v : G_{K'} \to X_K$ is G_K-equivariant, the schematic closure $\overline{Z}$ is stable by the diagonal G-action on $G_{S'} \times_S G_{S'}$. Hence the projection $p_1 : \overline{Z} \to G_{S'}$ is also G-equivariant.

We show that p_1 is surjective. Indeed, since p_1 is proper, and its image contains the generic fiber of $G_{S'}$ which is dense, hence it is surjective. Next we show that p_1 is finite. It suffices to prove that p_1 is quasi-finite, then finiteness follows from properness of $\overline{Z}$. Notice that we have

$$\dim \overline{Z}_k = \dim \overline{Z}_K = \dim G_{K'} = \dim G_{S'} \otimes k$$

since $\overline{Z}$ and $G_{S'}$ are both flat over S. Thus over an open dense subscheme of $G_{S'}$, p_1 is quasi-finite. By the G-action and the G-equivariance of p_1, it is therefore quasi-finite.

Finally we show the flatness of p_1. Let η' be a generic point of the special fiber of $G_{S'}$, then the local ring $\mathcal{O} = \mathscr{O}_{G_{S'},\eta'}$ is a discrete valuation ring. Notice that here we use the smooth condition of G, hence it satisfies Serre's $R1$ condition. We have the cartesian squares

$$\begin{tikzcd} \overline{Z}_{\mathcal{O}} \arrow[r, hook] \arrow[d] & G_{S'} \otimes \mathcal{O} \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}) \arrow[d, "\text{flat}"] \\ \overline{Z} \arrow[r, hook] & G_{S'} \times_S G_{S'} \arrow[r] & G_{S'} \end{tikzcd}$$

where $\overline{Z}_{\mathcal{O}}$ is the schematic closure of $\overline{Z}_K \otimes \mathrm{Frac}(\mathcal{O})$, because the formation of schematic closure commutes with flat base change $\mathrm{Spec}(\mathcal{O})/S'$ of discrete valuation rings. Thus $\overline{Z}_{\mathcal{O}}$ is flat over $\mathcal{O}$, the morphism p_1 is flat over an open subscheme. By G-equivariance of p_1, it is therefore flat. $\square$

2.3.1.2 Lemma. *Let Y be a flat S-scheme, and $\Gamma \subset Y \times_S Y$. Suppose that*

(1) Γ is the schematic closure of $\Gamma_K \subset Y_K \times_K Y_K$ and Γ_K defines a flat equivalence relation over Y_K;

(2) The two projections $\Gamma \rightrightarrows Y$ are flat.

Then Γ defines a flat equivalence relation over Y.

Proof : We need to check that for any S-scheme Z, $\Gamma(Z)$ defines an equivalence relation on $Y(Z)$. It is sufficient to check it for the universal case $Z = Y$. This is clear, since Γ is the schematic closure of Γ_K, and the set $\Gamma(Y)$ is determined by the set $\Gamma_K(Y_K)$ because of flatness, where the latter defines an equivalence relation on $Y_K(Z_K)$. $\square$

2.3.1.3 Proposition. *Let G be a proper smooth group scheme over S, and X_K is a G_K-torsor over K. Then there exists a maximal S-model X of the torsor X_K, it is proper over S and regular.*

Proof : The construction of a maximal S-model X of X_K is by taking the groupoid quotient

$$\overline{Z} \rightrightarrows G_{S'} \to X$$

where the right arrow is the required finite flat G-equivariant S-morphism in the definition of the minimal model. The properness of X follows from faithfully flat descent, and the regularity follows from [EGA] IV, Proposition 17.3.3. $\square$

2.3.2 The case of finite flat commutative group schemes

In this section, let G be a finite flat commutative S-group scheme. It is well-known that G can be embedded into an abelian S-scheme A (cf. [BBM82] Théorème 3.1.1), and one has an exact sequence

$$0 \longrightarrow G \longrightarrow A \longrightarrow B \longrightarrow 0$$

where $B := A/G$ is also an abelian scheme. Given a G-torsor P, one can form the induced A-torsor $(P \times A)/G$, which induces a trivial B-torsor. Conversely, given an A-torsor P' which induces a trivial B-torsor, the preimage of the unit section of

$$P' \longrightarrow (P' \times B)/A = B$$

gives a G-torsor. The processes are mutually inverse.

2.3.2.1 Proposition. *Let G be a finite flat commutative S-group scheme, X_K a G_K-torsor over K. Then there exists a maximal model X of X_K, which is finite flat over S. Moreover, X is a complete intersection over S.*

Proof : Let Y_K be the A_K-torsor induced by the G_K-torsor X_K. By Proposition 2.3.1.3, there is a maximal model Y of Y_K. Let X be the schematic closure of X_K in Y. The A-action on Y induces the extended G-action on X. We claim that X is the maximal model of the G_K-torsor X_K.

Let K'/K be a field extension which trivializes the A_K-torsor Y_K, and S' the normalization of S in K'. From the commutative diagram

$$\begin{tikzcd} G_{S'} \arrow[r, hook] \arrow[d, dashed] & A_{S'} \arrow[r] \arrow[d] & B_{S'} \arrow[d] \\ X \arrow[r, hook] & Y \arrow[r] & B \end{tikzcd}$$

it induces a finite flat G-equivariant S-morphism $G_{S'} \to X$. Therefore X is the maximal model of X_K. The fact that X/S is a complete intersection is indicated from that $X \subset Y$ is defined by the preimage of the unit section of $Y \to B$. □

2.3.2.2 Remark. From the proofs of Proposition 2.3.1.3 and Proposition 2.3.2.1, for any finite extension K'/K which trivializes the G_K-torsor X_K, it is always possible to construct a finite flat G-equivariant S-morphism $G_{S'} \to X$, where S' is the normalization of S, and X is the maximal model. Thus the requirement for K'/K in the definition of the minimal model is only to trivialize X_K, in the case of proper smooth group schemes and finite flat commutative group schemes.

2.4 Maximal model of torsors under a semi-direct product

Let $G = N \rtimes H$ be a flat S-group scheme, where $N \lhd G$ is a flat normal subgroup scheme. In this section, we study relations between maximal models under G and those under N and H.

2.4.0.1 Theorem. *Let X_K be an G_K-torsor over K, and $Y_K = \mathrm{Ind}_{G_K}^{H_K} X_K$ the induced H_K-torsor. If the maximal model Y of Y_K exists, then the following are equivalent*

(1) The maximal model X of X_K exists;

(2) The maximal model $X \to Y$ of the N_{Y_K}-torsor $X_K \to Y_K$ exists.

Moreover, X is the common maximal model in (1) and (2), if it exists.

Proof : $(1) \Rightarrow (2)$: We only need to check that the morphism $X \to Y$ is indeed the maximal model of the N_{Y_K}-torsor $X_K \to Y_K$. Let S'/S be a finite flat base change that verifies the definition of X and Y being the maximal models of X_K and Y_K respectively. Then we have the following N-equivariant diagram

$$\begin{tikzcd} G_{S'\times S'} \arrow[rr, "\text{finite flat}"] \arrow[d, equal] & & X \\ N_{S'} \times_S H_{S'} \arrow[d, "\text{finite flat}"'] & & \\ N_{S'} \times_S Y \arrow[rr, "\sim"] & & N_Y \times_Y Y_{S'} \arrow[uu] \end{tikzcd}$$

where the up left equality is as schemes with $N_{S'}$-action. The right vertical map is N_Y-equivariantly induced by

$$Y_{S'} \longrightarrow X_{S'} \longrightarrow X$$

where the first arrow is induced by an inclusion $H \hookrightarrow G$. By the fiberwise criterion of flatness, the morphism

$$N_Y \times_Y Y_{S'} \longrightarrow X$$

is finite flat. Hence $X \to Y$ is the maximal model of $X_K \to Y_K$.

(2) $\Rightarrow$ (1): Let h denote the morphism $Y \to S$. The N_Y-action on X as an Y-scheme induces an N-action on X as an S-scheme as follows

$$N \longrightarrow h_* N_Y \longrightarrow h_* \mathrm{Aut}_Y(X) \longrightarrow \mathrm{Aut}_S(X)$$

and clearly the generic fiber of this action is the N_K-action on X_K induced from the normal subgroup $N_K \lhd G_K$.

Let S'/S be a finite flat base change that verifies the definition of Y and $X \to Y$ being the maximal models of Y_K and $X_K \to Y_K$ respectively. Then we have a finite flat morphism

$$N \times_S Y_{S'} = N_Y \times_Y Y_{S'} \longrightarrow X$$

We claim that the H-action on the left (acting on N and Y) descends to X. Let Θ denote $(N \times_S Y_{S'}) \times_X (N \times_S Y_{S'})$. The two compositions $H \times_S \Theta \rightrightarrows X$ from the following diagram

$$\begin{tikzcd} H \times_S \Theta \arrow[r, shift left] \arrow[r, shift right] & H \times_S (N \times_S Y_{S'}) \arrow[r] \arrow[d, "H\text{-action}"] & H \times_S X \arrow[d, dashed] \\ & N \times_S Y_{S'} \arrow[r] & X \end{tikzcd}$$

coincide on their generic fibers, hence they agree. Since the top row is a flat groupoid because of flatness of H/S, it induces the dashed arrow, which is the descent H-action on X. Consequently, we obtain the G-action on X, whose generic fiber coincides with the G_K-action on X_K.

Finally we need to verify that X is indeed the maximal model of X. The following diagram

$$\begin{tikzcd} G_{S'\times S'} \arrow[rr] \arrow[d, equal] & & X \\ N_{S'} \rtimes_S H_{S'} \arrow[d] & & \\ N_{S'} \times_S Y \arrow[rr, "\sim"] & & N_Y \times_Y Y_{S'} \arrow[uu] \end{tikzcd}$$

is G-equivariant, and the top horizontal map is finite flat. □

2.4.0.2 Remark. It is crucial that G is a semi-direct product, rather than a general extension of flat S-group schemes. If the short exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0$$

does not split, then the right vertical map in the above diagram could not be defined. In other words, in general the N_H-torsor $G \to H$ is not the maximal model of its generic fiber, unless the sequence splits after a finite extension S'/S which is the normalization of a finite extension K'/K of local fields.

2.5 The ideal sheaf of different and transitivity formula

In this section, let G/S be either a proper smooth scheme or a finite flat commutative group scheme, X_K an G_K-torsor, and X the maximal model of its generic fiber X_K. Let $h : G \to S$ and $\pi : X \to S$ denote the structure morphisms. In the two cases, X is either regular or a local complete intersection, where for either case one can define the dualizing sheaf of X. Let $\omega_{G/S}, \omega_{X/S}$ be the dualizing sheaves of G/S and X/S, and ω an invertible sheaf on S such that $h^*\omega = \omega_{G/S}$.

Let us summarize all the necessary notations in the following commutative diagram

$$\begin{tikzcd}
S \arrow[r, bend right, "1"'] & G \arrow[l, "h"'] & \\
 & G \times_S X \arrow[u, "q_1"'] \arrow[d] \arrow[dl, "\sigma"] \arrow[dr, "q_2"] & \\
X \arrow[uu, "\pi"] \arrow[ur, bend left, "\epsilon"] & X \times_S X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X
\end{tikzcd}$$

where

- p_1, p_2 are projections from $X \times_S X$,
- q_1, q_2 are projections from $G \times_S X$,
- ϵ is induced from the unit section of G,
- σ is the morphism of G-action,
- λ is (σ, q_2).

There is a trace map from the composition $p_2 \circ \lambda = q_2$

$$\mathbf{Tr}_\lambda : \lambda_*\omega_{G\times_S X/X} \longrightarrow \omega_{X\times_S X/X} \simeq p_1^*\omega_{X/S}$$

where the structure morphisms of $G \times_S X/X$ and $X \times_S X/X$ are q_2 and p_2 respectively. Note that

$$\omega_{G\times_S X/X} = q_1^*\omega_{G/S} = q_1^*h^*\omega = q_2^*\pi^*\omega = \lambda^*p_2^*\pi^*\omega$$

hence the trace $\mathbf{Tr}_\lambda$ and the adjunction $1 \to \lambda_*\lambda^*$ induce

$$\begin{tikzcd} p_2^*\pi^*\omega \arrow[r, "\alpha"] \arrow[d] & p_1^*\omega_{X/S} \\ \lambda_*\lambda^*p_2^*\pi^*\omega \arrow[ur, "\mathbf{Tr}_\lambda"'] & \end{tikzcd}$$

Let $\varphi = (\lambda\epsilon)^*\alpha : \pi^*\omega \to \omega_{X/S}$, which is an isomorphism on the generic fiber. The ***different ideal sheaf*** of X/S is defined by $\delta_{X/S} := \pi^*\omega \otimes_{\mathscr{O}_X} \omega_{X/S}^{-1}$. It turns out that this ideal sheaf of different measures how far the maximal model X is from being a torsor under G. It plays a similar role as the usual different in the classical ramification theory of local fields.

2.5.0.1 Proposition. *$\delta_{X/S} \simeq \mathscr{O}_X$ if and only if X is a G-torsor over S.*

Proof : The "if" part is straightforward, since λ is an isomorphism and thus $\mathbf{Tr}_\lambda$ is an isomorphism as well.

If φ is an isomorphism, then α is an isomorphism along the diagonal. Notice that α is equivariant with respect to the G-action on the first factors of $G \times_S X$ and $X \times_S X$, therefore α is an isomorphism. It implies that the trace homomorphism $\mathbf{Tr}_\lambda$ is surjective. Then the proposition follows from the next lemma, applying to $T = X$, $Y = X \times_S X$ and $Y' = G \times_S X$. $\square$

2.5.0.2 Lemma. *Let T be a flat S-scheme, and $\lambda : Y' \to Y$ a finite morphism of T-schemes, and we assume that Y/T and Y'/T have locally free dualizing sheaves $\omega_{Y/T}$, $\omega_{Y'/T}$. Suppose that λ is an isomorphism over an open schematically dense subset of Y. If $\mathbf{Tr}_\lambda : \lambda_*\omega_{Y'/T} \to \omega_{Y/T}$ is surjective, then λ is an isomorphism.*

Proof : It suffices to show the following: Let $A \to B$ be a homomorphism of R-algebras, such that B is finite as an A-module, and $A_K \to B_K$ is an isomorphism. If the trace map

$$\mathbf{Tr} : \mathrm{Hom}_A(B, A) \longrightarrow A$$

is surjective, then $A \to B$ is an isomorphism.

Let us choose a presentation $A^{\oplus n} \to B$ of B as an A-module, and consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_{A_K}(B_K, A_K) \arrow[r, hook] & \mathrm{Hom}_{A_K}(A_K^{\oplus n}, A) \\ \mathrm{Hom}_A(B, A) \arrow[r, hook] \arrow[u] & \mathrm{Hom}_A(A^{\oplus n}, A) \arrow[u, hook] \end{tikzcd}$$

we see that the left vertical map is injective. Then, from the diagram of trace maps

$$\begin{tikzcd} \mathrm{Hom}_A(B, A) \arrow[r, hook] \arrow[d, "\mathbf{Tr}"] & \mathrm{Hom}_{A_K}(B_K, A_K) \arrow[d, "\mathbf{Tr}_K", "\sim"'] \\ A \arrow[r, hook] & A_K \end{tikzcd}$$

we deduce that $\mathbf{Tr}$ is injective, hence bijective. Therefore $A \to B$ is an isomorphism. $\square$

The next proposition is the transitivity formula for the different $\delta_{X/S}$.

2.5.0.3 Proposition. *Let*

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\beta} G'' \longrightarrow 1$$

be an exact sequence of S-group schemes (proper smooth schemes or finite flat commutative group schemes). Let $X_K \to X''_K$ be a morphism of torsors under G_K and G''_K (compatible with β_K), and $g : X \to X''$ the extended S-morphism of their maximal models. Then

(1) X is the maximal X''-model of the X''_K-torsor X_K under the group $G'_K \times_K X''_K$;

(2) $\delta_{X/S} \simeq \delta_{X/X''} \otimes g^* \delta_{X''/S}$.

Proof : Firstly, it is clear that $X_K \to X''_K$ is a torsor under the X''_K-group scheme $G'_K \times_K X''_K$. Let K'/K be a finite extension of fields which trivializes the torsors X_K and X''_K, and S' the normalization of S in K'. Then one has the following diagram

$$\begin{tikzcd}
1 \arrow[r] & G'_{S'} \arrow[r] & G_{S'} \arrow[r] \arrow[d] & G''_{S'} \arrow[r] \arrow[d] & 1 \\
 & & X \arrow[r, "g"] & X'' &
\end{tikzcd}$$

the composite morphism $G'_{S'} \to X$ gives the finite flat $(G' \times_S X'')$-equivariant X''-morphism

$$(G' \times_S X'')_{S'} = G'_{S'} \times_{S'} X''_{S'} \longrightarrow X$$

which proves (1). The transitivity formula (2) follows from the transitivity for dualizing sheaves. □

2.6 Examples of maximal models under finite flat group schemes of order p

Throughout this section, R is a complete discrete valuation ring with the perfect residue field k of characteristic $p > 0$, and the fraction field K. Let $\pi \in R$ be a uniformizer. Moreover, we assume that K contains a p-th root of unity whenever $\mathrm{char}(K) \neq p$. In the cases of μ_p and α_p, we assume that the characteristic of K is p. We will use the following lemma during this section.

2.6.0.1 Lemma. *Let G/S be a finite flat group scheme, and $P_K = \mathrm{Spec}(K')$ a G_K-torsor over K. Let P denote the normalization of S in K'. If the G_K-action on P_K extends to a G-action on P, then P/S is the maximal model of P_K/K.*

Proof : Indeed, the torsor P_K is tautologically trivialized by itself K'/K. We have the following diagram

$$\begin{tikzcd}
G_P \arrow[dr] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "\mathrm{pr}_2"'] & & \\
 & P_P \arrow[r] \arrow[d] & P \arrow[d] \\
 & P \arrow[r] & S
\end{tikzcd}$$

where m is the G-action morphism. In order that P/S is the maximal model, we need to show that m is finite flat. The morphism m decomposes as

$$G \times P \xrightarrow[\sim]{\mathrm{pr}_1 \times m} G \times P \xrightarrow{\mathrm{pr}_2} P$$

hence m is finite flat by the fact that pr_2 is finite flat. □

2.6.1 The étale-local group scheme $(\mathbb{Z}/p\mathbb{Z})_R$

A torsor over K under $\mathbb{Z}/p\mathbb{Z}$ is described by Kummer theory if $\mathrm{char}(K) \neq p$, by Artin–Schreier theory if $\mathrm{char}(K) = p$. Such a torsor is either a trivial one, or has the form $\mathrm{Spec}(L)$ where L/K is a cyclic Galois extension of order p.

The maximal model of a trivial torsor is the trivial torsor over R. In the nontrivial case, let R' be the integral closure of R in L. Since $\mathbb{Z}/p\mathbb{Z}$-action extends to the normalization $\mathrm{Spec}(R')$, it is therefore the maximal model by Lemma 2.6.0.1.

2.6.2 The local-étale group scheme $\mu_{p,R}$

By Kummer theory, a $\mu_{p,K}$-torsor over K has the following form

$$P_f = \mathrm{Spec}(K_f) := \mathrm{Spec}\big(K[X]/(X^p - f)\big)$$

where $f \in K^\times$, and we assume $f \notin (K^\times)^p$, which is equivalent to P_f being nontrivial. The $\mu_{p,K}$-action is given by

$$\begin{array}{ccc} \mu_{p,K} \times P_f & \longrightarrow & P_f \\ (z, x) & \longmapsto & z \cdot x \end{array}$$

We write $f = u\pi^i$ for $u \in R^\times$ and $0 \leqslant i \leqslant p-1$. The situation separates into two cases:

Case I. $i = 0$. In this case, P_f naturally extends to a $\mu_{p,R}$-torsor

$$\widetilde{P}_f := \mathrm{Spec}\big(R[X]/(X^p - f)\big)$$

which is trivialized by the normalization of R in K_f. Hence it is the maximal model of P_f, cf. Remark 2.1.0.3 (3). Note that only in this case, there could be maximal models which are not regular. If u modulo π is not a p-power, then $R[X]/(X^p - u)$ is a discrete valuation ring, and the maximal model is regular. If u is a p-power modulo π, let us write

$$u = \alpha^p + \pi^r \beta$$

for a maximal $r \geqslant 1$, where $\alpha, \beta \in R^\times$. The existence of such maximal r is clear, since otherwise u would be a p-power element in R. Let $Y = X - \alpha$, we see

$$\frac{R[X]}{(X^p - u)} = \frac{R[Y]}{(Y^p - \pi^r \beta)}$$

thus the maximal model is not regular if $r > 1$.

Case II. $i > 0$. In this case, we will see that the maximal model is always the normalization. Let $m, n \in \mathbb{N}$ with $mi - np = 1$, and $\tau := \pi^{-n} X^m \in K_f$. Then

$$R[\tau] = R[T]/(T^p - u^m \pi)$$

is a discrete valuation ring and it is the normalization of R in K_f. The μ_p-action naturally extends to $\mathrm{Spec}(R[\tau])$ by

$$\begin{array}{rcl} \mu_{p,R} \times \mathrm{Spec}(R[\tau]) & \longrightarrow & \mathrm{Spec}(R[\tau]) \\ (z, \tau) & \longmapsto & z^m \cdot \tau \end{array}$$

thus $\mathrm{Spec}(R[\tau])$ is the maximal model of P_f by Lemma 2.6.0.1.

2.6.3 The local-local group scheme $\alpha_{p,R}$

From the short exact sequence

$$0 \longrightarrow \alpha_{p,K} \longrightarrow \mathbb{G}_{a,K} \longrightarrow \mathbb{G}_{a,K} \longrightarrow 0$$

we know that $H^1(K, \alpha_p) \simeq K/K^p$, so an α_p-torsor over K has the form

$$P_f = \mathrm{Spec}(K_f) := \mathrm{Spec}\big(K[X]/(X^p - f)\big)$$

where $f \in K$ and we assume $f \notin K^p$. The $\alpha_{p,K}$-action is given by

$$\begin{array}{rcl} \alpha_{p,K} \times P_f & \longrightarrow & P_f \\ (a, x) & \longmapsto & x + a \end{array}$$

We write $f = u\pi^i$ for $u \in R^\times$ and $|i| \geqslant 1$, this is always possible by choosing an appropriate representative of $[f] \in K/K^p$, which represents the same isomorphism class of P_f. The situation also separates into two cases:

Case I. $i > 0$. In this case, P_f naturally extends to an $\alpha_{p,R}$-torsor

$$\widetilde{P}_f := \mathrm{Spec}\big(R[X]/(X^p - f)\big)$$

and it is the maximal model of P_f. Moreover, $\widetilde{P}_f$ is regular if and only if $i = 1$.

Case II. $i < 0$. First, we change the coordinate by $Y = X^{-1}$,

$$P_f = \mathrm{Spec}\big(K[Y]/(Y^p - u^{-1}\pi^{-i})\big)$$

and the $\alpha_{p,K}$-action goes by

$$\begin{array}{rcl} \alpha_{p,K} \times P_f & \longrightarrow & P_f \\ (a, y) & \longmapsto & \dfrac{y}{1 + ay} \end{array}$$

Let $m, n \in \mathbb{N}$ with $m \cdot (-i) - np = 1$, and let $\tau = \pi^{-n} Y^m$. Then

$$R[\tau] = R[T]/(T^p - u^{-m}\pi)$$

is the normalization of R in K_f. The $\alpha_{p,K}$-action extends to $\widetilde{P}_f := \operatorname{Spec}(R[\tau])$ by

$$\begin{array}{rcl} \alpha_{p,R} \times \widetilde{P}_f & \longrightarrow & \widetilde{P}_f \\ (a,\tau) & \longmapsto & \dfrac{\tau}{(1+au^n\tau^{-i})^m} \end{array}$$

therefore $\widetilde{P}_f$ is the maximal model of P_f by Lemma 2.6.0.1, and it is always regular.

2.6.4 The congruence group schemes H_λ

Let $\lambda \in R$ be an element satisfying

$$v_R(\lambda) \leqslant \frac{v_R(p)}{p-1}$$

in particular, the condition is void if R has characteristic p. The ***congruence** R**-group scheme** H_λ **of level** λ* has the underlying scheme structure

$$H_\lambda = \operatorname{Spec}\frac{R[x]}{\big((1+\lambda x)^p - 1\big)/\lambda^p}$$

and the group law is given by

$$x_1 \circ x_2 = x_1 + x_2 + \lambda \cdot x_1 x_2.$$

Let $\mu := p/\lambda^{p-1}$ (if $\lambda = 0$, then let $\mu \in R$), this is an element of R by looking at its valuation

$$v_R(\mu) = v_R(p) - (p-1)v_R(\lambda) \geqslant 0.$$

Then the group scheme H_λ fits into a Kummer-type sequence (cf. [AG07], Appendix A)

$$0 \longrightarrow H_\lambda \longrightarrow \mathcal{G}^\lambda \xrightarrow{\varphi^\lambda} \mathcal{G}^{\lambda^p} \longrightarrow 0$$

where $\mathcal{G}^\lambda$ is an affine smooth one-dimensional R-group scheme with the underlying scheme structure

$$\mathcal{G}^\lambda = \operatorname{Spec}\left(R\left[x, \frac{1}{1+\lambda x}\right]\right)$$

and the group law

$$x_1 \circ x_2 = x_1 + x_2 + \lambda \cdot x_1 x_2.$$

The isogeny φ^λ is defined explicitly by

$$\varphi^\lambda(x) := \big((1+\lambda x)^p - 1\big)/\lambda^p = x^p + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\lambda^{i-1}\mu x^i.$$

We have the following result on structure of the special fiber $(H_\lambda)_k$:

2.6.4.1 Lemma.[[AG07] Lemma 2.2]

$$(H_\lambda)_k = \begin{cases} \mu_{p,k} & \textit{if } v_R(\lambda) = 0; \\ \alpha_{p,k} & \textit{if } v_R(p) = \infty,\ v_R(\lambda) > 0; \\ \alpha_{p,k} & \textit{if } \infty > v_R(p) > (p-1)v_R(\lambda) > 0; \\ \mathbb{Z}/p\mathbb{Z} & \textit{if } \infty > v_R(p) = (p-1)v_R(\lambda). \end{cases}$$

In the following, we will calculate the maximal model under assumptions $0 < v_R(\lambda) < \infty$ and $v_R(p) = \infty$.

Under our assumptions, we have $\mu = 0$, hence $\varphi^\lambda(x) = x^p$, and the congruence group scheme H_λ is local. An $(H_\lambda)_K$-torsor P_f has the form

$$P_f := \operatorname{Spec}\big(K[W]/(W^p - f)\big)$$

with $f \in K$. We assume that P_f is nontrivial, hence $f \neq 0$. The action is given by

$$\begin{array}{rcl} (H_\lambda)_K \times P_f & \longrightarrow & P_f \\ (x, w) & \longmapsto & x + w + \lambda x w \end{array}$$

2.6.4.2 Lemma. *If* $v_R(f) + p v_R(\lambda) > 0$, *then* P_f *extends to an* H_λ*-torsor.*

Proof : If $v_R(f) \geqslant 0$, we claim that $\widetilde{P}_f = \operatorname{Spec}\big(R[W]/(W^p - f)\big)$ is an H_λ-torsor. It is clear that the $(H_\lambda)_K$-action on P_f extends to an H_λ-action on $\widetilde{P}_f$ via the same formula. We need to show that the following morphism is an isomorphism

$$\begin{array}{rcl} H_\lambda \times \widetilde{P}_f & \longrightarrow & \widetilde{P}_f \times \widetilde{P}_f \\ \dfrac{R[W_1, W_2]}{(W_1^p - f, W_2^p - f)} & \longrightarrow & \dfrac{R[x, W]}{(x^p, W^p - f)} \\ W_1 & \longmapsto & x + W + \lambda x W \\ W_2 & \longmapsto & W \end{array}$$

We can write an inverse morphism formally as

$$\begin{array}{rcl} W & \longmapsto & W_2 \\ x & \longmapsto & \dfrac{W_1 - W_2}{1 + \lambda W_2} \end{array}$$

Notice that

$$\frac{1}{1 + \lambda W_2} = \sum_{i=1}^{p-1} \Big(\sum_{k=0}^{\infty} (-\lambda)^{i+kp} f^k \Big) W_2^i$$

the valuation of each term in the coeafficient of W_2^i

$$v_R((-\lambda)^{i+kp} f^k) = i v_R(\lambda) + k(v_R(f) + p v_R(\lambda))$$

turns to $+\infty$ as $k \to +\infty$. Hence each coefficient converges, and this formal inverse is an actual inverse. Thus $\widetilde{P}_f$ is an H_λ-torsor which extends P_f.

If $v_R(f) < 0$, let us change the coordinate $W \mapsto U^{-1}$, and the morphism $H_\lambda \times \widetilde{P}_\lambda \to \widetilde{P}_f \times \widetilde{P}_f$ has the form

$$\begin{array}{rcl} H_\lambda \times \widetilde{P}_f & \longrightarrow & \widetilde{P}_f \times \widetilde{P}_f \\ \dfrac{R[U_1, U_2]}{(U_1^p - f^{-1}, U_2^p - f^{-1})} & \longrightarrow & \dfrac{R[x, U]}{(x^p, U^p - f^{-1})} \\ U_1 & \longmapsto & \dfrac{U}{1 + Ux + \lambda x} \\ U_2 & \longmapsto & U \end{array}$$

We write down the formal inverse of x

$$x \longmapsto \frac{U_2 - U_1}{U_1(U_2+\lambda)} = (U_1^{-1} - U_2^{-1}) \cdot \sum_{i=1}^{p-1} \Big(\sum_{k=0}^{\infty} (-\lambda)^{i+kp} f^k \Big) U_2^{-i}$$

substitute U_i^{-1} with fU_i^{p-1}, the right hand side becomes

$$\mathrm{RHS} = (U_1^{p-1} - U_2^{p-1}) \cdot \sum_{i=1}^{p-1} \Big(\sum_{k=0}^{\infty} (-\lambda)^{i+kp} f^{k+2} \Big) U_2^{p-i}$$

where the valuation of each term in the coefficient of U_2^{p-i}

$$v_R((-\lambda)^{i+kp} f^{k+2}) = iv_R(\lambda) + 2v_R(f) + k(v_R(f) + pv_R(\lambda))$$

turns to $+\infty$ as $k \to +\infty$. Hence this formal inverse is an inverse, and $\widetilde{P}_f$ is an H_λ-torsor which extends P_f. $\square$

Consequently, in the case $v_R(f) + pv_R(\lambda) > 0$, the extended H_λ-torsor $\widetilde{P}_f$ is the maximal model of P_f.

Next let us study the other case $v_R(f)+pv_R(\lambda) \leqslant 0$, in particualr $v_R(f) < 0$. Write $f = \delta\pi^{-i}$ for some $\delta \in R^\times$ and $i \in \mathbb{N}$, and let $m, n \in \mathbb{N}$ such that $mi - np = 1$. We change the coordinate $W \mapsto U^{-1}$ of P_f, and the $(H_\lambda)_K$-action on P_f goes like

$$\begin{array}{rcl} (H_\lambda)_K \times P_f & \longrightarrow & P_f \\ (x, u) & \longmapsto & \dfrac{u}{1 + ux + \lambda x} \end{array}$$

As in previous lemma, let $\widetilde{P}_f$ denote the R-scheme $\mathrm{Spec}\big(R[U]/(U^p - f^{-1})\big)$. Let $\mathcal{P}_f$ be the normalization of $\widetilde{P}_f$, which is

$$\begin{array}{rcl} \mathcal{P}_f & \longrightarrow & \widetilde{P}_f \\ R[U]/(U^p - \delta^{-1}\pi^i) & \longrightarrow & R[T]/(T^p - \delta^{-m}\pi) \\ U & \longmapsto & \delta^n T^i \end{array}$$

Then the H_λ-action on $\widetilde{P}_f$ extends to $\mathcal{P}_f$ as follows

$$\begin{array}{rcl} H_\lambda \times \mathcal{P}_f & \longrightarrow & \mathcal{P}_f \\ (x, \tau) & \longmapsto & \dfrac{\tau}{(1 + \delta^n \tau^i x + \lambda x)^m} \end{array}$$

Hence the normalization $\mathcal{P}^f$ is the maximal model of P_f in this case.

2.6.4.3 Remark. The two cases are not disjoint, it may happen that the R-scheme $\widetilde{P}_f$ is an H_λ-torsor and it is already regular. For example, this happens if $f = \pi$.

2.7 Appendix: A result of non-flat descent

In this appendix, we show a result on certain morphisms being effectively epimorphic. For the definition of effective epimorphism, see [FGA] 212-03.

Let us recall the definition of pure morphism. We fix a base scheme $S = \mathrm{Spec}(R)$, where R is a discrete valuation ring, K its fractional field, k its residue field. We denote the henselization of S by S^h, and correspondingly X^h by $X \times_S S^h$.

2.7.0.1 Definition. Let X be a S-scheme locally of finite type. It is called S-*pure* if the closure of any associated point of the generic fiber of X^h meets its special fiber.

2.7.0.2 Lemma. *Let $u : Z' \to Z$ be an S-morphism between finite flat S-schemes. If u is schematically dominant, then u is an effective epimorphism.*

Proof : Let A, A' be the function algebras of Z, Z' respectively. Since u is finite, by [SGA1] Exposé VIII, Proposition 5.1, it is an effective epimorphism if the sequence

$$A \longrightarrow A' \rightrightarrows A' \otimes_A A'$$

is exact. The first map is injective by assumption. Let $\{e_1 = 1, e_2, ..., e_n\}$ be a basis for A'. By the structure theorem for modules over a principal domain, there are natural numbers $a_1 \leqslant ... \leqslant a_m$ with $m \leqslant n$ such that $\{\pi^{a_1}e_1, ..., \pi^{a_m}e_m\}$ is a basis for A. Moreover, $\{e_1 \otimes_R e_j\}$ is a basis for $A' \otimes_R A'$, and $A' \otimes_A A'$ is a quotient of it by the submodule generated by elements of the form $\pi^{a_i}(e_i \otimes_R 1 - 1 \otimes_R e_i)$ for $1 \leqslant i \leqslant m$. Let $x = \sum_{i=1}^n x_i e_i \in A'$ such that $x \otimes_A 1 = 1 \otimes_A x$ in $A' \otimes_A A'$, then there are elements $y_1, ..., y_m \in R$ with

$$\sum_{i=1}^{n} x_i(e_i \otimes 1 - 1 \otimes e_i) = \sum_{i=1}^{m} y_i \pi^{a_i}(e_i \otimes 1 - 1 \otimes e_i),$$

theerefore $x = \sum_{i=1}^m y_i \pi^{a_i} e_i$ is in A. □

The main result of this appendix is the following

2.7.0.3 Proposition. *Let $u : Z' \to Z$ be a finite morphism between flat S-schemes of finite type, such that u_K is an effective epimorphism. Then u is an effective epimorphism, and it remains true after any flat finite type base change $T \to S$.*

Proof : It suffices to prove it over the henselization S^h of S, so we assume S is henselian and keep the same notations. Let $Z'' := Z' \times_Z Z'$ and $v\ Z'' \to Z$ denotes the canonical map. Since u is finite, we need to show that the sequence

$$\mathscr{O}_Z \longrightarrow u_*\mathscr{O}_{Z'} \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} v_*\mathscr{O}_{Z''}$$

is exact. Since Z, Z' are flat, and u_K is schematically dominant, the first arrow is injective.

Let us show the exactness in the middle. The question is Zariski local on Z. By [Rom12] Lemma 2.1.7 and Lemma 2.1.11, for any fixed point $z \in Z$ in the special fiber, we can find an affine open neighborhood which is S-pure. By restricting Z', Z'' to such an affine open, we reduce to the case where Z is affine and S-pure. Now, it suffices to prove that for any function $f' : Z' \to \mathbb{A}^1_R$ such that $\mathrm{pr}_1^* f' = \mathrm{pr}_2^* f'$, then f' descends to $f : Z \to \mathbb{A}^1_R$. By *op.cit.* Proposition 3.2.5, it is sufficient to check it on the generic fiber and all finite flat closed subschemes of Z. Indeed, since u_K is an effective epimorphism, we already have the descent function on Z_K. For finite flat closed subschemes, by restricting Z', Z'' to a finite flat subscheme of Z, we can apply Lemma 2.7.0.2 to descend f' to Z.

Finally, after any flat finite type base change $T \to S$, the generic fiber of the morphism $u_T : Z'_T \to Z_T$ is still an effective epimorphism. Hence by applying the result in the first part, we conclude that u_T is an effective epimorphism. □

2.7.0.4 Remark. Recall that in the definition of maximal mode, we have a finite flat morphism $f : G_{T_{S'}} \to X$, see Remark 2.1.0.3 (2). By Proposition 2.7.0.3, the induced non-flat morphism $g : G_{T_{S'}} \to X_{T_{S'}}$ is an effective epimorphism.

3

Coperfection and the étale fundamental pro-groupoid in positive characteristic

This chapter is a joint work with Giulio Orecchia and Matthieu Romagny.

3.1 Introduction

In this article, we study flat, finitely presented algebraic spaces and stacks in characteristic $p > 0$, and their maps to perfect ones (those whose relative Frobenius morphism is isomorphic). To put in context, observe that in simple situations, one can perfectize objects in two canonical ways. For example, an $\mathbb{F}_p$-algebra A has a ***perfection***

$$A^{\mathrm{pf}} = \lim \left(\cdots \ A \xrightarrow{\mathrm{F}_A} A \xrightarrow{\mathrm{F}_A} A \right)$$

and a ***coperfection***

$$A^{\mathrm{copf}} = \mathrm{colim} \left(A \xrightarrow{\mathrm{F}_A} A \xrightarrow{\mathrm{F}_A} A \ \cdots \right)$$

where the limit and the colimit are taken along the absolute Frobenius.

Note that there is no uniform use of the word "perfection" in the literature. For this reason we fix our convention: in any category with Frobenius, we will call ***perfection*** resp. ***coperfection*** the right adjoint, resp. the left adjoint, to the inclusion of the full subcategory of perfect objects in the ambient category. This choice is prompted by the fact that typically, in case of existence, perfection is given by a limit while coperfection is given by a colimit.

Our interest is in perfection of algebras and coperfection of algebraic spaces and stacks (for the study of perfection of schemes the reader is directed to Kato [Ka86]). This means that our setting is *relative* (over a non-perfect base) and *geometric* (with schemes, spaces and stacks). Both features introduce difficulties; we do not know if perfection of algebras and coperfection of algebraic spaces and stacks exist in general. Our results are better for morphisms with geometrically reduced fibres, also called *separable*. Let us now describe them.

Main results. For each algebraic space S and flat, finitely presented algebraic stack $\mathscr{X} \to S$, we construct its *étale fundamental pro-groupoid* $\Pi_1(\mathscr{X}/S)$. This is a 2-pro-object of the 2-category of étale stacks, with coarse moduli space the space of connected components $\pi_0(X/S)$, see [Rom11], seen as a constant 2-pro-object. When S is the spectrum of a field κ and $\mathscr{X}$ is geometrically connected, the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ is represented in the 2-category of stacks by the étale fundamental gerbe $\Pi^{\text{ét}}_{\mathscr{X}/\kappa}$ of Borne and Vistoli [BV15, § 8]. If S has characteristic p, we let

$$\mathrm{F}_i : \mathscr{X}^{p^i/S} \longrightarrow \mathscr{X}^{p^{i+1}/S}$$

denote the relative Frobenius of $\mathscr{X}^{p^i/S}$, the i-th Frobenius twist of $\mathscr{X}/S$.

Theorem A. (3.5.1.1, 3.5.3.5) *Let S be a noetherian algebraic space of characteristic p.*

(i) *Let $X \to S$ be a flat, finitely presented, separable morphism of algebraic spaces. The inductive system of relative Frobenii*

$$X \xrightarrow{\mathrm{F}_0} X^{p/S} \xrightarrow{\mathrm{F}_1} X^{p^2/S} \longrightarrow \dots$$

admits a colimit in the category of algebraic spaces over S. This colimit is the algebraic space of connected components $\pi_0(X/S)$; it is a coperfection of $X \to S$.

(ii) *Let $\mathscr{X} \to S$ be a flat, finitely presented, separable algebraic stack. The étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ is a coperfection of $\mathscr{X}/S$ in the 2-category of pro-algebraic stacks. Moreover, the inductive system of relative Frobenii*

$$\mathscr{X} \xrightarrow{\mathrm{F}_0} \mathscr{X}^{p/S} \xrightarrow{\mathrm{F}_1} \mathscr{X}^{p^2/S} \longrightarrow \dots$$

admits a colimit in the 2-category of pro-Deligne–Mumford stacks over S, which is the pro-étale stack $\Pi_1(\mathscr{X}/S)$.

Note that point (ii) includes point (i) as a special case, because $\Pi_1(\mathscr{X}/S)$ has coarse moduli space $\pi_0(\mathscr{X}/S)$. We include (i) for emphasis and also because the proof actually proceeds by deducing (ii) from (i).

Theorem A seems to suggest that taking coperfection in the higher category of pro-algebraic n-stacks - or of simplicial spaces - would eventually recover the whole relative étale homotopy of it. We plan to investigate this eventuality in a future article.

Within the category of algebras, the situation is somehow more subtle. Given a characteristic p ring R and an algebra $R \to A$, let

$$\mathrm{F}_i : A^{p^{i+1}/R} \to A^{p^i/R}$$

denote the relative Frobenius of $A^{p^i/R}$, the i-th Frobenius twist of A. Define the *preperfection*:

$$A^{p^\infty/R} = \lim\left(\cdots\ A^{p^2/R} \xrightarrow{\mathrm{F}_1} A^{p/R} \xrightarrow{\mathrm{F}_0} A\right).$$

The name is explained by a surprising fact: the algebra $A^{p^\infty/R}$ is not perfect in general, even if $R \to A$ is flat, finitely presented and separable. We give an example of this with R equal to the local ring of a nodal curve singularity (see 3.4.5.2). In our example the double preperfection is perfect but we do not know if iterated preperfections should converge to a perfect algebra in general. In the affine case $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, we write $\pi_0(A/R)$ instead of $\pi_0(X/S)$. What Theorem A implies in this case is that there is an isomorphism of R-algebras:

$$\mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Here $\mathcal{O}(-)$ is the functor of global functions. Given the bad properties of the rings under consideration, this could not really be anticipated: indeed, in general $\mathcal{O}(\pi_0(A/R))$ is not étale and $A^{p^\infty/R}$ is not perfect. Similarly as above, the structure of the proof is actually to first establish this isomorphism of algebras (see 3.4.3.2) and then deduce the geometric statement for spaces and stacks (Theorem A above).

This begs for a further study of perfection of algebras. Our general expectation is that for algebras of finite type, there should exist a largest étale subalgebra and this should be (at least close to) the perfection of $R \to A$. In striving to materialize this picture, we study étale hulls in more detail. We take up recent work of Ferrand [Fe19] and prove the following result which is not special to characteristic p.

Theorem B. (3.3.1.8, ??) *Let S be a noetherian algebraic space. Let $f : X \to S$ be a faithfully flat, finitely presented morphism of algebraic spaces.*

(i) *The category of factorizations $X \to E \to S$ such that $X \to S$ is schematically dominant and $E \to S$ is étale and affine is a lattice, that is, any two objects have a supremum and an infimum (for the obvious relation of domination). Moreover it has a maximal element $\pi^a(X/S)$.*

(ii) *If S is geometrically unibranch and without embedded points, the functor $X \mapsto \pi^a(X/S)$ is left adjoint to the inclusion of the category of étale, affine S-schemes into the category of faithfully flat, finitely presented S-algebraic spaces.*

The maximal element $\pi^a(X/S)$ is the relative spectrum of a sheaf of $\mathcal{O}_S$-algebras which is the largest étale subalgebra of $f_*\mathcal{O}_X$. When S is artinian or $X \to S$ is separable, so that $\pi_0(X/S)$ is an étale algebraic space, we have morphisms:

$$X \longrightarrow \pi_0(X/S) \longrightarrow \pi^a(X/S).$$

When $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, the largest étale subalgebra is written $A^{\text{ét}/R} \subset A$, that is $\pi^a(A/R) = \mathrm{Spec}(A^{\text{ét}/R})$. We then obtain the following positive results on perfection.

Theorem C. (3.4.2.1, 3.4.4.1) *Let $R \to A$ be a flat, finite type morphism of noetherian rings of characteristic p. Assume that one of the following holds:*

(1) *R is artinian,*

(2) *R is regular and $R \to A$ is separable,*

(2) *R is one-dimensional, reduced, geometrically unibranch, and $R \to A$ is separable.*

Then the natural maps give rise to isomorphisms:

$$A^{\text{ét}/R} \xrightarrow{\sim} \mathcal{O}(\pi(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

Overview. In Section 3.2 we start with basic facts on coperfection. In Section 3.3 which makes no characteristic assumption, we give complements on the functor π_0. We first focus the "étale hull" property, namely that π_0 is left adjoint to the inclusion of the category of étale finitely presented into the category of flat, finitely presented spaces; we prove Theorem 3.3.1.8 which gives existence of an *affine* étale hull. We then prove some results related to the definition of π_0 as a moduli space for connected components; this includes two crucial pushout results that allow to view $\pi_0(X/S)$ as glued from simpler pieces, (the simpler pieces being either π_0 of an atlas, Prop. 3.3.4.3, or a completion from a special fibre, Prop. 3.3.5.2). In Section 3.4 we study the commutative algebra of perfection, with the maps $A^{\text{ét}/R} \to \mathcal{O}(\pi(A/R)) \to A^{p^\infty/R}$ as main characters. The main results are Theorems 3.4.2.1and 3.4.3.2. Finally in Section 3.5 we derive the computation of the coperfection of algebraic spaces or stacks, first in the category of algebraic spaces (Theorem 3.5.1.1) and then in the 2-category of (pro-)algebraic stacks (Theorem 3.5.3.5). The construction of the étale fundamental pro-groupoid necessitates technical preparations to be found in Subsections 3.5.2 and 3.5.4.

3.2 Coperfection

In this section, we make some preliminary remarks on perfection, mainly of categorical nature.

3.2.1 Definitions

Let C be a category endowed with an endofunctor $C \to C$, $X \mapsto X^{(p)}$ which we call ***Frobenius twist***. Two relevant examples are the category of schemes over a fixed base scheme of characteristic p, and the category of algebras over a fixed ring of characteristic p. Let us call ***functorial Frobenius*** a functor $\mathrm{F} : C \to \mathrm{Arrow}(C)$ to the category of arrows of C of one of the following types:

(i) F_X is a morphism $X \to X^{(p)}$ for all $X \in C$ (example: schemes);

(ii) F_X is a morphism $X^{(p)} \to X$ for all $X \in C$ (example: rings).

Let $i : P \to C$ be the inclusion of the full subcategory of ***perfect*** objects, those such that F_X is an isomorphism. In case of existence, the right adjoint is called ***perfection*** and the left adjoint is called ***coperfection***. In the present article, we are mainly interested in coperfection of schemes, algebraic spaces and algebraic stacks; as most colimits in algebraic geometry, the coperfection depends very much on the category where it is considered.

3.2.2 Base restriction

The ***base restriction along*** $f : S' \to S$ is the functor that sends an S'-scheme X' to the S-scheme $X' \to S' \to S$. We denote by $f_! X'$ the base restriction. The functor $f_!$ is left adjoint to the pullback f^*. It should not be confused with the Weil restriction functor f_* which is right adjoint to f^*. We will need to use the fact that coperfection commutes with base restriction. This is a consequence of the simple categorical fact that if two functors commute and have left adjoints, then the left adjoints commute. Here is a precise statement in our context.

3.2.2.1 Lemma. *Let X, T, S be $\mathbb{F}_p$-algebraic spaces. Let $f : T \to S$ be a morphism which is relatively perfect, and $X \to T$ a morphism which admits a coperfection X^{copf}. Then $f_!(X^{\mathrm{copf}})$ is a coperfection for $f_! X$. In a formula, we obtain an isomorphism:*

$$f_!(X^{\mathrm{copf}}) \xrightarrow{\sim} (f_! X)^{\mathrm{copf}}.$$

Proof : Let Perf/S be the category of relatively perfect S-algebraic spaces, and $i_S : \mathrm{Perf}/S \to \mathrm{Sp}/S$ the inclusion. Since $f : T \to S$ is relatively perfect and relatively perfect morphisms are stable by composition, the functor $f_!$ maps Perf/T into Perf/S, that is, it commutes with i_S and

i_T. Similarly f^* maps Perf/S into Perf/T. For each $Y \in \mathrm{Perf}\,/S$ we have canonical bijections:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}/S}(f_!X, i_SY) &= \mathrm{Hom}_{\mathrm{Sp}/T}(X, f^*i_SY)\\ &= \mathrm{Hom}_{\mathrm{Sp}/T}(X, i_Tf^*Y)\\ &= \mathrm{Hom}_{\mathrm{Perf}/T}(X^{\mathrm{copf}}, f^*Y)\\ &= \mathrm{Hom}_{\mathrm{Perf}/S}(f_!X^{\mathrm{copf}}, Y).\end{aligned}$$

This shows that $f_!X^{\mathrm{copf}}$ is the coperfection of $f_!X$. □

The same result holds, with the same proof, for pairs of commuting adjoints in similar situations. For example it holds for schemes, or sheaves, or stacks instead of spaces. Also it holds for the inclusion of quasi-compact étale algebraic spaces in the category of faithfully flat, finitely presented, separable algebraic spaces; there the left adjoint "étalification" functor is given by the functor of connected components π_0 which we will review in Section 3.3.

3.2.3 Coperfection of sheaves and stacks

Let S be an algebraic space of characteristic $p > 0$. Here we wish to briefly emphasize the properties of the coperfection functor in the categories of sheaves and stacks. For simplicity we restrict to sheaves but our remarks hold just as well for coperfection of stacks. So let Sp/S be the category of S-algebraic spaces and Fun/S the category of set-valued functors $X : (\mathrm{Sp}/S)^\circ \to \mathrm{Set}$, also called simply ***functors***. The category Fun/S has a Frobenius-twist endofunctor

$$\mathrm{Fun}/S \to \mathrm{Fun}/S, \quad X \mapsto X^{p/S} := \mathrm{F}_S^* X$$

which respects the full subcategories of fppf sheaves, algebraic spaces, and schemes. Note that unless S is perfect, the Frobenius twist is not isomorphic to the identity functor. In this section we focus on the category Sh/S of fppf sheaves. If X is a sheaf, the inductive system

$$X \xrightarrow{\ \mathrm{F}_0\ } X^{p/S} \xrightarrow{\ \mathrm{F}_1\ } X^{p^2/S} \dots$$

is filtered. It follows that the presheaf colimit of this system satisfies the sheaf property for coverings of affine schemes $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$, and that its Zariski sheafification is an fppf sheaf. We denote the latter by $(X/S)^{\mathrm{copf}}$ or simply X^{copf} when the base S is clear from context. One checks the following facts:

(i) $(X/S)^{\mathrm{copf}}$ is perfect and is a coperfection of X/S;

(ii) $(X/S)^{\mathrm{copf}}$ is locally of finite presentation if X/S is;

(iii) the formation of $(X/S)^{\mathrm{copf}}$ commutes with all base changes $S' \to S$;

(iv) if X is an algebraic space, then X^{copf} is far from algebraic in general. For example if X is the affine line over $\mathbb{F}_p$ then for an $\mathbb{F}_p$-algebra A, the set $X^{\mathrm{copf}}(A)$ is equal to $(A/\mathbb{F}_p)^{\mathrm{copf}}$, the absolute coperfection of A. In particular, for $A = \mathbb{F}_p[[t]]$ the set $X^{\mathrm{copf}}(A) = \mathbb{F}_p[[t^{p^{-\infty}}]]$ is much bigger than $\lim X(A/t^n) = \mathbb{F}_p$.

Because of (iv), the sheaf coperfection of an algebraic space is not a satisfying object. It is the coperfection in the categories of schemes or algebraic spaces that we wish to understand. For this one the mere existence problem is delicate, and in case of existence the good properties are not granted; we expect neither of (i), (ii) or (iii) to be true in general.

3.3 Étale hulls and connected components

In this section, we provide some complements on the functor π_0 introduced in [Rom11]. Although these results hold for algebraic stacks, we restrict most of the time to algebraic spaces because this simplifies the treatment a little and is enough for our needs. There are two viewpoints on the functor π_0, and we consider both.

Firstly π_0 is a left adjoint to the inclusion of the category of étale quasi-compact spaces in the category of flat, finitely presented, separable spaces. In the study of such "étalification" functors, Ferrand [Fe19] recently highlighted the importance of the category of factorizations $X \to E \to S$ where the second arrow is étale. He proved that when the base S has finitely many irreducible components, there is a left adjoint π^s to the inclusion of étale, separated spaces into all flat, finitely presented spaces. In § 3.3.1 we prove that the category of factorizations as well as some interesting subcategories satisfy topological invariance (in the sense of [SGA4-2], Exp. VIII, Th. 1.1). Then we prove that the category of factorizations such that $X \to E$ is schematically dominant and $E \to S$ is étale and affine has a maximal element $\pi^a(X/S)$. When S is geometrically unibranch, the functor π^a is left adjoint to the inclusion of étale, affine spaces into all flat, finitely presented spaces. In § 3.3.2 we compare π^a with the affine hull of π_0.

Secondly π_0 is the functor of connected components of a relative space. In § 3.3.3 we provide basic properties on the behaviour of π_0 with respect to factorizations, which hold even when it is not representable. In § 3.3.4 and 3.3.5 we provide descriptions of π_0 which show how it is obtained by glueing simpler pieces. This will be used in later sections to study perfection of algebras and coperfection of spaces.

We sometimes impose some finiteness or regularity assumptions on the base S, but nothing on the characteristics; it is only in later sections that we specialize to characteristic p.

3.3.1 Étale affine hulls and largest étale subalgebras

Let us briefly recall what is known on étale hulls, also called étalification functors. Consider the following diagram with five fully faithful subcategories of the category of S-spaces ("fp" stands for finitely presented):

$$
\begin{tikzcd}
(\mathrm{EtAff}/S) \arrow[r, "i"'] & (\mathrm{EtSep}/S) \arrow[r, "j"'] & (\mathrm{Et}/S) \arrow[r, "k"'] & (\mathrm{Spb}/S) \arrow[l, bend right, "\pi_0"'] \arrow[r, "l"'] & (\mathrm{Flat}/S) \arrow[llll, bend right=40, "\pi^a\ (S \text{ unibranch})"'] \arrow[lll, bend right=35, "\pi^s"'] \arrow[ll, bend right=30, "\exists?\pi"'] \\
\text{étale affine} & \text{étale fp separated} & \text{étale fp} & \text{flat fp separable} & \text{flat fp}
\end{tikzcd}
$$

Here are some positive facts on the existence of these adjoints:

(i) π_0 is constructed in [Rom11]. It has a moduli description in terms of connected components. When $X \to S$ is flat, finitely presented, the functor $\pi_0(X/S)$ is representable by an algebraic space when either X is separable, or S is zero-dimensional, see [Rom11], 2.1.3. Its main properties (representability, adjointness, commutation with base change) hold with no assumption on S. The morphism $X \to \pi_0(X/S)$ is surjective with connected geometric fibres.

(ii) π^s is constructed in [Fe19] when S has finitely many irreducible components, and is not known to exist otherwise. It has no known moduli description. It has functoriality and base change properties available only in restricted cases. The morphism $X \to \pi^s(X/S)$ is surjective but its geometric fibres are usually not connected.

(iii) π^a is constructed is the present subsection when S is noetherian, geometrically unibranch, without embedded points. It shares the same features as those just listed for π^s, except that $X \to \pi^a(X/S)$ is schematically dominant but maybe not surjective.

Here are some negative facts:

(iv) π is not known to exist unless S is zero-dimensional (in which case $\pi = \pi_0$).

(v) π_0 *does* extend naturally to a functor $(\mathrm{Flat}/S) \to (\mathrm{Et}/S)$ but this is not a left adjoint to $l \circ k$. Indeed [Rom11], 2.1.3 implies that for all flat, finitely presented $X \to S$ the functor $\pi_0(X/S)$ defined as an étale sheaf is constructible, hence an étale quasi-compact algebraic space. Moreover, for each étale $E \to S$ there is a map $\mathrm{Hom}(X, E) \to \mathrm{Hom}(\pi_0(X/S), E)$. However, in general there is no map in the other direction; in particular there is no morphism $X \to \pi_0(X/S)$ and this prevents π_0 from being an adjoint of $l \circ k$. For instance, let S be the spectrum of a discrete valuation ring R with fraction field K and let $X = \mathrm{Spec}(R[x]/(x^2 - \pi x))$. Then $\pi_0(X/S) \simeq \mathrm{Spec}(K) \sqcup \mathrm{Spec}(K)$ and the map $\pi_0(X/S) \to S$ is not even surjective.

3.3.1.1 Topological invariance of the étale site. In this paragraph we briefly indicate bibliographical references on the topological invariance of the étale site. Let $f : S' \to S$ be a morphism of schemes or spaces which is integral, radicial and surjective. Then the pullback functor f^* induces an equivalence between the category of étale S-spaces and the category of étale S'-spaces, which preserves affine objects: see [SP19], Tag 05ZG and in particular Tag 07VW. The definitions of the étale sites are in Tag 03EB. Here *affine objects* are meant in the absolute sense, but working locally on S we see that the same statement holds with *affine* understood in the relative sense. In any case, the preservation of affine objects can be deduced easily from Chevalley's theorem on finite or integral images of affines, see [EGA] II.6.7.1 and [SP19], Tag 05YU. Note that if f is representable by schemes then the assumptions "integral, radicial, surjective" are equivalent to f being a universal homeomorphism ([EGA] IV.18.12.11) but a general universal homeomorphism of algebraic spaces may fail to be separated (see [SP19], Tag 05Z6) and in this case topological invariance fails [SP19], Tag 05ZI.

We recall the definition of the category of factorizations from [Fe19]. In order to make Theorem 3.3.1.8 possible, we modify the definition slightly by relaxing the assumption of surjectivity.

3.3.1.2 Definition. Let $X \to S$ be a morphism of algebraic spaces. The *category of factorizations* is the category $\mathsf{E}(X/S)$ whose objects are the factorizations $X \to E \to S$ such that $E \to S$ is étale, and whose morphisms are the commutative diagrams:

$$\begin{tikzcd} & E_1 \arrow[d] \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & S. \\ & E_2 \arrow[ur] & \end{tikzcd}$$

The category $\mathsf{E}^{\mathrm{surj}}(X/S)$, resp. $\mathsf{E}^{\mathrm{dom}}(X/S)$ is the full subcategory of factorizations such that $X \to E$ is surjective, resp. schematically dominant. The category $\mathsf{E}^{\mathrm{sep}}(X/S)$, resp. $\mathsf{E}^{\mathrm{aff}}(X/S)$ is the full subcategory of factorizations such that $E \to S$ is separated, resp. affine. We write $\mathsf{E}^{\mathrm{aff,dom}}(X/S) = \mathsf{E}^{\mathrm{aff}}(X/S) \cap \mathsf{E}^{\mathrm{dom}}(X/S)$ and similarly for other intersections.

We will often denote a factorization $X \to E \to S$ simply by using the letter E. We draw the attention of the reader to the fact that for the subcategories $\mathsf{E}^\sharp(X/S)$ defined above, the property "$\sharp$" applies either to $E \to S$ or to $X \to S$, depending on the case.

3.3.1.3 Lemma. *Let $X \to S$ be a morphism of algebraic spaces $X \to S$. Let $f : S' \to S$ be a morphism of spaces which is integral, radicial and surjective. Let $X' = X \times_S S'$.*

(1) *The pullback functor $f^* : \mathsf{E}(X/S) \to \mathsf{E}(X'/S')$ is an equivalence which preserves the subcategories $\mathsf{E}^{\mathrm{sep}}$, $\mathsf{E}^{\mathrm{aff}}$ and $\mathsf{E}^{\mathrm{surj}}$.*

(2) *If moreover S, S' are locally noetherian, f induces a bijection $\mathrm{Emb}(S') \to \mathrm{Emb}(S)$ of embedded points, and $X \to S$ is faithfully flat, then f^* preserves also the subcategory $\mathsf{E}^{\mathrm{dom}}$.*

Proof : (1) We prove that f^* is essentially surjective. Let $X' \to E' \to S'$ be a factorization. By topological invariance of the étale site, there exists an essentially unique $E \to S$ such that $E' \simeq E \times_S S'$. In order to descend $u' : X' \to E'$ to a morphism $u : X \to E$, by descent of morphisms to an étale scheme along universal submersions ([SGA1], Exp. IX, prop. 3.2) it is enough to prove that $\mathrm{pr}_1^* u' = \mathrm{pr}_2^* u'$ where $\mathrm{pr}_1, \mathrm{pr}_2 : S' \times_S S' \to S'$ are the projections. By [SGA1], Exp. IX, prop. 3.1 it is enough to find a surjective morphism $g : S''' \to S' \times_S S'$ such that the two maps agree after base change along g. We can take $S''' = S'$ and g the diagonal map. This proves essential surjectivity; we leave full faithfulness to the reader. We now prove that f^* preserves the indicated subcategories. Since the diagonal of $E \to S$ is a closed immersion if and only if the diagonal of $E' \to S'$ is a closed immersion, we see that f^* preserves $\mathsf{E}^{\mathrm{sep}}(X/S)$. The fact that f^* preserves $\mathsf{E}^{\mathrm{aff}}$ was recalled in 3.3.1.1. Finally f^* preserves $\mathsf{E}^{\mathrm{surj}}$ because f is a universal homeomorphism.

(2) Here the morphisms $X \to E$ in the factorizations are automatically flat. Thus such a morphism is schematically dominant if and only if its image contains the set of associated points $\mathrm{Ass}(E)$. Since $\mathrm{Ass}(E) = \cup_{s \in \mathrm{Ass}(S)} E_s$ by [EGA] IV.3.3.1, we see that $X \to E$ is schematically dominant if and only if the image of $X \to E$ contains all fibres E_s with $s \in \mathrm{Ass}(S)$. But f induces a bijection of the non-embedded associated points since it is a homeomorphism, and a bijection on embedded points by assumption. Hence it is equivalent to say that the image of $X' \to E'$ contains all fibres $E'_{s'}$ with $s' \in \mathrm{Ass}(S')$. □

3.3.1.4 Suprema and infima. We say that E_1 and E_2 have a *supremum* if the category of factorizations E mapping to E_1 and E_2 has a terminal element. In a picture:

$$\begin{tikzcd}
 & & & E_1 \arrow[dr] & \\
X \arrow[r] & E \arrow[urr, bend left] \arrow[drr, bend right] \arrow[r, dotted] & \sup(E_1, E_2) \arrow[ur] \arrow[dr] & & S. \\
 & & & E_2 \arrow[ur] &
\end{tikzcd}$$

We say that E_1 and E_2 have a *infimum* if the category of factorizations E receiving maps from E_1 and E_2 has an initial element. In a picture:

$$\begin{tikzcd}
 & E_1 \arrow[dr] \arrow[drr, bend left] & & \\
X \arrow[ur] \arrow[dr] & & \inf(E_1, E_2) \arrow[r, dotted] & E \arrow[r] & S. \\
 & E_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Note that in the three categories $\mathsf{E}^{\mathrm{surj}}(X/S)$, $\mathsf{E}^{\mathrm{sep,dom}}(X/S)$ and $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$, if there is a morphism between E_1 and E_2 then it is unique. In other words, these categories really are posets.

3.3.1.5 Corollary. *Let $\mathsf{E}^{\sharp}(X/S) \subset \mathsf{E}(X/S)$ be any subcategory with*

$$\sharp \in \{\varnothing, \mathrm{sep}, \mathrm{aff}, \mathrm{surj}, \mathrm{dom}\}.$$

Let $f : S' \to S$ be a morphism of spaces which is integral, radicial and surjective. In case $\sharp = \mathrm{dom}$ assume moreover that f and X satisfy the assumptions of 3.3.1.3(2). Then the following hold:

(1) *$\mathsf{E}^{\sharp}(X/S)$ has an initial element if and only if $\mathsf{E}^{\sharp}(X'/S')$ has one.*

(2) *Let E_1, E_2 be factorizations in $\mathsf{E}^{\sharp}(X/S)$ and E_1', E_2' their images in $\mathsf{E}^{\sharp}(X'/S')$. Then E_1, E_2 have a supremum, resp. a infimum, if and only if E_1', E_2' have a supremum, resp. a infimum.*

Proof : Suprema and infima are defined in terms of morphisms and are therefore preserved by the equivalences $f^* : \mathsf{E}^{\sharp}(X/S) \to \mathsf{E}^{\sharp}(X'/S')$. □

We arrive at the main existence result of this subsection. We prepare the proof with two lemmas. The first is classical; the proof given here was suggested to us by Daniel Ferrand.

3.3.1.6 Lemma. *Let $E \to S$ be an étale, quasi-compact, separated morphism of schemes. Then after an étale surjective base change $S' \to S$, the S-scheme E is a disjoint union of a finite number of open subschemes of S. If moreover $E \to S$ is surjective and birational, it is an isomorphism.*

Proof : Since $E \to S$ is of finite presentation, we can assume that S is affine noetherian. Let $m(E/S)$ be the maximum of the number of geometric connected components of the fibres of $E \to S$; this is finite by [EGA], IV_3.9.7.8 and noetherian induction. The base change $S_1'E \to S$ produces an open and closed section whose complement has m-number strictly less. By induction on m, we obtain a splitting of E as a disjoint union of finitely many opens, as asserted. The second claim follows because assuming birationality, the number of opens has to be one. □

3.3.1.7 Lemma. *Let S be a separated noetherian scheme, and $U \subset S$ a nonempty dense open. Then the set of opens V containing U and such that $V \to S$ is affine is finite and has a minimal element for inclusion.*

Proof : If V is such an open, the complement $S \backslash V$ is included in $S \backslash U$ and has pure codimension 1 in S by [EGA] IV.21.12.7. This proves that $S \setminus V$ is a union of one-codimensional irreducible components of $S \setminus U$. Since these are finite in number, we see the set of interest is finite. Since S is separated, the intersection of all its elements is again S-affine and is the minimal element. □

3.3.1.8 Theorem. *Let $f : X \to S$ be a faithfully flat, finitely presented morphism of algebraic spaces. Assume that S is noetherian, geometrically unibranch, without embedded points. Then the category $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$ is a lattice, that is, any two objects have a supremum and an infimum. Moreover $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$ has a largest element.*

A similar statement holds in the category $\mathsf{E}^{\mathrm{surj,sep}}(X/S)$ where existence of suprema and maxima are due to Ferrand [Fe19].

Proof : Throughout the proof we write $\mathsf{E} = \mathsf{E}^{\mathrm{aff,dom}}(X/S)$. Note that for each factorization $X \to E \to S$, the morphism $X \to E$ is flat and finitely presented.

We start with the proof that any two factorizations $E_1, E_2 \in \mathsf{E}$ have a supremum. By topological invariance of the étale site, we can assume that S is reduced. Let E be the schematic image of the morphism $X \to E_1 \times_S E_2$. As a closed subscheme of $E_1 \times_S E_2$, it is affine and unramified over S. By the theorem on unramified morphisms over unibranch schemes ([EGA], IV.18.10.1), it is enough to prove that for each $e \in E$ with image $s \in S$, the map of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{E,e}$ is injective. Let $\eta_1, \dots, \eta_n$ be the associated points of S and let $\mathcal{O}_{E,\eta_i}$ be the semi-local rings of the fibres of $E \to S$ at η_i. Like in the proof of 3.3.1.3, we have $\mathrm{Ass}(E) = E_{\eta_1} \cup \dots \cup E_{\eta_n}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{S,s} & \longrightarrow & \mathcal{O}_{E,e} \\ \big\downarrow & & \big\downarrow \\ \Pi_{i=1}^n \mathcal{O}_{S,\eta_i} & \hookrightarrow & \Pi_{i=1}^n \mathcal{O}_{E,\eta_i}. \end{array}$$

The left and right maps are injective. The bottom map is injective also because E_{η_i} is in the image of $X \to E$ and $X \to S$ is faithfully flat. Therefore $\mathcal{O}_{S,s} \to \mathcal{O}_{E,e}$ is injective and this concludes the argument.

Now we prove that there is a largest element. For each $E \in \mathsf{E}$, the image of $X \to E$ is an open subscheme $U \subset E$, étale, separated, quasi-compact over S, which we call the "image" of the factorization E. It is determined by the scheme $RX \times_E X = X \times_U X$ which is the graph in $X \times_S X$ of an open and closed equivalence relation: indeed, we recover U as the quotient algebraic space X/R. Because S is noetherian, there are finitely many open and closed equivalence relations ([Fe19], 3.2.1, 3.2.2) hence finitely many "images" U. By the existence of suprema in E, the poset of "images" forms a directed finite set, hence it has a largest element.

We fix $E \in \mathsf{E}$ whose "image" U is largest. It is now enough to prove that the directed set of maps $u : F \to E$ in E has a largest element $u^{max} \colon E^{max} \to E$. Since E is a directed set, E^{max} will automatically be a largest element for it, concluding the proof.

Given a map $u \colon E' \to E$, we observe that there is an induced isomorphism $U' \simeq U$ between the "images". Moreover $U \subset E$ and $U' \subset E'$ are schematically dense in E. It follows that the induced étale surjective separated morphism from E' onto its image $u(E') \subset E$ is birational, hence an isomorphism by Lemma 3.3.1.6. Since E' is affine over S, then so is $u(E')$; hence Lemma 3.3.1.7 applied to the open $U \subset E$ implies that the directed set of maps $F \to E$ stabilizes, so eventually an E^{max} is achieved.

Finally, we construct an infimum for E_1 and E_2. Let E_0 be the pushout of the diagram $E_1 \leftarrow X \to E_2$, that is, the quotient of $E_1 \sqcup E_2$ by the étale equivalence relation that identifies the image of $X \to E_1$ and the image of $X \to E_2$. Let E be the largest element of the category $\mathsf{E}^{\mathrm{aff,dom}}(E_0/S)$. This is the infimum of E_1 and E_2. □

3.3.1.9 Definition. With the notations and assumptions of Theorem 3.3.1.8, the largest element of the poset $\mathsf{E}^{\mathrm{aff,dom}}(X/S)$ is called the ***étale affine hull of*** X/S and denoted $\pi^a(X/S)$. Its $\mathcal{O}_S$-sheaf of functions is called the *largest (quasi-coherent) étale* $\mathcal{O}_S$*-subalgebra of* $f_*\mathcal{O}_X$.

Giving an existence proof which is more constructive than the one given above is not easy because of the limited formal properties of the étale affine hull (compatibility with base change, with the formation of products, etc). Such properties are of course essential in most situations where the etale affine hull is useful. A sample of base change results for the étale separated hull is given in [Fe19], § 7. Similar results can be proven for the étale affine hull.

3.3.1.10 Corollary. *Let S be a noetherian geometrically unibranch scheme without embedded points. Let $u : X \to Y$ be a morphism between faithfully flat, finitely presented S-algebraic spaces.*

(1) *There is an induced morphism of étale affine hulls $\pi^a(X/S) \to \pi^a(Y/S)$.*

(2) *The functor π^a is left adjoint to the inclusion of the category of étale, affine S-schemes into the category of faithfully flat, finitely presented S-algebraic spaces.*

Proof : (1) By topological invariance of the étale site (Lemma 3.3.1.3), we can assume that S is reduced. Let E be the schematic image of $X \to Y \to \pi^a(Y/S)$. It follows from the theorem on unramified morphisms over unibranch schemes ([EGA], IV.18.10.1) that $E \to S$ is étale. By the definition of $\pi^a(X/S)$ we obtain a morphism $\pi^a(X/S) \to \pi^a(Y/S)$.

(2) Let $u : X \to E$ be an S-morphism from a faithfully flat, finitely presented space to an étale, affine scheme. By (1) there is an induced morphism $\pi^a(X/S) \to \pi^a(E/S)$. Since $E \to \pi^a(E/S)$ is an isomorphism, we obtain a morphism $\pi^a(X/S) \to E$. □

3.3.2 Affine hull of π_0

Let S be a noetherian scheme and $X \to S$ a flat separable morphism of finite type. A priori, there is no reason to expect that $\pi_0(X/S)^{\mathrm{aff}} \to S$, the affine hull of $\pi_0(X/S) \to S$, be étale. There are two reasons for this: the first, is that a priori $\pi_0(X/S)^{\mathrm{aff}}$ may not be of finite type. The second reason is that, even when it is of finite type, it may well be ramified. This may happen already over a dimension 1 base with a nodal singularity, as Example 3.4.5.2 illustrates.

Here we describe a case where $\pi_0(X/S)^{\mathrm{aff}}$ is étale, for some geometrically unibranch reduced base schemes S. More precisely, in this situation the étale affine hull $\pi^a(X/S) \to S$ exists, and there is a natural map $\pi_0(X/S)^{\mathrm{aff}} \to \pi^a(X/S)$. We will prove that under some local factoriality-type conditions, this is an isomorphism.

3.3.2.1 Definition. A noetherian local ring R is called *geometrically set-theoretically factorial* if its strict henselization is integral, and each pure one-codimensional closed subscheme of $\mathrm{Spec}(R)$ has the same support as a principal closed subscheme.

Although a little ill-looking, this definition includes many examples of interest such as regular rings, $\mathbb{Q}$-factorial rings like the quadratic cone singularity $xy = z^2$, and all reduced unibranch curves. We note moreover that these examples are also S_2 and hence satisfy all the assumptions of the following statement.

3.3.2.2 Proposition. *Let $X \to S$ be a morphism of algebraic spaces which is flat, separable, and finitely presented. Assume that S is locally noetherian, S_2, with geometrically set-theoretically factorial local rings. Then the natural map $\pi_0(X/S)^{\mathrm{aff}} \to \pi^a(X/S)$ is an isomorphism.*

Proof : It is enough to prove that $\pi_0(X/S)^{\mathrm{aff}} \to S$ is étale. We prove more generally that for all étale, quasi-compact algebraic spaces $E \to S$ the map $E^{\mathrm{aff}} \to S$ is étale. For this, we can work étale-locally on S. First let us see that we can reduce to the case where $E \to S$ is separated. By Ferrand [Fe19], Th. 3.2.1 there is an étale separated hull $\pi^s(E/S) \to S$. By [Fe19], Prop. 8.1.2 the map $E \to \pi^s(E/S)$ is initial among maps to separated schemes; note that Ferrand assumes normality of S but really uses only the unibranch hypothesis (in *loc. cit.*, this is said explicitly before Lemma 6.1.1 which is the key to Lemma 8.1.1). Since $E^{\mathrm{aff}} \to S$ is separated, we obtain a factorization $E \to \pi^s(E/S) \to E^{\mathrm{aff}}$. Taking global sections, the map $\mathcal{O}(E^{\mathrm{aff}}) \to \mathcal{O}(\pi^s(E/S)) \to \mathcal{O}(E) = \mathcal{O}(E^{\mathrm{aff}})$ is the identity; since $E \to \pi^s(E/S)$ is dominant we see that E has the same affine hull as $\pi^s(E/S)$. Hence replacing E by $\pi^s(E/S)$ if necessary, we can assume that it is separated. By Lemma 3.3.1.6, working étale-locally around a fixed point $s \in S$ we can reduce to the case where S is affine and E is an open of S. Let us write the closed complement as $ZS \setminus E = Z_1 \cup Z'$ where Z_1 has pure codimension 1 in S and Z' has codimension at least 2. By the assumption that the strictly local ring of s is geometrically set-theoretically factorial, the 1-cycle Z_1 is set-theoretically principal on a small enough étale neighbourhood of s in S. We replace S by such a neighbourhood and let f be a local equation for Z_1. Then the morphism $\mathcal{O}(S) \to \mathcal{O}(S \setminus Z_1)$ is the localization-by-f map which is étale. Since moreover S has the S_2 property, the restriction $\mathcal{O}(S \setminus Z_1) \to \mathcal{O}(S \setminus Z) = \mathcal{O}(E)$ is an isomorphism. The result follows. □

In the one-dimensional case, removing the unibranch condition in 3.3.2.2 yields a weaker result:

3.3.2.3 Proposition. *Let S be a reduced noetherian excellent scheme of dimension $\leqslant 1$. Let $X \to S$ be a flat separable morphism of finite presentation. Then $\pi_0(X/S)^{\mathrm{aff}}$ is quasi-finite.*

Proof : Quasi-finiteness of $\pi_0(X/S)^{\mathrm{aff}}$ may be checked étale locally on S. So we let s be a geometric point of S and $(S', s') \to (S, s)$ an étale neighbourhood such that:

i) the irreducible components $S_1, \ldots, S_n$ of S' are geometrically unibranch;

ii) for every $i \neq j$, $S_i \cap S_j = \{s'\}$;

iii) the fibre $\pi_0(X/S)_{s'}$ is a disjoint union of copies of $\operatorname{Spec} k(s')$;

iv) $S' = \operatorname{Spec} R'$ is affine.

The reason why an étale neighbourhood satisfying condition i) exists, is that the regular locus of S is open dense by excellence, hence so is the geometrically unibranch locus. So we may replace S by S' and assume that $S = \operatorname{Spec} R$ satisfies the properties above.

Write $\pi = \pi_0(X/S)$. Then $\pi = \pi' \sqcup \pi^*$, where π^* is the union of those connected components that do not meet the fibre $\pi_{s'}$. Then π^* lives over $S' \setminus \{s'\}$ which by condition ii) is geometrically unibranch. By Proposition 3.3.2.2, the map $\pi^{\mathrm{aff}} \to S$ is étale, and in particular quasi-finite. It remains to check that $\pi'^{\,\mathrm{aff}}$ is quasi-finite.

Up to restricting S by a further étale neighbourhood of s, we may assume that the isomorphism $\bigsqcup_{i=1}^n \operatorname{Spec} k(s) \to \pi'_s$ extends to an open immersion $\alpha \colon \bigsqcup_{i=1}^n S \to \pi'$. We claim that α has dense image. Indeed, let Z be an irreducible component of π'. Then Z maps to some irreducible component S_i of S. By assumption, S_i is geometrically unibranch, so by [EGA], th. 18.10.1, $Z \to S_i$ is étale. In particular $Z \to \pi'_{S_i}$ is an étale, closed immersion, that is, Z is a connected component of π'_{S_i}. Thanks to condition ii), Z is also a connected component of π', and therefore meets the closed fibre. In particular it meets the image of α. This proves the claim.

The morphism α is dominant and induces an injective R-algebra morphism $\mathcal{O}(\pi') \hookrightarrow R^n$. It follows that $\mathcal{O}(\pi')$ is finite as an R-module. In particular $\pi'^{\,\mathrm{aff}} \to S$ is finite. □

3.3.3 Base restriction and base change

In this subsection we collect some results on π_0 that hold irrespective of whether it is representable or not.

3.3.3.1 Lemma. *Let $\mathscr{X}/S$ be an S-algebraic stack and let $S' \to S$ be a base change. Then we have a canonical isomorphism of S'-functors $\pi_0(\mathscr{X} \times_S S'/S') \xrightarrow{\sim} \pi_0(\mathscr{X}/S) \times_S S'$.*

Proof : This follows from the definition of π_0 because both sides of the map in the statement parametrize relative connected components of $\mathscr{X} \times_S T'$ for variable S'-schemes T'. $\square$

The second result is related to factorizations (in the sense of Definition 3.3.1.2).

3.3.3.2 Lemma. *Let $\mathscr{X} \xrightarrow{h} \mathscr{E} \xrightarrow{f} S$ be morphisms of algebraic stacks.*

(1) *If $\mathscr{E} \to S$ is an étale algebraic space, there is a morphism of S-functors*

$$f_!\pi_0(\mathscr{X}/\mathscr{E}) \longrightarrow \pi_0(f_!\mathscr{X}/S).$$

which is an isomorphism when $\mathscr{X} \to \mathscr{E}$ is universally open.

(2) *If $\mathscr{X}, \mathscr{E}$ are finitely presented over S and $\mathscr{X} \to \mathscr{E}$ is a universal submersion with connected geometric fibres, there is an isomorphism*

$$\pi_0(\mathscr{X}/S) \xrightarrow{\sim} \pi_0(\mathscr{E}/S).$$

Proof : Since (2) is easy to prove and not used in the paper, we only prove (1). Note that if $\mathscr{X} \to \mathscr{E}$ is flat, finitely presented and separable, this follows from Lemma 3.2.2.1. However, here we assume much less. The morphism in the statement is constructed as follows. For each S-scheme T, a point of $f_!\pi_0(\mathscr{X}/\mathscr{E})$ with values in T is a pair composed of an S-morphism $u : T \to \mathscr{E}$ and a T-relative connected component $\mathscr{C}' \subset \mathscr{X} \times_{\mathscr{E}} T$. Since $\mathscr{E} \to S$ is étale, the map $\mathscr{X} \times_{\mathscr{E}} T \to \mathscr{X} \times_S T$ is an open immersion globally and a closed immersion in the fibres, showing that $\mathscr{C}\mathscr{C}'$ is a T-relative connected component of $\mathscr{X} \times_S T$ i.e. a T-valued point of $\pi_0(f_!\mathscr{X}/S)$. Let us describe the inverse morphism, assuming $\mathscr{X} \to \mathscr{E}$ universally open. Let $\mathscr{C} \subset \mathscr{X} \times_S T$ be a T-relative connected component. By the assumption on $\mathscr{X} \to \mathscr{E}$, the image $\mathscr{D}$ of $\mathscr{C}$ in $\mathscr{E} \times_S T$ is open, hence étale over T with nonempty geometrically connected T-fibres. It follows that $\mathscr{D} \to T$ is an isomorphism. Using its inverse, we obtain a morphism $T \to \mathscr{D} \to \mathscr{E}$ and the pair $(T \to \mathscr{E}, \mathscr{C})$ is a T-point of $f_!\pi_0(\mathscr{X}/\mathscr{E})$. These constructions are inverse to each other. $\square$

3.3.4 Description via nice atlases

In this subsection, we explain how to describe the space of connected components of an algebraic space in terms of a "nice" étale atlas (see Definition 3.3.4.2). The starting point is a pushout property which is a consequence of the right exactness of the functor π_0.

3.3.4.1 Lemma. *Let $X \to S$ be a flat, finitely presented morphism of algebraic spaces with geometrically reduced fibres and let $U \to X$ be an fppf surjective morphism.*

(1) *Let $R \subset U \times U$ be the fppf equivalence relation defined by $U \to X$, so that X is identified with the coequalizer* $\mathrm{coeq}(R \rightrightarrows U)$. *Then we have* $\pi_0(X/S) = \mathrm{coeq}(\pi_0(R/S) \rightrightarrows \pi_0(U/S))$.

(2) *The diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ \pi_0(U/S) \arrow[r] & \pi_0(X/S) \end{tikzcd}$$

is a pushout in the category of sheaves.

We warn the reader that $\pi_0(R/S) \to \pi_0(U/S) \times_S \pi_0(U/S)$ may fail to be injective; e.g. R may be disconnected in a connected U.

Proof : Throughout, we write $\pi_0(X)$ instead of $\pi_0(X/S)$ and we omit S from fibred products.

(1) Let $\tilde{\pi}_0(R)$ denote the equivalence relation generated by the image of $\pi_0(R) \to \pi_0(U) \times \pi_0(U)$. Let us prove that the formation of $\tilde{\pi}_0(R)$ commutes with fppf surjective refinements $f : U' \to U$. That is, if $f^*R \subset U' \times U'$ is the preimage of the equivalence relation R under $U' \times U' \to U \times U$ and $f^*(\tilde{\pi}_0(R))$ is the preimage of the relation $\tilde{\pi}_0(R)$ under $\pi_0(U') \times \pi_0(U') \to \pi_0(U) \times \pi_0(U)$ then we want to prove that the natural map

$$\tilde{\pi}_0(f^*R) \longrightarrow f^*\tilde{\pi}_0(R)$$

is an isomorphism. For this it is enough to prove that $\pi_0(f^*R) \to f^*\pi_0(R)$ is surjective. Since the spaces are étale, we may assume that S is the spectrum of an algebraically closed field, and we can represent each connected component by a point lying on it. Since $f : U' \to U$ is surjective, a point of

$$f^*\pi_0(R) = \pi_0(R) \times_{\pi_0(U)\times\pi_0(U)} \pi_0(U') \times \pi_0(U')$$

can be represented by a triple $(r, u_1, u_2) \in R(k) \times U'(k) \times U'(k)$, which is what we wanted to prove. Since any two atlases for X have a common refinement, it follows that the quotient space $\pi_0(U)/\tilde{\pi}_0(R)$ does not depend on the choice of U up to a canonical isomorphism, and taking $U = X$ and $U' = U$ we see that

$$\pi_0(U)/\tilde{\pi}_0(R) \simeq \pi_0(X).$$

(2) Let Y be a sheaf and let $a : X \to Y$, $b : \pi_0(U) \to Y$ be maps that coincide on U. Denote by $u : U \to X$ the chosen atlas and $s, t : R \to U$ the projections. Let σ, τ be the maps $\pi_0(s), \pi_0(t) : \pi_0(U) \to \pi_0(X)$. Using that $R \to \pi_0(R)$ is an epimorphism of sheaves, from $aus = aut$ we deduce $b\sigma = b\tau$. Then (1) implies that b factors through a map $\pi_0(X) \to Y$. □

3.3.4.2 Definition. Let $X \to S$ be a flat, finitely presented, separable morphism of algebraic spaces. We call an étale surjective morphism $U \to X$ a ***nice atlas*** if U and $\pi_0(U/S)$ are both affine over S.

The following proposition about existence of nice étale atlases will be key in the proof of theorem 3.5.1.1.

3.3.4.3 Proposition. *Any flat, finitely presented, separable morphism $X \to S$ of algebraic spaces admits a nice atlas.*

Proof : Since $X \to S$ is finitely presented and the formation of $\pi_0(X/S)$ commutes with base change, we may assume that S is noetherian. Restricting to an open affine of S and taking an atlas of X, we may then assume both S and X are affine.

We claim that we may reduce to the case where S is the spectrum of a local strictly henselian ring. Let $\bar{s} \to S$ be a geometric point, $S^{\mathrm{sh}} \to S$ the associated strict henselization. By assumption there exists an étale, affine surjective morphism $\tilde{f}\colon V \to X_{S^{\mathrm{sh}}}$ such that $\pi_0(V/S^{\mathrm{sh}})$ is affine. As both V and $X_{S^{\mathrm{sh}}}$ are of finite type, there exists by [EGA] IV$_3$, 8.8.2 an étale, affine neighbourhood $S' \to S$ of $\bar{s}$ and a morphism of S'-schemes $f\colon U \to X_{S'}$ such that $f \times_{S'} S^{\mathrm{sh}} = \tilde{f}$. By [EGA] IV$_3$, 8.10.5 there is a further étale, affine neighbourhood $S'' \to S'$ of $\bar{s}$, such that $U_{S''}$ and $\pi_0(U_{S''}/S'')$ are affine, and $U_{S'} \to X_{S'}$ is étale and surjective.

Now, for every geometric point $\bar{s}$ of S let $S(\bar{s}) \to S$ be the étale neighbourhood just constructed. Because S is quasi-compact, finitely many of these neighbourhoods cover S. Taking their disjoint union produces an affine étale surjective morphism $W \to S$. From the construction in the previous paragraph we obtain an étale, affine cover $U \to X \times_S W \to X$, with $\pi_0(U/W)$ affine. By 3.3.3.2 the space $\pi_0(U/S)$ is also affine. This proves the claim.

In particular, we can reduce to the case where S has finite dimension. We proceed therefore by induction on the dimension. If S has dimension zero, it is the disjoint union of finitely many spectra of Artin local rings. In this case, $\pi_0(X/S)$ itself is a finite S-scheme and therefore affine.

Let us now consider S of dimension $d \geqslant 1$. By the previous reduction step we can also assume that S is strictly henselian local, with closed point s and maximal ideal $\mathfrak{m}$. We cover the complement of the closed point $S \setminus \{s\}$ with an étale affine cover V. As V has dimension $d-1$, by inductive hypothesis X_V admits an étale cover $\tilde{U} \to X_V$ with $\tilde{U}$ and $\pi_0(\tilde{U}/V)$ affine. Hence $\pi_0(\tilde{U}/S)$ is affine as well.

It remains to cover the closed fibre X_s, so let $x \in X_s$ and write $Y_1, Y_2, \ldots, Y_n$ for the irreducible components of X_s containing x. There exists an open neighbourhood W of x in X, such that the fibre W_s is contained in $Y_1 \cup Y_2 \ldots \cup Y_n$. By [Rom12], Lemmas 2.1.11 and 2.1.7, there exists an affine open $U \subset W$ containing x and such that U is pure. Notice that $x \in U_s \subset Y_1 \cup Y_2 \ldots \cup Y_n$, hence U_s is connected. Therefore, $\pi_0(U_s/s)$ consists of one point mapping étale to s. As $k(s)$ is separably closed, $\pi_0(U_s/s) \to s$ is an isomorphism. Its inverse extends uniquely to a section $\alpha \colon S \to \pi_0(U/S)$, which is automatically an étale monomorphism, hence an open immersion. Now, by *loc. cit.* Theorem 2.2.1 (ii), all fibres of $U \to S$ are connected, therefore all fibres of the morphism $\pi_0(U/S) \to S$ consist of one point. It follows that α is surjective, hence an isomorphism.

For every point $x \in X_s$, we construct such an affine open $U(x) \subset X$; finitely many of these opens suffice to cover X_s, and we let U be their disjoint union. Now, the étale, affine cover $\tilde{U} \cup U \to X$ is the desired cover. □

3.3.5 Description over a complete local base

Let S be the spectrum of a complete noetherian local ring R with maximal ideal $\mathfrak{m}$. Our purpose in this subsection is to give a description of π_0 in terms of its completion $\widehat{\pi}$, which will be crucial for the proof of Theorem 3.4.3.2.

3.3.5.1 Completion of π_0 along the closed fibre. For each $n \geqslant 0$ let $S_n = \operatorname{Spec} R/\mathfrak{m}^{n+1}$. By [EGA] IV.18.5.15, restriction to S_0 yields an equivalence $/S \simeq /S_0$ between the categories of finite étale algebras. In particular, given $X \to S$ flat of finite type and separable, there exists a unique finite étale scheme $\widehat{\pi}/S$ restricting to $\pi_0(X \times_S S_n/S_n)$ over each S_n. Alternatively, one can see $\widehat{\pi}$ as the algebraization of the formal completion of $\pi_0(X/S)$, which explains the choice of notation $\widehat{\pi}$. As $\widehat{\pi}$ is finite over S, it is a product of complete local rings. By [SP19], Tag 0AQH there is a natural morphism of S-algebraic spaces

$$\psi \colon \widehat{\pi} \to \pi_0(X/S), \tag{3.1}$$

which restricts to an isomorphism over each S_n.

3.3.5.2 Proposition. *Let R be a complete noetherian ring, A a flat separable R-algebra of finite type. Write $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, s for the closed point of S, and let $V = S \setminus \{s\}$. The commutative diagram of S-algebraic spaces*

$$\begin{tikzcd} \widehat{\pi}_V \arrow[r, hook] \arrow[d, "\psi_V"] & \widehat{\pi} \arrow[d, "\psi"] \\ \pi_V \arrow[r, hook] & \pi_0(X/S) \end{tikzcd}$$

is a pushout in the category of fppf sheaves over S.

Proof : In the proof we write $\pi := \pi_0(X/S)$. In order to prove the claim, it suffices to show that any diagram of solid arrows

$$\begin{tikzcd} \widehat{\pi}_V \arrow[r] \arrow[d, "\psi_V"] & \widehat{\pi} \arrow[d, "\psi"] \arrow[ddr, bend left, "a"] & \\ \pi_V \arrow[r] \arrow[drr, bend right, "b"'] & \pi \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

where Z is an S-sheaf, admits a unique dashed arrow making the diagram commute.

First of all, notice that $\psi\colon \widehat{\pi} \to \pi$ is étale; writing $U = \pi_V \sqcup \widehat{\pi}$, it follows that $U \to \pi$ is faithfully flat of finite presentation, hence it is a coequalizer for $U \times_\pi U \to U$. Therefore, in order to obtain a unique dashed arrow, it suffices to check that $a \circ p_1 = a \circ p_2$, where p_1, p_2 are the projections $\widehat{\pi} \times_\pi \widehat{\pi} \to \widehat{\pi}$.

The S-scheme $\widehat{\pi}$ is finite étale, hence the map $\psi\colon \widehat{\pi} \to \pi$ is separated and quasi-finite, and so is also the base change $p_1\colon \widehat{\pi} \times_\pi \widehat{\pi} \to \widehat{\pi}$. Moreover, we know that $\widehat{\pi}$ is a finite disjoint union of spectra of completed local rings; by the classification of separated quasi-finite schemes over henselian local rings, $\widehat{\pi} \times_\pi \widehat{\pi}$ decomposes into a disjoint union $P^f \sqcup P'$ such that $p_1\colon P^f \to \widehat{\pi}$ is finite (and étale), and $P' = P'_V$ has empty closed fibre. One obtains a similar decomposition for the map p_2, let us say $\widehat{\pi} \times_\pi \widehat{\pi} = Q^f \sqcup Q'$. However, the compositions $\widehat{\pi} \times_\pi \widehat{\pi} \xrightarrow{p_i} \widehat{\pi} \to \pi \to S$ are the same map for $i = 1, 2$, and are both quasi-finite, separated; so both P^f and Q^f are equal to the finite part of the composition, and we find $P^f = Q^f$.

The restriction of ψ to the closed fibre, $\psi_s\colon \widehat{\pi}_s \to \pi_s$, is an isomorphism by construction of $\widehat{\pi}$, and therefore so is $P^f_s = (\widehat{\pi} \times_\pi \widehat{\pi})_s \xrightarrow{p_1} \widehat{\pi}_s$. The isomorphism extends uniquely to an isomorphism $P^f \to \widehat{\pi}$.

Consider the diagram of solid arrows

$$\begin{tikzcd} \widehat{\pi} \sqcup P' \arrow[r, "p_2"] \arrow[d, "p_1"] & \widehat{\pi} \arrow[d, "a"] \\ \widehat{\pi} \arrow[r, "a"] & Z \end{tikzcd}$$

where we have identified P^f with $\widehat{\pi}$. We want to show that it is commutative.

For $i = 1, 2$, the morphism p_i is the identity on $\widehat{\pi}$, so we really only need to show that $a \circ p_1$ agrees with $a \circ p_2$ on P'. As P' is contained in $(\widehat{\pi} \times_\pi \widehat{\pi})_V$, we have $a \circ (p_1)_V = b \circ \psi_V \circ (p_1)_V = b \circ \psi_V \circ (p_2)_V = a \circ (p_2)_V$ and the proof is complete. $\square$

3.4 Perfection of algebras

The commutative algebra developed in this section has independent interest but is also fruitfully introduced with an eye towards the geometric applications of the next section. Let $X \to S$ be a flat, finitely presented morphism of algebraic spaces of characteristic p. In order to study the coperfection of X in the category of S-algebraic spaces, we will use the étale algebraic spaces $\pi_0(X/S)$ and $\pi^a(X/S)$ (assuming they exist). Since étale implies relatively perfect, the morphism $X \to \pi_0(X/S)$ extends to the direct Frobenius system and we have a diagram:

$$\Big(X \xrightarrow{\mathrm{F}_0} X^{p/S} \xrightarrow{\mathrm{F}_1} \dots\Big) \longrightarrow \pi_0(X/S) \longrightarrow \pi^a(X/S).$$

The present section is devoted to the case where $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. The main question is whether there exists a ***perfection functor***, right adjoint to the inclusion of perfect R-algebras into all R-algebras. In such generality we do not know if such perfection exists. At least an obvious approximation should be the ***preperfection***:

$$A^{p^\infty/R} := \lim A^{p^i/R} = \lim\big(\cdots\ A^{p^2/R} \xrightarrow{\mathrm{F}_A} A^{p/R} \xrightarrow{\mathrm{F}_A} A\big).$$

The above diagram of spaces provides a diagram of algebras

$$A^{\text{ét}\,/R} \longrightarrow \mathcal{O}(\pi_0(A/R)) \longrightarrow A^{p^\infty/R}$$

where $A^{\text{ét}\,/R} = \mathcal{O}(\pi^a(A/R))$ is the largest étale subalgebra of A, see Definition 3.3.1.9. Our goal is roughly to get at close as possible to the ideal situation where both maps above are isomorphisms.

We start in § 3.4.1 with preliminary material on base change in the formation of the preperfection. Then we prove that the situation is indeed ideal when R is artinian and $R \to A$ is of finite type, see § 3.4.2, or R is regular and $R \to A$ is of finite type and separable, see § 3.4.4. Over a general noetherian ring, only the map $\mathcal{O}(\pi_0(A/R)) \to A^{p^\infty/R}$ is an isomorphism, see § 3.4.3. This is already remarkable, given the poor properties of both algebras: in general $\mathcal{O}(\pi_0(A/R))$ is not étale and $A^{p^\infty/R}$ is not perfect, even when $R \to A$ is flat, of finite type and separable. One may expect that after iterating the preperfection functor $(-)^{p^\infty/R}$ a finite (sufficiently high) number of times, one reaches a perfect R-algebra. With the hope that this might be true, we establish in § 3.4.4 some finiteness properties of $A^{p^\infty/R}$. We conclude the section with counterexamples.

3.4.1 Base change in preperfection

For each morphism of $\mathbb{F}_p$-algebras $R \to A$ and each base change morphism $R \to R'$ we have a natural base change map for preperfection:

$$\phi = \phi_{R,R',A} : A^{p^\infty/R} \otimes_R R' \longrightarrow (A \otimes_R R')^{p^\infty/R'}.$$

It is important to understand this map for at least two reasons. The first is that the study of $A^{p^\infty/R}$ with the usual tools (localization, completion on R...) involves many base changes. The second is that the base change map along Frobenius $\mathrm{F} : R \to R$ controls the success or failure of $A^{p^\infty/R}$ to be perfect; we elaborate on this in Remark 3.4.1.4. Before stating the first lemma devoted to properties of ϕ, we recall a result of T. Dumitrescu.

3.4.1.1 Theorem. *Let $R \to A$ be a morphism of noetherian commutative rings. Let $\mathrm{F}_{A/R} : A^{p/R} \to A$ be the relative Frobenius morphism. Then the following are equivalent:*

(i) *$R \to A$ is flat and separable,*

(ii) *$\mathrm{F}_{A/R}$ is injective and its cokernel is a flat R-module.*

Proof : See [Du95], Theorem 3. □

3.4.1.2 Remark. If we do not assume that R and A are noetherian but $R \to A$ is of finite presentation, then (i) ⇒ (ii) is true. Indeed $R \to A$ is the base change of a map $R_0 \to A_0$ along a morphism $R_0 \to R$ with R_0 noetherian and we may choose $R_0 \to A_0$ flat and separable, see [EGA] IV_3, 11.2.7 and 12.1.1(vii). Then by the noetherian case, it follows that F_{A_0/R_0} is injective with R_0-flat cokernel. By base change $\mathrm{F}_{A/R}$ is injective with R-flat cokernel.

3.4.1.3 Lemma. *The base change map* $\phi_{R,R',A} : A^{p^\infty/R} \otimes_R R' \longrightarrow (A \otimes_R R')^{p^\infty/R'}$ *is:*

(1) *an isomorphism if* $R \to R'$ *is finite locally free.*

(2) *injective in each of the following cases:*

 (i) $R \to R'$ *is projective.*

 (ii) $R \to R'$ *is flat and* $R \to A$ *is flat, finitely presented, with reduced geometric fibres.*

 (iii) $R' = \operatorname{colim} R$ *is the absolute coperfection of a ring* R *such that* $\mathrm{F} : R \to R$ *is projective.*

Proof : Note that since $(A \otimes_R R') \otimes_{R',\mathrm{F}^i} R' = A^{p^i/R} \otimes_R R'$, the map $\phi_{R,R',A}$ is just a special case for the R-module $M := R'$ of the map $\phi_{R,M,A}$ that appears as the upper horizontal row in the following commutative diagram:

$$\begin{array}{ccc} (\lim A^{p^i/R}) \otimes_R M & \xrightarrow{\phi_{R,M,A}} & \lim(A^{p^i/R} \otimes_R M) \\ \downarrow & & \downarrow \\ \left(\prod_{i \geqslant 0} A^{p^i/R}\right) \otimes_R M & \xrightarrow{\psi_{R,M,A}} & \prod_{i \geqslant 0}(A^{p^i/R} \otimes_R M). \end{array}$$

In the sequel we assume that M is flat, so the left-hand vertical map is injective. If M is free, resp. free of finite rank, then $\psi_{R,M,A}$ is injective, resp. an isomorphism. It follows that also $\phi_{R,M,A}$ is injective, resp. an isomorphism. If M is projective, one reaches the same conclusions by embedding it in a free module, resp. a free module of finite rank, and using the facts that $\phi_{R,M,A}$ and $\psi_{R,M,A}$ are additive in M. This settles cases (1) and (2.i).

In case (2.ii), by Dumitrescu's theorem 3.4.1.1 all the maps $A^{p^i/R} \to A^{p^{i+1}/R}$ are injective; it follows that $\lim A^{p^i/R} \to A^{p^j/R}$ is injective for each fixed j. By flatness of $R \to R'$ the tensored map $(\lim A^{p^i/R}) \otimes_R R' \to A^{p^j/R} \otimes_R R'$ is injective. Therefore $\phi_{R,R',A}$ is also injective.

In case (2.iii) we can write the coperfection as $R' = \operatorname{colim} R^{p^{-j}}$. Since the absolute Frobenius of R is projective, it is in fact faithfully flat. It follows that the maps $R^{p^{-j}} \to R^{p^{-(j+1)}}$ are faithfully flat, hence universally injective. Thus for each i, j the map

$$A^{p^i/R} \otimes R^{p^{-j}} \longrightarrow A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective. Then for each i

$$A^{p^i/R} \otimes R^{p^{-j}} \longrightarrow \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective. Taking limits

$$\lim_i \, (A^{p^i/R} \otimes R^{p^{-j}}) \longrightarrow \lim_i \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective, which implies that

$$\operatorname*{colim}_j \lim_i \, (A^{p^i/R} \otimes R^{p^{-j}}) \longrightarrow \lim_i \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}} = \lim_i A^{p^i/R} \otimes R'$$

is injective. Since also by (2.i) the map

$$(\lim_i A^{p^i/R}) \otimes R' = \operatorname*{colim}_j (\lim_i A^{p^i/R}) \otimes R^{p^{-j}} \longrightarrow \operatorname*{colim}_j \lim_i (A^{p^i/R} \otimes R^{p^{-j}})$$

is injective, by composition we obtain the result. □

3.4.1.4 Remarks. (1) Let $R \to A$ be a map of rings of characteristic $p > 0$. When inquiring whether the preperfection $A^{p^\infty/R}$ is perfect, we are led to ask if the Frobenius of the preperfection ("Frobenius of the limit") is an isomorphism. In general it is not; an example is given in in 3.4.5.2. In contrast, the morphism obtained as the limit of the Frobenius maps of the individual rings of the system ("the limit of Frobenius") *is* an isomorphism: it is essentially a shift by one in the indices, which is invisible in the infinite system. In fact, "Frobenius of the limit" and "the limit of Frobenius" are the two edges of a commutative triangle whose third edge, the base change map in preperfection, serves to compare them:

$$\begin{tikzcd}
A^{p^\infty/R} \otimes_{R,\mathrm{F}} R \arrow[dr, bend left, "\mathrm{F}_{A^{p^\infty}/R}"] \arrow[d, "\phi_{R,R,A}"'] & \\
& A^{p^\infty/R} \\
(A \otimes_{R,\mathrm{F}} R)^{p^\infty/R} \arrow[ur, bend right, "\lim \mathrm{F}"'] &
\end{tikzcd}$$

Since $\lim \mathrm{F}$ is an isomorphism, we see that $A^{p^\infty/R}$ is a perfect R-algebra if and only if the base change map $\phi_{R,R,A}$ is an isomorphism. According to Lemma 3.4.1.3(1), this happens when R is regular and F-finite, for then absolute Frobenius is finite locally free (see [Ku69]).

(2) In case (2.ii), it will be a consequence of Theorem 3.5.1.1 that the base change map is in fact an isomorphism.

(3) Here is an example where the base change map is not surjective. Let k be a field of characteristic p and k' an infinite-dimensional field extension. Let

$$A = k[\epsilon_0, \epsilon_1, \dots]/(\epsilon_0^p, \epsilon_{i+1}^p - \epsilon_i)$$

and $A' = A \otimes_k k'$. Let $t_0, t_1, \dots$ be an infinite family of elements of k' that is k-linearly independent. Let $x_i' = \epsilon_0 t_i + \epsilon_1 t_{i-1} + \dots + \epsilon_i t_0 \in (A')^{p^i/k'}$. Then $\mathrm{F}_{A'/k'}(x_{i+1}') = x_i'$ so $x' = (x_i')$ is an element of $(A')^{p^\infty/k'}$ which obviously does not come from $A^{p^\infty/k} \otimes k'$.

The following lemma is a case where preperfection commutes with base change; it is included for completeness but is not used in the paper.

3.4.1.5 Lemma. *Let A be an R-algebra, flat of finite presentation, such that the induced morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ has geometrically reduced fibres. Let $f \in R$ be a non-zero divisor, with R/fR reduced. The natural map $\phi\colon A^{p^\infty/R} \otimes_R R_f \to (A \otimes_R R_f)^{p^\infty/R_f}$ is an isomorphism.*

Proof : By Dumitrescu's theorem 3.4.1.1, the maps $A^{p^\infty/R} \hookrightarrow A$ and $g\colon (A_f)^{p^\infty/R_f} \hookrightarrow A_f$ are injective. Moreover $h\colon A^{p^\infty/R} \otimes_R R_f \hookrightarrow A_f$ is also injective. As $h = g \circ \phi$, we see that ϕ is injective.

For surjectivity, let $G = \mathrm{F}^m\colon R \to R$ be the m-th iterate of Frobenius. Write $B = A \otimes_{R,G} R$. Consider the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B_f & \longrightarrow & A_f \end{array}$$

where the top horizontal map is the m-th relative Frobenius of A over R, and the bottom horizontal map is obtained by inverting f. We have $B_f = (A \otimes_{R,G} R) \otimes_R R_f = A_f \otimes_{R_f,G} R_f$ so that in fact the bottom horizontal map is the m-th relative Frobenius for A_f over R_f. As both B and A are R-flat, with geometrically reduced fibres, all four maps in the diagram are injective.

Assume for a moment that the diagram is cartesian, that is, $B = A \cap B_f$. Let $x \in (A_f)^{p^\infty/R_f} = \bigcap_n \left(A_f \otimes_{R_f,F^n} R_f\right) \subset A_f$. Then, for some $k > 0$, $f^k x \in A$, and in particular it belongs to $A \cap \bigcap_n \left(A_f \otimes_{R_f,F^n} R_f\right)$, which by our assumption is $\bigcap_n (A \otimes_{R,F^n} R) = A^{p^\infty/R}$. Hence $x \in A^{p^\infty/R} \otimes_R R_f$, which proves the desired surjectivity.

It remains to show that the diagram is cartesian. We enlarge it to a bigger commutative diagram

$$\begin{array}{ccccc} B & \hookrightarrow & A & \hookrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ B_f & \hookrightarrow & A_f & \hookrightarrow & B_f \end{array}$$

where the maps $A \to B$ and $A_f \to B_f$ are given by base-changing the m-th absolute Frobenius $G\colon R \hookrightarrow R$ via $R \to A$ and $R \to A_f$, and are therefore also injective. Now, the compositions $B \hookrightarrow A \hookrightarrow B$ and $B_f \hookrightarrow A_f \hookrightarrow B_f$ are the absolute m-th Frobenii, which we call again G. It suffices to show that the biggest diagram is cartesian, that is, that $B = B_f \times_{G,B_f} B$. Write an element of B_f as b/f^k, with $b \in B$ and with k minimal, so that f does not divide b in B. Let $\beta \in B$ such that $\beta = G(b/f^k)$ in B_f. Then $\beta = b^{p^n}/f^{kp^n}$. Because f is not a zero-divisor in R, it is also not a zero-divisor in the flat R-module B. This means $\beta f^{kp^n} = b^{p^n}$. Assume now that $k > 0$. Reducing modulo f, we find $b^{p^n} = 0$ in B/fb. But B/fB is reduced, as it is flat with reduced fibres over the reduced ring R/fR. Hence $b = 0$ in B/fB, or in other words, f divides b, contradiction. This shows that $k = 0$ and that b/f^k belongs to B, which concludes the proof. □

3.4.2 Perfection over artinian rings

In this subsection we consider the case where R is an artinian ring. For such a ring, Theorem 3.3.1.8 implies that any flat, finitely generated algebra $R \to A$ has a largest étale subalgebra $A^{\text{ét}}$. Below we prove that the natural map $A^{\text{ét}} \to A^{p^\infty}$ to the preperfection is an isomorphism. In particular, the preperfection is perfect, hence a perfection. We point out that in this special situation the separability of $R \to A$ is not needed.

3.4.2.1 Theorem. *Let R be an artinian local ring of characteristic p, and let A be a flat R-algebra of finite type. Then the maps $A^{\text{ét}} \to \mathcal{O}(\pi_0(A)) \to A^{p^\infty}$ are isomorphisms.*

Proof : It follows from [Rom11], 2.1.3 that $\pi_0(A)$ is an étale quasi-compact R-algebraic space. Since R is artinian, this space is finite. In particular it is affine and the map $A^{\text{ét}} \to \mathcal{O}(\pi_0(A))$ is an isomorphism. It remains to prove that $A^{\text{ét}} \to A^{p^\infty}$ is an isomorphism. The proof of this is in five steps.

Step 1. We reduce to the case where $R = k$ is a field. Let $\mathfrak{m}$ resp. k be the maximal ideal resp. residue field. Let $\mathrm{F} : R \to R$ be the absolute Frobenius and e an integer such that $\mathfrak{m} = \ker \mathrm{F}^e$. Then F^e induces a ring map $\alpha : k \to R$ which we use to view R as a k-algebra. We compute the perfection of A using the cofinal system of indices $e\mathbb{N} \subset \mathbb{N}$. For each $i \geqslant 0$ the morphism $\mathrm{F}^{ei} : R \to R$ has a factorization:

$$R \twoheadrightarrow k \xrightarrow{\mathrm{F}^{e(i-1)}} k \xrightarrow{\alpha} R.$$

Writing $A_0 = A \otimes_R k$, it follows that $A^{p^{ei}/R} = A_0^{p^{e(i-1)}/k} \otimes_k R$. Passing to the limit and using 3.4.1.3 (1), we deduce an isomorphism:

$$\lambda : A_0^{p^\infty/k} \otimes_k R \xrightarrow{\sim} A^{p^\infty/R}.$$

On the other hand, the e-fold absolute Frobenius $\mathrm{F}_A^e : A_0^{\text{ét}/k} \to A^{\text{ét}/R}$ extends the map $\alpha : k \to R$, providing an isomorphism:

$$\mu : A_0^{\text{ét}/k} \otimes_{k,\alpha} R \xrightarrow{\sim} A^{\text{ét}/R}.$$

Since λ and μ fit together in a commutative square, the reduction step follows.

Step 2. We reduce to the case where k is algebraically closed. Let k' be an algebraic closure of k, and $A' := A \otimes_k k'$. We have injections

$$A^{\text{ét}/k} \otimes_k k' \hookrightarrow A^{p^\infty/k} \otimes_k k' \hookrightarrow (A')^{p^\infty/k'}$$

where the first is deduced from $A^{\text{ét}/k} \hookrightarrow A^{p^\infty/k}$ and the second comes from case (2.i) of Lemma 3.4.1.3. It is classical that $A^{\text{ét}/k} \otimes_k k' = (A')^{\text{ét}/k'}$, see [Wa79], Th. 6.5. It follows that if $(A')^{\text{ét}/k'} \to (A')^{p^\infty/k'}$ is an isomorphism, then $A^{\text{ét}/k} \otimes_k k' \hookrightarrow A^{p^\infty/k} \otimes_k k'$ is an isomorphism and hence $A^{\text{ét}/k} \to A^{p^\infty/k}$ is an isomorphism.

Step 3. We reduce to the case where A is reduced. Let A_{red} be the reduced quotient. On the separable closure side, since $A^{\text{ét}/k}$ does not meet the nilradical $\mathrm{Nil}(A)$ and all separable elements of A_{red} lift to A, we have an isomorphism $A^{\text{ét}/k} \xrightarrow{\sim} (A_{\text{red}})^{\text{ét}/k}$. On the preperfection side, we use the isomorphisms $A^{p^i/k} \xrightarrow{\sim} A$, $a \otimes \lambda \mapsto a\lambda^{p^{-i}}$ to obtain an isomorphism of rings $A^{p^\infty/k} \xrightarrow{\sim} A^{p^\infty/\mathbb{F}_p}$, and similarly for A_{red}. Since $\mathrm{Nil}(A)$ is finitely generated, there is $e \geqslant 0$ such that $\mathrm{Nil}(A) = \ker \mathrm{F}^e$ where $\mathrm{F} : A \to A$ is the absolute Frobenius. Then the computation of the perfection can be carried out along the cofinal system of indices $e\mathbb{N} \subset \mathbb{N}$, showing that the projection $A^{p^\infty/\mathbb{F}_p} \to (A_{\text{red}})^{p^\infty/\mathbb{F}_p}$ is an isomorphism. Contemplating the commutative diagram below, we see that if $(A_{\text{red}})^{\text{ét}/k} \to (A_{\text{red}})^{p^\infty/k}$ is an isomorphism then $A^{\text{ét}/k} \to A^{p^\infty/k}$ also.

$$\begin{tikzcd}
A^{\text{ét}/k} \arrow[r] \arrow[d, "\simeq"] & A^{p^\infty/k} \arrow[r, "\simeq"] & A^{p^\infty/\mathbb{F}_p} \arrow[d, "\simeq"] \\
(A_{\text{red}})^{\text{ét}/k} \arrow[r] & (A_{\text{red}})^{p^\infty/k} \arrow[r, "\simeq"] & (A_{\text{red}})^{p^\infty/\mathbb{F}_p}
\end{tikzcd}$$

Step 4. We reduce to the case where A has connected spectrum. This is straightforward, because if $A = A_1 \times \cdots \times A_d$ is the decomposition of A as a product of rings with connected spectrum, we have $(\prod A_i)^{\text{ét}/k} \simeq \prod A_i^{\text{ét}/k}$ and $(\prod A_i)^{p^\infty/k} \simeq \prod A_i^{p^\infty/k}$.

Step 5. We conclude that $A^{\text{ét}/k} \to A^{p^\infty/k}$ is surjective. Let x be an element of the ring

$$A^{p^\infty/k} \simeq A^{p^\infty/\mathbb{F}_p} = \cap_{n \geqslant 0} A^{p^n},$$

with $x = x_n^{p^n}$ and $x_n \in A$, for each n. By noetherianity, the increasing sequence of ideals (x_i) stabilizes at some N. It follows that yx_N satisfies $(y) = (y^p)$, in particular $(y) = (y^2)$.

Since $X = \mathrm{Spec}(A)$ is connected, we deduce that $y = 0$ or y is a unit; therefore $x = 0$ or x is a unit. Let A_i be the quotients of A by the minimal primes. Again by connectedness, the injection $A \hookrightarrow A_1 \times \cdots \times A_n$ induces a morphism of groups of units modulo constants $A^\times/k^\times \hookrightarrow (A_1^\times/k^\times) \cdots \times (A_n^\times/k^\times)$ which is *injective.* It is a classical result of Rosenlicht ([Ros57], lemma to Prop. 3) that each $A_i^\times/k^\times$ is a finitely generated free abelian group; hence the same is true for $A^\times/k^\times$. In particular the class of x in this group cannot be infinitely p-divisible, so $x \in k^\times$ and this proves the claim. □

3.4.3 Preperfection over noetherian rings

The aim of this section is to generalize the statement that $\mathcal{O}(\pi_0(A)) \to A^{p^\infty}$ is an isomorphism to the case of a general noetherian base ring R, in the case of *separable* algebras. The proof proceeds by thickening from an artinian base to a complete local base, then a Zariski-local base and then to a general base by induction on the dimension.

3.4.3.1 Lemma. *Let R be a complete noetherian local ring and A a flat separable R-algebra of finite type. Write $\widehat{A}$ for the completion of A with respect to the maximal ideal of R, and write $\widehat{\pi}$ for the finite étale R-scheme built from $\pi_0(A/R)$ as in 3.3.5.1. Then the natural map $\mathcal{O}(\widehat{\pi}) \to (\widehat{A})^{p^\infty/R}$ is an isomorphism.*

Proof : Let $\mathfrak{m}$ be the maximal ideal of R. Write $B = \mathcal{O}(\widehat{\pi})$. For every $n \geqslant 0$, let $R_n = R/\mathfrak{m}^{n+1}$, $A_n = A \otimes_R R_n$, $B_n = B \otimes_R R_n$. As $B_n = \mathcal{O}(\pi_0(A_n/R_n))$, for every n we have an inclusion $B_n \hookrightarrow A_n$. Taking the limit over n, and noticing that B is finite over R hence complete, we obtain an inclusion $B \hookrightarrow \widehat{A}$. As B is also étale over R, it is in fact contained in $(\widehat{A}/R)^{p^\infty}$.

On the other hand, a section to the inclusion $B \hookrightarrow \widehat{A}^{p^\infty}$ is given by the map

$$\widehat{A}^{p^\infty} = \lim_i \widehat{A}^{p^i} = \lim_i (\lim_n A_n)^{p^i} \to \lim_i \lim_n (A_n^{p^i}) = \lim_n \lim_i (A_n^{p^i}) = \lim_n A_n^{p^\infty} = \lim_n B_n = B.$$

Here, the second-to-last equality comes from Theorem 3.4.2.1. To complete the proof it suffices to show that $\widehat{A}^{p^\infty} \to B$ is injective, or that $(\lim_n A_n)^{p^i} \to \lim_n (A_n^{p^i})$ is injective. The latter is the completion morphism:

$$(\widehat{A})^{p^i} \to \widehat{(\widehat{A})^{p^i}}.$$

As R and $\widehat{A}$ are both noetherian, the hypotheses of Theorem 3.4.1.1 are satisfied, and we deduce that $(\widehat{A})^{p^i}$ is a subalgebra of $\widehat{A}$. As the latter is $\mathfrak{m}$-adically separated (that is, $\bigcap_{i=1}^n \mathfrak{m}^i \widehat{A} = 0$), so is its subalgebra $(\widehat{A})^{p^i}$. Hence the completion morphism above is injective and we conclude. □

3.4.3.2 Theorem. *Let R be a noetherian ring and A a flat, separable R-algebra of finite type. Then the natural map*

$$\phi\colon \mathcal{O}(\pi_0(A/R)) \longrightarrow A^{p^\infty/R}$$

is an isomorphism.

Proof : As a first step, we claim that we may reduce to the case of R complete local. Indeed, let $R \to R'$ be the completion of the local ring at some prime $\mathfrak{p} \subset R$. The morphism $R \to R'$ is flat. We have a map

$$\mathcal{O}(\pi_0(A \otimes_R R'/R')) = \mathcal{O}(\pi_0(A/R)) \otimes_R R' \to A^{p^\infty} \otimes_R R' \hookrightarrow (A \otimes_R R'/R')^{p^\infty}.$$

The first equality is compatibility of global sections and flat base change, the second arrow is $\phi \otimes_R R'$, while the last arrow is injective by 3.4.1.3. We see that if the composition is an isomorphism, then also the central arrow $\phi \otimes_R R'$ is an isomorphism. As $R_{\mathfrak{p}} \to R'$ is faithfully flat, the map $\phi \otimes_R R_{\mathfrak{p}}$ is also an isomorphism. Repeating the argument for all $\mathfrak{p} \subset R$, we find that ϕ is an isomorphism. This proves the claim.

We argue by induction on the dimension of R. If R is of dimension zero, it is a product of finitely many artinian local rings; we reduce to R local and the result follows by Theorem 3.4.2.1.

Now let d be the dimension of R, and assume the result true for base rings of dimension at most $d-1$. We may assume R local and complete with respect to its maximal ideal. Let s be the closed point of $\operatorname{Spec} R$, and $V = S \setminus \{s\}$. Notice that V is of dimension $d-1$. Cover V with open affines $U_i = \operatorname{Spec} R_i$. Consider the commutative diagram of solid arrows

$$\begin{tikzcd}
 & A \arrow[r, dashed] \arrow[d, dashed] \arrow[dl, dashed, bend right] & \widehat{A} \arrow[d] \\
 & \prod_i A \otimes_R R_i \arrow[r] \arrow[d] & \prod_i \widehat{A} \otimes_R R_i \arrow[d] \\
0 \arrow[r] & \prod_{i,j} A \otimes_R R_i \otimes_R R_j \arrow[r] & \prod_{i,j} \widehat{A} \otimes_R R_i \otimes_R R_j
\end{tikzcd}$$

Clearly, A admits natural compatible maps towards the diagram, represented by dashed arrows in the diagram.

Next, we take the preperfection of the diagram. By Lemma 3.4.3.1 we have $\widehat{A}^{p^\infty} = \mathcal{O}(\widehat{\pi})$. Moreover, for every R-algebra R', there is a natural map $\mathcal{O}(\widehat{\pi} \otimes_R R') = \widehat{A}^{p^\infty} \otimes_R R' \to (\widehat{A} \otimes_R R')^{p^\infty}$. Finally, by the induction hypothesis $(A \otimes R_i)^{p^\infty} = \mathcal{O}(\pi(X_{U_i}/U_i))$. We get a commutative diagram

$$\begin{tikzcd}
 & A^{p^\infty} \arrow[r, dashed] \arrow[d, dashed] \arrow[dl, dashed, bend right] & \mathcal{O}(\widehat{\pi}) \arrow[d] \\
 & \prod_i \mathcal{O}(\pi_0(X_{U_i}/U_i)) \arrow[r] \arrow[d] & \prod_i \mathcal{O}(\widehat{\pi}_{U_i}) \arrow[d] \\
0 \arrow[r] & \prod_{i,j} \mathcal{O}(\pi_0(X_{U_{ij}}/U_{ij})) \arrow[r] & \prod_{i,j} \mathcal{O}(\widehat{\pi}_{U_{ij}})
\end{tikzcd}$$

where the horizontal arrows are those induced by the natural morphism $\psi \colon \widehat{\pi} \to \pi_0(X/S)$ of section 3.3.5. The limit of the diagram of solid arrows coincides with the limit of the subdiagram of solid arrows in the commutative diagram

$$\begin{tikzcd}
A^{p^\infty} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{O}(\widehat{\pi}) \arrow[d] \\
\mathcal{O}(\pi_0(X_V/V)) \arrow[r] & \mathcal{O}(\widehat{\pi}_V)
\end{tikzcd} \tag{3.2}$$

Taking global sections in the pushout diagram of lemma 3.3.5.2, we see that $\mathcal{O}(\pi_0(X/S))$ is a fibre product for the subdiagram (3.2) of solid arrows. Therefore we get a natural map $\chi \colon A^{p^\infty} \to \mathcal{O}(\pi_0(X/S))$. The maps

$$A^{p^\infty} \xrightarrow{\chi} \mathcal{O}(\pi_0(X/S)) \xrightarrow{\phi} A^{p^\infty}$$

are compatible with the natural inclusions of A^{p^∞} and $\mathcal{O}(\pi_0(X/S))$ into A. Hence ϕ is injective, and because $\phi \circ \chi$ is the identity, it is also surjective, as we wished to show. $\square$

With the notation of 3.4.3.2, the algebraic space $\pi_0(X/S)$ is étale; however, its R-algebra of global sections $\mathcal{O}(\pi_0(X/S))$ may fail to be unramified (and therefore étale and perfect); see for instance example 3.4.5.2. In particular, the preperfection $A^{p^\infty/R}$ needs not to be perfect.

3.4.4 Perfection over regular or unibranch one-dimensional rings

Recall from Remark 3.4.1.4 that if R is regular and F-finite, then for all $R \to A$ the preperfection A^{p^∞} is perfect. For the separable R-algebras that we have been studying in this section, Theorem 3.4.3.2 provides an explicit description of A^{p^∞} which allows to find more cases when preperfection is perfect.

3.4.4.1 Corollary. *Let R be a noetherian $\mathbb{F}_p$-algebra and A a flat and separable R-algebra of finite type.*

(1) *If R is either*

- *geometrically $\mathbb{Q}$-factorial (e.g. regular), or*
- *integral, geometrically unibranch and one-dimensional,*

then we have isomorphisms:

$$A^{\text{ét}} \xrightarrow{\sim} \mathcal{O}(\pi_0(A)) \xrightarrow{\sim} A^{p^\infty}.$$

In particular A^{p^∞} is étale, hence perfect and of finite type.

(2) *If R is reduced, excellent, of dimension $\leqslant 1$, then A^{p^∞} is quasi-finite, and in particular of finite type.*

Proof : (1) We proved that the map $A^{\text{ét}} \to \mathcal{O}(\pi_0(A))$ is an isomorphism in Proposition 3.3.2.2, and that the map $\mathcal{O}(\pi_0(A)) \to A^{p^\infty}$ is an isomorphism in Theorem 3.4.3.2.

(2) This follows immediately from Proposition 3.3.2.3. $\square$

3.4.5 Examples

It should be obvious to the reader that the coperfection of the spectrum of an algebra is not the spectrum of its perfection. In fact, in the flat and separable case the coperfection of an affine scheme is π_0 and may be non-separated. Here is an example.

3.4.5.1 Lemma. *Let $R = \mathbb{F}_p[[u]]$ and consider the R-algebra*

$$A = \frac{R[x, y, (x-y)^{-1}]}{(xy-u)}.$$

Then $A^{p^\infty} = R$ while $\pi_0(A/R)$ is the non-separated scheme obtained by glueing two copies of $\operatorname{Spec}(R)$ *along the generic fibre.*

Proof : Let $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$. The fibre of $X \to S$ over the closed point has two connected components, while the generic fibre is connected. The two sections $s_1, s_2 \colon S \to X$,

$s_1 = \{x = u, y = 1\}$ and $s_2 = \{x = 1, y = u\}$ meet all components of all fibres; it follows that the composition

$$S \sqcup S \xrightarrow{s_1, s_2} X \to \pi_0(X/S)$$

is given by glueing the two copies of S along the generic fibre. Therefore $\pi_0(X/S)$ is non-separated. From 3.4.3.2 it follows that $A^{p^\infty} = \mathcal{O}(\pi_0(X/S))$, which is equal to R. $\square$

The following is the most basic example of a non-perfect preperfection, that is, an R-algebra A which is flat, separable, of finite presentation, for which the preperfection $A^{p^\infty/R}$ is not perfect. The ring R is one-dimensional; we remark that, in accordance with Proposition 3.3.2.2, we need to choose R with multiple branches. Since the preperfection is not perfect, it is natural to ask what happens if we take the preperfection once more. Here is the answer.

3.4.5.2 Lemma. *Let $R = \mathbb{F}_p[[u, v]]/(uv)$ and $A = R[x, y, (x - y)^{-1}]/(xy - u)$. If $p \neq 2$, we have:*

(1) $A^{p^\infty} \simeq \frac{R[\alpha]}{(u\alpha, v^2 - \alpha^2)}$ *mapping to A by $\alpha \mapsto v\frac{x+y}{x-y}$,*

(2) $(A^{p^\infty})^{p^\infty} \simeq R$.

Notice that the restriction of $R \to A^{p^\infty}$ to the branch $\{u = 0\}$ is $\mathbb{F}_p[[v]] \to \mathbb{F}_p[[v]][\alpha]/(v^2 - \alpha^2)$ which is not formally étale. Therefore ϕ itself is not formally étale and in particular not relatively perfect. The restriction $p \neq 2$ allows a simpler presentation of A^{p^∞} but is inessential.

Proof : Once for all we set $k = \mathbb{F}_p$.

(1) Let $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$. The open complement $V = S \setminus \{s\}$ of the closed point of S is affine, with $\mathcal{O}(V) = R_u \times R_v$. It is easy to see that $\mathcal{O}(\pi_0(X_V/V)) = R_u \times R_v \times R_v$. The inclusion $\mathcal{O}(\pi_0(X_V/V) \hookrightarrow \mathcal{O}(X_V) = A_u \times A_v$ maps the elements $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ to $(1, 0)$, $(0, \frac{x}{x-y})$, $(0, \frac{y}{y-x})$ respectively.

Applying the global sections functor to the pushout diagram of Lemma 3.3.5.2, and noticing that $\widehat{\pi} = \operatorname{Spec}(R \times R)$, we obtain a cartesian diagram

$$\begin{tikzcd} \mathcal{O}(\pi_0(X/S)) \arrow[r] \arrow[d] & R \times R \arrow[d] \\ R_u \times R_v \times R_v \arrow[r] & (R_u \times R_v) \times (R_u \times R_v) \end{tikzcd} \tag{3.3}$$

The lower horizontal map sends $(f(u), g(v), h(v))$ to $(f(u), g(v), f(u), h(v))$. The right vertical map sends $(\alpha(u, v), \beta(u, v))$ to $(\alpha(u, 0), \beta(u, 0), \alpha(0, v), \beta(0, v))$. It follows that the fibre product $\mathcal{O}(\pi_0(X/S))$ is the subring of $R_u \times R_v \times R_v$ generated as an R-submodule by $(1, 1, 1)$, $(u, 0, 0)$, $(0, v, 0)$, $(0, 0, v)$.

Since $p \neq 2$, we may choose instead $(1, 1, 1)$, $(u, 0, 0)$, $(0, v, v)$ and $(0, v, -v)$ as generators. We find that

$$A^{p^\infty/R} \cong \frac{k[[u, v]][\alpha]}{uv, u\alpha, \alpha^2 - v^2} = \frac{R[\alpha]}{u\alpha, \alpha^2 - v^2}$$

via the map $(u, 0, 0) \mapsto u$, $(0, v, v) \mapsto v$, $(0, v, -v) \mapsto \alpha$.

Finally, notice that the element $(0, v, -v) \in R_u \times R_v \times R_v$ is mapped to $v\frac{x+y}{x-y}$ in $A_u \times A_v$. This proves the claim.

(2) Let $B = A^{p^\infty}$. Notice first that any element of B can be written uniquely as $f + g\alpha$, with $f \in R$ and $g \in R/u$. Therefore, any element of $B^{(p^n)} = B \otimes_{R,F^n} R$ takes either the form $1 \otimes f$ with $f \in R$ or $\alpha \otimes g$ with $g \in R/u^{p^n}$. In fact, the map of R-modules

$$\begin{array}{ccc} B^{(p^n)} & \longrightarrow & R \oplus R/u^{p^n} \\ 1 \otimes f & \longmapsto & (f, 0) \\ \alpha \otimes g & \longmapsto & (0, g) \end{array}$$

is an isomorphism, which we will use to rewrite the preperfection diagram of B. The n-th map in the diagram is $B^{(p^n)} \to B^{(p^{n-1})}$ sending $1 \otimes f$ to $1 \otimes f$ and $\alpha \otimes g$ to $\alpha^p \otimes g = v^{p-1}\alpha \otimes g = \alpha \otimes v^{p^n - p^{n-1}} g$. Using the isomorphism of R-modules above, this becomes the map of R-modules

$$G_n \colon R \oplus R/u^{p^n} \to R \oplus R/u^{p^{n-1}}$$

sending (f, g) to $(f, gv^{p^n - p^{n-1}})$. Consider now the preperfection diagram

$$\ldots \xrightarrow{G_{n+1}} R \oplus R/u^{p^n} \xrightarrow{G_n} R \oplus R/u^{p^{n-1}} \xrightarrow{G_{n-1}} \ldots \xrightarrow{G_1} R \oplus R/u.$$

Let $H_n = G_1 \circ \ldots \circ G_n \colon R \oplus R/u^{p^n} \to R \oplus R/u$ and let $(\ldots, a_n, a_{n-1}, \ldots, a_0)$ be an element of the limit of the diagram. We can of course consider the limit in the category of R-modules, as it will automatically have an R-algebra structure making it into the limit in the category of R-algebras. Now, the image of $(f, g) \in R \oplus R/u^{p^n}$ via H_n is $(f, gv^{p^n - 1})$. Hence $a_0 = (f_0, g_0)$ is such that for every $n \geqslant 1$, g_0 is in the ideal of R/u generated by $v^{p^n - 1}$. Therefore $g_0 = 0$. One can use the same argument to show that for every $a_n = (f_n, g_n)$, g_n vanishes. Therefore the limit is simply the limit of the diagram:

$$\ldots \xrightarrow{\mathrm{id}} R \xrightarrow{\mathrm{id}} R \xrightarrow{\mathrm{id}} \ldots \xrightarrow{\mathrm{id}} R.$$

This shows that $B^{p^\infty} = R$. $\square$

3.5 Coperfection of algebraic spaces and stacks

In this section, our goal is to describe the coperfection of a flat, finitely presented, separable algebraic stack in the 2-category of Deligne–Mumford stacks. In order to achieve this goal, in 3.5.1 we first investigate the coperfection of a flat, finitely presented, separable algebraic space, based on our study of preperfection of algebras. We prove that the space π_0 of connected components provides a construction of the coperfection for a flat, finitely presented, separable algebraic space in the category of algebraic spaces.

Next we proceed to the case of algebraic stacks. Let $\mathscr{X}/S$ be a flat, finitely presented, separable algebraic stack. In [Rom11], it is proved that $\pi_0(\mathscr{X}/S)$ is representable by a quasi-compact étale algebraic space. In particular, if $\mathscr{X}$ is a tame algebraic stack with finite diagonal, admitting a relative coarse moduli space X/S (in the sense of Abramovich–Olsson–Vistoli, cf. [AOV11] Section 3), then the natural morphism

$$\pi_0(\mathscr{X}/S) \longrightarrow \pi_0(X/S)$$

is an isomorphism. However, $\pi_0(\mathscr{X}/S)$ does not provide a construction of the coperfection in the 2-category of Deligne–Mumford stacks. Instead, we give a construction in this case, built on the result of π_0, which is called the ***étale fundamental pro-groupoid*** $\Pi_1(\mathscr{X}/S)$ of $\mathscr{X}/S$. We define $\Pi_1(\mathscr{X}/S)$ as a pro-étale stack, using the formalism of 2-pro-objects, see [DD14]. The construction of Π_1 involves the notion of groupoid closure of a pregroupoid, we provide details in Appendix 3.5.4. In case S is a field, we also discuss the relation with the étale fundamental gerbe of Borne–Vistoli [BV15].

3.5.1 Coperfection as an algebraic space

Let R be a noetherian ring and A a flat, finite type separable R-algebra. Taking the preperfection of A, i.e., the limit of relative Frobenius morphisms

$$\dots \longrightarrow A^{p^2/R} \xrightarrow{\mathrm{F}_{A^p/R}} A^{p/R} \xrightarrow{\mathrm{F}_{A/R}} A$$

does not guarantee to produce a perfect object, as illustrated in 3.4.5.2. The next theorem shows the analogous construction carried out in the category of algebraic spaces yields an étale algebraic space. At this stage we give a statement for an algebraic space X/S but the later Theorem 3.5.3.5 will prove that the same holds for an algebraic stack $\mathscr{X}/S$.

3.5.1.1 Theorem. *Let S be a noetherian $\mathbb{F}_p$-algebraic space and $f\colon X \to S$ a flat, separable morphism of finite type. The inductive system of relative Frobenii*

$$X \xrightarrow{\mathrm{F}_{X/S}} X^{p/S} \xrightarrow{\mathrm{F}_{X^p/S}} X^{p^2/S} \longrightarrow \dots$$

admits a colimit in the category of algebraic spaces over S. The colimit is the algebraic space $\pi_0(X/S)$, and it is a coperfection of $X \to S$.

Proof : Since $\pi_0(X/S)$ is étale, it is relatively perfect; thus the map $X \to \pi_0(X/S)$ induces maps $\alpha_n\colon X^{p^n/S} \to \pi_0(X/S)$ for each $n \geqslant 0$. Now let $Y \to S$ be an algebraic space and $\beta_n\colon X^{p^n/S} \to Y$, $n \geqslant 0$ a system of compatible maps. We want to construct a morphism $f\colon \pi_0(X/S) \to Y$ such that $f\alpha_n = \beta_n$ for all n. By the faithful flatness of α_0, there will automatically be at most one such f.

We leave it to the reader to check that we may reduce to S affine. We claim that we may also reduce to Y affine. Let $g\colon V \to Y$ be an étale surjection with V an affine scheme. Consider the new diagram

$$\begin{tikzcd}
X \times_{\alpha_0,Y} V \arrow[r] \arrow[d] & X^{p/S} \times_{\alpha_1,Y} V \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & X^{p^n/S} \times_{\alpha_n,V} V \arrow[r] \arrow[d] & \dots \\
X \arrow[r] & X^{p/S} \arrow[r] & \dots \arrow[r] & X^{p^n/S} \arrow[r] & \dots
\end{tikzcd}$$

The upper row has compatible maps $\alpha'_n\colon X^{p^n} \times_{\alpha_n,Y} V \to V$ obtained by base changing α_n.

Now, for any étale morphism $U \to X$, consider the commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & U^{p/S} \arrow[r] \arrow[d] & U \arrow[d] \\
X \arrow[r] & X^{p/S} \arrow[r] & X
\end{tikzcd}$$

where the horizontal maps in the left square are relative Frobenius, and the horizontal maps in the right square are pullback by absolute Frobenius on S. The compositions $U \to U$ and $X \to X$ are absolute Frobenius. The right square is cartesian, and so is the big rectangle, by étaleness of $U \to X$. As the map $X^{p/S} \to X$ induces an equivalence of étale sites, the left square is also cartesian.

By applying this result with $U = X \times_{\alpha_0,Y} V$, we see that the diagram

$$\begin{tikzcd}
X \times_{\alpha_0,Y} V \arrow[r] \arrow[d] & (X \times_{\alpha_0,Y} V)^{p/S} \arrow[d] \\
X \arrow[r] & X^{p/S}
\end{tikzcd}$$

is cartesian.

On the other hand, the diagram

$$\begin{tikzcd}
X \times_{\alpha_0,Y} V \arrow[r] \arrow[d] & X^{p^n/S} \times_{\alpha_n,Y} V \arrow[d] \\
X \arrow[r] & X^{p/S}
\end{tikzcd}$$

is also cartesian. The two rightmost vertical maps in the latter two diagrams are both étale, and pull back to the same map via the universal homeomorphism $X \to X^{p/S}$. By topological invariance of the étale site, the two maps are canonically isomorphic; that is, there exists a canonical isomorphism $(X \times_{\alpha_0,Y} V)^{p/S} \to X^{p/S} \times_{\alpha_0,Y} V$ compatible with the maps to $X^{p/S}$.

We have therefore obtained a Frobenius system

$$X \times_{\alpha_0,Y} V \longrightarrow (X \times_{\alpha_0,Y} V)^{p/S} \longrightarrow \dots$$

with compatible maps α'_n towards the affine scheme V. Because V is affine, we use our assumption, and find that these maps factor uniquely via $\pi_0(X \times_{\alpha_0,Y} V/S)$.

Using the pushout property of lemma 3.3.4.1 and the fact that $\pi_0(X^{p^n/S}/S) = \pi_0(X/S)$, we see that the diagram

$$\begin{tikzcd}
(X \times_{\alpha_0,Y} V)^{p^n/S} \arrow[r] \arrow[d] & X^{p/S} \arrow[d] \arrow[dr, bend left] & \\
\pi_0((X_V)^{p/S}/S) \arrow[r] \arrow[rr, bend right] & \pi_0(X/S) \arrow[r, dotted] & Y
\end{tikzcd}$$

admits a unique dashed arrow making everything commute. The maps thus obtained are compatible and the proof of the claim is finished.

We assume therefore that S and Y are affine. Let $U \to X$ be a nice atlas as in Definition 3.3.4.2; this exists by Proposition 3.3.4.3. We get a morphism of inductive systems:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & U^p \arrow[r] \arrow[d] & U^{p^2} \arrow[r] \arrow[d] & \dots \\
X \arrow[r] & X^p \arrow[r] & X^{p^2} \arrow[r] & \dots
\end{tikzcd}$$

As both Y and U are affine, the induced morphism from the system of relative Frobenii $U \to U^p \to U^{p^2} \dots$ to Y factors uniquely via $\operatorname{Spec} \mathcal{O}(U)^{p^\infty}$, the spectrum of the preperfection of U. By theorem 3.4.3.2 and the definition of nice atlas, $\operatorname{Spec} \mathcal{O}(U)^{p^\infty} = \operatorname{Spec} \mathcal{O}(\pi_0(U/S)) = \pi_0(U/S)$. Now, we have a commutative diagram of solid arrows

$$\begin{tikzcd}
U^{p/S} \arrow[r] \arrow[d] & X^{p/S} \arrow[d] \arrow[dr, bend left] & \\
\pi_0(U/S) \arrow[r] \arrow[rr, bend right] & \pi_0(X/S) \arrow[r, dashed] & Y
\end{tikzcd}$$

The square diagram is once again a pushout (Lemma 3.3.4.1) and we obtain a unique dashed arrow $\pi_0(X/S) \to Y$. This proves that the Frobenius inductive system admits $\pi_0(X/S)$ as a representable colimit. Starting from a relatively perfect $Y \to S$ and a morphism $X \to S$, the Frobenius maps $Y^{p^n/S} \to Y^{p^{n+1}/S}$ are isomorphisms and we deduce a system of compatible maps $X^{p^n/S} \to Y$, $n \geqslant 0$. As we have seen, there is a unique induced morphism $\pi_0(X/S) \to Y$ and this is the universal property of the coperfection. □

3.5.2 The étale fundamental pro-groupoid

From this subsection on, we work with algebraic stacks over a fixed base S. All 2-morphisms between morphisms of algebraic stacks are 2-isomorphisms. Let $\mathscr{X}/S$ be a flat, finitely presented, separable algebraic stack. Before we introduce $\Pi_1(\mathscr{X}/S)$, let us briefly explain why $\pi_0(\mathscr{X}/S)$ is not a suitable object for coperfection in the 2-category of algebraic stacks.

Let G be a finite group, view as a finite constant group scheme over S, then the classifying stack BG is an étale Deligne–Mumford stack over S. We know that $\pi_0(BG/S) \simeq S$. However, there is no factorization

$$\begin{tikzcd} & S \arrow[dr, dashed] & \\ BG \arrow[ur] \arrow[rr, "\text{identity}"] & & BG \end{tikzcd}$$

namely, there is no morphism $S \to BG$ which could fit to the dashed arrow. In this very situation, BG itself is more likely a candidate for the universal property of factorization of maps to étale algebraic stacks.

The étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ is defined as a limit of algebraic stacks, which is naturally a ***2-pro-object*** of the 2-category of algebraic stacks. Let us recall the definition of a 2-pro-object. For more details, we refer to the paper [DD14].

3.5.2.1 Definition. A nonempty 2-category $\mathcal{I}$ is ***2-cofiltered*** if it satifies the following conditions:

(1) Given two objects $i, j \in \mathcal{I}$, there is an object $k \in \mathcal{I}$ and arrows $k \to i$, $k \to j$;

(2) Given two arrows $f, g : j \to i$, there is an arrow $h : k \to j$ and a 2-isomorphism $\alpha : fh \to gh$;

(3) Given two 2-arrows $\alpha, \beta : f \to g$, where $f, g \in \mathscr{H}om_{\mathcal{I}}(j, i)$, there is an arrow $h : k \to j$ such that $\alpha h = \beta h$.

Clearly, a nonempty 1-category is cofiltered if and only if it is 2-cofiltered when seen as a 2-category.

3.5.2.2 Definition. A ***2-pro-object*** of a 2-category $\mathcal{C}$ is a 2-functor $F : \mathcal{I} \to \mathcal{C}$ from a small 2-cofiltered 2-category $\mathcal{I}$. The 2-category of 2-pro-objects of $\mathcal{C}$ is denoted by 2-Pro$(\mathcal{C})$. The category of morphisms between two 2-pro-objects $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ is

$$\mathscr{H}om_{2\text{-Pro}(\mathcal{C})}(F, G) := \lim_{j \in \mathcal{J}} \operatorname{colim}_{i \in \mathcal{I}} \mathscr{H}om_{\mathcal{C}}(F(i), G(j))$$

where lim (resp. colim) is the pseudolimit (resp. pseudocolimit) for strict 2-categories, cf. [DD14], Prop. 2.1.5. In particular, by a ***pro-algebraic stack*** we mean a 2-pro-object of the 2-category **AlgStack** of algebraic stacks.

The index 2-category for defining Π_1 will be a 2-category of factorizations similar to that of Definition 3.3.1.2, with the difference that the étale part $\mathscr{E} \to S$ is allowed to be an algebraic stack rather than an algebraic space. For simplicity, we use again the notation $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ although to be fully consistent, the category defined in 3.3.1.2 should be denoted $\mathsf{E}^{\mathrm{surj,rep}}(\mathscr{X}/S)$ to indicate that $\mathscr{E} \to S$ is representable by algebraic spaces. No confusion is likely to occur since the former definition is not used anymore in the present section of the article.

3.5.2.3 Definition. Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. We define $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ to be the following 2-category:

- objects are factorizations $\mathscr{X} \xrightarrow{h} \mathscr{E} \to S$ where $\mathscr{E}/S$ is an étale, finitely presented algebraic stack and h is surjective;
- 1-arrows $(\mathscr{X} \xrightarrow{h} \mathscr{E} \to S) \to (\mathscr{X} \xrightarrow{h'} \mathscr{E}' \to S)$ are pairs (f, α), with $f\colon \mathscr{E} \to \mathscr{E}'$ and $\alpha\colon fh \to h'$ giving a 2-commutative diagram:

$$\begin{tikzcd} & \mathscr{E} \arrow[dd, "f"] \arrow[dr] & \\ \mathscr{X} \arrow[ur, "h"] \arrow[dr, "h'"'] & & S; \\ & \mathscr{E}' \arrow[ur] & \end{tikzcd}$$

- 2-arrows $(f, \alpha) \to (g, \beta)$ are 2-morphisms $u\colon f \to g$ giving a commutative diagram:

$$\begin{tikzcd} fh \arrow[r, "uh"] \arrow[dr, "\alpha"'] & gh \arrow[d, "\beta"] \\ & h'. \end{tikzcd}$$

We emphasize that for a factorization $\mathscr{X} \to \mathscr{E} \to S$ in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, the requirement that $\mathscr{E} \to S$ be quasi-separated will be crucial in the sequel, cf. Remark 3.5.3.6 (1). On the contrary, the condition of quasi-compactness of $\mathscr{E} \to S$ is automatic from the same property for $\mathscr{X} \to S$.

3.5.2.4 Lemma. *Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. The 2-category $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is small and 2-cofiltered. Moreover, it is equivalent to a 1-category.*

Proof : Since $\mathscr{X}$ and $\mathscr{E}$ in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ are all finitely presented, it is standard to deduce that $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is a small 2-category. Moreover, $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is nonempty, because it contains the image of $\mathscr{X}$ in S, which is open in S hence étale over S. Next, we check the three conditions for 2-cofilteredness.

(1) Given two factorizations $h : \mathscr{X} \to \mathscr{E}$ and $h' : \mathscr{X} \to \mathscr{E}'$, there is the common refinement $\mathscr{X} \to \mathscr{E} \times_S \mathscr{E}'$ and 2-commutative diagram

$$\begin{tikzcd} \mathscr{X} \arrow[dr] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\ & \mathscr{E} \times \mathscr{E}' \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ & \mathscr{E}' \arrow[r] & S \end{tikzcd}$$

Take the image $\mathscr{E}''$ of $\mathscr{X} \to \mathscr{E} \times \mathscr{E}'$. Then $\mathscr{E}''$ is again an étale finitely presented S-stack and $h'' : \mathscr{X} \to \mathscr{E}''$ is a common refinement of h and h' in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$.

(2) Given two morphisms (f, α) and (g, β)

$$\begin{tikzcd}
 & \mathscr{X} \arrow[ld, dashed, bend right, "h''"'] \arrow[d, "h"] \arrow[rd, bend left, "h'"] & \\
\mathscr{E}'' \arrow[r, dashed, "{(k,\gamma)}"] & \mathscr{E} \arrow[r, shift left, "{(f,\alpha)}"] \arrow[r, shift right, "{(g,\beta)}"'] & \mathscr{E}'
\end{tikzcd}$$

we want to find a third morphism $(k, \gamma) : \mathscr{E}'' \to \mathscr{E}$ and a 2-isomorphism $u : fk \to gk$. For this we consider the 2-fibred product:

$$\begin{tikzcd}
\mathscr{E}'' \arrow[r] \arrow[d, "k"] & \mathscr{E}' \arrow[d, "\Delta"] \\
\mathscr{E} \arrow[r, "{(f,g)}"] & \mathscr{E}' \times_S \mathscr{E}'.
\end{tikzcd}$$

Then u is given by definition. Moreover, the morphisms $h : \mathscr{X} \to \mathscr{E}$ and $h' : \mathscr{X} \to \mathscr{E}'$ and the 2-commutativity isomorphisms

$$fh \xrightarrow{\alpha} h' \xrightarrow{\beta^{-1}} gh$$

provide a morphism $(h, h') : \mathscr{X} \to \mathscr{E}''$.

(3) Given two morphisms (f, α), (g, β) and two 2-morphisms $u, v : (f, \alpha) \to (g, \beta)$:

$$\begin{tikzcd}
 & \mathscr{X} \arrow[ld, dashed, bend right, "h''"'] \arrow[d, "h"] \arrow[rd, bend left, "h'"] & \\
\mathscr{E}'' \arrow[r, dashed, "{(k,\gamma)}"] & \mathscr{E} \arrow[r, bend left, "{(f,\alpha)}"] \arrow[r, bend right, "{(g,\beta)}"'] \arrow[r, phantom, "\Downarrow u \ \Downarrow v"] & \mathscr{E}'
\end{tikzcd}$$

we want to find a third morphism $(k, \gamma) : \mathscr{E}'' \to \mathscr{E}$ such that $uk = vk$. For this we view f and g as $\mathscr{E}$-valued points of the stack $\mathscr{E}'$ and u, v as sections of the Isom functor $I_{\mathscr{E}}(f, g) \to \mathscr{E}$, that is $u, v : \mathscr{E} \to {}_{\mathscr{E}}(f, g)$. Since the diagonal of $\mathscr{E}'$ is an étale morphism, the map $I \to \mathscr{E}$ is representable and étale, so its diagonal is an open immersion. We consider the fibred product:

$$\begin{tikzcd}
\mathscr{E}'' \arrow[r] \arrow[d, hook] & I \arrow[d, hook, "\Delta"] \\
\mathscr{E} \arrow[r, "{(u,v)}"] & I \times_{\mathscr{E}} I.
\end{tikzcd}$$

The 2-commutativity isomorphisms

$$fh \xrightarrow{\alpha} h' \xrightarrow{\beta^{-1}} gh$$

provide a morphism $\mathscr{X} \to I$. Moreover, the conditions $\beta \circ uh = \beta \circ vh = \alpha$ ensure that $(uh, vh) = (\beta^{-1}\alpha, \beta^{-1}\alpha)$, that is, we have a commutative square:

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\beta^{-1}\alpha"] \arrow[d, "h"'] & I \arrow[d, hook, "\Delta"] \\
\mathscr{E} \arrow[r, "{(u,v)}"] & I \times_{\mathscr{E}} I.
\end{tikzcd}$$

We deduce a morphism $h'' : \mathscr{X} \to \mathscr{E}''$. Moreover, since we have the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "h''"] \arrow[dr, two heads, "h"'] & \mathscr{E}'' \arrow[d, hook, "k"] \\ & \mathscr{E} \end{tikzcd}$$

where the map h is surjective, the vertical inclusion is in fact an isomorphism. Hence the two 2-morphisms u, v are equalized by an isomorphism $k : \mathscr{E}'' \to \mathscr{E}$. In particular, it means that for any such two morphisms (f, α) and (g, β), there is at most one 2-isomorphism between them, thus $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is equivalent to a 1-category. □

3.5.2.5 Definition. Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. We define the ***étale fundamental pro-groupoid*** $\Pi_1(\mathscr{X}/S)$ of $\mathscr{X}$ to be the pro-algebraic stack

$$\begin{array}{rcl} \Pi_1(\mathscr{X}/S) : \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S) & \longrightarrow & \mathbf{AlgStack}_S \\ \{\mathscr{X} \to \mathscr{E}\} & \longmapsto & \mathscr{E} \end{array}$$

The pro-algebraic stack $\Pi_1(\mathscr{X}/S)$ is pro-étale by definition, and it comes with a canonical morphism $\mathscr{X} \to \Pi_1(\mathscr{X}/S)$ which is unique up to a unique 2-isomorphism. This object defines a 2-functor

$$\Pi_1 : \mathbf{AlgStack}_S \longrightarrow 2\text{-Pro}(\mathbf{EtStack}_S)$$

from the 2-category of flat, finitely presented algebraic stacks over S to the 2-category of pro-étale stacks over S. It is tautological from its definition that the 2-functor $\Pi_1(-/S)$ is pro-left adjoint to the inclusion $\mathbf{EtStack}_S \hookrightarrow \mathbf{AlgStack}_S$.

3.5.2.6 Π_1 **via smooth atlases**. Now let us assume that $\mathscr{X}$ is separable. Let $U \to \mathscr{X}$ be a smooth atlas with U finitely presented, and $R = U \times_{\mathscr{X}} U$. Note that, because of quasi-compactness and quasi-separation of $\mathscr{X}$, we can always choose $U \to \mathscr{X}$ to be quasi-compact and quasi-separated. Indeed, we can find a quasi-compact algebraic space U_0 as a smooth atlas of $\mathscr{X}$, then by [SP19] Tag 050Y, $U_0 \to \mathscr{X}$ is quasi-compact. Taking an affine Zariski covering $U \to U_0$ provides an atlas which is quasi-compact and quasi-separated over $\mathscr{X}$. Now since $\mathscr{X}$ is finitely presented, a quasi-compact quasi-separated $U \to \mathscr{X}$ is also finitely presented, hence we can take π_0 of U and R. In the sequel, for simplicity let us write $\pi_0(U)$ for $\pi_0(U/S)$. Therefore the groupoid presentation $R \rightrightarrows U$ of $\mathscr{X}$ induces a 2-commutative diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \pi_0(R) \arrow[r, shift left] \arrow[r, shift right] & \pi_0(U) \arrow[r] & {[\pi_0(U)/\pi_0(R)]} \end{tikzcd}$$

where $[\pi_0(U)/\pi_0(R)]$ is the quotient stack of the groupoid closure of the pregroupoid $(\pi_0(R) \rightrightarrows \pi_0(U))$. For details on pregroupoids and groupoid closures, see Section 3.5.4. The construction of groupoid closures works well for pregroupoids in objects of the category of étale S-algebraic spaces, cf. Remark 3.5.4.10. In particular, the groupoid closure $(\pi_0(R)^{\mathrm{gpd}} \rightrightarrows \pi_0(U))$ is an étale groupoid, and the quotient $[\pi_0(U)/\pi_0(R)]$ is an étale stack over S, see Corollary 3.5.4.11. Since moreover we have a surjection $R \to \pi_0(R)$, the quasi-compactness of R is inherited by $\pi_0(R)$ and this implies that $[\pi_0(U)/\pi_0(R)]$ is finitely presented. Hence the factorization $\mathscr{X} \to [\pi_0(U)/\pi_0(R)]$ is an object of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$.

3.5.2.7 Definition. Let $\mathscr{X}/S$ be a flat, finitely presented, separable algebraic stack. We define $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ to be the full subcategory of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, which consists of objects of the form

$$\mathscr{X} \to [\pi_0(U/S)/\pi_0(R/S)],$$

where $U \to \mathscr{X}$ is a smooth atlas with U finitely presented and $RU \times_{\mathscr{X}} U$.

3.5.2.8 Lemma. *The inclusion functor* $i : \mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S) \hookrightarrow \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ *is initial. In particular, the full subcategory* $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ *is cofiltered.*

Proof : For the definition of initial functor, see [SP19], Tag 09WN. Since i is fully faithful, by the dual version of Prop. 8.1.3 (c) in [SGA4-1] Exposé I, we only need to verify that any object of $\mathsf{E}^{\mathrm{surj}}$ can be dominated by an object of $\mathsf{E}^{\mathrm{cov}}$, according to condition F 1) in *loc. cit.*

Let $\{\mathscr{X} \to \mathscr{E}\} \in \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$. Choose an étale finitely presented atlas $E \to \mathscr{E}$, and a smooth finitely presented atlas $U \to \mathscr{X} \times_{\mathscr{E}} E$. Let $R = U \times_{\mathscr{X}} U$ and $F = E \times_{\mathscr{E}} E$. Since E, F are étale S-spaces, the two morphisms $U \to E$ and $R \to F$ factor through their π_0. Taking groupoid closures and using functoriality of stack quotients ([SP19], Tag 04Y3), we obtain a 2-commutative diagram:

$$\begin{tikzcd}
R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
\pi_0(R) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \pi_0(U) \arrow[r] \arrow[d] & {[\pi_0(U)/\pi_0(R)]} \arrow[d] \\
F \arrow[r, shift left] \arrow[r, shift right] & E \arrow[r] & \mathscr{E}
\end{tikzcd}$$

The right column is a morphism in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, hence $\mathrm{Hom}_{\mathsf{E}^{\mathrm{surj}}}\big(i([\pi_0(U)/\pi_0(R)]), \mathscr{E}\big) \neq \emptyset$ and i is an initial functor. $\square$

Therefore the cofiltered category $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$, seen as a 2-cofiltered 2-category, defines the same object $\Pi_1(\mathscr{X}/S)$ inside the 2-category 2-Pro($\mathbf{EtStack}_S$):

$$\Pi_1(\mathscr{X}/S) \quad \lim_{\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)} \mathscr{E} \;=\; \lim_{\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)} [\pi_0(U)/\pi_0(R)].$$

Note that the stacks $[\pi_0(U)/\pi_0(R)]$ are étale gerbes over the algebraic space $\pi_0(U)/\pi_0(R) = \pi_0(\mathscr{X}/S)$. The expression as a limit over $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ is sometimes useful for computing Π_1.

3.5.2.9 Proposition. *Let G be a smooth group scheme over S. Then we have a canonical isomorphism* $\Pi_1(BG/S) \simeq B(\pi_0(G)/S)$. *In particular, the formation of* Π_1 *commutes with base change in the special case of classifying stacks.*

Proof : Let $U \to BG$ be a finitely presented smooth atlas, this determines a G-torsor $P \to U$. Consider the refinement $P \to U$ of atlases

$$\begin{tikzcd}
P \times_U P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & S \arrow[d] \\
P \arrow[r] & U \arrow[r] & BG
\end{tikzcd}$$

since $P \times_U P \simeq G \times_S P$, the left vertical arrow is a trivial G-torsor. Hence any smooth atlas of BG is refined by an atlas corresponding to a trivial torsor, we may therefore assume that

$U \to BG$ corresponds to a trivial G-torsor. Equivalently, it means that there is a factorization $U \to S \to BG$. From the following cartesian squares

$$\begin{tikzcd} U \times_S U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & U \arrow[d] \\ U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & G \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & S \arrow[d] \\ U \arrow[r] & S \arrow[r] & BG \end{tikzcd}$$

we have $U \times_{BG} U \simeq U \times_S U \times_S G$. Hence the groupoid presentation of BG

$$U \times U \times G \rightrightarrows U \longrightarrow BG$$

gives rise to the quotient stack

$$[\pi_0(U)/\pi_0(U \times U \times G)] \simeq [\pi_0(U)/\pi_0(U) \times \pi_0(U) \times \pi_0(G)] \simeq B(\pi_0(G)/S)$$

Since these atlases of trivial torsors are initial among all smooth atlases of BG, and the corresponding étale quotient stacks are initial in $\mathsf{E}^{\mathrm{cov}}(BG/S)$, we deduce the canonical isomorphism $\Pi_1(BG/S) \simeq B(\pi_0(G)/S)$. $\Box$

In the final part of this subsection, we explain the relation between $\Pi_1(\mathscr{X}/S)$ and the étale fundamental gerbe of Borne and Vistoli [BV15], when the base $S = k$ is a field. In *loc. cit.*, the authors introduced the notion of ***inflexible stack*** over a field k. This notion extends immediately to the case when the base is a finite product of fields, e.g. a finite reduced k-scheme. In particular, a separable geometrically connected stack of finite type over a reduced k-scheme is inflexible and has an étale fundamental gerbe ([BV15], Prop. 5.5, Th. 5.7).

3.5.2.10 Proposition. (1) *Let S be an artinian local scheme. Then in the 2-category of stacks, the pro-algebraic stack $\Pi_1(\mathscr{X}/S)$ is representable by a stack which is an fpqc affine gerbe over $\pi_0(\mathscr{X}/S)$.*
(2) *Let k be a field, and $\mathscr{X}$ a separable k-stack of finite type. Let $\Pi^{\text{ét}}_{\mathscr{X}/k} \to \pi_0(\mathscr{X}/k)$ denote the étale fundamental gerbe of $\mathscr{X} \to \pi_0(\mathscr{X}/k)$ as defined in* [BV15], § 8. *Then $\Pi^{\text{ét}}_{\mathscr{X}/k}$ is the fpqc affine gerbe that represents the pro-algebraic stack $\Pi_1(\mathscr{X}/k)$ in the 2-category of stacks.*

Proof : (1) For each smooth atlas $U \to \mathscr{X}$ of finite presentation, the étale stack $[\pi_0(U)/\pi_0(R)]$ has coarse moduli space $\pi_0(\mathscr{X}/S)$. If S is local artinian, then each quasi-finite S-space is in fact a finite S-scheme. In particular, $\pi_0(\mathscr{X}/S)$ is artinian and $[\pi_0(U)/\pi_0(R)]$ is an affine flat gerbe over it. It follows from [BV15], Prop. 3.7 that the stack which represents the projective system $\Pi_1(\mathscr{X}/S)$ is an fpqc affine gerbe over $\pi_0(\mathscr{X}/S)$.
(2) Let Π be the fpqc affine gerbe that represents the $\Pi_1(\mathscr{X}/k)$. From the fact that $\Pi_1(\mathscr{X}/k)$ has coarse moduli space $\pi_0(\mathscr{X}/k)$, the same follows for Π. Then we see that both $\mathscr{X} \to \Pi$ and $\mathscr{X} \to \Pi^{\text{ét}}_{\mathscr{X}/k}$ are universal among morphisms from $\mathscr{X}$ to an étale $\pi_0(\mathscr{X}/k)$. $\Box$

3.5.3 Coperfection as an algebraic stack

Now let us turn to characteristic p. In the case of algebraic spaces, we have proved in Theorem 3.5.1.1 that for a flat finitely presented separable algebraic space X/S, the colimit of relative Frobenii is representable by the étale algebraic space $\pi_0(X/S)$, and it is the coperfection in the category of algebraic spaces. In this section, we will show that Π_1 is the coperfection in the 2-category of (pro-)Deligne–Mumford stacks, and it represents the colimit of Frobenii.

3.5.3.1 Lemma. *Let $\mathscr{X}/S$ be a flat, finitely presented, separable algebraic stack. Then $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ is initial for morphisms from $\mathscr{X}$ to unramified S-algebraic spaces.*

Proof : Let $f : \mathscr{X} \to I$ be a morphism to an unramified S-algebraic space I. According to [Rom11], Th. 2.5.2 the algebraic space $\pi_0(\mathscr{X}/S)$ is the quotient of $\mathscr{X}$ by the open equivalence relation whose graph $\mathscr{R} \subset \mathscr{X} \times_S \mathscr{X}$ is the open connected component of the diagonal. Therefore, in order to obtain a factorization $\pi_0(\mathscr{X}/S) \to I$ it is enough to prove that $f\,\mathrm{pr}_1 = f\,\mathrm{pr}_2$ where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{R} \to \mathscr{X}$ are the projections. Let $\mathscr{Z} \to \mathscr{R}$ be the equalizer of $f\,\mathrm{pr}_1$ and $f\,\mathrm{pr}_2$. Since I is unramified, $\mathscr{Z}$ is an open substack of $\mathscr{R}$. Moreover, in each fibre above a point $s \in S$, we have $\mathscr{Z}_s = \mathscr{R}_s$ because I_s is étale over the residue field $k(s)$ and $\mathscr{X}_s \to \pi_0(\mathscr{X}_s/k(s))$ is initial for maps to étale $k(s)$-spaces (note that the formation of π_0 commutes with arbitrary base change). Therefore $\mathscr{Z} = \mathscr{R}$, so $f\,\mathrm{pr}_1 = f\,\mathrm{pr}_2$ and we are done. □

3.5.3.2 Lemma. *Let $\mathscr{X}/S$ be a flat, finitely presented, separable algebraic stack and $U \to \mathscr{X}$ a faithfully flat, finitely presented, separable atlas (e.g. a smooth surjective atlas of finite presentation). Let $R \rightrightarrows U$ be the corresponding groupoid presentation of $\mathscr{X}$. Consider the 2-commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{X} \arrow[d] \\ \pi_0(U) \arrow[r] & {[\pi_0(U)/\pi_0(R)]} \end{tikzcd}$$

and let $\mathscr{M} \to S$ a Deligne–Mumford stack. Then the natural functor

$$\mathscr{H}om([\pi_0(U)/\pi_0(R)], \mathscr{M}) \longrightarrow \mathscr{H}om(\mathscr{X}, \mathscr{M}) \underset{\mathscr{H}om(U,\mathscr{M})}{\times} \mathscr{H}om(\pi_0(U), \mathscr{M})$$

is an equivalence of categories.

Proof : Throughout the proof we write $\mathscr{Q} = [\pi_0(U)/\pi_0(R)]$ the quotient stack of the pregroupoid $\pi_0(R) \rightrightarrows \pi_0(U)$. First we explain precisely what is the functor F of the statement. The target of F is the category with objects the triples $(v : \mathscr{X} \to \mathscr{M}, f : \pi_0(U) \to \mathscr{M}, \delta : v\pi \xrightarrow{\sim} fh)$, or in other words the 2-commutative diagrams

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\delta"] & \mathscr{X} \arrow[d, "v"] \\ \pi_0(U) \arrow[r, "f"'] & \mathscr{M}. \end{tikzcd}$$

For $\mathscr{M} = \mathscr{Q}$, we have a canonical particular object of this category (see 3.5.2.6):

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\gamma"] & \mathscr{X} \arrow[d, "v_0"] \\ \pi_0(U) \arrow[r, "f_0"'] & \mathscr{Q}. \end{tikzcd}$$

Here is how the functor F is defined. For a morphism $g : \mathscr{Q} \to \mathscr{M}$, we have:

$$F(g) = \big(v = gv_0, f = gf_0, \delta = g\gamma : gv_0\pi \to gf_0h\big).$$

To construct a quasi-inverse for F, we will construct a functor G such that $GF = \mathrm{id}$, and an isomorphism $\epsilon : FG \xrightarrow{\sim} \mathrm{id}$. This means that given (v, f, δ), we seek to construct functorially a morphism $g\colon \mathscr{Q} \to \mathscr{M}$ and 2-isomorphisms $a\colon gf_0 \to f$, $b\colon gv_0 \to v$ filling in a 2-commutative diagram:

$$\begin{tikzcd}
U \arrow[r, "\pi"] \arrow[d, "h"'] & \mathscr{X} \arrow[d, "v_0"] \arrow[dl, Rightarrow, "\gamma"'] \arrow[ddr, bend left, "v"] & \\
\pi_0(U) \arrow[r, "f_0"'] \arrow[drr, bend right, "f"'] & \mathscr{Q} \arrow[dr, dashed, "g"] \arrow[r, phantom, "b"] \arrow[d, phantom, "a"] & \\
 & & \mathscr{M}. \arrow[ul, phantom, "\delta"]
\end{tikzcd}$$

We use the usual notations as in Section 3.5.4 for the groupoid $R \rightrightarrows U$, and we complete the picture by adding in the bottom row the pregroupoid $\pi_0(R) \rightrightarrows \pi_0(U)$.

$$\begin{tikzcd}
R \underset{s,U,t}{\times} R \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "c" description] \arrow[r, shift right=2, "\mathrm{pr}_2"'] \arrow[d, "l"] & R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "k"] & U \arrow[r, "\pi"] \arrow[d, "h"] & \mathscr{X} \arrow[d, "v_0"] \arrow[dr, bend left, "v"] & \\
\pi_0\left(R \underset{s,U,t}{\times} R\right) \arrow[r, shift left=2, "p_1"] \arrow[r, "d" description] \arrow[r, shift right=2, "p_2"'] & \pi_0(R) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \pi_0(U) \arrow[r, "f_0"] \arrow[rr, bend right, "f"'] & \mathscr{Q} & \mathscr{M}
\end{tikzcd}$$

First we construct the pair (g, a) using Corollary 3.5.4.11 on the coequalizer property of the stack quotient $\pi_0(U) \to [\pi_0(U)/\pi_0(R)]$ on objects. Consider $x = f\sigma$ and $y = f\tau$ viewed as $\pi_0(R)$-points of $\mathscr{M}$, and $I(x, y)$. Let $\alpha : \pi s \to \pi t$ and $\alpha_0 : f_0\sigma \xrightarrow{\sim} f_0\tau$ be the canonical 2-isomorphisms. The composition

$$f\sigma k = fhs \xrightarrow{\delta^{-1}s} v\pi s \xrightarrow{v\alpha} v\pi t \xrightarrow{\delta t} fht = f\tau k$$

is an isomorphism $\widetilde{\beta}\colon k^*x \xrightarrow{\sim} k^*y$, that is, a point $\widetilde{\beta}\colon R \to I$.

We claim that $\widetilde{\beta}$ factors uniquely via $\pi_0(R/S)$. Since $\mathscr{M} \to S$ is Deligne–Mumford, $I \to \pi_0(R)$ is unramified, hence so is $I \to S$. Lemma 3.5.3.1 implies that $\widetilde{\beta}$ factors uniquely as

$$R \xrightarrow{k} \pi_0(R) \xrightarrow{\beta} I.$$

We have obtained an isomorphism $\beta\colon x \xrightarrow{\sim} y$. Now we check that $\beta d = \beta p_1 \circ \beta p_2$ holds.

Consider the equality $\alpha c = \alpha \operatorname{pr}_1 \circ \alpha \operatorname{pr}_2$:

$$
\begin{tikzcd}[column sep=huge]
R \times_{s,U,t} R \arrow[r, bend left=50, "\pi s \operatorname{pr}_2 = \pi s c"{name=A}] \arrow[r, "\pi t \operatorname{pr}_2 = \pi s \operatorname{pr}_1"{name=B}] \arrow[r, bend right=50, "\pi t \operatorname{pr}_1 = \pi t c"'{name=C}] & \mathscr{X}
\arrow[from=A, to=B, phantom, "\Downarrow \alpha \operatorname{pr}_2"]
\arrow[from=B, to=C, phantom, "\Downarrow \alpha \operatorname{pr}_1"]
\arrow[from=A, to=C, phantom, "\Downarrow \alpha c", shift left=8]
\end{tikzcd}
$$

This gives $v\alpha c = (v\alpha \operatorname{pr}_1) \circ (v\alpha \operatorname{pr}_2)$ which, using the three relations $t \operatorname{pr}_1 = tc$, $s \operatorname{pr}_1 = t \operatorname{pr}_2$, $s \operatorname{pr}_2 = sc$, we can write:

$$(\delta tc) \circ (v\alpha c) \circ (\delta^{-1} sc) = (\delta t \operatorname{pr}_1) \circ (v\alpha \operatorname{pr}_1) \circ (\delta^{-1} s \operatorname{pr}_1) \circ (\delta t \operatorname{pr}_2) \circ (v\alpha \operatorname{pr}_2) \circ (\delta^{-1} s \operatorname{pr}_2).$$

Now, by definition $\widetilde{\beta} = \delta t \circ v\alpha \circ \delta^{-1} s$ so the above equality becomes $\widetilde{\beta} c = \widetilde{\beta} \operatorname{pr}_1 \circ \widetilde{\beta} \operatorname{pr}_2$ which in turn can be rewritten as $\beta dl = \beta p_1 l \circ \beta p_2 l$. Finally, because l is faithfully flat hence an epimorphism of spaces, we obtain:

$$\beta d = \beta p_1 \circ \beta p_2.$$

Then Corollary 3.5.4.11 applies and provides a pair (g, a) and a 2-commutative diagram:

$$
\begin{tikzcd}
\pi_0(R) \arrow[r, "\sigma"] \arrow[d, "\tau"'] & \pi_0(U) \arrow[d, "f_0"] \arrow[ddr, bend left, "f"] & \\
\pi_0(U) \arrow[r, "f_0"] \arrow[drr, bend right, "f"'] & \mathscr{Q} \arrow[dr, dashed, "g"] \arrow[dr, phantom, "\Downarrow \delta", shift left=3] & \\
& & \mathscr{M}.
\end{tikzcd}
$$

Now we construct $b : gv_0 \to v$ using Corollary 3.5.4.11 on the coequalizer property of $U \to [U/R]$ on morphisms. Define $c\delta^{-1} \circ (bh) \circ (g\gamma)$ and consider the solid diagram:

$$
\begin{tikzcd}
gv_0\pi s \arrow[r, "g\gamma s"] \arrow[d, "gv_0\alpha"'] \arrow[rrr, dotted, bend left, "cs"] & gf_0hs \arrow[r, "bhs"] \arrow[d, "g\alpha_0 k"'] & fhs \arrow[r, "\delta^{-1}s"] \arrow[d, "\widetilde{\beta}"'] & v\pi s \arrow[d, "v\alpha"'] \\
gv_0\pi t \arrow[r, "g\gamma t"'] \arrow[rrr, dotted, bend right, "ct"'] & gf_0ht \arrow[r, "bht"'] & fht \arrow[r, "\delta^{-1}t"'] & v\pi t.
\end{tikzcd}
$$

The first square is commutative because $\alpha_0 k \circ \gamma s = \gamma t \circ v_0 \alpha$ by the functoriality of quotient stacks for the morphism of pregroupoids $(R \rightrightarrows U) \longrightarrow (\pi_0(R) \rightrightarrows \pi_0(U))$. The second square is commutative by the compatiblity between α_0 and b that results from Corollary 3.5.4.11. The third square is commutative by definition of $\widetilde{\beta}$. Therefore the outer rectangle is commutative. That is, with the words of Corollary 3.5.4.11, the arrow c is a morphism from $(f_1, \beta_1) = (gv_0\pi, gv_0\alpha)$ to $(f_2, \beta_2) = (v\pi, v\alpha)$ in the equalizer category

$$(\mathscr{H}om(U, \mathscr{M}) \rightrightarrows \mathscr{H}om(R, \mathscr{M})).$$

The quoted corollary gives existence of a 2-isomorphism $b : gv_0 \to v$ such that $c = b\pi$. This concludes the proof of the lemma. $\square$

3.5.3.3 Remark. (1) Lemma 3.5.3.2 does not hold if $\mathscr{M}$ is not Deligne–Mumford. Here is a simple counterexample: let us consider $\mathscr{X} = B\mathbb{G}_m$, and let $U = S$ be the base. Note that $\Pi_1(S/S) = S$, so the pushout of previous diagram should be

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B\mathbb{G}_m \arrow[d] \\ \Pi_1(U/S) \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

However, by Proposition 3.5.2.9, we have $\Pi_1(B\mathbb{G}_m/S) \simeq S$.

(2) A pushout property for Π_1 (analogous to Lemma 3.3.4.1) can be easily deduced from the previous lemma, namely, the square

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{X} \arrow[d] \\ \Pi_1(U/S) \arrow[r] & \Pi_1(\mathscr{X}/S) \end{tikzcd}$$

satisfies the 2-pushout property for morphisms to Deligne–Mumford stacks.

Before considering the coperfection, we make a few remarks on perfect stacks. The definition of relative Frobenius and perfect stack carry no mystery. It turns out that perfect algebraic stacks have a very simple structure.

3.5.3.4 Lemma. *Let S be an algebraic space of characteristit p and let $\mathscr{X}$ be an S-algebraic stack. Consider the following conditions:*

(1) *$\mathscr{X}$ is a perfect S-stack.*

(2) *There exists an étale, surjective morphism $U \to \mathscr{X}$ from a perfect S-algebraic space.*

(3) *$\mathscr{X}$ is an étale gerbe over a perfect S-algebraic space.*

Then we have the implications (1) $\iff$ (2) $\Leftarrow$ (3). *Moreover, if the diagonal of $\mathscr{X} \to S$ is locally of finite presentation then all three conditions are equivalent. In particular, all perfect algebraic stacks are Deligne-Mumford.*

To obtain an example of a perfect algebraic stack that does not satisfy (3), take a scheme X over a perfect field k with a non-free action of a finite group G, form the perfection

$$X^{\mathrm{pf}/k} = \lim \big(\cdots\ X \xrightarrow{\mathrm{F}} X \xrightarrow{\mathrm{F}} X\big).$$

and let $\mathscr{X} = [X^{\mathrm{pf}}/G]$.

Proof : First we recall some basics on Frobenius. If $f : X \to Y$ and $g : Y \to S$ are morphisms of $\mathbb{F}_p$-stacks, then we have

$$\mathrm{F}_{X/S} = f^* \mathrm{F}_{Y/S} \circ \mathrm{F}_{X/Y}\,.$$

Here $f^*\,\mathrm{F}_{Y/S}$ denotes the pullback of $\mathrm{F}_{Y/S} : Y \to Y^{p/S}$ along $f^{p/S} : X^{p/S} \to Y^{p/S}$.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{F}_{X/Y}"] \arrow[rr, bend left, "\mathrm{F}_{X/S}"] & X^{p/Y} \arrow[r, "f^*\,\mathrm{F}_{Y/S}"] \arrow[d] & X^{p/S} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & Y \arrow[r, "\mathrm{F}_{Y/S}"] & Y^{p/S} \arrow[r] \arrow[d] & Y \arrow[d] \\
 & & S \arrow[r, "\mathrm{F}_S"] & S
\end{tikzcd}$$

This shows that the composition of perfect morphisms is perfect; any morphism between perfect S-stacks is perfect; and if $X \to S$ is perfect and f is perfect and faithfully flat quasi-compact, then $Y \to S$ is perfect.

(1) $\Rightarrow$ (2) If $\mathscr{X} \to S$ is perfect, then so is $\mathscr{X} \times_S \mathscr{X} \to S$ and hence also the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$. In particular Δ is formally unramified. Being locally of finite type ([SP19], Tag 04XS)), it is unramified in the sense of [Ra70] and [SP19]. It follows that $\mathscr{X}$ is Deligne–Mumford ([SP19], Tag 06N3). Let $U \to \mathscr{X}$ be an étale surjective morphism from an algebraic space; then $U \to \mathscr{X}$ is perfect and it follows that U is perfect.

(2) $\Rightarrow$ (1) As the preliminary remarks showed, if U is perfect and $U \to \mathscr{X}$ is étale surjective then $\mathscr{X}$ is perfect.

(3) $\Rightarrow$ (1) This is clear because an étale gerbe is perfect.

(1) $\Rightarrow$ (3) If Δ is locally of finite presentation, it is formally étale hence étale. It follows that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is étale and therefore there is an algebraic space X and an étale gerbe morphism $\mathscr{X} \to X$, see [SP19], Tag 06QJ. $\square$

3.5.3.5 Theorem. *Let S be a noetherian $\mathbb{F}_p$-scheme and $\mathscr{X}/S$ a flat finitely presented separable algebraic stack. Then the pro-étale stack $\Pi_1(\mathscr{X}/S)$ represents the colimit of the inductive system of relative Frobenii*

$$\mathscr{X} \xrightarrow{F_{\mathscr{X}/S}} \mathscr{X}^{p/S} \xrightarrow{F_{\mathscr{X}^{p}/S}} \mathscr{X}^{p^2/S} \longrightarrow \dots$$

in the 2-category of quasi-separated Deligne–Mumford stacks over S, and it is a coperfection of $\mathscr{X}/S$.

Proof : Let $\mathscr{M}/S$ be a Deligne-Mumford stack and $f_i : \mathscr{X}^{p^i/S} \to \mathscr{M}$ a series of morphisms from the relative Frobenii of $\mathscr{X}$ to $\mathscr{M}$, compatible in the usual sense:

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\mathrm{F}"] \arrow[drrrr, "f_0"'] & \mathscr{X}^{p/S} \arrow[r, "\mathrm{F}"] \arrow[drrr, "f_1"] & \mathscr{X}^{p^2/S} \arrow[r, "\mathrm{F}"] \arrow[drr, "f_2"] & \dots \arrow[r] & \Pi_1(\mathscr{X}/S) \arrow[d, dashed] \\
 & & & & \mathscr{M}
\end{tikzcd}$$

We would like to construct the dashed arrow in the diagram, and prove its 2-uniqueness.

Let $V \to \mathscr{M}$ be an étale atlas. Applying the same argument as in Theorem 3.5.1.1, we have the following diagram:

$$\begin{tikzcd}
\mathscr{X}_V \arrow[r] \arrow[d] \arrow[drrrr] & (\mathscr{X}_V)^{p/S} \arrow[r] \arrow[d] \arrow[drrr] & (\mathscr{X}_V)^{p^2/S} \arrow[r] \arrow[d] \arrow[drr] & \dots & \\
\mathscr{X} \arrow[r, "\mathrm{F}"] \arrow[drrrr, "f_0"'] & \mathscr{X}^{p/S} \arrow[r, "\mathrm{F}"] \arrow[drrr, "f_1"] & \mathscr{X}^{p^2/S} \arrow[r, "\mathrm{F}"] \arrow[drr, "f_2"] & \dots & V \arrow[d] \\
 & & & & \mathscr{M}
\end{tikzcd}$$

Now we choose a finitely presented atlas $U \to \mathscr{X}_V$. Taking Frobenius twists and composing with $(\mathscr{X}_V)^{p^i/S} \to V$, it induces a series of maps $U^{p^i/S} \to V$:

$$\begin{tikzcd}
U \arrow[r, "\mathrm{F}"] \arrow[drrrr] & U^{p/S} \arrow[r, "\mathrm{F}"] \arrow[drrr] & U^{p^2/S} \arrow[r, "\mathrm{F}"] \arrow[drr] & \dots \arrow[r] & \pi_0(U/S) \arrow[d, dashed] \\
 & & & & V
\end{tikzcd}$$

By Theorem 3.5.1.1, $\pi_0(U/S)$ represents the colimit of relative Frobenii of U/S, the above diagram induces a unique morphism from $\pi_0(U/S)$ to V. Using the pushout property of Lemma 3.5.3.2

$$\begin{tikzcd}
U \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d] \arrow[dddr, bend left] & \\
 & \Pi_1(\mathscr{X}/S) \arrow[d] \arrow[ddr, dashed] & \\
\pi_0(U) \arrow[r] \arrow[dr] & {[\pi_0(U)/\pi_0(R)]} \arrow[dr, dashed] & \\
 & V \arrow[r] & \mathscr{M}
\end{tikzcd}$$

we obtain a factorization $\mathscr{X} \to \Pi_1(\mathscr{X}/S) \to \mathscr{M}$.

Now let us show the independence of atlases U, V and the 2-uniqueness of the factorization. Let U' (resp. V') be a refinement of U (resp. V) with $R' = U' \times_{\mathscr{X}} U'$, such that we have a

2-commutative diagram as before:

$$\begin{tikzcd}
U' \arrow[r] \arrow[dd] & \mathscr{X} \arrow[d] \arrow[dddr, bend left] & \\
& \Pi_1(\mathscr{X}/S) \arrow[d] & \\
\pi_0(U') \arrow[r] \arrow[dr] & {[\pi_0(U')/\pi_0(R')]} \arrow[dr, dashed] & \\
& V' \arrow[r] & \mathscr{M}
\end{tikzcd}$$

Combining with the diagram from U, V, we obtain a 2-commutative diagram

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \arrow[ddr, bend left] & \\
\pi_0(U') \arrow[r] \arrow[dr] & \pi_0(U) \arrow[r] \arrow[dr] & {[\pi_0(U)/\pi_0(R)]} \arrow[dr, dashed] & \\
& V' \arrow[r] & V \arrow[r] & \mathscr{M}
\end{tikzcd}$$

where the top squares are all cocartesian by Proposition 3.3.4.1 and Lemma 3.5.3.2, hence the dashed arrow is unique up to a unique 2-isomorphism. Consequently, the two factorizations induced by U, V and U', V' are compatible, in the sense that theyfit into the 2-commutative diagram

$$\begin{tikzcd}
& \mathscr{X} \arrow[d] \\
& \Pi_1(\mathscr{X}/S) \arrow[d] \arrow[dl] \\
{[\pi_0(U')/\pi_0(R')]} \arrow[r] \arrow[dr] & {[\pi_0(U)/\pi_0(R)]} \arrow[d] \\
& \mathscr{M}
\end{tikzcd}$$

hence they differ by a unique 2-isomorphism, and the factorization $\mathscr{X} \to \Pi_1(\mathscr{X}/S) \to \mathscr{M}$ is independent of the choice of atlases U, V. By the fact that any morphism from the pro-étale stack $\Pi_1(\mathscr{X}/S)$ to $\mathscr{Y}$ factors through some quotient stack $[\pi_0(U)/\pi_0(R)]$, we deduce that the factorization is unique up to a unique 2-isomorphism. □

3.5.3.6 Remark. Here the assumption of quasi-separation is crucial. If one considers possibly non-quasi-separated Deligne-Mumford stacks in the statement of Theorem 3.5.3.5, one may not be able to find a finitely presented étale atlas $V \to \mathscr{M}$, and hence the atlas $U \to \mathscr{X}$ in the theorem may not be chosen with U/S being finitely presented, consequently one cannot apply π_0 to U/S. Here is a counterexample: let $\mathscr{X} = C$ be a nodal irreducible curve over a field k. It is known that there exists an infinite étale cover of C which does not come from finite étale covers, corresponding to a morphism $C \to B\mathbb{Z}$ to the perfect stack $B\mathbb{Z}$ which is not quasi-separated. However, by Proposition 3.5.2.10 the stack $\Pi_1(C/k) \simeq \Pi^{\text{ét}}_{C/k}$ is profinite, while the morphism $C \to B\mathbb{Z}$ does not factor through any finite étale stack, hence it does not factor through $\Pi_1(C/k)$.

3.5.4 Appendix: the groupoid closure of a pregroupoid

In this appendix, we give the construction of the groupoid closure of a pregroupoid.

3.5.4.1 Groupoids. A *groupoid* is a small category where every morphism is an isomorphism. Alternatively, it is given by a set of objects U, a set of arrows R, and morphisms source and target $s, t : R \to U$, composition $c : R \times_{s,U,t} R \to R$, identity $e : U \to R$, inverse $i : R \to R$, satisfying the following axioms:

(1) Associativity: $c \circ (1, c) = c \circ (c, 1)$,

(2) Identity: $s \circ e = t \circ e = 1$ and $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$,

(3) Inverse: $s \circ i = t$, $t \circ i = s$, $c \circ (i, 1) = e \circ s$ and $c \circ (1, i) = e \circ t$.

In a groupoid, the maps e and i are uniquely determined, i is an involution, and $i \circ e = e$. In particular the quintuple (U, R, s, t, c) suffices to describe the groupoid.

3.5.4.2 Symmetry. Inversion i extends to n-tuples of composable arrows:

$$(R/U)^n R \times_{s,U,t} R \times_{s,U,t} \cdots \times_{s,U,t} R \quad , \quad (\alpha_1, \dots, \alpha_n) \mapsto (\alpha_n^{-1}, \dots, \alpha_1^{-1}).$$

This can be used to shrink the number of axioms. Indeed, we have:

$$c = i \circ c \circ i, \quad (1, c) = i \circ (c, 1) \circ i, \quad (e \circ t, 1) = i \circ (1, e \circ s) \circ i, \quad (1, i) = i \circ (i, 1) \circ i.$$

Using this, the axioms $t \circ e = 1$, $c \circ (e \circ t, 1) = 1$ and $c \circ (1, i) = e \circ t$ follow from $s \circ e = 1$, $c \circ (1, e \circ s) = 1$ and $c \circ (i, 1) = e \circ s$, respectively. Of course the reduced system of axioms has the drawback that it is not symmetric. In the sequel, for legibility we will prefer to give full, symmetric lists of axioms, but we will use symmetry to reduce the number of constructions. Namely, when we want to construct a (pre)groupoid and maps $\lambda = (1, e \circ s)$ and $\lambda^* = (e \circ t, 1)$ have to be provided, then we know that it is enough to construct λ since then $\lambda^* = i \circ \lambda \circ i$.

3.5.4.3 Pregroupoids: motivation. Put in a nutshell, a pregroupoid is a structure which resembles that of a groupoid, but where composition is only partially defined and associativity holds only partially. More on the technical side, our working definition will be that a pregroupoid is what you obtain when you apply a functor to a groupoid. Since this produces a lot of data, we will describe the motivating example first in order to make the ensuing Definition 3.5.4.5 readable. We simplify notations by allowing the omission of the "$\circ$" sign for compositions.

3.5.4.4 Example. Assume that $(U_0, R_0, s_0, t_0, c_0)$ is a groupoid in objects of a category $\mathscr{C}_0$. Let $F : \mathscr{C}_0 \to \mathscr{C}$ be a functor. If F transforms fibred products into fibred products, then

$$(F(U_0), F(R_0), F(s_0), F(t_0), F(c_0))$$

is a groupoid in objects of $\mathscr{C}$. In general however, all data and axioms involving fibred products are altered. We will now describe the result precisely.

We first look at the data that do not involve fibred products, that is $U_0, R_0, s_0, t_0, e_0, i_0$. By taking their images under F, we obtain:

(1) objects U, R and maps $s, t : R \to U$, $e : U \to R$, $i : R \to R$ such that $se = te = 1$, $i^2 = 1$, $si = t$, $ti = s$.

Now we look at the data involving double fibred products and composition. By taking the images of $D_0(R_0/U_0)^2$, the involution $i_0 : D_0 \to D_0$, the projections $\mathrm{pr}_1, \mathrm{pr}_2 : D_0 \to R_0$, we obtain:

(2) an object D and maps $i : D \to D$, $p_1 : D \to R$, $p_2 : D \to R$ such that $sp_1 = tp_2$ and $p_1 i = ip_2$.

By taking the images of the composition $c_0 : D_0 \to R_0$ and the maps $\lambda_0(1, e_0 s_0) : R_0 \to D_0$, $\lambda_0^+(e_0 t_0, 1) : R_0 \to D_0$, $\mu_0(i_0, 1) : R_0 \to D_0$, $\mu_0^+(1, i_0) : R_0 \to D_0$ we obtain:

(3) maps $c : D \to R$, $\lambda, \lambda^+ : R \to D$, $\mu, \mu^+ : R \to D$ such that

(3.a) $p_1\lambda = 1$, $p_2\lambda = es$, $p_1\lambda^+ = et$, $p_2\lambda^+ = 1$, $\lambda i = i\lambda^+$, $c\lambda = c\lambda^+ = 1$,

(3.b) $p_1\mu = i$, $p_2\mu = 1$, $p_1\mu^+ = 1$, $p_2\mu^+ = i$, $\mu i = \mu^+$, $c\mu = es$, $c\mu^+ = et$.

Finally we look at the data involving triple fibred products and associativity. By taking the images of $E_0(R_0/U_0)^3$, the involution $i_0 : E_0 \to E_0$, and the projections $\mathrm{pr}_{12}, \mathrm{pr}_{23} : E_0 \to D_0$, we obtain:

(4) an object E and maps $i : E \to E$, $q_{12}, q_{23} : E \to D$ such that $p_2 q_{12} = p_1 q_{23}$ and $q_{12} i = i q_{23}$.

We define $q_1 p_1 q_{12}$, $q_2 p_1 q_{23}$, $q_3 p_2 q_{23}$. By taking the images of $\nu_0(1, c_0) : E_0 \to D_0$ and $\nu_0^+(c_0, 1) : E_0 \to D_0$ we obtain:

(5) maps $\nu, \nu^+ : E \to D$ such that $p_1\nu = q_1$, $p_2\nu = cq_{23}$, $p_1\nu^+ = cq_{12}$, $p_2\nu^+ = q_3$, $\nu i = i\nu^+$ and $c\nu = c\nu^+$.

The axioms of a groupoid survive in modified guise: associativity is in (5); identity is in (1) and (3.a); inverse is in (1) and (3.b). Using symmetry, this set of data is determined by the subcollection $P(U, (R, i), (D, i), (E, i), s, c, e, p_1, \lambda, \mu, q_{12}, \nu)$.

3.5.4.5 Definition. A *pregroupoid* (over U) is given by a collection of objects and maps

$$P = \big(U, (R, i), (D, i), (E, i), s, c, e, p_1, \lambda, \mu, q_{12}, \nu\big)$$

satisfying the conditions (1) to (5) in 3.5.4.4. A *morphism of pregroupoids* $f : P \to P'$ is given by a quadruple of maps $U \to U'$, $R \to R'$, $D \to D'$, $E \to E'$ that are compatible with all the structure maps of the pregroupoids P and P'.

3.5.4.6 Remarks. (1) Each groupoid (U, R, s, t, c) defines a unique pregroupoid such that $D = (R/U)^2$ and $E = (R/U)^3$. This gives rise to a faithful embedding of categories:

$$\iota : (\text{Groupoid}\,/U) \hookrightarrow (\text{Pregroupoid}\,/U).$$

(2) A pregroupoid is a groupoid if and only if the following two maps are isomorphisms:

$$(p_1, p_2) : D \to (R/U)^2 \quad \text{and} \quad (q_{12}, q_{23}) : E \to D \times_{p_2, R, p_1} D.$$

(3) A pregroupoid is a truncated simplicial set:

$$\begin{tikzcd} E \arrow[r, shift left=4.5, "q_{12}"] \arrow[r, shift left=1.5, "\nu"] \arrow[r, shift right=1.5, "\nu^+"] \arrow[r, shift right=4.5, "q_{23}"'] & D \arrow[r, shift left=3, "p_1"] \arrow[r, "c"] \arrow[r, shift right=3, "p_2"'] & R \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & U. \end{tikzcd}$$

With the usual notations d_n^i and s_n^i for faces and degenerations, we have $d_3^0 = q_{12}$, $d_3^1 = \nu$, $d_3^2 = \nu^+$, $d_3^3 = q_{23}$, $s_2^0 = (\mathrm{id}, e)$, $s_2^2 = (e, \mathrm{id})$, $d_2^0 = p_1$, $d_2^1 = c$, $d_2^2 = p_2$, $s_1^0 = \lambda$, $s_1^1 = \lambda^+$, $d_1^0 = t$, $d_1^1 = s$, $s_0^0 = e$.

3.5.4.7 Groupoid closure. We now construct a left adjoint to the inclusion ι. This will be called the *groupoid closure*, since it is analogous to the transitive closure of an equivalence relation. Let $P = (U, R, D, E, \dots)$ be a pregroupoid. We wish to enlarge R and D in a universal way so that the vertical maps in the diagrams below become isomorphisms:

$$\begin{tikzcd} D \arrow[r, "c"] \arrow[d, "(p_1,p_2)"'] & R \\ R \underset{s,U,t}{\times} R & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\nu_1"] \arrow[d, "(q_{12},q_{23})"'] & D \\ D \underset{p_2,R,p_1}{\times} D & \end{tikzcd}$$

With this idea in mind, we seek to define a new pregroupoid P':

- $R' = (R \times_{s,U,t} R) \amalg R$ is the pushout:

$$\begin{tikzcd} D \arrow[r, "c"] \arrow[d, "(p_1,p_2)"'] \arrow[dr, phantom, "\ulcorner"] & R \arrow[d, "\rho"] \\ R \underset{s,U,t}{\times} R \arrow[r, "c'"] & R'. \end{tikzcd}$$

- $i' = c'i \amalg \rho i : R' \to R'$ with $c'i : R \times_{s,U,t} R \to R'$ and $\rho i : R \to R'$.
- $D' = R \times_{s,U,t} R$ and $i' : D' \to D'$ is the inversion of $(R/U)^2$.
- $E' = D \times_{p_2,R,p_1} D$.
- $i' = \mathrm{sw} \circ (i, i) : E' \to E'$ where sw swaps the two D factors.
- $s' = s\,\mathrm{pr}_2 \amalg s : R' \to U$ with $s\,\mathrm{pr}_2 : R \times_{s,U,t} R \to U$ and $s : R \to U$.
- $c' : D' \to R'$ is the map in the pushout defining R'.
- $e' = \rho e : U \to R'$.
- $p_1' = \rho\,\mathrm{pr}_1 : D' \to R'$.
- $\lambda' = (p_1, p_2)\lambda$, $\mu' = (p_1, p_2)\mu$ as maps $R \to D \to D'$.
- $q_{12}' = (p_1, p_2)\,\mathrm{pr}_1 : E' \to D \to D'$.
- $\nu' = (1, c) : E' \to D'$.

Should λ' and μ' be defined on R' instead of merely on R, the data P' would be a pregroupoid, and the four maps

$$1 : U \to U, \quad \rho : R \to R', \quad (p_1, p_2) : D \to D', \quad (q_{12}, q_{23}) : E \to E'$$

would define a morphism of pregroupoids $P \to P'$. Nevertheless we can define:

$$\begin{aligned} &\phi(P) = P' \\ &P_n = \phi^n(P) = (U, R_n, D_n, E_n) \\ &P^{\mathrm{gpd}} = \operatorname{colim} P_n. \end{aligned}$$

The underlying sets of P^{gpd} are $U^{\mathrm{gpd}} = U$, $R^{\mathrm{gpd}} = \operatorname{colim} R_n$, $D^{\mathrm{gpd}} = \operatorname{colim} D_n$, $E^{\mathrm{gpd}} = \operatorname{colim} E_n$. Passing to the limit, the maps $\lambda, \mu : R_n \to D_{n+1}$ yield maps $\lambda^{\mathrm{gpd}}, \mu^{\mathrm{gpd}} : R^{\mathrm{gpd}} \to D^{\mathrm{gpd}}$ so that the problem concerning the domain of definition of these maps disappears at infinity.

3.5.4.8 Remark. It is possible to modify the definition of P_n so as to have $\lambda', \mu' : R_n \to D_n$, making P_n a pregroupoid. For this, it is enough to replace $D' = R \times_{s,U,t} R$ by a suitable subset of $R' \times_{s,U,t} R'$ where c' can be defined. The description of D' is made a little cumbersome by the fact that $R' \times_{s,U,t} R'$ is an amalgam of four sets. Since this complication can be avoided by passing to the limit, we preferred to do it this way.

3.5.4.9 Proposition. *With notation as before, the collection P^{gpd} is a groupoid. Moreover, the morphism $P \to P^{\mathrm{gpd}}$ is universal for morphisms from P to a groupoid. Thus the functor $P \mapsto P^{\mathrm{gpd}}$ is left adjoint to the embedding* $(\mathrm{Groupoid}\,/U) \hookrightarrow (\mathrm{Pregroupoid}\,/U)$.

Proof : The proof is straighforward; we merely give the idea. We start from a pregroupoid

$$P = \big(U, (R, i), (D, i), (E, i), s, c, e, p_1, \lambda, \mu, q_{12}, \nu\big),$$

a groupoid $\mathscr{P} = (\mathscr{U}, \mathscr{R}, s, t, c)$ and a morphism of pregroupoids $P \to \mathscr{P}$. Let $\mathscr{D} = \mathscr{R} \times_{s,\mathscr{U},t} \mathscr{R}$. We have a cube:

$$\begin{tikzcd}
& D \arrow[rr] \arrow[dl] \arrow[dd] & & R \arrow[dl] \arrow[dd] \\
R \underset{s,U,t}{\times} R \arrow[rr] \arrow[dd] & & R' \arrow[dd, dashed] & \\
& \mathscr{D} \arrow[rr] \arrow[dl, equal] & & \mathscr{R} \arrow[dl, equal] \\
\mathscr{R} \underset{s,\mathscr{U},t}{\times} \mathscr{R} \arrow[rr] & & \mathscr{R} &
\end{tikzcd}$$

By commutativity of the diagram in solid arrows, we can find a dotted arrow completing the cube with an arrow from the pushout R'. The contruction of a morphism $P' \to \mathscr{P}$ proceeds along the same lines. Iterating this construction gives morphisms $P_n \to \mathscr{P}$ for all n and finally a morphism $P^{\mathrm{gpd}} \to \mathscr{P}$. □

3.5.4.10 Remark. The construction of the groupoid closure works similarly for pregroupoids in objects of a category $\mathscr{C}$ with the following properties: $\mathscr{C}$ has fibred products, pushouts, colimits indexed by $\mathbb{N}$, and the latter colimits commute with fibred products. Examples of categories satisfying these properties are the category of sets; the category of sheaves on a site; the category of algebraic spaces étale over a fixed algebraic space S.

We finish this Appendix with the 2-coequalizer property of the quotient stack of a pregroupoid in algebraic spaces.

3.5.4.11 Corollary. *Let S be an algebraic space. All notations being as in 3.5.4.4, let*

$$P = (U, R, D, E, \dots)$$

be a pregroupoid in algebraic spaces over S. Assume that the groupoid closure

$$P^{\mathrm{gpd}} = (U, R^{\mathrm{gpd}}, D^{\mathrm{gpd}}, E^{\mathrm{gpd}}, \dots)$$

exists and is an fppf groupoid (this holds for example if P is a pregroupoid in étale algebraic spaces, or if P is an fppf groupoid). Let $\pi : U \to [U/R]$ be the quotient stack of the groupoid closure P^{gpd} and

$\alpha\colon \pi s \to \pi t$ *the canonical 2-isomorphism, such that* $\alpha c = \alpha p_1 \circ \alpha p_2$. *For each* S*-stack in groupoids* $\mathscr{X}$, *let*

$$\big(\,\mathscr{H}om(U,\mathscr{X}) \overset{s^*}{\underset{t^*}{\rightrightarrows}} \mathscr{H}om(R,\mathscr{X})\,\big)$$

be the "equalizer" category described as follows:

(i) *objects are pairs* (f,β) *composed of a* 1*-morphism* $f\colon U \to \mathscr{M}$ *and a* 2*-isomorphism* $\beta\colon fs \to ft$ *such that* $\beta c = \beta p_1 \circ \beta p_2$.

(ii) *morphisms* $(f_1,\beta_1) \to (f_2,\beta_2)$ *are 2-isomorphisms* $\varphi : f_1 \to f_2$ *such that* $\beta_2 \circ \varphi s = \varphi t \circ \beta_1$.

Then the functor

$$\begin{aligned} \mathscr{H}om([U/R],\mathscr{X}) &\longrightarrow \big(\mathscr{H}om(U,\mathscr{X}) \overset{s^*}{\underset{t^*}{\rightrightarrows}} \mathscr{H}om(R,\mathscr{X})\big) \\ g &\longmapsto (f = g\pi, \beta = g\alpha) \end{aligned}$$

is an equivalence of categories.

Before we pass to the proof, here are pictures for the 2-morphisms β and φ:

$$\begin{tikzcd}[column sep=huge]
D \arrow[r, bend left=50, "fsp_2=fsc"] \arrow[r, "ftp_2=fsp_1" description] \arrow[r, bend right=50, "ftp_1=ftc"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \beta p_2"] \arrow[r, phantom, shift right=3ex, "\Downarrow \beta p_1"] \arrow[r, phantom, shift left=1ex, "\Downarrow \beta c" near end] & \mathscr{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
f_1 s \arrow[r, "\varphi s"] \arrow[d, "\beta_1"'] & f_2 s \arrow[d, "\beta_2"] \\
f_1 t \arrow[r, "\varphi t"] & f_2 t
\end{tikzcd}$$

Proof : Set $\mathscr{H} = \mathscr{H}(P) = \mathscr{H}om([U/R],\mathscr{X})$ and $\mathscr{E} = \mathscr{E}(P) = (\mathscr{H}om(U,\mathscr{X}) \rightrightarrows \mathscr{H}om(R,\mathscr{X}))$. Let $F : \mathscr{H} \to \mathscr{E}$ be the functor in the statement.

Suppose first that P is a groupoid. For each pair (f,β), Lemma Tag 044U in [SP19] produces functorially a morphism $g : [U/R] \to \mathscr{X}$ and a 2-isomorphism $\epsilon : g\pi \xrightarrow{\sim} f$. That is, we have a functor $G : \mathscr{E} \to \mathscr{H}$ and an isomorphism $\epsilon : FG \xrightarrow{\sim} \mathrm{id}$. Moreover the proof of *loc. cit.* shows that GF is equal to the identity; hence F and G are quasi-inverse equivalences.

Suppose now that P is a pregroupoid with a groupoid closure P^{gpd} which is an fppf groupoid. The morphism of pregroupoids $P \to P^{\mathrm{gpd}}$ induces a functor $\mathscr{E}(P^{\mathrm{gpd}}) \to \mathscr{E}(P)$ which we claim is an equivalence. To show this, note that an object $(f,\beta) \in \mathscr{E}(P)$ is the same thing as a morphism of prestacks of pregroupoids from P to $\mathscr{X}$, namely:

- the map from U to objects of $\mathscr{X}$ is given by the 1-morphism f, namely each object $u \in U(T)$ is mapped to $f(u) \in \mathscr{X}(T)$,
- the map from R to arrows of $\mathscr{X}$ is given by the 2-morphism β, namely each arrow $r \in R(T)$ is mapped to the arrow $\beta(r) : fs(r) \to ft(r)$ in $\mathscr{X}(T)$,
- the maps from D to pairs of composable arrows of $\mathscr{X}$, and from E to triples of composable arrows of $\mathscr{M}$, are determined by the previous ones because $\mathscr{X}$ is a stack in groupoids, see Remark 3.5.4.6(2). Namely, the former is $(\beta p_1, \beta p_2)$ and the latter is $(\beta q_1, \beta q_2, \beta q_3)$,
- the condition $\beta c = \beta p_1 \circ \beta p_2$ ensures that the map on arrows is compatible with composition; one sees easily that is also implies compatibility with associativity.

Eventually the universal property of the groupoid closure (Proposition 3.5.4.9) shows that the functor $\mathscr{E}(P^{\mathrm{gpd}}) \to \mathscr{E}(P)$ is an equivalence. Since $\mathscr{H}(P) \to \mathscr{E}(P^{\mathrm{gpd}})$ is an equivalence, so is $\mathscr{H}(P) \to \mathscr{E}(P)$. $\square$

4

Geometry of torsors over curves arising from the moduli space of Galois p-covers

4.1 Introduction

Modular curves, which classify elliptic curves equipped with various level structures, are classical geometric objects which play significant roles in arithmetics and algebraic geometry. They are intensively studied over the last few decades, and are well-understood by now. Especially, the study of integral structure of modular curves over the integer $\mathbb{Z}$, and their reductions modulo bad primes, is accomplished thanks to the works of Igusa, Deligne–Rapoport [DR73], Katz–Mazur [KM85], and eventually Conrad [Con07]. The reduction modulo a bad prime of a modular curve turns out to be the union of its irreducible components with crossings at supersingular points.

Here we are interested in the generalization of the problem to the case of higher genus. For example, the modular curve $\mathcal{X}_1(p)$ classifies generalized elliptic curves equipped with a $\Gamma(p)$-structure, that is basically pairs $(E/S, \varphi)$ where E/S is an elliptic curve and φ stands for a cyclic subgroup scheme G of E/S of order p together with a generator in the sense of Katz–Mazur [KM85]. Alternatively, such a pair is also a Galois p-cover $E \to E/G$ between elliptic curves together with a generator of its Galois group G. Thus, to generalize the problem in case of higher genus, it aims to do the following things:

(1) Construct a proper moduli stack over $\mathbb{Z}$ which contains an open substack classifying Galois p-covers $Y \to X$ between semistable curves of genus g and h respectively, possibly (tamely or wildly) ramified;

(2) Identify a flat model of the generic fiber of this moduli stack;

(3) Study the reduction modulo p of this moduli stack.

Thus the classical problem of reductions modulo bad primes of modular curves becomes the special (unramified) case of genus one.

The Step (1) has been constructed by Abramovich–Romagny [AR12], using the idea of twisted curves appeared in the pioneer works of Abramovich–Vistoli [AV02] on the compactification of Kontsevich's moduli stacks of stable maps, and Abramovich–Olsson–Vistoli [AOV11] its generalization to positive characteristics. Roughly speaking, the idea of Abramovich–Romagny to construct a proper moduli stack required in (1) is to replace the bottom curve X of a Galois p-cover $Y \to X$ with a twisted curve $\mathscr{X}$ as follows

$$\begin{tikzcd} & Y \arrow[dl, dashed] \arrow[d] \\ \mathscr{X} \arrow[r] & X \end{tikzcd}$$

such that the resulting lifting $Y \to \mathscr{X}$ is a torsor under some finite flat group scheme G of order p who lives over $\mathscr{X}$, which is a *cyclic group scheme of order* p, namely, there is a generator

$$\gamma : (\mathbb{Z}/p\mathbb{Z})_{\mathscr{X}} \longrightarrow G$$

in the sense of Katz–Mazur. If moreover, we put the condition that Y is a stable curve in the sense of Deligne–Mumford–Knudsen and $\mathscr{X}$ is semistable, then the resulting torsor is called a *stable p-torsor.* Now the idea is to classify these stable p-torsors as a generalized concept of Galois p-cover, this gives us a proper moduli stack $\mathbf{ST}_{p,g,h,n}$, called the moduli stack of n-marked genus (g,h) stable p-torsors.

However, Step (2) is nontrivial, because $\mathbf{ST}_{p,g,h,n}$ is in general not flat over $\mathbb{Z}$. Actually, not only $\mathbf{ST}_{p,g,h,n}$ is not flat in general, it has some extraneous disjoint components in characteristic p, which are very interesting because they classify wild p-covers in characteristic p, and they are proper over $\mathbb{F}_p$.

An interesting appendix in [AR12] (Appendix A) suggests the condition of having a *cogenerator* to a finite flat group scheme G/S of order p. A cogenerator of G is a morphism $\kappa : G \to \mu_{p,S}$ of group schemes such that the Cartier dual $\kappa^D : (\mathbb{Z}/p\mathbb{Z})_S \to G^D$ is a generator of G^D. This condition arises naturally if we consider the moduli stack $\mathbf{ST}_{p,g,h,n} \otimes \mathbb{Z}[\zeta_p]$ where ζ_p is a p-th root of unity, since in this case the generic fiber classifies $\mathbb{Z}/p\mathbb{Z}$-covers where the group $\mathbb{Z}/p\mathbb{Z}$ has a generator and a cogenerator. Thus in order to identify the flat model, it is reasonable to require that the group scheme $G/\mathscr{X}$ is equipped with a generator and a cogenerator. When a group scheme G/S has a cogenerator $\kappa : G \to \mu_{p,S}$, Theorem A.2 of *op.cit.* shows that κ embeds into a Kummer-type sequence

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d, "\kappa"] & \mathcal{G}^{\lambda} \arrow[r, "\varphi^{\kappa}"] \arrow[d] & \mathcal{G}^{\lambda^p} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mu_{p,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r, "(-)^p"] & \mathbb{G}_{m,S} \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{G}^{\lambda}, \mathcal{G}^{\lambda^p}$ are smooth one-dimensional group schemes over S, and the subscript λ denotes a global section of the inverse of an invertible sheaf $\mathscr{L}$ over the base. We will recall the construction of $\mathcal{G}^{\lambda}$ in Section 4.3.4. Notably, these one-dimensional group schemes also play significant roles in the works of Sekiguchi–Oort–Suwa [SOS89] on deforming Artin–Schreier theory to Kummer theory.

The problems of finding the flat model of $\mathbf{ST}_{p,g,h,n}$ and the study of its reduction are very difficult. In this chapter, we wish to shed some light, by investigating relevant moduli spaces of torsors arising from $\mathbf{ST}_{p,g,h,n}$. Especially, the moduli stacks of torsors under groups $G, \mathcal{G}^{\lambda}$ are central objects of this chapter. Let us give an overview.

Overview. In Section 4.2, we construct explicitly the local twisted lifting of Artin–Schreier covers and Frobenius of the projective line, and make a remark on the existence of extra components of moduli stack of stable p-torsors in characteristic p. In Section 4.3, we study moduli spaces of torsors over proper curves under group schemes $\mathcal{G}^\lambda$ and cocyclic group schemes G. In particular, we prove some representability and flatness results of these moduli spaces in certain cases:

4.1.0.1 Theorem.[Proposition 4.3.4.2] *Assume that λ is fiberwise regular. Then the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable and smooth of dimension $g+d-1$ over S, where $d = \deg D$ and g is the genus of X.*

4.1.0.2 Theorem.[Theorem 4.3.4.3] *Assume that λ is fiberwise regular. Then the moduli stack $\mathbf{TORS}_X(G)$ is representable by a finite flat group scheme, of degree p^{2g+d-1}.*

4.1.0.3 Theorem.[Theorem 4.3.5.3] *Let S be a discrete valuation ring, X/S a proper, geometrically connected and generically irreducible curve, $\mathscr{L}$ an invertible sheaf over X, and $\lambda \in H^0(X, \mathscr{L}^{-1})$ satisfying the condition* (*). *Then the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable by a smooth S-group scheme of dimension $g+d-1$.*

In Section 4.4, we discuss the relation between moduli spaces of $\mathcal{G}^\lambda$-torsors over proper curves and the generalized Jacobians of open curves. In Section 4.5, we prove a categorical classification of $\mathcal{G}^\lambda$-torsors which slightly generalizes a result of Andreatta–Gasbarri [AG07].

4.2 Local twisted lifting of Artin-Schreier covers and the Frobenius of $\mathbb{P}^1$

In this section, we work over an algebraically closed field k of characteristic $p > 0$. Here the examples are *Artin-Schreier covers* and the *Frobenius* of $\mathbb{P}^1$, which are ramified torsors, namely, they are torsors under certain finite group schemes over k except for finitely many points. To describe the *twisted lifting problem*, let $Y \to \mathbb{P}^1$ be a ramified cyclic torsor over k, under a constant finite cyclic k-group scheme G of order p. We aim to find a lifting

$$\begin{tikzcd} & \mathscr{X} \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & \mathbb{P}^1 \end{tikzcd}$$

where $\mathscr{X}$ is a twisted curve whose coarse moduli space is $\mathbb{P}^1$, such that $Y \to \mathscr{X}$ is a torsor under an $\mathscr{X}$-group scheme (or stack) $\mathscr{G}$, which admits a global generator and is isomorphic to the constant group scheme $G \times_k \mathscr{X}$ outside of the twisted points.

If the twisted lifting problem is solvable for $Y \to \mathbb{P}^1$, and moreover that Y is a stable curve in the sense of Deligne–Mumford-Knudsen, then the resulting $\mathscr{G}$-torsor $Y \to \mathscr{X}$ is a *stable p-torsor* in the sense of Abramovich-Romagny [AR12]. Though, through out the note, we only need to deal with smooth curves Y.

4.2.1 The Artin-Schreier case

Consider an Artin-Schreier cover $Y \to \mathbb{P}^1$ with only one branched point at 0 with conductor $r \geqslant 1$.[1] Over the affine open subset $\mathbb{P}^1 \backslash \{\infty\}$, the cover is defined by the following equation

$$y^p + y^{p-1}x^r - x^r = 0$$

where x is a local parameter of $\mathbb{P}^1$ at 0. The $\mathbb{Z}/p$-action on Y is given by

$$\sigma : y \longmapsto \frac{y}{y+1}.$$

Note that the function ring of Y is the normalization of $k[x, y]/(y^p + y^{p-1}x^r - x^r)$.

Our goal is to lift this ramified cover $Y \to \mathbb{P}^1$ to a p-torsor under certain group scheme over a twisted curve, in our case it should be $\mathscr{X} = \mathbb{P}^1(\sqrt[p]{0})$. Let D be the universal μ_p-torsor over $\mathscr{X}$,

$$\begin{tikzcd} D \arrow[d, "/\mu_p"] & k[d] \\ \mathscr{X} \arrow[d, "\text{coarse moduli}"'] & \\ \mathbb{P}^1 & k[x] \arrow[uu, "x \mapsto d^p"'] \end{tikzcd}$$

we have $\mathscr{X} \cong [D/\mu_p]$ by definition. Let $k[d]$ be the affine coordinate ring of D near the origin.

Here is the general strategy. A morphism $Y \to \mathscr{X}$ means to give a μ_p-torsor $E \to Y$ together with a μ_p-equivariant morphism to D. Such a morphism exists if and only if the following diagram commutes

$$\begin{tikzcd} E \arrow[r] \arrow[d] & D \arrow[d] \\ Y \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

Indeed, if such a commutative diagram exists, then $Y \to \mathscr{X}$ is nothing but the morphism $E \to D$ modulo μ_p-actions

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\mu_p"'] & D \arrow[d, "/\mu_p"] \\ Y \arrow[r, dashed] \arrow[dr] & \mathscr{X} \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

Moreover, the condition on the torsor structure is as follows. Firstly we have a $\mathbb{Z}/p$-action on Y, which is free apart from the origin $y = 0$. After choosing certain factorization for the equation defining the cover $Y \to \mathbb{P}^1$, it induces a $\mathbb{Z}/p$-action on E over D. Then we take the schematic image of $(\mathbb{Z}/p)_D \to \mathrm{Aut}_D(E)$, we obtain a D-group scheme $\mathscr{G}_D$ which will necessarily endow a μ_p-action to make the $\mathscr{G}_D$-action μ_p-equivariant on E. If $E \to D$ is a $\mathscr{G}_D$-torsor, then the twisted lifting problem is solved.

[1] Note that we always have $(r, p) = 1$.

For convenience, we restrict the question to the étale local rings, i.e., the strict henselizations. The expected diagram becomes

$$\begin{tikzcd} E \arrow[r] \arrow[d] & D \arrow[d] \\ \operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{Y,0}) \arrow[r] & \operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{\mathbb{P}^1,0}) \end{tikzcd}$$

Now the Artin-Schreier cover is given by a function $f \in \mathscr{O}^{\mathrm{sh}}_{Y,0}$, namely, the image of the local parameter x of $\mathscr{O}^{\mathrm{sh}}_{\mathbb{P}^1,0}$. Taking global sections, we obtain

$$\begin{tikzcd} \mathscr{O}^{\mathrm{sh}}_{Y,0}[w]/(w^p - u) & \mathscr{O}^{\mathrm{sh}}_{\mathbb{P}^1,0}[d]/(d^p - x) \arrow[l] \\ \mathscr{O}^{\mathrm{sh}}_{Y,0} \arrow[u] & \mathscr{O}^{\mathrm{sh}}_{\mathbb{P}^1,0} \arrow[l] \arrow[u] \end{tikzcd}$$

where $u \in \left(\mathscr{O}^{\mathrm{sh}}_{Y,0}\right)^{\times}$ is a unit element. The top morphism is μ_p-equivariant, hence d maps to hw for some $h \in \mathscr{O}^{\mathrm{sh}}_{Y,0}$. The commutativity of the above diagram implies that $f = h^p u$, which is equivalent to the existence of the local lifting of the Artin-Schreier cover $Y \to \mathbb{P}^1$ to the twisted curve.

Let A be the strict henselization of $k[x,y]/(y^p + y^{p-1}x^r - x^r)$ at the origin, then $\mathscr{O}^{\mathrm{sh}}_{Y,0}$ is the normalization of A in its fraction field. Let $t \in \mathscr{O}^{\mathrm{sh}}_{Y,0}$ be a uniformizer, then $y = u_0 t^r$ for some unit element $u_0 \in (\mathscr{O}^{\mathrm{sh}}_{Y,0})^{\times}$. By Hensel Lemma, there exists $u_1 \in (\mathscr{O}^{\mathrm{sh}}_{Y,0})^{\times}$ such that $u_0 = u_1^r$. The function f satisfies

$$f^r = \frac{y^p}{1 - y^{p-1}}$$

thus $f = (u_1 t)^p \cdot u_2$ for some $u_2 \in (\mathscr{O}^{\mathrm{sh}}_{Y,0})^{\times}$, which is the required factorization, where we take $h = u_1 t$ and $u = u_2$.

Finally we need to find the group stack $\mathscr{G}$ over $\mathscr{X}$. It should be the quotient of $\mathscr{G}_D = \mathscr{G} \times_{\mathscr{X}} D$ by the μ_p-action on D, where $\mathscr{G}_D$ is the scheme-theoretic image of $(\mathbb{Z}/p)_D$ in $\mathrm{Aut}_D(E)$. Intuitively, the $\mathbb{Z}/p$-action on E is given by

$$\sigma^i(w) \;=\; (1 + i \cdot y)^{\frac{1}{r}} w.$$

Write $(\mathbb{Z}/p)_k = \operatorname{Spec}\bigl(k[e]/(e^p - e)\bigr)$. Let $F(e,t) = \sum_{i=0}^{\infty} \binom{1/r}{i} e^i y^i$, this is an element in the ring $\mathscr{O}^{\mathrm{sh}}_{E,0}[e]/(e^p - e)$ which satisfies $F(e,t)^r = 1 + ey$. Then the $\mathbb{Z}/p$-action on E is given by the morphism

$$\begin{array}{rcl} E \times_D E & \longleftarrow & (\mathbb{Z}/p)_k \times_k E \\ \dfrac{\mathscr{O}^{\mathrm{sh}}_{E,0}[t', w']}{\bigl(w'^p - u, t'w' - d\bigr)} & \longrightarrow & \dfrac{\mathscr{O}^{\mathrm{sh}}_{E,0}[e]}{(e^p - e)} \\ t' & \longmapsto & t \cdot F(e,t)^{-1} \\ w' & \longmapsto & w \cdot F(e,t) \end{array}$$

The above morphism is not an isomorphism, which prevents $E \to D$ from being a $\mathbb{Z}/p$-torsor. Indeed, over the origin $d = 0$ of D, we have

$$w' \;\mapsto\; w \cdot F(e,t) = w + \sum_{i=1}^{\infty} \binom{1/r}{i} e^i y^{i-\frac{1}{r}} \cdot d = w,$$

therefore the image of $w' - w$ is zero.

Now we are ready to find the explicit equation for the D-group scheme $\mathscr{G}_D$, and sequentially the $\mathscr{X}$-group stack $[\mathscr{G}_D/\mu_p]$. To make the calculation easier, we find the equation after a faithfully flat base change. Consider the following finite faithfully flat morphism[2]

$$D' := \operatorname{Spec}\Big(\mathscr{O}^{\mathrm{sh}}_{D,0}[s]/(s^{rp} - d^{r(p-1)}s^r - 1)\Big) \;\longrightarrow\; D.$$

Let $E' \to D'$ be the base change of $E \to D$ along $D' \to D$. We claim that $E' \to D'$ is a torsor under the following group scheme $\mathscr{G}_{D'}$

$$\mathscr{G}_{D'} \;=\; \operatorname{Spec}\Big(\mathscr{O}^{\mathrm{sh}}_{D',0}[a]\Big/\big((a^p + s^p)^r - s^{rp} - d^{r(p-1)}((a+s)^r - s^r)\big)\Big)^{\mathrm{sh}}_{(a=0)}$$

and the morphism of schematic image

$$\begin{tikzcd} (\mathbb{Z}/p)_{D'} \arrow[rr] \arrow[dr] & & \mathrm{Aut}_{D'}(E') \\ & \mathscr{G}_{D'} \arrow[ur, hook, "\text{schematic image of } \mathbb{Z}/p"'] & \end{tikzcd}$$

is given by

$$a \;\longmapsto\; s \cdot \big(F(e,t) - 1\big) = s \cdot \sum_{i=1}^{\infty} e^i d^{ir} s^{-ir}.$$

Indeed, the fiber of $\mathscr{G}_{D'}$ over the origin $d = 0$, $s = 1$ is

$$\mathscr{G}_{D'} \times_{D'} k[d = 0, s = 1] \;=\; \operatorname{Spec}\big(k[a]/((a^p+1)^r - 1)\big)^{\mathrm{sh}}_{(a=0)}$$

which is isomorphic to $\alpha_{p,k}$, and $\mathscr{G}_{D'}$ is generically $\mathbb{Z}/p$.

The morphism $\mathscr{G}_{D'} \times_{D'} E' \to E' \times_{D'} E'$, in terms of global functions,

$$\begin{array}{ccc} \dfrac{\mathscr{O}^{\mathrm{sh}}_{E',0}[t', w']}{\big(w'^p - u, t'w' - d\big)} & \longrightarrow & \dfrac{\mathscr{O}^{\mathrm{sh}}_{E',0}[a]_{(a=0)}}{(\text{the long equation for } a)} \\ t' & \longmapsto & \dfrac{tw}{w+a} \\ w' & \longmapsto & w + a \end{array}$$

is an isomorphism, i.e., E' is a $\mathscr{G}_{D'}$-torsor over D'. Moreover, the μ_p-action on $\mathscr{G}_{D'}$ is given by $a \mapsto \xi \cdot a$, $d \mapsto \xi \cdot d$ and $s \mapsto \xi \cdot s$, hence the above morphism is μ_p-equivariant as required. Over the punctured disc $\mathscr{O}^{\mathrm{sh}}_{D',0}[d^{-1}]$, it reduces to the initial $\mathbb{Z}/p$-action on E'.

[2]Intuitively, we could keep in mind that "$s = d \cdot y^{\frac{1}{r}} = d \cdot u_1 t$".

By faithfully flat descent,[3] we obtain a D-group scheme $\mathscr{G}_D$ such that $\mathscr{G}_D \times_D D' \cong \mathscr{G}_{D'}$, and the $\mathscr{G}_{D'}$-torsor $E' \to D'$ descends to the $\mathscr{G}_D$-torsor $E \to D$, together with the μ_p-action on $\mathscr{G}_D$ and the μ_p-equivariance of the $\mathscr{G}_D$-action on E. Consequently, $Y \to \mathscr{X}$ is a p-torsor under the group stack $[\mathscr{G}_D/\mu_p]$.

4.2.2 The Frobenius case

The twisted lifting problem for the Frobenius F on $\mathbb{P}^1$ is even more explicit. Following the same strategy, we actually have the diagram

$$\begin{array}{ccc} E & \xrightarrow{m} & \mathbb{P}^1 \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow{\scriptstyle F} \\ \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 \end{array}$$

where E is $\mu_p \times \mathbb{P}^1$, and m is given by the multiplication $(\xi, x) \mapsto \xi \cdot x$. We restrict to affine open subsets near the origin. Then the above diagram becomes

$$\begin{array}{ccc} k[y,w]/(w^p-1) & \longleftarrow & k[d] \\ \big\uparrow & & \big\uparrow \\ k[y] & \longleftarrow & k[x] \end{array}$$

where the bottom map sends x to $f = y^p$, and the top map sends d to hw for some $h \in k[y]$. By commutativity, we have $h = y$.

The induced α_p-action on E gives the morphism $\alpha_p \times E \to E \times_{k[d]} E$

$$\begin{array}{ccc} \dfrac{\mathscr{O}_E[y', w']}{(w'^p - 1, y'w' - d)} & \longrightarrow & \mathscr{O}_E[a]/(a^p) \\ y' & \longmapsto & \dfrac{y}{1+ay} \\ w' & \longmapsto & (1+ay)w \end{array}$$

which is not an isomorphism over $d = 0$ by similar reason. Now, as before if we take the schematic image of α_p in $\mathrm{Aut}_{k[d]}(E)$, we still get α_p since any α_p-bundle over $\mathbb{A}^1$ is trivial. However, we get a different α_p-action. Let us denote the new parameter of α_p by a'. Then the schematic image is given by

$$\begin{array}{ccc} k[d][a']/(a'^p) & \longrightarrow & k[d][a]/(a^p) \\ a' & \longmapsto & ad \end{array}$$

and the morphism $\alpha_p \times E \to E \times_{k[d]} E$ is[4]

$$\begin{array}{ccc} \dfrac{\mathscr{O}_E[y', w']}{(w'^p - 1, y'w' - d)} & \longrightarrow & \mathscr{O}_E[a']/(a'^p) \\ y' & \longmapsto & \dfrac{d}{w+a'} \\ w' & \longmapsto & w+a' \end{array}$$

[3] The isomorphism between $\mathrm{pr}_1^*\mathscr{G}_{D'}$ and $\mathrm{pr}_2^*\mathscr{G}_{D'}$ is given by $a \mapsto a + s_1 - s_2$, where s_1, s_2 stand for parameters of two pieces of D'. It clearly satisfies the cocycle condition, hence a descent datum for $\mathscr{G}_{D'}$.

[4] As we can see, over $d = 0$, this new α_p-action effects only on the μ_p-fiber over $y = 0$ which is μ_p-equivariant, and it descends to certain stacky action on the point $y = 0$ which is the fixed point of the original α_p-action.

where we also have a μ_p-action on α_p, given by $a' \to \xi \cdot a'$. The new α_p-action on E is visibly μ_p-equivariant, and the above morphism is evidently an isomorphism. Therefore, the Frobenius morphism F can be lifting to $\mathbb{P}^1 \to \mathbb{P}^1(\sqrt[p]{0})$ as an $[\alpha_p/\mu_p]$-torsor.

4.2.3 Remark on moduli stack of stable p-torsors

As a consequence of the result of this section, if $Y \to \mathbb{P}^1_k$ is an Artin–Schreier $\mathbb{Z}/p$-cover, then it lifts to a torsor over a twisted curve

$$\begin{tikzcd} & \mathbb{P}^1(x_1^{1/p}, ..., x_r^{1/p}) \arrow[d] \\ Y \arrow[ur, dashed] \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

where $x_1, ..., x_r$ are branch points on $\mathbb{P}^1$. Let us denote $\Sigma_i := x_i^{1/p}$ for $1 \leqslant i \leqslant r$. Then it gives rise to a stable p-torsor in the sense of Abramovich–Romagny [AR12], if we mark suitable untwisted points $\Sigma_{r+1}, ..., \Sigma_n$ on $\mathbb{P}^1(x_1^{1/p}, ..., x_r^{1/p})$ such that Y is a stable curve with respect to the marking divisors $P_i = \Sigma_i \times_{\mathbb{P}^1(...)} Y$.

In paticular, if $Y \to \mathbb{P}^1$ is an Artin–Schreier cover ramified at only one point $\infty \in \mathbb{P}^1$, then

(1) If $g(Y) = 0$, then after marking one more point x on $\mathbb{P}^1(\infty^{1/p})$, it gives rise to a stable p-torsor in the moduli stack $\mathbf{ST}_{p,0,0,2}$;

(2) If $g(Y) > 0$, we don't mark any other points. The twisted lifting gives rise to a stable p-torsor in the moduli stack $\mathbf{ST}_{p,g(Y),0,1}$.

Note that for $\mathbf{ST}_{p,0,0,2}$, its generic geometric fiber only has one object, the Kummer cover of genus 0 with two twisted markings. The two marking divisors on the top curve have degree 1 over the base field. Since we do require that in a family of stable p-torsors, the marking divisors on the top curve need to be étale over the base (cf. Definition 1.3.0.3), hence the stable p-torsor in (1) does not lift to characteristic 0. In other words, the stable p-torsor in (1) lives on some extra components of $\mathbf{ST}_{p,0,0,2}$. As for the moduli stack $\mathbf{ST}_{p,g(Y),0,1}$ where $g(Y) \geqslant 1$, its generic fiber $\mathbf{ST}_{p,g(Y),0,1} \otimes \mathbb{Q}$ is empty, simply because there are no Galois p-covers over $\mathbb{P}^1$ ramified at one point in characteristic 0 (or just other than p).

For the case of Frobenius, it is an unstable p-torsor. But after we blow up some point, it also gives an unliftable stable p-torsor.

In particular, the statement

> *"The number m of twisted markings is determined by $(2g-2) = p(2h-2) + m(p-1)$ and it is equivalent to fix h or m, ..."*

in [AR12] page 758, is not true. Here the number m stands for the number of twisted markings. Let m_0 denote

$$m_0 = \frac{(2g-2) - p(2h-2)}{p-1}.$$

In characteristic p, we may very well have some extra components of $\mathbf{ST}_{p,g,h,n}$ with smaller $m < m_0$. Hence it has a natural stratification according to the index m

$$\mathbf{ST}_{p,g,h,n} = \mathbf{ST}^{\text{tame}}_{p,g,h,n} \coprod \left(\coprod_{1 \leqslant m < m_0} \mathbf{ST}^{\text{wild},m}_{p,g,h,n} \right)$$

where the tame component $\mathbf{ST}^{\text{tame}}_{p,g,h,n}$ classifies stable p-torsors which can lift to characteristic 0, the wild components $\mathbf{ST}^{\text{wild},m}_{p,g,h,n}$ only live in characteristic p which each of them contains an open (could be empty) substack classifying wild Galois p-covers, and they are mutually disjoint.

4.3 Geometry of moduli spaces $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ and $\mathbf{TORS}_X(G)$

In this section, we study moduli spaces of torsors over a curve, under the group scheme $\mathcal{G}^\lambda$ and cocyclic group schemes G. These group schemes arise from the moduli space $\mathbf{ST}_{p,g,h,n}$ of stable p-torsors.

4.3.1 The 2-category of commutative group stacks and exact sequences

In this subsection, we review some aspects of commutative group stacks. For details of the theory of commutative group stacks, we refer the reader to [SGA4-3] exposé XVIII and Brochard's paper [Br14]. Throughout this section, we fix a base scheme X.

4.3.1.1 Definition. A *Picard category* G is a groupoid together with a bifunctor

$$+ : G \times G \longrightarrow G,$$

and two functorial isomorphisms

$$\lambda_{x,y,z} : (x+y)+z \xrightarrow{\sim} x+(y+z)$$

$$\tau_{x,y} : x+y \xrightarrow{\quad\sim\quad} y+x$$

satisfying:

(1) the pentagon axiom and the hexagon axiom (cf. [SGA4-3] XVIII, 1.4.1);

(2) for any objects x, y, one has $\tau_{y,x} \circ \tau_{x,y} = \mathrm{id}_{x+y}$ and $\tau_{x,x} = \mathrm{id}_{x+x}$;

(3) for any object x, the functor $y \mapsto x+y$ is an equivalence of categories.

4.3.1.2 Definition. A *commutative X-group stack* is a stack G over the category of X-schemes, together with a bifunctor $+ : G \times_X G \to G$, and functorial 2-isomorphisms λ, τ as above, such that for any X-scheme T, $G(T)$ with $+, \lambda, \tau$ form a Picard category.

Besides abelian sheaves, the simplest example of a commutative group stack is a classifying stack of an abelian sheaf. Let G be an abelian sheaf over X. Let us recall the definition of the classifying stack B_XG of G/X, as a fibred category over the category of X-schemes. For any X-scheme T, $B_XG(T)$ is the groupoid of G_T-torsors over T, where $G_T = G \times_X T$. Morphisms are G_T-equivariant isomorphisms. In particular, B_XG is the quotient stack $[X/G]$ with the trivial G-action on X.

Let us specify the group law of the group object B_XG in the category of X-stacks. Let $T \to X$ be a X-scheme, the group multiplication

$$B_XG(T) \times B_XG(T) \longrightarrow B_XG(T)$$

sends a pair of G_T-torsors P_1, P_2 to the G_T-torsor $(P_1 \times P_2)/G_T$, where the G_T-action on the product $P_1 \times P_2$ is given by

$$g : (p_1, p_2) \longmapsto (p_1 g, g^{-1} p_2)$$

we write the action in a set-theoretic style for convenience, but it is clear how to transform it into the categorical one. The unit element of $B_XG(T)$ is the trivial torsor G_T, and the inverse of a G_T-torsor P is $\mathrm{Hom}_{B_XG(T)}(P, G_T)$.

Let $\mathbf{CGS}_X$ denote the 2-category of commutative X-group stacks, and $\mathbf{CGS}_X^\flat$ its underlying category, namely, the objects are commutative group stacks and morphisms are isomorphism classes of homomorphisms in $\mathbf{CGS}_X$. Then there is the following equivalence of categories:

4.3.1.3 Theorem.[[SGA4-3] XVIII, 1.4.15] *The functor* $\mathrm{ch}(-)$ *from* $D^{[-1,0]}(X, \mathbb{Z})$ *the derived category of length one complexes of abelian sheaves to* $\mathbf{CGS}_X^\flat$ *by sending* $G^\bullet = [G^{-1} \to G^0]$ *to the quotient stack* $\mathrm{ch}(G^\bullet) = [G^0/G^{-1}]$*, is an equivalence of categories.*

4.3.1.4 Definition. Let G be a commutative group stack over X. We define $H^0(G)$ to be its coarse moduli sheaf and $H^{-1}(G)$ the automorphism group of a neutral section of G. In particular, for any length one complex $G^\bullet$ of abelian sheaves, one has $H^i(G^\bullet) \simeq H^i(\mathrm{ch}(G^\bullet))$ for $i = -1, 0$.

Now let us define exact sequences of commutative group stacks:

4.3.1.5 Definition. A sequence of commutative group stacks

$$G' \longrightarrow G \longrightarrow G''$$

is called exact if the sequences of abelian sheaves

$$H^i(G') \longrightarrow H^i(G) \longrightarrow H^i(G'')$$

are exact for $i = -1, 0$.

Let $\mathbf{CGSh}_X$ denote the category of sheaves of abelian group over X. Then $B_X(-)$ defines a 2-functor from $\mathbf{CGSh}_X$ to $\mathbf{CGS}_X$. If $\varphi : H \to G$ is a homomorphism of abelian sheaves over X, then the morphism $B_X(\varphi) : B_XH \to B_XG$ sends a H-torsor Q to the induced G-torsor

$\mathrm{Ind}_G^H(Q) := (G \times Q)/H$.

Moreover, the functor $B_X(-)$ is exact, namely, if

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

is a short exact sequence of X-group schemes, then we have the exact sequence

$$0 \longrightarrow BG' \longrightarrow BG \longrightarrow BG'' \longrightarrow 0$$

Indeed, the corresponding object of BG (resp. BG', BG'') in $D^{[-1,0]}(X,\mathbb{Z})$ is $\{G \to 0\}$ (resp. $\{G' \to 0\}$, $\{G'' \to 0\}$), and clearly we have

$$0 \longrightarrow H^i(BG') \longrightarrow H^i(BG) \longrightarrow H^i(BG'') \longrightarrow 0$$

for $i = -1, 0$. Though, one should be careful that, it does not mean that $BG' \to BG$ (resp. $BG \to BG''$) is a monomorphism (resp. epimorphism) as in the 2-category of stacks. For example, an exact sequence

$$0 \longrightarrow \mathscr{H} \longrightarrow \mathscr{G}$$

of commutative group stacks gives a monomorphism $\mathscr{H} \to \mathscr{G}$ if and only if $H^{-1}(\mathscr{H}) \to H^{-1}(\mathscr{G})$ is an isomorphism.

4.3.2 Syntomic property of flat algebraic group stacks

It is well-known that any flat group scheme of finite presentation $G \to S$ is a local complete intersection, or equivalently, syntomic. For example, see Corollaire 5.5.1 in [SGA3-1] Exp. $\mathrm{VII_B}$. In this section, we will show that the conclusion remains true for any flat algebraic group stack $\mathcal{G} \to S$.

In this section, any algebraic (not necessarily commutative) group stack $\mathcal{G}/S$ is finitely presented.

4.3.2.1 Lemma. *Let $G \to S$ be a flat group scheme. Then the classifying stack $B_SG \to S$ is a local complete intersection.*

Proof : It is known that any group scheme and torsors are local complete intersections. Notice that we have the following diagram

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[dr, equal] & B_SG \arrow[d] \\ & S \end{tikzcd}$$

the top arrow u is the universal G-torsor, which is a local complete intersection. Thus B_SG is syntomic-locally syntomic over S, hence globally syntomic by [SP19, Tag 036S]. □

4.3.2.2 Proposition. *Let $\mathcal{G}$ be an algebraic group stack over a field k. Then $\mathcal{G}$ is a local complete intersection.*

Proof : Let $e : \mathrm{Spec}(k) \to \mathcal{G}$ be the unit morphism of the group. Recall that the *inertia stack* $I_{\mathcal{G}}$ of $\mathcal{G}$ is by definition the pullback of the diagonal along itself

$$\begin{tikzcd} I_{\mathcal{G}} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "\Delta_{\mathcal{G}/k}"] \\ \mathcal{G} \arrow[r, "\Delta_{\mathcal{G}/k}"] & \mathcal{G} \times \mathcal{G} \end{tikzcd}$$

as a category, an object of $I_{\mathcal{G}}$ over a k-scheme U is a pair (x, α), where x is an object of $\mathcal{G}(U)$ and α is an automorphism of x in $\mathcal{G}(U)$. Let H be the automorphism k-group scheme of the unit e, namely, it is the pullback

$$\begin{tikzcd} H \arrow[r] \arrow[d] & I_{\mathcal{G}} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "e"] & \mathcal{G} \end{tikzcd}$$

We claim that $I_{\mathcal{G}} \simeq H \times \mathcal{G}$. Let

$$t_x : \mathcal{G} \longrightarrow \mathcal{G}$$

be the left translation by a point $x : \mathrm{Spec}(k) \to \mathcal{G}$. Then we can define a morphism

$$\begin{array}{rcl} \Phi : I_{\mathcal{G}} & \longrightarrow & H \times \mathcal{G} \\ (x, \alpha) & \longmapsto & (t_x^* \alpha, x) \end{array}$$

where the pullback $t_x^*\alpha$ is an automorphism of the unit e

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[rr, equal] \arrow[dr, "t_x^*\alpha"] \arrow[dd, "e"'] & & \mathrm{Spec}(k) \arrow[dr, "\alpha"] \arrow[dd] & \\ & \mathrm{Spec}(k) \arrow[rr, equal] \arrow[dl, "e"] & & \mathrm{Spec}(k) \arrow[dl, "x"] \\ \mathcal{G} \arrow[rr, "t_x"] & & \mathcal{G} & \end{tikzcd}$$

It is clear that Φ has an inverse

$$\begin{array}{rcl} \Psi : H \times \mathcal{G} & \longrightarrow & I_{\mathcal{G}} \\ (\beta, x) & \longmapsto & (x, t_{x^{-1}}^* \alpha) \end{array}$$

which concludes our claim $I_{\mathcal{G}} \simeq H \times \mathcal{G}$.

In particular, the morphism of algebraic stacks $I_{\mathcal{G}} \to \mathcal{G}$ is flat. By [SP19, Tag 06QJ], $\mathcal{G}$ is a H-gerbe over an algebraic space G, and moreover G is also the coarse moduli space of $\mathcal{G}$. Now we claim that G is a group algebraic space, hence a group scheme, cf. [Ar69].

We will use the following lemma:

4.3.2.3 Lemma. *Let $\mathcal{X} \to X$ be a gerbe. Then $\mathcal{X} \times \mathcal{X}$ is a gerbe over $X \times X$.*

Proof : The formation of gerbes commutes with any base change,[5] hence $\mathcal{X} \times X$ is a gerbe over $X \times X$, and $\mathcal{X} \times \mathcal{X}$ is a gerbe over $\mathcal{X} \times X$. Therefore the composition

$$\mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times X \longrightarrow X \times X$$

[5]See [SP19, Tag 06QE]

is a gerbe as well. □

Thus $\mathcal{G}\times\mathcal{G}$ has the coarse moduli space $G\times G$, and is a gerbe over it. Using the universal property of coarse moduli spaces, one obtains a morphism of multiplication on G

$$\begin{tikzcd}
\mathcal{G}\times\mathcal{G} \arrow[r, "m_{\mathcal{G}}"] \arrow[d] & \mathcal{G} \arrow[d] \\
G\times G \arrow[r, dashed, "m_G"] & G
\end{tikzcd}$$

and a morphism of inversion

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "i_{\mathcal{G}}"] \arrow[d] & \mathcal{G} \arrow[d] \\
G \arrow[r, dashed, "i_G"] & G
\end{tikzcd}$$

moreover one has the unit morphism given by the composition

$$\mathrm{Spec}(k) \xrightarrow{\ e\ } \mathcal{G} \longrightarrow G$$

These morphisms satisfy the axioms for group objects. For example, for the associativity, one has the diagram induced by the universal property of coarse moduli spaces as follow

$$\begin{tikzcd}
& \mathcal{G}\times\mathcal{G}\times\mathcal{G} \arrow[rr, "\mathrm{id}_{\mathcal{G}}\times m_{\mathcal{G}}"] \arrow[dl, "m_{\mathcal{G}}\times\mathrm{id}_{\mathcal{G}}"'] \arrow[dd] & & \mathcal{G}\times\mathcal{G} \arrow[dl, "m_{\mathcal{G}}"'] \arrow[dd] \\
\mathcal{G}\times\mathcal{G} \arrow[rr, "m_{\mathcal{G}}"] \arrow[dd] & & \mathcal{G} \arrow[dd] & \\
& G\times G\times G \arrow[rr, dashed, "\mathrm{id}_G\times m_G"] \arrow[dl, dashed, "m_G\times\mathrm{id}_G"'] & & G\times G \arrow[dl, dashed, "m_G"] \\
G\times G \arrow[rr, dashed, "m_G"] & & G &
\end{tikzcd}$$

Indeed, because the morphism

$$\mathcal{G}\times\mathcal{G}\times\mathcal{G} \longrightarrow G\times G\times G$$

is an epimorphism, thus the coincidence of the two compositions

$$\mathcal{G}\times\mathcal{G}\times\mathcal{G} \twoheadrightarrow G\times G\times G \underset{m_G\circ(\mathrm{id}_G\times m_G)}{\overset{m_G\circ(m_G\times\mathrm{id}_G)}{\rightrightarrows}} G$$

implies that the bottom dotted square is commutative, which verifies the associativity for group algebraic spaces. The laws of inverse and identity are similar. Thus we conclude that the coarse moduli space G of $\mathcal{G}$ is a group algebraic space, hence a group scheme over k.

The morphism $\mathcal{G}\to G$ is flat and surjective, see [SP19, Tag 06QI], therefore by Lemma 4.3.2.1, $\mathcal{G}$ is syntomic over G. Since G is a group scheme, which is a lci, thus $\mathcal{G}$ is a lci over k. □

4.3.2.4 Corollary. *Let $\mathcal{G}\to S$ be a flat algebraic group stack of finite presentation. Then $\mathcal{G}$ is syntomic over S.*

4.3.3 Weil restriction of classifying stacks

Let T be a scheme, G/T a commutative group scheme over T, and $f : T \to S$ a morphism of schemes. In this section, we study the commutative S-group stack $f_*(B_TG)$, which is the Weil restriction of the T-gerbe B_TG via f. In particular, we will see in which cases the Weil restriction of a classifying stack is still a classifying stack. From this section on, we only work with the *small syntomic sites*[6], unless mentioned otherwise.

4.3.3.1 Lemma. *The automorphism group of a neutral section of $f_*(B_TG)$ is the Weil restriction f_*G over S. Moreover, there is a canonical short exact sequence*

$$0 \longrightarrow B_S(f_*G) \longrightarrow f_*(B_TG) \longrightarrow R^1f_*G \longrightarrow 0$$

Proof : Let $S' \to S$ be a syntomic S-scheme, the commutative group stack $f_*(B_TG)$ gives

$$f_*(B_TG)(S') = B_TG(T \times_S S') = \big\{G_{S'}\text{-torsors over } T_{S'}\big\}$$

the unit object is the trivial torsor $G_{S'} \to T_{S'}$, whose $T_{S'}$-automorphism (as a torsor) group scheme is

$$\mathrm{Aut}_T(G/T)(S') = G_{S'} = f_*G(S').$$

Hence the automorphism group of a neutral section of $f_*(B_TG)$ is f_*G. The coarse moduli sheaf of $f_*(B_TG)$ is R^1f_*G, so we have the exact sequence

$$0 \longrightarrow B_S(f_*G) \longrightarrow f_*(B_TG) \longrightarrow R^1f_*G \longrightarrow 0$$

via the canonical short exact sequence of a commutative group stack, cf. [Br14] Example 2.12.

More explicitly, the monomorphism

$$B_S(f_*G) \hookrightarrow f_*(B_TG)$$

is given by firstly sending a $(f_*G)_{S'}$-torsor $P \to S'$ to its inverse image along f

$$f^{-1}P \longrightarrow T_{S'}$$

We claim that this is a $(f^{-1}f_*G)_{S'}$-torsor. Indeed, since the inverse image functor f^{-1} is exact, we have the isomorphism

$$(f^{-1}f_*G)_{S'} \times f^{-1}P \simeq f^{-1}\big((f_*G)_{S'} \times P\big) \xrightarrow{f^{-1}(m_P \times \mathrm{pr}_2)} f^{-1}(P \times P) \simeq f^{-1}P \times f^{-1}P$$

induced by the $G_{S'}$-torsor structure m_P on P, it gives the $(f^{-1}f_*G)_{S'}$-action and the torsor structure on $f^{-1}P$. Then we take the induced $G_{S'}$-torsor via the adjunction $f^{-1}f_*G \to G$

$$\mathrm{Ind}^G_{f^{-1}f_*G}(f^{-1}P) := f^{-1}P \times^{f^{-1}f_*G} G \xrightarrow{G_{S'}} T_{S'}$$

the monomorphism $B_S(f_*G)(S') \hookrightarrow f_*(B_TG)(S')$ is thus defined by $\mathrm{Ind}^G_{f^{-1}f_*G} \circ f^{-1}$. □

Let us now consider the following conditions:

[6]For the definition of small syntomic site, cf. [SP19], Tag 03XB.

(1) The morphism $f : T \to S$ is finite and the group scheme is smooth over T. In this case we may work over small étale sites, since the natural functor $X_{\mathrm{syn}} \to X_{\text{ét}}$ is acyclic, cf. [Mi80] Theorem III 3.9;

(2) The morphism $f : T \to S$ is a closed immersion, G arbitrary.

4.3.3.2 Proposition. *Under either of the above assumptions, we have the isomorphism*

$$\mathrm{Ind}^G_{f^{-1}f_*G} \circ f^{-1} : \ B_S(f_*G) \xrightarrow{\ \sim\ } f_*(B_T G)$$

of commutative group stacks. The inverse map is given by the direct image functor f_*.

Proof : In both cases above, the pushforward f_* is exact,[7] hence $R^1 f_* G = 0$ and we obtain the isomorphism via the exact sequence of Lemma 4.3.3.1.

Let us specify the inverse morphism. Let S' be a S-scheme. Let $E \to T_{S'}$ be a $G_{S'}$-torsor, its image along the inverse morphism

$$f_*(B_T G) \xrightarrow{\ \sim\ } B_S(f_*G)$$

is the sheaf $f_*E \to S'$. Indeed, because f_* is left exact, it commutes with finite products, hence

$$(f_*G)_{S'} \times f_*E \simeq f_*(G_{S'} \times E) \xrightarrow{f_*(m_E \times \mathrm{pr}_2)} f_*(E \times E) \simeq f_*E \times f_*E$$

which gives the $(f_*G)_{S'}$-action on f_*E. We claim that f_*E is locally nonempty.

Case (1): Let S' be an étale S-scheme, the pullback along f is denoted by $T_{S'}$, and the morphism $T_{S'} \to S'$ is denoted by f'. We shrink S' such that f' is finite locally free. Let $T' \to T_{S'}$ be an étale morphism such that E restricted on T' is a trivial torsor.

$$\begin{tikzcd} T \arrow[r, "f"] & S \\ T_{S'} \arrow[u, "\mathrm{t}"] \arrow[r, "f'"] & S' \arrow[u, "\mathrm{t}"'] \\ T' \arrow[u, "\mathrm{t}"] & \end{tikzcd}$$

Now let us consider the Weil restriction $f'_*T' \to S'$, which is representable and étale (cf. [CGP15] Proposition A.5.2). Its pullback $f'^* f'_* T'$ fits into the following commutative diagram

$$\begin{tikzcd} T_{S'} \arrow[r, "f'"] & S' \\ T' \arrow[u, "\mathrm{t}"'] & \\ f'^* f'_* T' \arrow[u, "N"'] \arrow[uu, bend left, "\mathrm{t}"] \arrow[r] & f'_* T' \arrow[uu, "\mathrm{t}"'] \end{tikzcd}$$

[7] For Case (1), see [Mi80] Chapter 2, Corollary 3.6; and for Case (2), see [Et09] Théorème 1.5 (2).

where $N : f'^* f'_* T' \to T'$ is the adjunction map, which is étale since both T' and $f'^* f'_* T'$ are étale over $T_{S'}$. Thus

$$f_* E(f'_* T') = E(f' * f'_* T') = \mathrm{res}_{f' * f'_* T', T'}(E(T')) \neq \emptyset$$

is nonempty.

Case (2): Recall that in this case, the topology is syntomic, and f is a closed immersion. Let $U \to T$ be a syntomic morphism such that $E|_U$ is a trivial torsor. Using the lifting property of syntomic morphisms from a closed subscheme, we may shrink U (Zariski-locally) such that there exists a syntomic morphism $V \to S$ such that the diagram

$$\begin{tikzcd} T \arrow[r, hook, "f"] & S \\ U \arrow[u, "\mathrm{syn}"] \arrow[r, hook] & V \arrow[u, "\mathrm{syn}"'] \end{tikzcd}$$

is cartesian. Thus

$$f_* E(V) = E(U) \neq \emptyset$$

is nonempty.

Therefore in both cases, $f_* E \to S'$ is a well-defined $(f_* G)_{S'}$-torsor. It is clear that f_* is the inverse of $\mathrm{Ind}^G_{f^{-1} f_* G} \circ f^{-1}$. $\square$

4.3.3.3 Remark. (i) The inverse morphism defined in previous proposition does not apply to general case. For example, let $f : X \to k$ be a connected proper scheme X over a field k, and let $\mathcal{L}$ be a nontrivial invertible sheaf on X. Then $\mathcal{L} \backslash \{0\}$ is a $\mathbb{G}_m$-torsor over X, and the pushforward $f_*(\mathcal{L} \backslash \{0\})$ is the abelian group of nonvanishing global sections of $\mathcal{L}$, which is empty as long as $\mathcal{L}$ is not isomorphic to $\mathcal{O}_X$. Thus $f_*(\mathcal{L} \backslash \{0\})$ is not a torsor under $f_* \mathbb{G}_{m,X} = \mathbb{G}_{m,k}$, i.e., the inverse morphism

$$\mathscr{P}ic_{X/k} = f_*(B_X \mathbb{G}_{m,X}) \dashrightarrow B_k(f_* \mathbb{G}_{m,X}) = B_k \mathbb{G}_{m,k}$$

is not well-defined. Indeed, the existence of such morphism would imply that the $\mathbb{G}_m$-gerbe structure on $\mathscr{P}ic_{X/k}$ is trivial, which is not the case.

(ii) In Case (2), if we only require that f is a finite morphism, then the conclusion of previous proposition is false in general. In other words, the direct image functor along a finite morphism is in general not exact in syntomic topology. Here is a counterexample: let us consider a scheme X over a field k of characteristic $p > 0$, and let $k[\varepsilon]$ be the k-algebra of dual numbers. We set

$$f : X' := X_{k[\varepsilon]} \longrightarrow X$$

to be the natural projection, and the group scheme that we shall consider is $\mu_{p,X'}$. By Kummer theory, we have the short exact sequence in syntomic topology

$$0 \longrightarrow \mu_{p,X'} \longrightarrow \mathbb{G}_{m,X'} \xrightarrow{F} \mathbb{G}_{m,X'} \longrightarrow 0$$

Applying the pushforward f_*

$$0 \longrightarrow \mu_{p,X} \oplus \mathcal{O}_X \cdot \varepsilon \longrightarrow \mathcal{O}_X^\times \oplus \mathcal{O}_X \cdot \varepsilon \xrightarrow{f_*(F)} \mathcal{O}_X^\times \oplus \mathcal{O}_X \cdot \varepsilon \longrightarrow R^1 f_* \mu_{p,X'}$$

the cokernel of $f_*(F)$ is visibly $\mathscr{O}_X \cdot \varepsilon \simeq \mathscr{O}_X$, which is mapped into $R^1 f_* \mu_{p,X'}$. In particular, $R^1 f_* \mu_{p,X'}$ is not trivial, therefore the monomorphism

$$B_X(f_* \mu_{p,X'}) \hookrightarrow f_*(B_{X'} \mu_{p,X'})$$

is not isomorphic.

4.3.4 Representability and flatness of $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ and $\mathbf{TORS}_X(G)$ in regular case

Let us fix a base scheme S. Let X/S be a proper, geometrically connected, and generically irreducible curve over S, and $\mathscr{L}$ an invertible sheaf on X. Suppose that there is a global section λ of $\mathscr{L}^{-1}$, such that λ is fiberwise regular, namely, for any closed point $s \in S$, the homomorphism

$$\lambda_s : \mathscr{L}|_{X_s} \hookrightarrow \mathscr{O}_{X_s}$$

is injective. Let D be the effective Cartier divisor defined by λ, its ideal sheaf is $\mathscr{L} = \mathscr{O}_X(-D)$. Throughout this subsection, we assume that D is nontrivial.

4.3.4.1 Lemma. *Under the assumption that λ is fiberwise regular, the divisor D is a relative effective Cartier divisor of X/S.*

Proof : For the notion of relative Cartier divisors, we refer to Section 1.1 of the book by Katz–Mazur [KM85]. The ideal sheaf $\mathscr{L}$ is already an invertible sheaf. By [Mi80] Chapter I, Proposition 2.5, the flatness of D/S is implied by fiberwise regularity of the section λ. $\square$

The pair $(\mathscr{L}, \lambda)$ gives rise to a smooth one-dimensional group scheme $\mathcal{G}^\lambda$ over X. We briefly recall the construction.[8] The section λ gives a morphism

$$\lambda : \mathbb{V}(\mathscr{L}) \longrightarrow \mathbb{G}_{a,X}$$

where $\mathbb{V}(\mathscr{L}) = \operatorname{Spec}\big(\operatorname{Sym}(\mathscr{L}^{-1})\big)$ is the geometric line bundle associated to the invertible sheaf $\mathscr{L}$. Then $\mathcal{G}^\lambda$ is defined via the cartesian square

$$\begin{tikzcd} \mathcal{G}^\lambda \arrow[r, "1+\lambda"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathbb{G}_{m,X} \arrow[d, hook] \\ \mathbb{V}(\mathscr{L}) \arrow[r, "1+\lambda"] & \mathbb{G}_{a,X} \end{tikzcd}$$

As a sheaf on X, its points with values in a X-scheme Y are

$$\mathcal{G}^\lambda(Y) = \{u \in H^0(Y, \mathscr{L} \otimes \mathscr{O}_Y);\ 1 + \lambda \otimes u \in H^0(Y, \mathscr{O}_Y)^\times\}$$

and the group multiplication is given by

$$(u_1, u_2) \longmapsto u_1 + u_2 + \lambda \otimes u_1 \otimes u_2$$

[8] The construction does not depend on the specific base X and the regularity condition of λ. For more details, we refer to Appendix A of [AR12].

Let us fix the notations for structure morphisms:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "g"'] & X \arrow[d, "h"] \\ & S \end{tikzcd}$$

For the morphism $1+\lambda$ in the definition of $\mathcal{G}^\lambda$, one deduces the following exact sequence of sheaves of abelian groups on the small syntomic site X_{syn}

$$0 \longrightarrow \mathcal{G}^\lambda \xrightarrow{1+\lambda} \mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,D} \longrightarrow 0$$

Applying the direct image functor h_*, one obtains a long exact sequence of abelian sheaves

$$\begin{tikzcd} 0 \arrow[r] & h_*\mathcal{G}^\lambda \arrow[r] & h_*\mathbb{G}_{m,X} = \mathbb{G}_{m,S} \arrow[r] & g_*\mathbb{G}_{m,D} \arrow[dll] \\ & R^1h_*\mathcal{G}^\lambda \arrow[r] & \underline{\mathrm{Pic}}_{X/S} \arrow[r] & R^1h_*(i_*\mathbb{G}_{m,D}) \arrow[r] & \dots \end{tikzcd}$$

We claim that $h_*\mathcal{G}^\lambda$ is trivial. For any syntomic S-scheme S', we have

$$\begin{aligned} h_*\mathcal{G}^\lambda(S') &= \mathcal{G}^\lambda(X') \\ &= \{u \in H^0(X', \mathscr{L}_{X'});\ 1+\lambda\otimes u \in H^0(X', \mathscr{O}_{X'})^\times\} \end{aligned}$$

where $X' = X \times S'$. To show that $h_*\mathcal{G}^\lambda(S')$ is trivial, we may assume that S' is strict henselian. Let κ denote the residue field of S'. If X'_κ is irreducible, then $H^0(X'_\kappa, \mathscr{L}_\kappa) = 0$, since D_κ is nontrivial. By Theorem 12.11 in [Ha83], we have

$$H^0(X', \mathscr{L}_{X'}) \otimes \kappa \xrightarrow{\sim} H^0(X'_\kappa, \mathscr{L}_\kappa) = 0$$

By Nakayama, $H^0(X', \mathscr{L}_{X'}) = 0$. Hence any section in $h_*\mathcal{G}^\lambda(S')$ vanishes over the locus of irreducible fibers, therefore it vanishes, i.e., the abelian sheaf $h_*\mathcal{G}^\lambda$ is trivial. In particular, it means that a $\mathcal{G}^\lambda$-torsor over X has no nontrivial automorphisms. Let $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ denote the moduli S-stack of $\mathcal{G}^\lambda$-torsors over X. It is a commutative group stack, which is nothing but $h_*B_X\mathcal{G}^\lambda$. As we have seen from Lemma 4.3.3.1, there is a canonical exact sequence of commutative group stacks

$$0 \longrightarrow B_S h_*\mathcal{G}^\lambda \longrightarrow \mathbf{TORS}_X(\mathcal{G}^\lambda) \longrightarrow R^1h_*\mathcal{G}^\lambda \longrightarrow 0$$

hence we have $\mathbf{TORS}_X(\mathcal{G}^\lambda) \simeq R^1h_*\mathcal{G}^\lambda$. As for $R^1h_*(i_*\mathbb{G}_{m,D})$, thanks to Proposition 4.3.3.2, we have

$$\begin{aligned} R^1h_*(i_*\mathbb{G}_{m,D}) &= H^0(h_*B_X(i_*\mathbb{G}_{m,D})) \\ &= H^0(g_*B_D\mathbb{G}_{m,D}) = H^0(B_S g_*\mathbb{G}_{m,D}) = \{0\} \end{aligned}$$

thus the previous long exact sequence becomes

$$0 \longrightarrow U \longrightarrow \mathbf{TORS}_X(\mathcal{G}^\lambda) \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

where $U := g_*\mathbb{G}_{m,D}/\mathbb{G}_{m,S}$.

4.3.4.2 Proposition. *The moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable by a smooth S-scheme of relative dimension $g+d-1$, where $d = \deg D$ and g is the genus of X.*

Proof : From the short exact sequence

$$0 \longrightarrow U \longrightarrow \mathbf{TORS}_X(\mathcal{G}^\lambda) \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

we see that $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is a U-torsor over the smooth group scheme $\underline{\mathrm{Pic}}_{X/S}$ of relative dimension g over S. It suffices to show that U is a smooth group scheme of dimension $d-1$ over S. This is clear, since the Weil restriction $g_*\mathbb{G}_{m,D}$ is representable and smooth of dimension d, and U is the quotient of $g_*\mathbb{G}_{m,D}$ by its one-dimensional smooth subgroup $\mathbb{G}_{m,S}$, hence it is smooth as well, of dimension $d-1$. □

Next we consider the moduli stack $\mathbf{TORS}_X(G)$, where G is the kernel of the isogeny $\varphi : \mathcal{G}^\lambda \to \mathcal{G}^{\lambda^p}$. To ensure the existence of such isogeny, from now on, we assume the following condition: there exists a section $\mu \in H^0(X, \mathscr{L}^{p-1})$ such that $\lambda^{p-1} \otimes \mu = p$. In fact, together with fiberwise regularity of λ, we are now automatically in characteristic p. Indeed, if there is some geometric point s of the base S with residual characteristic different from p, then λ_s is invertible by the condition $\lambda_s^{p-1} \otimes \mu_s = p$, hence D is trivial which is out of our consideration. Note that in characteristic p, regularity of λ also implies $\mu = 0$.

The finite flat group scheme G comes with a canonical cogenerator $\kappa : G \to \mu_{p,X}$, which is induced from the morphism $1+\lambda$ on $\mathcal{G}^\lambda$. Moreover, the cogenerator κ fits into the Kummer sequence

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\kappa"] & \mathcal{G}^\lambda \arrow[r, "\varphi"] \arrow[d, "1+\lambda"] & \mathcal{G}^{\lambda^p} \arrow[r] \arrow[d, "1+\lambda^p"] & 0 \\
0 \arrow[r] & \mu_{p,X} \arrow[r] & \mathbb{G}_{m,X} \arrow[r, "F"] & \mathbb{G}_{m,X} \arrow[r] & 0
\end{tikzcd}$$

see Theorem A.2 in [AR12]. Since all the three vertical morphisms are monomorphisms of abelian sheaves on X_{syn}, we can further complete it as follows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d, "\kappa"] & \mathcal{G}^\lambda \arrow[r, "\varphi"] \arrow[d, "1+\lambda"] & \mathcal{G}^{\lambda^p} \arrow[r] \arrow[d, "1+\lambda^p"] & 0 \\
0 \arrow[r] & \mu_{p,X} \arrow[r] \arrow[d] & \mathbb{G}_{m,X} \arrow[r, "F"] \arrow[d] & \mathbb{G}_{m,X} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & i_*\mathbb{G}_{m,D} \arrow[r] \arrow[d] & i'_*\mathbb{G}_{m,D'} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $K = \mathrm{coker}(\kappa)$, $D' := V(\lambda^p) = p \cdot D$, and the closed immersion $D' \hookrightarrow X$ is denoted by i'.

Let us once more fix the notations for structure morphisms:

$$\begin{tikzcd}
D \arrow[r, hook, "\alpha"] \arrow[rr, hook, bend left, "i"] \arrow[drr, "g"'] & D' \arrow[r, hook, "i'"] \arrow[dr, "g'"] & X \arrow[d, "h"] \\
& & S
\end{tikzcd}$$

Applying the direct image functor h_* to the short exact sequence $0 \to G \to \mu_{p,K} \to K \to 0$, we obtain the long exact sequence

$$0 \longrightarrow \mu_{p,S} \longrightarrow h_*K \longrightarrow \mathbf{TORS}_X(G) \longrightarrow \underline{\mathrm{Pic}}_{X/S}[p] \longrightarrow R^1h_*K \longrightarrow \dots$$

In order to analyze R^1h_*K, let us denote $K_0 := i^{-1}K$. Note that $i_*K_0 = K$, by Théorème 1.5 (6) in [Et09], hence we deduce that

$$R^1h_*K = H^0(h_*B_Xi_*K_0) = H^0(B_Sg_*K_0) = \{0\}.$$

Thus the previous sequence reduces to

$$0 \longrightarrow U_0 \longrightarrow \mathbf{TORS}_X(G) \longrightarrow \underline{\mathrm{Pic}}_{X/S}[p] \longrightarrow 0$$

where $U_0 := g_*K_0/\mu_{p,S}$.

4.3.4.3 Theorem. *The moduli stack $\mathbf{TORS}_X(G)$ is representable by a finite flat group scheme of degree p^{2g+d-1}, where d is the degree of D and g is the genus of X.*

Proof : First we show that $\mathbf{TORS}_X(G)$ is representable. Applying h_* to the exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{G}^{\lambda} \longrightarrow \mathcal{G}^{\lambda^p} \longrightarrow 0$$

and recall that $h_*\mathcal{G}^{\lambda}, h_*\mathcal{G}^{\lambda^p}$ and h_*G are trivial, hence we have an exact sequence of groups

$$0 \longrightarrow \mathbf{TORS}_X(G) \longrightarrow \mathbf{TORS}_X(\mathcal{G}^{\lambda}) \longrightarrow \mathbf{TORS}_X(\mathcal{G}^{\lambda^p})$$

which shows that $\mathbf{TORS}_X(G)$ is the kernel of a group homomorphism. Consequently, $\mathbf{TORS}_X(G)$ is represented by the fiber product

$$\mathbf{TORS}_X(G) = \mathbf{TORS}_X(\mathcal{G}^{\lambda}) \times_{\mathbf{TORS}_X(\mathcal{G}^{\lambda^p})} S.$$

The question concerning flatness is fppf-local, so that we may assume that $S = \mathrm{Spec}(R)$ and that, moreover,

$$D \simeq \mathrm{Spec}\big(R[T]/F(T)\big) = \mathrm{Spec}\big(R[T]/(T-s_1)...(T-s_r)\big).$$

In order to explicitly describe the Weil restriction g_*K_0, let us specify the homomorphism $g_*\mathbb{G}_{m,D} \to g'_*\mathbb{G}_{m,D'}$. Let $\mathrm{Spec}(A)$ be an affine syntomic R-scheme. The homomorphism between A-valued points is given by

$$\begin{array}{ccc} \big(A[T]/F(T)\big)^{\times} & \longrightarrow & \big(A[T]/F(T)^p\big)^{\times} \\ a & \longmapsto & \widetilde{a}^p \end{array}$$

where $\widetilde{a}$ is a lifting of a to $A[T]/F(T)^p$. Therefore the set of A-values of g_*K_0 is

$$g_*K_0(A) = \left\{ \sum_{j=0}^{d-1} b_j T^j \in \big(A[T]/F(T)\big)^\times \;\middle|\; b_0^p = 1,\ b_1^p = ... = b_{d-1}^p = 0 \right\}$$

Thus it is clear that g_*K_0 is representable by a group scheme, whose underlying scheme is

$$g_*K_0 = \mathrm{Spec}\big(R[B_0, ..., B_{d-1}]/(B_0^p - 1, B_1^p, ..., B_{d-1}^p)\big)$$

which is clearly finite flat over S, of degree p^d. In particular, as the quotient of g_*K_0 by its flat subgroup scheme $\mu_{p,S}$, U_0 is a finite flat group scheme of degree p^{d-1} over S. Therefore, as a torsor over the finite flat group scheme $\underline{\mathrm{Pic}}_{X/S}[p]$ of order p^{2g} under the finite flat group scheme U_0, the moduli scheme $\mathbf{TORS}_X(G)$ is finite flat over S, of degree p^{2g+d-1}. □

4.3.4.4 Remark. In the proof, the representability of the Weil restriction g_*K_0 is rather special. A priori, K_0 is a subsheaf of $\mu_{p,D}$ which is not representable, and g_*K_0 is a subsheaf of $g_*\mu_{p,D}$. It is known that the Weil restriction of a smooth group scheme along a finite flat morphism is representable by a smooth scheme, cf. [BLR90] Theorem 7.6.4 and Proposition 7.6.5, and in particular it is flat over the base. While here $\mu_{p,D}$ is not smooth in characteristic p, although $g_*\mu_{p,D}$ is representable, in general it is not flat. For example, let us consider the following simple case of $g : D \to S$

$$g : D = \mathrm{Spec}\big(R[T]/T(T-\pi)\big) \longrightarrow S = \mathrm{Spec}(R)$$

where we take R to be a discrete valuation ring, and π is a uniformizer of R. Then for any syntomic R-algebra A,

$$g_*\mu_{p,D}(A) = \left\{ b_0 + b_1 T \in (A[T]/T(T-\pi))^\times \;\middle|\; (b_0 + b_1 T)^p = b_0^p + \pi^{p-1} b_1^p T = 1 \right\}$$

thus the underlying scheme of $g_*\mu_{p,D}$ is

$$g_*\mu_{p,D} = \mathrm{Spec}\big(R[B_0, B_1]/(B_0^p - 1, \pi^{p-1}B_1^p)\big)$$

which is apparently not flat over R. However, it turns out, as we have seen, that g_*K_0 is flat. In fact here g_*K_0 is exactly the closure of the generic fiber of $g_*\mu_{p,D}$.

4.3.5 Weil restriction of $\mathbb{G}_m$ along a twig

In this section, we study the Weil restriction of the multiplicative group $\mathbb{G}_m$ along a nonflat morphism. Let $S = \mathrm{Spec}(R)$ be the base scheme, where R is a discrete valuation ring with the maximal ideal $\mathfrak{m} = (\pi)$ and the residual field k.

4.3.5.1 Definition. A *twig* is a S-scheme $g : D \to S$ where D is the union of X_n a proper curve over $S_n = \mathrm{Spec}(R/\mathfrak{m}^n)$ and D_0 a (nonempty) finite flat scheme over S, such that D can be embedded into a proper curve X/S with D_0 being an effective Cartier divisor of X which is an infinitesimal neighborhood of an étale divisor.

A key result of this section is the following:

4.3.5.2 Proposition. *Let $g : D \to S$ be a twig. Then the Weil restriction $g_*\mathbb{G}_{m,D}$ is representable by a smooth group scheme over S.*

Proof : The question is local on the base, so we may assume that D_0 has the form

$$D_0 = \mathrm{Spec}\big(R[T]/F(T)\big)$$

where $F(T) = \prod_{i=1}^{r}(T - s_i)^{a_i}$ is a splitting polynomial with disjoint roots, in other words, by Chinese Remainder Theorem, we have

$$D_0 = \mathrm{Spec}\Big(\prod_{i=1}^{r} R[T]/(T - s_i)^{a_i}\Big) = \coprod_{i=1}^{r} \mathrm{Spec}\big(R[T]/(T - s_i)^{a_i}\big)$$

Let A be a syntomic R-algebra. The set of A-values of $g_*\mathbb{G}_{m,D}$ is given by

$$g_*\mathbb{G}_{m,D}(A) = \Big\{ f \in (A[T]/F(T))^\times \;\Big|\; f(s_1) \equiv f(s_2) \equiv ... \equiv f(s_r) \mod \mathfrak{m}^n \Big\}$$

Moreover, an element $f \in (A[T]/F(T))^\times$ can be written as $(f_1, ..., f_r)$ for certain $f_i \in \big(A[T]/(T- s_i)^{a_i}\big)$ via the Chinese Remainder Theorem. If we write

$$f_i(T) = \sum_{j=0}^{a_i-1} b_{i,j}T^j$$

for $1 \leqslant i \leqslant r$, then the congruence condition is

$$b_{1,0} \equiv b_{2,0} \equiv ... \equiv b_{r,0} \mod \mathfrak{m}^n.$$

Thus the Weil restriction $g_*\mathbb{G}_{m,D}$ is representable by the following scheme

$$g_*\mathbb{G}_{m,D} \simeq \mathrm{Spec}\Big(R\Big[\{B_{i,j}\}_{1\leqslant i\leqslant r}^{1\leqslant j\leqslant a_i-1}, B_{1,0}^{\pm}, A_{2,0}, ..., A_{r,0}\Big]\Big)$$

where an element $f \in (A[T]/F(T))^\times$ with the above notations is given by the morphism

$$\begin{array}{rcl} R\Big[\{B_{i,j}\}_{1\leqslant i\leqslant r}^{1\leqslant j\leqslant a_i-1}, B_{1,0}^{\pm}, A_{2,0}, ..., A_{r,0}\Big] & \longrightarrow & A \\ B_{i,j} & \longmapsto & b_{i,j} \\ A_{i,0} & \longmapsto & (b_{i,0} - b_{1,0})/\pi^n \end{array}$$

Therefore, $g_*\mathbb{G}_{m,D}$ is representable by a smooth group scheme of dimension $d = \deg D$. □

Now let us go back to the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$. In Section 4.3.4, we have studied it under the condition that λ is fiberwise regular over the base. Here let us assume that the base $S = \mathrm{Spec}(R)$ is a discrete valuation ring, and λ satisfies the following assumption:

> (*) λ is degenerate on some irreducible component X_0 of the special fiber X_k, with multiplicity one. And the horizontal part of $D = V(\lambda)$ is a relative Cartier divisor of X/S. In other words, D is a twig over S.

4.3.5.3 Theorem. *Let S be a discrete valuation ring scheme, X/S a proper, geometrically connected and generically irreducible curve, $\mathscr{L}$ an invertible sheaf over X, and $\lambda \in H^0(X, \mathscr{L}^{-1})$ satisfying the condition (*). Then the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable by a smooth S-group scheme of dimension $g + d - 1$.*

Proof : The cohomological long exact sequence of

$$0 \longrightarrow \mathcal{G}^\lambda \longrightarrow \mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,D} \longrightarrow 0$$

is now different

$$0 \longrightarrow U \longrightarrow \mathbf{TORS}_X(\mathcal{G}^\lambda) \longrightarrow \underline{\mathrm{Pic}}_{X/S} \xrightarrow{i^*} \underline{\mathrm{Pic}}_{D/S}$$

since in the current case, the Picard functor $\underline{\mathrm{Pic}}_{D/S}$ of the twig D is no longer trivial. Let N be the kernel $\ker i^*$, be cautious that N is the kernel of a homomorphism of sheaves on the small syntomic site S_{syn}, which is different from the kernel of those on the big syntomic site S_{SYN}.

First we claim that $\underline{\mathrm{Pic}}_{D/S}$ is isomorphic to $\iota_*\underline{\mathrm{Pic}}_{X_0/k}$, where ι denotes the closed immersion $\mathrm{Spec}(k) \hookrightarrow S$. This is clear, because D is the union of X_0 with a relative effective Cartier divisor D_0, whose coarse Picard scheme is trivial (cf. Proposition 4.3.3.2), thus an isomorphism class of invertible sheaves over D is completely determined by its restriction on X_0, and hence the claim.

Let H_k be the kernel of the following homomorphism

$$\underline{\mathrm{Pic}}_{X_k/k} \longrightarrow \underline{\mathrm{Pic}}_{X_0/k}$$

where X_k is the special fiber of X, and the homomorphism is given by restriction to X_0. This is clearly a smooth closed subgroup scheme of $\underline{\mathrm{Pic}}_{X_k/k}$, which is isomorphic to the Picard scheme of the proper curve given by pinching the component X_0 of X_k.

Now the kernel N can be identified with the dilatation of the S-group scheme $\underline{\mathrm{Pic}}_{X/S}$ along the smooth subgroup scheme H_k of its special fiber, via the universal property of dilatation. By Theorem 1.7 [WW80], N is representable by a smooth group scheme over S. Therefore, from the short exact sequence

$$0 \longrightarrow U \longrightarrow \mathbf{TORS}_X(\mathcal{G}^\lambda) \longrightarrow N \longrightarrow 0$$

the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is a torsor over the smooth S-group scheme N under the group scheme U, which is smooth by Proposition 4.3.5.2. Hence $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is representable and smooth over S, of relative dimension $g+d-1$, since N has the same dimension of $\underline{\mathrm{Pic}}_{X/S}$. $\square$

4.4 $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ and the generalized Jacobian

In this section, let X/S be a smooth proper curve with connected geometric fibers, and D a relative effective Cartier divisor of X with $\deg D = d > 0$. Let X_0 denote $X\backslash D$, and D_n denote the effective Cartier divisor nD for $n \in \mathbb{Z}_{\geqslant 1}$. Recall that the one-dimensional smooth X-group scheme $\mathcal{G}^\lambda$ is defined by the line bundle $\mathscr{L} = \mathscr{O}_X(D)$ and a section $\lambda \in H^0(X, \mathscr{L}^{-1})$, or in other words, it is the dilatation of $\mathbb{G}_{m,X}$ along the trivial subgroup supported by D. Similarly, the group scheme $\mathcal{G}^{\lambda^n}$ is the dilatation of $\mathbb{G}_{m.X}$ along the trivial subgroup supported over D_n. The notations for structure morphisms are the following

$$\begin{tikzcd} D_n \arrow[r, hook, "i_n"] \arrow[dr, "g_n"'] & X \arrow[d, "h"] \\ & S \end{tikzcd}$$

The following construction is classical, see [Se88], [CC79], [CC90]. The singular curve $X^{(n)}/S$ is defined by the pushout

$$\begin{tikzcd} X \arrow[r, "\pi^{(n)}"] & X^{(n)} \arrow[d, shift left, "h^{(n)}"] \\ D_n \arrow[u, "i_n"] \arrow[r, "g_n"] & S \arrow[u, shift left, "e_n"] \end{tikzcd}$$

i.e., $X^{(n)} := X \coprod_{D_n} S$. There is a natural short exact sequence of sheaves of abelian groups over $X^{(n)}$

$$0 \longrightarrow \mathbb{G}_{m,X^{(n)}} \longrightarrow \pi^{(n)}_* \mathbb{G}_{m,X} \longrightarrow \pi^{(n)}_* \mathbb{G}_{m,X}/\mathbb{G}_{m,X^{(n)}} \simeq e_{n*}(g_{n*}\mathbb{G}_{m,D}/\mathbb{G}_{m,S}) \longrightarrow 0$$

let U_n denote $g_{n*}\mathbb{G}_{m,D}/\mathbb{G}_{m,S}$. Applying the pushforward $h^{(n)}_*$ to the sequence, one obtains

$$0 \longrightarrow h^{(n)}_* \mathbb{G}_{m,X^{(n)}} \xrightarrow{\sim} h_*\mathbb{G}_{m,X} \xrightarrow{0} U_n \longrightarrow \underline{\mathrm{Pic}}_{X^{(n)}/S} \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

or just

$$0 \longrightarrow U_n \longrightarrow \underline{\mathrm{Pic}}_{X^{(n)}/S} \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

4.4.0.1 Proposition. *The moduli scheme* $\mathbf{TORS}_X(\mathcal{G}^{\lambda^n})$ *is isomorphic to the Picard scheme* $\underline{\mathrm{Pic}}_{X^{(n)}/S}$.

Proof : For the group scheme $\mathcal{G}^{\lambda^n}$, we have the following short exact sequence on the small étale site of X

$$0 \longrightarrow \mathcal{G}^{\lambda^n} \longrightarrow \mathbb{G}_{m,X} \longrightarrow i_{n*}\mathbb{G}_{m,D_n} \longrightarrow 0$$

From this short exact sequence, we can describe objects of $\mathbf{TORS}_X(\mathcal{G}^{\lambda^n})$ as follows: let T be a S-scheme, an object of $\mathbf{TORS}_X(\mathcal{G}^{\lambda^n})(T)$ is a line bundle L over X_T such that the induced $i_{n*}\mathbb{G}_{m,D_n}$-torsor (via $\mathbb{G}_{m,X} \to i_{n*}\mathbb{G}_{m,D_n}$) is trivial together with a trivialization, in other words, a given trivialization of the restriction $\mathscr{L}|_{D_n \times T}$.

Applying h_* to the above short exact sequence, one has

$$0 \longrightarrow h_*\mathcal{G}^{\lambda^n} \longrightarrow h_*\mathbb{G}_{m,X} \longrightarrow g_{n*}\mathbb{G}_{m,D_n} \longrightarrow \mathbf{TORS}_X(\mathcal{G}^{\lambda^n}) \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

where $h_*\mathcal{G}^{\lambda^n} = 0$ and $\mathbb{G}_{m,S} \simeq h_*\mathbb{G}_{m,X}$, thus the above sequence reduces to

$$0 \longrightarrow U_n \longrightarrow \mathbf{TORS}_X(\mathcal{G}^{\lambda^n}) \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow 0$$

Let us define a morphism

$$\mathbf{TORS}_X(\mathcal{G}^{\lambda^n}) \longrightarrow \underline{\mathrm{Pic}}_{X^{(n)}/S}$$

by sending an object $(L, \varphi : L|_{D_n} \simeq \mathscr{O}_{D_n})$ of $\mathbf{TORS}_X(\mathcal{G}^{\lambda^n})$ to the line bundle $L \coprod_\varphi 1_S$, where 1_S is the trivial line bundle over S, and we treat φ as the gluing condition

$$\varphi : \; i_n^* L = L|_{D_n} \xrightarrow{\sim} \mathscr{O}_{D_n} = g_n^* 1_S$$

This morphism turns out to be an equivalence of categories, according to Ferrand's result [Fe03] Théorème 2.2. More directly, the above morphism yields the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_n \arrow[r] \arrow[d, equal] & \mathbf{TORS}_X(\mathcal{G}^{\lambda^n}) \arrow[r] \arrow[d, "\sim"] & \underline{\mathrm{Pic}}_{X/S} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_n \arrow[r] & \underline{\mathrm{Pic}}_{X^{(n)}/S} \arrow[r] & \underline{\mathrm{Pic}}_{X/S} \arrow[r] & 0
\end{tikzcd}$$

which forces the middle vertical morphism to be an isomorphism. □

4.4.0.2 Remark. Note that the assumption $d = \deg D > 0$ is important. If $d = 0$, the moduli stack $\mathbf{TORS}_X(\mathcal{G}^\lambda)$ is the Picard stack $\mathscr{P}ic_{X/S}$, which is not representable, and certainly not isomorphic to its coarse moduli space $\underline{\mathrm{Pic}}_{X/S}$.

For any $m \geqslant n \geqslant 1$, there is the natural morphism

$$\mathbf{TORS}_X(\mathcal{G}^{\lambda^m}) \longrightarrow \mathbf{TORS}_X(\mathcal{G}^{\lambda^n})$$

by sending $(L, \varphi : L|_{D_m} \simeq \mathscr{O}_{D_m})$ to $(L, \varphi|_{D_n})$. The projective limit of $\{\mathbf{TORS}_X(\mathcal{G}^{\lambda^n})\}_{n\geqslant 1}$ gives rise to the generalized Jacobian of the open relative curve X_0/S:

$$J^\bullet_\infty(X_0/S) := \lim_n \underline{\mathrm{Pic}}_{X^{(n)}/S} \simeq \lim_n \mathbf{TORS}_X(\mathcal{G}^{\lambda^n}).$$

via the isomorphism of Proposition 4.4.0.1.

4.5 A categorical classification of $\mathcal{G}^\lambda$-torsors

In this section, we give a categorical classification of $\mathcal{G}^\lambda$-torsors, which slightly generalizes a result of Andreatta and Gasbarri [AG07].

We fix a base scheme S. A finite flat group scheme G/S with a cogenerator is equivalently given by a datum (M, λ, μ),[9] where M is an invertible sheaf on S, with $\lambda \in H^0(S, M^{-1})$ and $\mu \in H^0(S, M^{p-1})$ such that $\lambda^{p-1} \otimes \mu = p \cdot 1_{\mathscr{O}_S}$. The group scheme G can be embedded into a smooth 1-dimensional group scheme $\mathcal{G}^\lambda/S$, which is defined by

$$\begin{tikzcd}
\mathcal{G}^\lambda \arrow[r, "1+\lambda\otimes -"] \arrow[d, hook] & \mathbb{G}_{m,S} \arrow[d, hook] \\
\mathbb{V}(M) \arrow[r, "1+\lambda\otimes -"] & \mathbb{G}_{a,S}
\end{tikzcd}$$

as a group functor, $\mathcal{G}^\lambda(T)$ for some S-scheme $T \to S$ is given by

$$\mathcal{G}^\lambda(T) = \{u \in H^0(T, M_T);\ 1 + \lambda \otimes u \in H^0(T, \mathscr{O}_T)^\times\}.$$

[9] cf. [AR12] Appendix I.

Furthermore, we have the following diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d] & \mathcal{G}^{\lambda} \arrow[r, "\varphi"] \arrow[d, "1+\lambda\otimes -"] & \mathcal{G}^{\lambda^p} \arrow[r] \arrow[d, "1+\lambda^p\otimes -"] & 0 \\ 0 \arrow[r] & \mu_{p,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r, "F"] & \mathbb{G}_{m,S} \arrow[r] & 0 \end{tikzcd}$$

where the left vertical arrow is the cogenerator, F is the Frobenius of $\mathbb{G}_{m,S}$, and φ is explicitly given by

$$\varphi(u) = u^p + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\lambda^{i-1} \otimes \mu \otimes u^i.$$

which shows that the embedding of G into $\mathcal{G}^{\lambda}$ is compatible with Kummer theory via the cogenerator.

Let $\mathbf{TORS}_S(G)$ (resp. $\mathbf{TORS}_S(\mathcal{G}^{\lambda})$) be the category of G-torsors (resp. $\mathcal{G}^{\lambda}$-torsors) over S. Let $\mathfrak{CD}^{M,\lambda}(S)$ be the category of *classifying data* of $\mathcal{G}^{\lambda}$-torsors, consisting of triples (L, E, Ψ), where

1. L is an invertible sheaf on S;
2. E is an extension of L by $\mathcal{O}_S$;
3. $\Psi : E \to E \otimes M^{-1}$ is an $\mathcal{O}_S$-linear morphism such that
   3.1 $\ker \Psi = \mathcal{O}_S$;
   3.2 the induced map $E/\ker \Psi \to (E/\ker \Psi) \otimes M^{-1}$ is multiplication by λ on L;
   3.3 the quotient $E \otimes M^{-1}/\Psi(E)$ is an invertible sheaf on S;

Moreover, the morphisms between classifying data are those satisfying some natural commutative diagrams, see the details after Definition 3.1 [AG07]. We describe the trivial datum. Let $L^{\mathrm{triv}} = M^{-1}$, and $E^{\mathrm{triv}} = \mathcal{O}_S \oplus M^{-1}$. We denote A_S^{λ} to be the $\mathcal{O}_S$-algebra such that $\mathcal{G}^{\lambda} = \mathrm{Spec}(A_S^{\lambda})$, more explicitly, it is

$$A_S^{\lambda} = \mathrm{Sym}(M^{-1})\left[\frac{1}{1+\lambda}\right].$$

Then it is clear that $L^{\mathrm{triv}}, E^{\mathrm{triv}} \subset A_S^{\lambda}$. The $\mathcal{O}_S$-linear morphism Ψ^{triv} is given by

$$\begin{array}{rcl} \Psi^{\mathrm{triv}} : \mathcal{O}_S \oplus M^{-1} & \longrightarrow & M^{-1} \oplus M^{-2} \\ 1 & \longmapsto & 0 \\ e & \longmapsto & e + \lambda e \end{array}$$

where e is a local section of M^{-1}. We will see in the proof of the next theorem that $(L^{\mathrm{triv}}, E^{\mathrm{triv}}, \Psi^{\mathrm{triv}})$ is the trivial datum which corresponds to the trivial $\mathcal{G}^{\lambda}$-torsor over S. Let

$$c^{\lambda} : A_S^{\lambda} \longrightarrow A_S^{\lambda} \otimes_{\mathcal{O}_S} A_S^{\lambda}$$

be the comultiplication. Then the preimage of $A_S^{\lambda} \otimes E^{\mathrm{triv}}$ under c^{λ} is E^{triv}, the proof is no different than Lemma 3.4 in [AG07].

4.5.0.1 Theorem. *The category $\mathbf{TORS}_S(\mathcal{G}^{\lambda})$ of $\mathcal{G}^{\lambda}$-torsors over S is equivalent to the category $\mathfrak{CD}^{M,\lambda}(S)$ of classifying data.*

Proof : (1) Let $(L, E, \Psi) \in \mathfrak{C}\mathfrak{D}^{M,\lambda}(S)$ be a classifying datum. From the inclusion $\mathscr{O}_S \subset E$, it gives

$$\mathrm{Sym}(\mathscr{O}_S) \simeq \mathscr{O}_S[T] \subset \mathrm{Sym}(E).$$

Let $Z \to S$ be the S-subscheme of $\mathrm{Spec}(\mathrm{Sym}(E))$ defined by $T = 1$, in other words, Z is the preimage of the constant section 1 of the quotient map

$$\mathbb{V}(L^{-1}) \hookrightarrow \mathbb{V}(E^{-1}) \twoheadrightarrow \mathbb{V}(\mathscr{O}_S)$$

The induced map $\Psi : L = E/\ker\Psi \hookrightarrow E \otimes M^{-1}$ gives a map of S-schemes $f : Z \to \mathbb{V}(L^{-1} \otimes M^{-1})$. Now let P be defined by the cartesian diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] & \mathbb{V}(L^{-1} \otimes M^{-1})\backslash\{0\} \arrow[d, hook] \\
Z \arrow[r, "f"] & \mathbb{V}(L^{-1} \otimes M^{-1})
\end{tikzcd}$$

We claim that P/S is a $\mathcal{G}^\lambda$-torsor.

Firstly let us check the trivial datum $(L^{\mathrm{triv}}, E^{\mathrm{triv}}, \Psi^{\mathrm{triv}})$. It is immediate that the S-scheme Z^{triv} is given by

$$Z^{\mathrm{triv}} = \mathrm{Spec}(\mathrm{Sym}(M^{-1})) = \mathbb{V}(M)$$

which is the geometric line bundle associated to the invertible sheaf M. The previous defining diagram for P^{triv} becomes

$$\begin{tikzcd}
P^{\mathrm{triv}} \arrow[r, "1+\lambda"] \arrow[d] & \mathbb{V}(\mathscr{O}_S)\backslash\{0\} \arrow[d, hook] \\
\mathbb{V}(M) \arrow[r, "1+\lambda"] & \mathbb{V}(\mathscr{O}_S)
\end{tikzcd}$$

comparing the above diagram with the one at the beginning of this section, it is clear that $P^{\mathrm{triv}} \simeq \mathcal{G}^\lambda$, the trivial torsor.

In general, let (L, E, Ψ) be a classifying datum, and $P \to S$ defined as above. We show that it is a $\mathcal{G}^\lambda$-torsor. Notice that we have the following map

$$\mathrm{Sym}(E) \longrightarrow \mathrm{Sym}(E) \otimes_{\mathscr{O}_S} \mathrm{Sym}(M^{-1})$$

induced from Ψ. Let us restrict the above map on the locally closed subscheme P of $\mathbb{V}(E^{-1})$, then we can form the following map

$$\begin{array}{rcl} c_P : \mathscr{O}_P & \longrightarrow & \mathscr{O}_P \otimes_{\mathscr{O}_S} A_S^\lambda \\ e & \longmapsto & e \otimes 1 + \Psi(e) \end{array}$$

where e is a local section of $\mathscr{O}_P$, and we use the same notation Ψ for the restriction on P. The map c_P is the coaction corresponding to an action

$$m_P : \mathcal{G}^\lambda \times_S P \longrightarrow P,$$

this is the action that makes P/S a $\mathcal{G}^\lambda$-torsor. The fact that it is a torsor can be verified (fppf) locally, by trivializing E, L and M, the rest is the same as in 3.6 [AG07]. Note that under the morphism $(1+\lambda, f)$, the action m_P transforms to the scalar multiplication of the geometric line bundle $\mathbb{V}(L^{-1} \otimes M^{-1})$

$$\begin{tikzcd} \mathcal{G}^{\lambda} \times P \arrow[r, "m_P"] \arrow[d, "(1+\lambda, f)"'] & P \arrow[d, "f"] \\ \mathbb{G}_{m,S} \times \mathbb{V}(L^{-1} \otimes M^{-1})\backslash\{0\} \arrow[r, "m"] & \mathbb{V}(L^{-1} \otimes M^{-1})\backslash\{0\} \end{tikzcd}$$

where the bottom morphism makes the complement of the zero section of $\mathbb{V}(L^{-1} \otimes M^{-1})$ a $\mathbb{G}_m$-torsor, which is classical.

(2) Let $P \to S$ be a $\mathcal{G}^{\lambda}$-torsor. Since P is affine over S, let $P = \mathrm{Spec}(\mathscr{B})$. Let

$$c_P : \mathscr{B} \longrightarrow \mathscr{B} \otimes_{\mathscr{O}_S} A^{\lambda}$$

be the coaction of the $\mathcal{G}^{\lambda}$-torsor P, and let E be the preimage of $\mathscr{B} \otimes E^{\mathrm{triv}}$ under c_P

$$\begin{tikzcd} \mathscr{B} \arrow[r, "c_P"] & \mathscr{B} \otimes_{\mathscr{O}_S} A^{\lambda} \\ E \arrow[r] \arrow[u, hook] & \mathscr{B} \otimes_{\mathscr{O}_S} E^{\mathrm{triv}} \arrow[u, hook] \end{tikzcd}$$

By local triviality of $P \to S$, E is a locally free $\mathscr{O}_S$-module of rank 2, and it contains an $\mathscr{O}_S$ as a submodule. Let L be the quotient $E/\mathscr{O}_S$, which is an invertible sheaf. Due to local triviality of the torsor P, it is clear that c_P maps E to $E \otimes E^{\mathrm{triv}}$. Define an $\mathscr{O}_S$-linear map $\Psi : E \to E \otimes M^{-1}$ by

$$\Psi := c_P|_E - \mathrm{id} \otimes 1$$

A priori Ψ maps E to $E \otimes E^{\mathrm{triv}}$, still by local triviality, it is immediate to check that the image is in $E \otimes M^{-1}$. The triple (L, E, Ψ) is the classifying data of P.

(3) Finally, to show that the functors in (1) and (2) are mutually inverse, it reduces to local case, hence we apply directly the argument in [AG07] 3.7. $\square$

5
On monadic spaces

5.1 Definition

Let (A, A^+) be a Huber pair. The usual adic space $\mathrm{Spa}(A, A^+)$, as a set, consists of equivalent classes of continuous valuations with values in totally ordered abelian groups. Especially, when we take a Huber pair (A, A^+) where A equips with the discrete topology and $A^+ = A$, then the adic space $\mathrm{Spa}(A, A)$ recovers the scheme $\mathrm{Spec}(A)$. In this chapter, we also consider valuations with values in totally ordered commutative monoids with 0 as a minimal element. The underlying commutative monoids of such valuation monoids are also known as $\mathbb{F}_1$-algebras, which from now on we will adapt this name.

A ***totally ordered commutative monoid*** (***tomonoid***) Γ satisfies the condition that any translation by an element $\gamma \in \Gamma$ preserves the total ordering, i.e., if we have $\gamma_1 \leqslant \gamma_2$ in Γ and $\gamma \in \Gamma$, then

$$\gamma \cdot \gamma_1 \leqslant \gamma \cdot \gamma_2.$$

Moreover, we require that [1]

- Γ has an 0 element, namely, $0 \cdot \gamma = 0$ for any $\gamma \in \Gamma$;
- 0 is the minimal element of Γ, and $0 \neq 1$.

5.1.0.1 Definition. Let (A, A^+) be a Huber pair. The ***monadic space*** $\mathrm{MSpa}(A, A^+)$ consists of equivalent classes of continuous valuations of A with values in commutative monoids with 0 as the minimal element, such that $|f| \leqslant 1$ for any valuations and any $f \in A^+$. The topology is given by the same way of adic spaces, namely, it is generated by subsets of the form $\{|\cdot| : |f| \leqslant |g| \neq 0\}$ for $f, g \in A$.

[1] cf. [Hu96] page 38. Note that the condition of minimality of 0 is necessary to state, we might very well have negative elements in a totally ordered commutative monoid. For example, the commutative monoid $\{0, \epsilon, 1\}$ ($\epsilon^2 = 0$) can be given the ordering $\epsilon < 0 < 1$.

5.1.0.2 Remark. Comparing the classification of points in an adic disc, here we have a new kind of points, these are points with values in $\mathbb{F}_1$-algebras which cannot be equivalent to any group-valued valuations. Such points could exist, as we will see from the examples below. We will call this kind of points the ***ghost points***. However, it is easy to see that none of the ghost points is closed, in case of scheme theory (i.e., A has discrete topology and $A^+ = A$), they are sort of "generic points" of non-reduced irreducible subschemes.

5.2 Examples

5.2.1 Schemes as monadic spaces

Let k be an algebraically closed field. Through out this subsection, we only consider the discrete topology on the Huber pair (A, A^+), with $A^+ = A$.

5.2.1.1 Example. Let $A = k[t]/(t^n)$ be an Artinian k-algebra. Then the monadic space $\mathrm{MSpa}(A, A)$ consists of n points, more precisely, they are the following valuations

$$\begin{array}{rcl} x_i : k[t]/(t^n) & \longrightarrow & \mathbb{F}_1[\gamma]/(\gamma^i) \\ t & \longmapsto & \gamma \end{array}$$

where $1 \leqslant i \leqslant n$, and in the $\mathbb{F}_1$-algebra we set $0 < \gamma < 1$ which is the only possible case. The point x_1 is the trivial valuation, which is exactly the only closed point in the scheme $\mathrm{Spec}(A)$. All the other $n-1$ points are ghost points, the open subsets are of the form $\{x_n, x_{n-1}, ..., x_{i+1}, x_i\}$ for some $1 \leqslant i \leqslant n$, indeed, we have

$$\{x_n, x_{n-1}, ..., x_{i+1}, x_i\} = \big\{ |\cdot| : |0| \leqslant |t^{i-1}| \neq 0 \big\}.$$

In particular, we have that $\overline{\{x_i\}} = \{x_i, x_{i-1}, ..., x_1\}$ which is exactly the i-th infinitesimal neighborhood of the closed point x_1, hence it is the generic point of the non-reduced irreducible subscheme $\mathrm{Spec}(k[t]/(t^i))$. And x_n is the only open point, the generic point of $\mathrm{MSpa}(A, A)$.

5.2.1.2 Example. Let $A = k[t]$ be the affine line over k. We know that $\mathrm{Spec}(A)$ is a subspace of $\mathrm{MSpa}(A, A)$, but we will see that there are much more points than the scheme $\mathbb{A}^1_k$. Given any monoid-valued valuation

$$|\cdot| : \; A \longrightarrow \Gamma,$$

mod out the nilradical of Γ and groupify $(\Gamma/\text{nilrad}(\Gamma))\backslash\{0\}$, we obtain a group-valued valuation[2]

$$|\cdot|' : \; A \longrightarrow \text{Grp}((\Gamma/\text{nilrad}(\Gamma))\backslash\{0\}) \cup \{0\},$$

which is a usual point in $\mathrm{Spec}(A)$. If x' is a ghost point, and x is a schematic point which is obtained by the previous process, then we call x the support of x'. Now let x' be a ghost point supported at the point $x_1 = x = (t)$. Then x' stands for a valuation such that the value of t is nilpotent, i.e., $x'(t)$ is in the kernel of the groupification homomorphism

$$\Gamma \longrightarrow \text{Grp}((\Gamma/\text{nilrad}(\Gamma))\backslash\{0\}) \cup \{0\},$$

[2] Here the nilradical is the same as it for commutative rings, namely $\text{nilrad}(\Gamma)$ consists of nilpotent elements in Γ which is certainly an ideal. With our assumption that Γ is totally ordered and 0 is the minimal element, the quotient $\Gamma/\text{nilrad}(\Gamma)$ is in fact the union of 0 and an integral commutative monoid. *Proof: if $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\gamma_1 \cdot \gamma_2 = 0$, assume $\gamma_1 \leqslant \gamma_2$, then $\gamma_1^2 \leqslant \gamma_1 \cdot \gamma_2 = 0$ hence γ_1 is nilpotent. Therefore after modulo the nilradical, there will be no zero divisors.* □

which is clearly nilpotent. Therefore x' has the following representative

$$\begin{array}{rcl} x' : k[t] & \longrightarrow & \mathbb{F}_1[\gamma]/(\gamma^i) \\ t & \longmapsto & \gamma \end{array}$$

with $i \geqslant 2$ the order $0 < \gamma < 1$. This is the ghost point $x_i = x'$ as the generic point of the i-th infinitesimal neighborhood of x_1. If we let $i = +\infty$, then the valuation monoid is $\mathbb{F}_1[\gamma]$ which can be embedded into the rank one totally ordered abelian group $\mathbb{R}_{\geqslant 0}$, i.e., it is the point of t-adic valuation, we denote this point by x_∞. Finally we have the points η of trivial valuation

$$\begin{array}{rcl} \eta : k[t] & \longrightarrow & \mathbb{F}_1 \\ \text{nonzero elements} & \longmapsto & 1 \end{array}$$

Thus as a set, we know that

$$\mathrm{MSpa}(k[t], k[t]) = \{x_i\}_{x_1 = x \in \mathrm{Spec}(k[t]), 1 \leqslant i \leqslant \infty} \cup \{\eta\}$$

where the non-ghost points are

$$\{x_1, x_\infty\}_{x \in \mathrm{Spec}(k[t])} \cup \{\eta\}$$

in other words, these are the points in the Berkovich space of $k[t]$. What are the open subsets of $\mathrm{MSpa}(k[t], k[t])$? Essentially we only need to check $\big\{|\cdot| : |f| \neq 0\big\}$ for some $f \in A$, and moreover we may assume that f is a power of some prime element, or simply just $f = t^k$ for some integer $k \geqslant 1$. Then it is straightforward that

$$U(t^k) := \big\{|\cdot| : |t^k| > 0\big\} = \{x_i\}_{x_1 \neq (t), 1 \leqslant i < \infty} \cup \{(t)_i\}_{k+1 \leqslant i \leqslant \infty} \cup \{\eta\},$$

where the notation $U(t^k)$ is reasonable, because set-theoretically $U(t^k)$ is exactly the complement of the k-th infinitesimal neighborhood $V(t^k) = \mathrm{Spec}(k[t]/(t^k))$. Except that here $V(t^k)$ is different from $V(t)$. Thus the open subsets of $\mathrm{MSpa}(k[t], k[t])$ are complements of closed subschemes of $\mathbb{A}^1_k$, where the fatness of the closed subschemes can be recognized.

5.2.1.3 Remark. Notice that we have a unique generic point for every distinct irreducible subscheme of the monadic affine line, these are fat points of finite length. Thus the new points are exactly all the generic points of fat points of finite length.

5.2.1.4 Example. Let $A = k[u, v]/(u^2, uv)$ be the affine line with a fat origin. As a scheme, $\mathrm{Spec}(A)$ is irreducible. However it has an embedded component given by the associated prime $\mathrm{ann}_A(u) = (u, v)$, this is the part that we cannot see it set-theoretically from the scheme theory. As a monadic space, $\mathrm{MSpa}(A, A)$ has the following points[3]

(1) Valuations with $|u| = 0$. For convenience, we set the notations for the points supported at the origin $x_{1,i} : k[u, v]/(u^2, uv) \to \mathbb{F}_1[\gamma]/(\gamma^i)$, where $u \mapsto 0$ and $v \mapsto \gamma$, $1 \leqslant i \leqslant \infty$;

(2) $x_{2,i}^{\sim} : k[u, v]/(u^2, uv) \to \mathbb{F}_1[\gamma_u, \gamma_v]/(\gamma_u^2, \gamma_u\gamma_v, \gamma_v^i)$, where $\sim$ is an equivalent class of ordering of the right hand side $\mathbb{F}_1$-algebra, and $1 \leqslant i < \infty$;

(3) $x_{2,\infty} : k[u, v]/(u^2, uv) \to \mathbb{F}_1[\gamma_u, \gamma_v]/(\gamma_u^2, \gamma_u\gamma_v)$, where the total ordering is given by $0 < \gamma_u < \gamma_v^m < 1$ for any $m \in \mathbb{N}_+$.

[3]Without specifications, we always mean $* \mapsto \gamma_*$ for indeterminants.

The points in (1) come from the closed immersion

$$i : \mathrm{MSpa}(k[v], k[v]) \hookrightarrow \mathrm{MSpa}(A, A)$$

of the adic affine line, since all the valuations $x_{1,i}$ factor through the quotient $k[u,v]/(u^2, uv) \twoheadrightarrow k[v]$. The single point $x_{2,\infty}$ in (3) is "the" generic point of $\mathrm{MSpa}(A, A)$, and notice that we also have a generic point $x_{1,\infty}$ which is the usual generic point of the scheme $\mathrm{Spec}(A)$, however the closure $\overline{\{x_{1,\infty}\}}$ is the adic affine line, but $\overline{\{x_{2,\infty}\}}$ is really the whole $\mathrm{MSpa}(A, A)$.

The points in (2) are more complicated. If $i = 1$, then $x_{2,1}^{\sim} = x_{2,1}$ is unique, it is the image of the only ghost point of $\mathrm{MSpa}(k[u]/(u^2), k[u]/(u^2))$ under the closed immersion

$$i' : \mathrm{MSpa}(k[u]/(u^2), k[u]/(u^2)) \hookrightarrow \mathrm{MSpa}(A, A).$$

The closure $\overline{\{x_{2,1}\}}$ is visibly the image of the above immersion, in other words, $x_{2,1}$ is (one of) the generic point(s) of "the" embedded component.[4] In case $i \geqslant 2$, there could be different classes of orderings $\sim$ for the $\mathbb{F}_1$-algebra. Let us consider the simplest case $i = 2$. Now we need to find all possible orderings on the $\mathbb{F}_1$-algebra $\mathbb{F}_1[\gamma_u, \gamma_v]/(\gamma_u^2, \gamma_u\gamma_v, \gamma_v^2)$. It has only four elements $\{0, \gamma_u, \gamma_v, 1\}$, the only ordered relation we need to set is between γ_u and γ_v, obviously there are only 3 possibilities

$$\gamma_u \ <,\ =,\ \text{or}\ > \gamma_v,$$

and mutually non-equivalent. We should be careful that in the case $\sim$ is $=$, we can have different valuations into $\mathbb{F}_1[\gamma]/(\gamma^2)$ as following

$$\begin{array}{rcl} k[u,v]/(u^2, uv) & \longrightarrow & \mathbb{F}_1[\gamma]/(\gamma^2) \\ u,\ v & \longmapsto & \gamma \\ u - \lambda v & \longmapsto & 0, \text{ or } \gamma \end{array}$$

where $\lambda \in k^*$. Indeed, we cannot have at the same time that $|u - \lambda v| = |u - \lambda' v| = 0$ for different $\lambda, \lambda' \in k^*$, otherwise we would have

$$|v| = |(\lambda - \lambda^*)v| \leqslant \max\{u - \lambda v, u - \lambda' v\} = 0$$

and similarly also $|u| = 0$. So when we take $|u| = |v| = \gamma$ (i.e., when $\sim$ is $=$), we have the set of points

$$x_{2,2}^{\lambda}\ (\lambda \in k^*),\ x_{2,2}^{=}$$

where $x_{2,2}^{=}$ is the valuation with $|u - \lambda v| = \gamma$ for all $\lambda \in k^*$. Thus in conclusion, when $i = 2$, we have the following set of points

$$x_{2,2}^{<},\ x_{2,2}^{>},\ x_{2,2}^{=},\ x_{2,2}^{\lambda}\ (\lambda \in k^*)$$

Notice that these points are exactly from the immersion of monadic spaces

$$i'' : \mathrm{MSpa}(k[u,v]/(u^2, uv, v^2), k[u,v]/(u^2, uv, v^2)) \hookrightarrow \mathrm{MSpa}(A, A),$$

where $x_{2,2}^{=}$ is the generic point of the image in $\mathrm{MSpa}(A, A)$. Next we analyze the topology among these points with respect to the immersion i'', or equivalently, we are going to analyze the topology of the monadic space $\mathrm{MSpa}(k[u,v]/(u^2, uv, v^2), k[u,v]/(u^2, uv, v^2))$, hence temporarily we just

[4] Since the primary decomposition is not unique, the embedded component is also "not unique", in the sense that we can have different embeddings from $\mathrm{Spec}(k[u]/(u^2))$ to $\mathrm{Spec}(A)$ supported at the same underlying reduced subscheme. This fact will be reflexed by the points with index $i = 2$.

pretend $A = k[u,v]/(u^2, uv, v^2)$. By our previous observation, the space has the following set of points

$$x_{1,1},\ x_{1,2},\ x_{2,1},\ x_{2,2}^{<},\ x_{2,2}^{>},\ x_{2,2}^{=},\ x_{2,2}^{\lambda}\ (\lambda \in k^*).$$

Here $x_{1,1}$ is the only schematic point, namely the origin. We list here some open subsets (these generates the topology):

(1) $\{0 < |u|\} = \{x_{2,1},\ x_{2,2}^{>,=,<},\ \{x_{2,2}^{\lambda}\}\}$

(2) $\{0 < |v|\} = \{x_{1,2},\ x_{2,2}^{>,=,<},\ \{x_{2,2}^{\lambda}\}\}$

(3) $\{|v| \leqslant |u| \neq 0\} = \{x_{2,1},\ x_{2,2}^{>,=},\ \{x_{2,2}^{\lambda}\}\}$

(4) $\{|u| \leqslant |v| \neq 0\} = \{x_{1,2},\ x_{2,2}^{=,<},\ \{x_{2,2}^{\lambda}\}\}$

(5) $\{0 < |u - \lambda_0 v|,\ \lambda_0 \in k^*\} = \{x_{1,2},\ x_{2,1},\ x_{2,2}^{>,=,<},\ \{x_{2,2}^{\lambda}\}_{\lambda \neq \lambda_0}\}$

Here we have a nice explanation of the non-uniqueness of the embedded component. The complements of (2), (5) are[5]

$$\{x_{1,1},\ x_{2,1}\},\ \{x_{1,1},\ x_{2,2}^{\lambda}\}\ (\lambda \in k^*)$$

which correspond to different embeddings of $\mathrm{Spec}(k[u]/(u^2))$ into $\mathrm{Spec}(A)$

$$\begin{array}{rcl} k[u,v]/(u^2, uv, v^2) & \longrightarrow & k[u]/(u^2) \\ v & \longmapsto & \lambda u \end{array}$$

where $\lambda \in k$. When $\lambda = 0$, the generic point of the embedded subspace is $x_{2,1}$, otherwise it is $x_{2,2}^{\lambda}$. It is also clear that the closure of $x_{2,2}^{>}$ (resp. $x_{2,2}^{<}$) is

$$\{x_{1,1},\ x_{2,1},\ x_{2,2}^{>}\}\ (\text{resp. } \{x_{1,1},\ x_{1,2},\ x_{2,2}^{<}\})$$

We have seen that $x_{2,2}^{=}$ is the generic point of the total space. Thus the topology is completely clear for points of index $i = 2$.

The cases for $i \geqslant 3$ are not more complicated, though it looks like that we have much more options on the ordering of the $\mathbb{F}_1$-algebra $\mathbb{F}_1[\gamma_u, \gamma_v]/(\gamma_u^2, \gamma_u\gamma_v, \gamma_v^i)$. However we do have an easy constraint, namely, what we need to settle is the following relation for some integer $1 \leqslant j \leqslant i$

$$\gamma_v^j \leqslant \gamma_u < \gamma_v^{j-1},$$

multiplying γ_v we have $\gamma_v^{j+1} = 0$. Hence $j = i-1$ or i, i.e., we have either $0 < \gamma_u < \gamma_v^{i-1}$ or $\gamma_v^{i-1} \leqslant \gamma_u < \gamma_v^{i-2}$. Therefore similarly, the points of index i are

$$x_{2,i}^{>},\ x_{2,i}^{<},\ x_{2,i}^{=},\ x_{2,i}^{\lambda}\ (\lambda \in k^*)$$

where the upper index $>, <$ depends on $\gamma_u > \gamma_v^{i-1}$ or $\gamma_u < \gamma_v^{i-1}$, and $x_{2,i}^{\lambda}$ is given by

$$\begin{array}{rcl} k[u,v]/(u^2, uv) & \longrightarrow & \mathbb{F}_1[\gamma]/(\gamma^2) \\ u & \longmapsto & \gamma^{i-1} \\ v & \longmapsto & \gamma \\ u - \lambda v^{i-1} & \longmapsto & 0 \end{array}$$

and $x_{2,i}^{=}$ is given by sending all $u - \lambda v^{i-1}$ to γ^{i-1}. The topology is also similar.

[5]Thought here we also have that the closure of $x_{1,2}$ is $\{x_{1,1},\ x_{1,2}\}$, this does not correspond to the embedded component, it is the second infinitesimal neighborhood of the origin in the v-axis direction.

5.2.1.5 Remark.

- The above example for $A = k[u, v]/(u^2, uv)$ already contains all the information of local schemes of the form $\mathrm{Spec}(k[u, v]/(u^2, uv, v^i))$. The description via points for a general local scheme of embedded dimension 2 could be very complicated.

- It is a bit surprising that if we want to describe the monadic space of the node $\mathrm{Spec}(k[u, v]/(uv))$, then Example 5.2.1.4 already gives all that we need. It suffices to describe all local schemes of the form $\mathrm{Spec}(k[u, v]/(u^i, uv, v^j))$ (assume $i, j \geqslant 2$). The orderings on the $\mathbb{F}_1[\gamma_u, \gamma_v]/(\gamma_u^i, \gamma_u\gamma_v, \gamma_v^j)$ are easy to classify: by our observation on the easy constraint, we have that for any integers $1 \leqslant r \leqslant i - 1$ and $1 \leqslant s \leqslant j - 1$

$$0 < \gamma_u^r < \gamma_v^{j-1}, \text{ or } \gamma_v^{j-1} \leqslant \gamma_u^r < \gamma_v^{j-2}$$

and

$$0 < \gamma_v^s < \gamma_u^{i-1}, \text{ or } \gamma_u^{i-1} \leqslant \gamma_v^s < \gamma_u^{i-2}$$

let $r = s = 1$, we find that in any case we always have $\gamma_u < \gamma_v^{j-2}$ and $\gamma_v < \gamma_u^{i-2}$. If one of i, j is strictly larger than 2, say i, then multiplying γ_v on the second inequality, we have $\gamma_v^2 = 0$. Hence it reduces to the case in Example 5.2.1.4.

5.2.1.6 Example. Let $A = k[e]/(e^p - e)$ be the finite étale k-algebra where $\mathrm{char}(k) \neq p - 1$, it is the function algebra of $(\mathbb{Z}/p)_k$. We show that there are no ghost points in $\mathrm{MSpa}(A, A)$. Let x be any monoid-valued valuation of A

$$\begin{aligned} x = |\cdot| : k[e]/(e^p - e) &\longrightarrow \Gamma \\ e &\longmapsto \gamma \end{aligned}$$

If $\gamma > 1$, then $|e^{p-1} - 1| \leqslant \max\{\gamma^{p-1}, 1\}$ hence $|e^{p-1} - 1| = \gamma^{p-1}$. But then

$$0 = |e^p - e| = \gamma \cdot \gamma^{p-1} = \gamma$$

a contradiction. If $\gamma < 1$, then the same argument shows also $\gamma = 0$, which means that x is the schematic point (e). If $\gamma = 1$, let $\zeta \in k$ be a primary $(p-1)$-th root of unity. If for some integer $1 \leqslant i \leqslant p - 1$, we have $|e - \zeta^i| = 0$, then the valuation of any other element of A is determined and the valuation visibly gives the schematic point $(e - \zeta^i)$. If all of $|e - \zeta^i|$ are nonzero, by change of coordinate $e - \zeta^i \mapsto e$ and our previous discussion, $|e - \zeta^i|$ can only be 1. But then

$$0 = |e^p - e| = |e| \cdot \prod_{i=1}^{p-1} |e - \zeta^i| = 1$$

a contradiction. Therefore the monadic space $\mathrm{MSpa}(A, A)$ is identified with the scheme $(\mathbb{Z}/p)_k$ as a topological space (in particular set-theoretically).

5.2.1.7 Example. Now we examine an easy case of fiber products. Let $A = k[\![t]\!][\epsilon]/(\epsilon^2)$. Notice that any element of A with nonzero constant must have valuation 1, since these are invertible. As before, denote $\gamma_t, \gamma_\epsilon$ to be the valuations of t and ϵ in an $\mathbb{F}_1$-algebra Γ. We separate the situation into two cases.

I. γ_t is not nilpotent. We must have $\gamma_t^i \leqslant \gamma_\epsilon < \gamma_t^{i-1}$ for some integer $1 \leqslant i$ or $0 \leqslant \gamma_\epsilon < \gamma_t^i$ for any $i \geqslant 1$. In the first case, we have $\gamma_t^i\gamma_\epsilon = 0$, but then we have $\gamma_t^{2i} \leqslant \gamma_t^i\gamma_\epsilon = 0$, a contradiction. Hence we are in the second case. Moreover, we cannot have $\gamma_\epsilon\gamma_t^i \geqslant \gamma_t^j$ for any $i, j \geqslant 0$, otherwise we would have $\gamma_t^j\gamma_\epsilon \leqslant 0$ which leads to the same contradiction. Therefore for any element in A, we can express it as $f + g\epsilon$, where $f, g \in k[\![t]\!]$, then (assume $f \neq 0$)

$$|f + g\epsilon| \leqslant \max\{|f|, |g|\gamma_\epsilon\} = |f|,$$

hence $|f + g\epsilon| = |f|$, i.e., the valuation is uniquely determined on A if we determine all the algebraic relations in the valuation $\mathbb{F}_1$-algebra Γ.[6] The possible algebraic relations are easy to classify.

First of all, if $\gamma_t = 1$, we obtain two points η_1, η_2, where the former takes $|\epsilon| = 0$ and the latter takes $|\epsilon| > 0$. Now assume $\gamma_t < 1$. We cannot have a relation $|f + g\epsilon| = 0$ for $f \neq 0$, otherwise γ_t would be nilpotent. So the only possibility is $|t^i\epsilon| = 0$ for some $0 \leqslant i \leqslant \infty$.[7] We denote the subset of points corresponding to the relation $\gamma_t^i\gamma_\epsilon$ to be $S_{\infty,2,i}$ for $i \geqslant 1$ and the point $x_{\infty,1}$ for $i = 0$. Note that each subset $S_{\infty,2,i}$ has more than one element, it depends on how we precise the ordering on the following sequence

$$0 = \gamma_\epsilon\gamma_t^i < \gamma_\epsilon\gamma_t^{i-1} \leqslant \gamma_\epsilon\gamma_t^{i-2} \leqslant \dots \leqslant \gamma_\epsilon\gamma_t \leqslant \gamma_\epsilon$$

in $\mathbb{F}_1[\gamma_t, \gamma_\epsilon]/(\gamma_t^i\gamma_\epsilon, \gamma_\epsilon^2)$. It is easy to see that the cardinal of $S_{\infty,2,i}$ is i, since once we have an $=$ at some place, all the orderings of its left hand side would be $=$ as well, by the translation invariance. Thus the first case gives us the following points [8]

$$\eta_1,\ \eta_2,\ x_{\infty,1},\ S_{\infty,2,i} = \{x_{\infty,2,i,j}\}_{0\leqslant j\leqslant i-1}\ (1 \leqslant i \leqslant \infty)$$

II. $\gamma_t^n = 0$ for some integer $n \geqslant 1$.[9] Now we are meant to describe the closed monadic subspace

$$\mathrm{MSpa}(k[t,\epsilon]/(t^n,\epsilon^2), k[t,\epsilon]/(t^n,\epsilon^2)) \subset \mathrm{MSpa}(A, A)$$

The "prototype" valuation $\mathbb{F}_1$-algebra is $\mathbb{F}_1[\gamma_t, \gamma_\epsilon]/(\gamma_t^n, \gamma_\epsilon^2)$, we need to classify all the ordered quotients. Unlike Example 5.2.1.4, things are getting very complicated when n is large. If $n = 2$, the $\mathbb{F}_1$-algebra has five elements, and the possible orderings are

- $0 \leqslant \gamma_t\gamma_\epsilon < \gamma_t < \gamma_\epsilon < 1$
- $0 \leqslant \gamma_t\gamma_\epsilon < \gamma_\epsilon < \gamma_t < 1$
- $0 = \gamma_t\gamma_\epsilon < \gamma_t = \gamma_\epsilon < 1$

If $n = 3$, we got

- $0 = \gamma_t^2\gamma_\epsilon = \gamma_t\gamma_\epsilon < \gamma_\epsilon < \gamma_t^2 < \gamma_t < 1$
- $0 = \gamma_t^2\gamma_\epsilon = \gamma_t\gamma_\epsilon < \gamma_\epsilon = \gamma_t^2 < \gamma_t < 1$

[6] Or in other words, which quotient of $\mathbb{F}_1[\gamma_t, \gamma_\epsilon]/(\gamma_\epsilon^2)$ can be equipped with a compatible ordering such that it is a valuation $\mathbb{F}_1$-algebra of A?

[7] $i = \infty$ means no relations.

[8] Precisely, the point $x_{\infty,2,i,j}$ gives the ordering

$$0 = \gamma_\epsilon\gamma_t^i = \dots = \gamma_\epsilon\gamma_t^j < \gamma_\epsilon\gamma_t^{j-1} < \dots < \gamma_\epsilon\gamma_t < \gamma_\epsilon\ (j \geqslant 1)$$

and all equal if $j = 0$.

[9] We always assume that n is minimal, in case it does not reduce to the cases with smaller n.

- $0 = \gamma_t^2\gamma_\epsilon = \gamma_t\gamma_\epsilon < \gamma_t^2 < \gamma_\epsilon < \gamma_t < 1$
- $0 = \gamma_t^2\gamma_\epsilon < \gamma_t\gamma_\epsilon \leqslant \gamma_t^2 < \gamma_\epsilon < \gamma_t < 1$
- $0 = \gamma_t^2\gamma_\epsilon < \gamma_t^2 \leqslant \gamma_t\gamma_\epsilon < \gamma_t < \gamma_\epsilon < 1$
- $0 < \gamma_t^2\gamma_\epsilon < \gamma_t\gamma_\epsilon < \gamma_\epsilon < \gamma_t^2 < \gamma_t < 1$

which is already complicated. The number of all possible orderings on $\mathbb{F}_1[\gamma_t, \gamma_\epsilon]/(\gamma_t^n, \gamma_\epsilon)$ is a combinatorial problem, we stop here.

5.2.2 Some rigid analytic spaces and adic spaces

5.2.2.1 Example. We consider the Huber pair $(\mathbb{Z}_p, \mathbb{Z}_p)$, with the p-adic topology. Its usual adic space $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ consists of two points, namely, a generic point $x_\infty = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and a closed point x_1 (the valuation pull-back from the trivial valuation on $\mathbb{F}_p = \mathbb{Z}_p/p$). Because here we consider the p-adic topology, hence the trivial valuation on $\mathbb{Z}_p$ (send p to 1) is not continuous.

If instead, we consider the discrete topology on $\mathbb{Z}_p$, then the structure of $\mathrm{MSpa}(\mathbb{Z}_p, \mathbb{Z}_p)$ is basically the same as for $k[\![t]\!]$ in Example 5.2.1.2. Except for the trivial valuation on $\mathbb{Z}_p$, others are all continuous with respect to the p-adic topology. Therefore as a set, $\mathrm{MSpa}(\mathbb{Z}_p, \mathbb{Z}_p)$ is

$$\{x_1,\ x_2,\ \ldots,\ x_\infty\}$$

where

$$\begin{array}{rcl} x_i : \mathbb{Z}_p & \longrightarrow & \mathbb{F}_1[\gamma]/(\gamma^i) \\ p & \longmapsto & \gamma \end{array}$$

The topology is also the same as for $k[\![t]\!]$.

5.2.2.2 Example. Let us consider the adic unit disc $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathscr{O}_{\mathbb{C}_p}\langle T\rangle)$. It is known that there is a classification of points on adic unit disc, divided into five types. For the monadic unit disc $\mathrm{MSpa}(\mathbb{C}_p\langle T\rangle, \mathscr{O}_{\mathbb{C}_p}\langle T\rangle)$, we have the sixth type of points, namely the ghost points. For example, there are the points as below

$$|\cdot| : \mathbb{C}_p\langle T\rangle \twoheadrightarrow \mathbb{C}_p[T]/(T^n) \longrightarrow \Gamma$$

which is supported at the origin $f \mapsto |f(0)|$. There are also ghosts points supported at other types of points, but the classification seems to be very complicated.

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