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Schémas en groupes et familles de courbes

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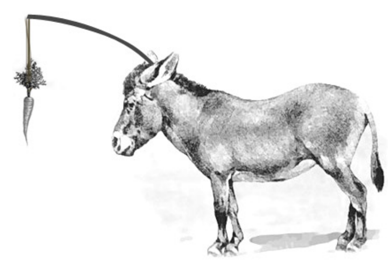
Introduction

Introduction to a carrot. What makes the donkey keep going is the carrot. We are all familiar with the picture of a donkey trotting on, with a stick tied to its back and a carrot hanging from it, at a fixed distance from its muzzle. The donkey will doubtless never catch up with the carrot, but still, the animal is moving on. The truth is even more difficult to bear than that : the donkey will never catch up with the carrot, but this is the very reason why he keeps moving, for ever and ever. What makes the mathematician move on is the dream of new – mathematical – fronteers. Mathematicians will never catch up with their dreams, but still, the dreams are what makes them move on. One little difference between donkeys and mathematicians is that the latter are often aware of the fact that their exertions are vain, but the motive power is about just as powerful. The dreams of mathematicians are varied. The complex geometer is running after the classification of smooth projective complex varieties up to birational equivalence. The arithmetician is running after the intimate structure of the absolute Galois group of the field of rational numbers. If there exists somewhere in the universe a God of mathematicians, he is doubtless thinking: "Keep running!". But the mathematician will not listen.



The modest mathematician – or donkey? – author of the present report wishes to tell you about a dream – a carrot ? – that gives motivation for his efforts. Let us choose a natural number g, a finite group G, and a prime p. The Galois covers $C \to \mathbb{P}^1$ of the projective line by a projective, smooth, geometrically connected algebraic curve of genus g, with Galois group $\mathrm{Gal}(C/\mathbb{P}^1) = G$, are classified by a "variety" that we shall denote $\mathscr{H}_{g,G}$, or simply $\mathscr{H}$ in honor of Hurwitz for his article [Hu]. It has a natural compactification build using stable curves, and we shall denote it $\bar{\mathscr{H}}$. If we take the word "variety" to mean "algebraic stack" then $\bar{\mathscr{H}}$ is proper and smooth. This is a marvellous object, at the crossroads of two topics that travelled down two centuries of mathematics: Galois theory and the moduli space of curves $\mathscr{M}_g$. The stack $\bar{\mathscr{H}}$ may be defined – in many ways – by equations with rational coefficients and hence, after chasing denominators, with integral coefficients. A question that motivates my efforts is:

What is the reduction of $\bar{\mathscr{H}}$ at p?

This means to consider all sets of equations with integral coefficients defining $\bar{\mathscr{H}}$, and to look for a set of equations whose reduction modulo p gives the nicest possible variety $\bar{\mathscr{H}} \otimes \mathbb{F}_p$. The situation is totally different according to whether p divides the order of G or not. The problems that arise are different. They are all very interesting.

Introduction to the contents of this report. The question above is quite general and lead me to consider several subtopics. In this text, I chose to organize them into three main themes corresponding to the three parts of the report: the general problem of quotients in Algebraic Geometry, important for the construction of moduli spaces; the investigation of integral models of group schemes, which is a necessary step for the reduction of Galois coverings when p divides the order of G; the construction and study of moduli spaces of coverings of curves. Discarding a few exceptions, what is set out here may be found in the following six articles:

[4] *Champs de Hurwitz* (with J. Bertin),

[6] *On the adjoint quotient of Chevalley groups over arbitrary base schemes* (with P.-E. Chaput),

[7] *Effective models of group schemes*,

[8] *Composantes connexes et irréductibles en familles*,

[9] *Moduli of Galois p-covers in mixed characteristics* (with D. Abramovich),

[11] *Models of group schemes of roots of unity* (with A. Mézard and D. Tossici).

The numbers in brackets refer to the list of my articles in the end of the report. Let us now say a word about the "few exceptions" we discarded above. This report is a survey of previous work and is not supposed to contain any new result, nor any proof. But it does in fact contain two proofs. The first is about a representability result used in the article [8] with only a very rough proof sketch; this is the opportunity to take a more general as well as more illuminating point of view and to give a full proof. The second is about a case of representability for the functor of connected components that complements the results of the article [8]. This case was indicated to me by Laurent Fargues and I am happy to thank him for that. I take the opportunity to thank Dajano Tossici for the explanation of example 8.5. Finally, a last remark on the contents of the report is essential. The length of the text devoted here to each of the six individual articles above is very far from giving a faithful picture of their respective importance and interest. It is only due to the choice of such-or-such point that I thought would be more useful to examine, or to the prominence of my current concerns.

Notation. Throughout the text, we write $R = (R, K, k, \pi)$ to denote a discrete valuation ring R with fraction field K, residue field k, and a fixed uniformizer π.

Part I
Quotients

In this part, we wish to present some problems of representability of covariant functors. In practice, such problems usually come down to taking a quotient by a group action, or by an equivalence relation, or more generally by a groupoid. For us, these questions appear in two situations: when some group scheme is involved in a particular geometric situation, or when we want to construct a moduli space. Here we present three problems of quotient. The one discussed in Section 1 is quite independent from the other works in this report, but those discussed in Sections 2 and 3 have very close relationship with the results of the second and third parts.

The construction of quotients in Algebraic Geometry raises several questions :

(1) What is the definition of the quotient we are trying to find? In particular, what is the category where it should be looked for?

(2) Does the quotient have nice properties?

(3) Does its formation commute with base change?

When they exist, quotients in different categories are in general different. The choice of the category where the existence of a given quotient will be investigated has in fact strong consequences on its properties. For example, the quotient of the affine plane minus the origin $\mathbb{A}^2_k \setminus \{0\}$ by the action by homotheties of the multiplicative group $\mathbb{G}_{m,k}$ is the point $\mathrm{Spec}(k)$ in the category of quasi-affine k-schemes, but of course it is the projective line $\mathbb{P}^1_k$ in the category of all k-schemes. As it turns out, the quotient is again $\mathbb{P}^1_k$ is bigger "categories" like the 2-category of algebraic stacks. What is more, the morphism $\mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$ is a torsor under the group $\mathbb{G}_{m,k}$. These facts may be seen to mean that $\mathbb{P}^1_k$ is the "good" quotient. These phenomena will be discussed in the examples of this part.

According to Grothendieck's philosophy, although an object that represents a covariant functor is defined in terms of morphisms of which it is the *source*, it should be constructed using a description of morphisms of which it is the *target*, that is to say its points. We do this with special care in Sections 2 and 3 where the quotients are somehow less concrete.

1 Quotient of the adjoint action of reductive groups

To start with, we revisit a famous example where the choice of the category where the quotient should live is imposed by the problem itself.

Let S be a scheme, G a split reductive S-group scheme with Lie algebra $\mathfrak{g}$, and T a maximal torus with Lie algebra $\mathfrak{t}$. The Lie algebra $\mathfrak{g}$ is endowed with the so-called *ajoint action* of G and the algebra $\mathfrak{t}$ is endowed with an induced action of the Weyl group W, defined as the quotient of the normalizer of T by its centralizer. The *adjoint quotient* of $\mathfrak{g}$ denoted $\mathfrak{g}/G$ is the object that represents the covariant functor defined on the category of schemes affine over S by

$$F(X) = \mathrm{Hom}_G(\mathfrak{g}, X),$$

where X is seen as a G-scheme with trivial action. It is easy to see that $\mathfrak{g}/G$ is represented by the spectrum of the $\mathcal{O}_S$-algebra of G-invariant functions of $\mathfrak{g}$. The morphism

$$\chi : \mathfrak{g} \to \mathfrak{g}/G$$

is an extremely important object playing a crucial role in the construction of the Hitchin fibration, a starting point in B. C. Ngô's approach to the proof of the fundamental lemma (see [Ngô]). The quotient $\mathfrak{t}/W$ is defined in a similar fashion and is an affine S-scheme. The inclusion $\mathfrak{t} \hookrightarrow \mathfrak{g}$ induces a morphism $\pi : \mathfrak{t}/W \to \mathfrak{g}/G$. The question whether π is an isomorphism is a natural one, and a positive answer has several important consequences for $\mathfrak{g}/G$: first of all it reduces its computation to a computation of a quotient by a finite group; moreover it allows to identify it with the set of semisimple conjugation classes of $\mathfrak{g}$ – because the points of $\mathfrak{t}/W$ are indeed in one-to-one correspondence with this set. When the base is the spectrum of a field of odd characteristic, the question was studied in the classic work of Springer and Steinberg [SpSt] and more recently by Levy [Le]. When G is simple (a case which leads easily to a treatment in the semisimple case), we give an answer without any assumption on the base in the article *On the adjoint quotient of Chevalley groups over arbitrary base schemes* in collaboration with P.-E. Chaput.

Theorem. *Assume that G is split simple on an arbitrary base scheme S. Then, the morphism $\pi : \mathfrak{t}/W \to \mathfrak{g}/G$ is an isomorphism, except in the following case: $G = \mathrm{Sp}_{2n}$ and the structure sheaf $\mathcal{O}_S$ of the base scheme has 2-torsion elements.*

Moreover we study in full detail the exceptional case: we compute the schemes $\mathfrak{t}/W$, $\mathfrak{g}/G$ and the morphism π. These results may be found in [6], 3.11 and 6.6.

The question whether the formation of the quotient $\chi : \mathfrak{g} \to \mathfrak{g}/G$ commutes with base change is a matter of case-by-case considerations. We give an answer for the four classical groups in types A, B, C, D. Denote by $S[2]$ the closed subscheme of S defined by the ideal of functions killed by 2. Then:

(i) if $G = \mathrm{SL}_n$ or $G = \mathrm{Sp}_{2n}$, the formation of the quotient commutes with base change;

(ii) if $G = \mathrm{SO}_{2n+1}$ or $G = \mathrm{SO}_{2n}$, the formation of the quotient commutes with the base change $f : S' \to S$ if and only if $f^*S[2] = S'[2]$.

These results may be found in [6], 4.8, 4.9, 5.3, 6.6.

2 Quotient by a groupoid in algebraic stacks

We now present a result which is used in the proof of Théorème 2.5.2 in the article *Composantes connexes et irréductibles en familles* [8] to be introduced in Section 3 below. In *loc. cit.* is given only a very rough sketch of proof (see [8] 2.5.1). The quotient stack of a groupoid is described only as the stackification of a certain prestack, which makes it somehow uneasy to show that it is algebraic. Now we take the opportunity to give a more complete statement and to give a full proof. In particular, we give a direct description of the sections of the quotient stack.

In contrast with the situation of Section 1, it may be said that this quotient problem is set in a category that is chosen in such a way that the quotient exists and is as nice as possible. To be more accurate, it is in fact a 2-category: the 2-category of algebraic (1-)stacks. One may indeed consider that algebraic stacks were invented in order to obtain quotients with the best possible properties.

In the sequel, we always write *algebraic stack* to mean an algebraic stack in the sense of Artin. We fix an algebraic space S and all morphisms are S-morphisms.

2.1 A reminder on groupoids. We briefly recall a couple of definitions in order to fix our notations and make the text coherent. A *groupoid in S-algebraic spaces* or simply *groupoid space* is given by two S-algebraic spaces R, X and five morphisms $s, t : R \to X$, $e : X \to R$, $c : R \times_{s,X,t} R \to R$, $i : R \to R$ satisfying the well-known axioms expressing the fact that X is the set of objects and R is the set of arrows of a small category. The groupoid is often denoted $s, t : R \rightrightarrows X$ or simply $R \rightrightarrows X$. We say that the groupoid is *fppf* if s, or equivalently t, is an fppf morphism. We say that a morphism $g : X \to Z$ is *invariant* under the groupoid if $g \circ s = g \circ t$. In this case, the groupoid may be seen as a groupoid in Z-spaces. A *quotient* for $R \rightrightarrows X$ is a morphism $f : X \to Y$ which is universal among all invariant morphisms. If it exists, we way that the quotient is *effective* if the morphism $R \to X \times_Y X$ is an isomorphism. A *morphism of groupoids* $(R \rightrightarrows X) \to (R' \rightrightarrows X')$ is a pair of morphisms $R \to R'$, $X \to X'$ which are compatible with the structure morphisms of the two groupoids in an obvious way. A groupoid such that $j = (t, s) : R \to X \times_S X$ is a monomorphism, resp. a strict epimorphism, is called *free*, resp. *transitive*. In particular, a groupoid that has an effective quotient defines a free and transitive groupoid in Y-spaces. A free groupoid is also called an *equivalence relation*. If a groupoid is free and transitive, then j is a monomorphism and a strict epimorphism, hence an isomorphism. The *stabilizer* of the groupoid is the fibred product $j^{-1}(\Delta_{X/S})$ seen as an X-space defined by the diagramme :

$$\begin{tikzcd} j^{-1}(\Delta_{X/S}) \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\ R \arrow[r, "j"] & X \times X. \end{tikzcd}$$

The action is free if and only if the stabilizer is trivial, that is, $j^{-1}(\Delta_{X/S}) \to X$ is an isomorphism. A fundamental example is given by the action of an S-group algebraic space G over an S-algebraic space X: there is then a groupoid $G \times_S X \rightrightarrows X$ with s equal to the second projection, t the action morphism, and e, c, i induced by the neutral section, the composition and the inversion of the group. This groupoid is free transitive if and only if X is a pseudo-G-torsor (a formally principal homogeneous space).

A *groupoid in S-algebraic stacks* is a groupoid $\mathscr{R} \rightrightarrows \mathscr{X}$ where $\mathscr{X}$ and $\mathscr{R}$ are S-algebraic stacks, the structure morphisms s, t, e, c, i are morphisms of S-algebraic stacks, and all the diagrammes of the groupoid axioms are 2-*commutative*. The 2-isomorphisms of commutativity are part of the structure, which makes it a little heavy. For more details on this point in the case of groupoids defined by a group action, we refer to [2], Section 1. The various notions defined before for groupoid spaces extend without any change. Let us point out that in the present context, an *invariant morphism* is a pair composed of a morphism $g : \mathscr{X} \to \mathscr{Z}$ and a 2-isomorphism $\beta : g \circ s \Rightarrow g \circ t$. Thus a *quotient* is a pair $f : \mathscr{X} \to \mathscr{Y}$, $\alpha : f \circ s \Rightarrow f \circ t$ such that for all invariant morphisms (g, β) there exists a unique pair $h : \mathscr{Y} \to \mathscr{Z}$, $\gamma : h \circ f \Rightarrow g$ making the following diagramme commutative :

$$\begin{tikzcd} h \circ f \circ s \arrow[r, Rightarrow, "\gamma * s"] \arrow[d, Rightarrow, "h*\alpha"'] & g \circ s \arrow[d, Rightarrow, "\beta"] \\ h \circ f \circ t \arrow[r, Rightarrow, "\gamma * t"'] & g \circ t. \end{tikzcd}$$

2.2 Statement of the theorem. The result whose proof we want to explain is the following.

Theorem. *Let S be an algebraic space.*

(1) *Let $f : \mathscr{X} \to \mathscr{Y}$ be an fppf morphism of S-algebraic stacks. Then, the groupoid*

$$\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \rightrightarrows \mathscr{X}$$

is an fppf groupoid with representable stabilizer, with quotient $f : \mathscr{X} \to \mathscr{Y}$.

(2) *Let $s, t : \mathscr{R} \rightrightarrows \mathscr{X}$ be an fppf groupoid in S-algebraic stacks whose stabilizer is representable. Then, there exists a quotient $f : \mathscr{X} \to \mathscr{Y}$, $\alpha : f \circ s \Rightarrow f \circ t$ which is an fppf morphism of S-algebraic stacks. This quotient is effective and its formation commutes with base change.*

In short, the theorem says that in the 2-category of S-algebraic stacks, fppf groupoids with representable stabilizer are the same thing as groupoids defined by the fibres of an fppf morphism. Let us make a few side comments.

(i) The first part is a descent statement: it simply means that f is a strict epimorphism. The second part is a statement of existence of a quotient.

(ii) One can replace *fppf* by *smooth* or *étale* in the statement.

(iii) This theorem is a generalization of the statements of the paper *Group actions on stacks and applications* [2] on the existence of quotients of stacks by algebraic group actions.

(iv) It may be worth pointing out that the proof of (2) uses (1).

2.3 Comments on the homotopical context. Before we come to the proof, it is useful to recast the above result in a more general framework. In [TV], Toën and Vezzosi construct an $(n+1)$-*category of Artin* n-*stacks*. Let us denote it by C_{n+1}. For small values of n, this gives the set $C_0 = \{S\}$, the category C_1 of S-algebraic spaces, and the 2-category C_2 of algebraic stacks in the sense this phrase has in this report. For all $n \geqslant 0$, the Yoneda functor $C_n \hookrightarrow C_{n+1}$ allows to speak about the n-stacks that are (representable by) $(n-1)$-stacks. Then we have the following assertions:

(1) Let $f : \mathscr{X} \to \mathscr{Y}$ be an fppf morphism of Artin n-stacks. Then the groupoid

$$\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \rightrightarrows \mathscr{X}$$

is an fppf groupoid with stabilizer an Artin $(n-1)$-stack, with quotient $f : \mathscr{X} \to \mathscr{Y}$.

(2) Let $s, t : \mathscr{R} \rightrightarrows \mathscr{X}$ be an fppf groupoid in Artin n-stacks whose stabilizer is an Artin $(n-1)$-stack. Then, there exists a quotient $f : \mathscr{X} \to \mathscr{Y}$, $\alpha : f \circ s \Rightarrow f \circ t$ which is an fppf morphism of S-stacks. This quotient is effective and its formation commutes with base change. It is also a quotient among Artin m-stacks, for all $m \geqslant n$.

It may be that [TV] contains complete or partial proofs of these facts. For $n = 0$, point (1) is a classical statement of fppf descent theory and point (2) is Artin's theorem on the existence of quotients for flat equivalence relations. For $n = 1$, statements (1) and (2) make up Theorem 2.2. Of course, Toën and Vezzosi's results are stated in the framework of Homotopical Algebraic Geometry which, like many geometers, I am not too familiar with. To start with, in the homotopical world it is not straightforward to simply *recognize* the objects we need in (ordinary) Algebraic Geometry. This makes it uncomfortable to understand and use the results we are concerned with: the reader may have a look at Sections 1.3.4 and 1.3.5 in [TV] to make up her (his) own opinion. It is therefore useful to formulate and prove these results in our own terms: this is what we do in the following lines, for $n = 1$.

2.4 Proof of point (1) of the theorem. We are given an fppf morphism of S-algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$. The fact that the stabilizer of the groupoid $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \rightrightarrows \mathscr{X}$ is representable is a consequence of the fact that the diagonal $\Delta_{\mathscr{Y}/S}$ is representable, in view of the 2-cartesian squares :

$$\begin{tikzcd}
j^{-1}(\Delta_{\mathscr{X}/S}) \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\Delta_{\mathscr{X}/S}"] \\
\mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \times_S \mathscr{X} \arrow[d] \\
\mathscr{Y} \arrow[r, "\Delta_{\mathscr{Y}/S}"] & \mathscr{Y} \times_S \mathscr{Y}.
\end{tikzcd}$$

Now let us show that $f : \mathscr{X} \to \mathscr{Y}$ is the quotient of the groupoid it gives birth to, or in other words is a strict epimorphism. Note that by definition of the fibred product, we have a canonical 2-isomorphism $\alpha : f \circ \mathrm{pr}_1 \Rightarrow f \circ \mathrm{pr}_2$. Let $\mathscr{Z}$ be an S-algebraic stack, $g : \mathscr{X} \to \mathscr{Z}$ a morphism and $\beta : g \circ \mathrm{pr}_1 \Rightarrow g \circ \mathrm{pr}_2$ a 2-isomorphism. Let $U \to \mathscr{Y}$

and $U' \to U \times_{\mathscr{Y}} \mathscr{X}$ be smooth presentations by algebraic spaces; let $V = U \times_{\mathscr{Y}} U$ and $V' = U' \times_{\mathscr{X}} U'$. We have a 2-commutative diagramme:

$$\begin{tikzcd}
V' \times_V V' \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, bend right=60, "y"'] & V' \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & V \arrow[d, shift left] \arrow[d, shift right] & \\
U' \times_U U' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "w"] & U' \arrow[r, "\varphi"] \arrow[d, "v"] & U \arrow[d, "u"] \arrow[dr, dotted, "h_1"] & \\
\mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{X} \arrow[r, "f"] \arrow[rr, bend right, "g"'] & \mathscr{Y} \arrow[r, dotted, "h"] & \mathscr{Z}.
\end{tikzcd}$$

By composition of the 2-isomorphism $\beta : g \circ \mathrm{pr}_1 \Rightarrow g \circ \mathrm{pr}_2$ with w and using the 2-isomorphisms $\mathrm{pr}_i \circ w \Rightarrow v \circ \mathrm{pr}_i$ with $i = 1, 2$, we get a 2-isomorphism $\beta * w : g \circ v \circ \mathrm{pr}_1 \Rightarrow g \circ v \circ \mathrm{pr}_2$. Therefore we obtain a 2-commutative diagramme

$$\begin{tikzcd}
U' \times_U U' \arrow[r, shift left] \arrow[r, shift right] & U' \arrow[r, "g \circ v"] & \mathscr{Z} .
\end{tikzcd}$$

Since $\mathscr{Z}$ is a stack, the morphism $g \circ v$ factors through U i.e. there is a unique pair (h_1, β_1) where $h_1 : U \to \mathscr{Z}$ and $\beta_1 : h_1 \circ \varphi \Rightarrow g \circ v$. By the same argument applied to the first line of the diagramme instead of the second, the assertion of unicity provides a descent datum for h_1 relative to u. It follows that h_1 factors through $\mathscr{Y}$ i.e. there is a unique pair (h, β_2) where $h : \mathscr{Y} \to \mathscr{Z}$ and $\beta_2 : h \circ u \Rightarrow h_1$. By composition of β_2 and β_1 we get a 2-isomorphism $h \circ f \circ v \Rightarrow g \circ v$. Finally, looking at the preimages along $V' \rightrightarrows U'$ and using the stack properties of $\mathscr{X}$, we see that this 2-isomorphism descends to a unique 2-isomorphism $\beta : h \circ f \Rightarrow g$. This finishes the proof of (1).

2.5 Proof of point (2) of the theorem. We are given an fppf groupoid $s, t : \mathscr{R} \rightrightarrows \mathscr{X}$. We shall concentrate on the construction of the quotient stack $\mathscr{Y}$ which is the main difficulty of the proof. For this, let us first assume that $\mathscr{Y}$ exists and draw some conclusions. If $\mathscr{Y}$ has the properties required in the statement of (2), the morphism $\mathscr{R} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is an isomorphism, i.e. the groupoid $\mathscr{R} \rightrightarrows \mathscr{X}$ seen as a groupoid in algebraic stacks over $\mathscr{Y}$ is free and transitive. For all schemes S'/S, by base change we have a free and transitive fppf S'-groupoid:

$$\begin{tikzcd}
\mathscr{R}' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] \arrow[d] & \mathscr{X}' \arrow[r, "\pi'"] \arrow[d, "\beta"] & S' \arrow[d, "\alpha"] \\
\mathscr{R} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathscr{X} \arrow[r, "\pi"] & \mathscr{Y}.
\end{tikzcd}$$

Let us write $\mathscr{R}_s$, resp. $\mathscr{R}_t$, for the stack $\mathscr{R}$ seen as an $\mathscr{X}$-stack via s, resp. via t. Similarly let us write $\mathscr{R}_s^2$, resp. $\mathscr{R}_t^2$, for the stack $\mathscr{R} \times_{s,\mathscr{X},t} \mathscr{R}$ seen as an $\mathscr{X}$-stack via $s \circ \mathrm{pr}_2$, resp. via $t \circ \mathrm{pr}_1$. The operation of composition in the groupoid gives rise to two $\mathscr{X}$-morphisms $c_s = c : \mathscr{R}_s^2 \to \mathscr{R}_s$ and $c_t = c : \mathscr{R}_t^2 \to \mathscr{R}_t$. We shall write with a "$'$" the same objects attached to the groupoid obtained by base change. By transitivity of the fibred product, for $a \in \{s, t\}$, we have canonical isomorphisms $\varphi_a : \mathscr{R}'_a \to \beta^* \mathscr{R}_a$ and $\psi_a : \mathscr{R}'^2_a \to \beta^* \mathscr{R}^2_a$ satisfying

$$\varphi_a \circ c'_a = \beta^* c_a \circ \psi_a.$$

Since a groupoid is determined by the structure maps s, t, c, these equalities determine all the information in $\mathscr{R}'$ coming from $\mathscr{R}$ by the base change $\beta : \mathscr{X}' \to \mathscr{X}$. Now we change our point of view and we *define* $\mathscr{Y}$ as the stack whose sections over S' are the objects $\mathscr{T}'$ composed of a diagramme

$$\begin{tikzcd} \mathscr{R}' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{X}' \arrow[r] \arrow[d, "\beta"] & S' \\ \mathscr{R} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{X} & \end{tikzcd}$$

and isomorphisms $\varphi_a : \mathscr{R}'_a \to \beta^*\mathscr{R}_a$ and $\psi_a : \mathscr{R}'^2_a \to \beta^*\mathscr{R}^2_a$ for $a \in \{s, t\}$, such that:

(a) $\mathscr{X}'$ is an arbitrary fppf S'-algebraic stack,

(b) $\mathscr{R}' \rightrightarrows \mathscr{X}'$ is an fppf S'-groupoid which is free and transitive,

(c) $\varphi_a \circ c'_a = \beta^* c_a \circ \psi_a$ for $a \in \{s, t\}$.

The morphisms between two objects $\mathscr{T}'_1$ and $\mathscr{T}'_2$ are the 2-commutative diagrammes

$$\begin{tikzcd} \mathscr{R}'_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, "\lambda"] & \mathscr{X}'_1 \arrow[dd, "\mu"] \arrow[dr, "\beta_1"] & \\ & & \mathscr{X} \\ \mathscr{R}'_2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{X}'_2 \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

satisfying the obvious compatibility conditions with the structure morphisms of the groupoids and the other data $\varphi_{a,i}$, $\psi_{a,i}$ for $a \in \{s, t\}$ and $i \in \{1, 2\}$.

We have to show that $\mathscr{Y}$ is algebraic. By definition of a morphism of groupoids, in the above diagramme the map μ is determined by λ. In this way we see that the fibre of $\Delta_{\mathscr{Y}/S}$ at some point $(\mathscr{T}'_1, \mathscr{T}'_2)$, i.e. the S'-stack $\mathrm{Isom}_{\mathscr{Y}}(\mathscr{T}'_1, \mathscr{T}'_2)$, is a sucategory of the S'-stack $\mathrm{Hom}_{\mathscr{X}}(\mathscr{R}'_1, \mathscr{R}'_2)$. Using the assumption that the stabilizer of the groupoid $\mathscr{R} \rightrightarrows \mathscr{X}$ is representable, we see that $\mathrm{Hom}_{\mathscr{X}}(\mathscr{R}'_1, \mathscr{R}'_2)$ is represented by an S'-algebraic space, and the fibre of $\Delta_{\mathscr{Y}/S}$ at the point of interest is a subspace ; here we omit a couple of tedious but routine details. Thus $\Delta_{\mathscr{Y}/S}$ is representable. Now algebraicity of of $\mathscr{Y}$ will follow once the quotient $f : \mathscr{X} \to \mathscr{Y}$ is constructed and proven to be fppf. In order to construct f, for all points $c : S' \to \mathscr{X}$ we have to provide a point $f(c) : S' \to \mathscr{Y}$. Let us consider the fibred product $\mathscr{R}' = \mathscr{R} \times_{s,\mathscr{X},c} S'$, which is none other than the orbit of c under the groupoid, and let us denote by $w : \mathscr{R}' \to \mathscr{R}$ the projection. The trivial groupoid $\mathscr{R}' \times_{S'} \mathscr{R}' \rightrightarrows \mathscr{R}'$ takes place in a diagramme

$$\begin{tikzcd} \mathscr{R}' \times_{S'} \mathscr{R}' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "w \circ \mathrm{pr}_1"'] & \mathscr{R}' \arrow[r] \arrow[d] & S' \\ \mathscr{R} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{X} & \end{tikzcd}$$

that defines the sought-for point $f(c) \in \mathscr{Y}(S')$. It is clear that the morphism f is fppf, because its fibre at an object $\mathscr{T}'$ as above is representable by the S'-stack $\mathscr{X}'$ which is fppf by construction.

It only remains to prove that $f : \mathscr{X} \to \mathscr{Y}$ is a quotient of $\mathscr{R} \rightrightarrows \mathscr{X}$. Let $g : \mathscr{X} \to \mathscr{Z}$ be a morphism of S-algebraic stacks and $\beta : g \circ s \Rightarrow g \circ t$ a 2-isomorphism. Let $\mathscr{T}'$ be an objet of $\mathscr{Y}(S')$. By definition of $\mathscr{Y}$, we have a 2-commutative diagramme:

$$\begin{tikzcd}
\mathscr{R}' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] \arrow[d, "w"] & \mathscr{X}' \arrow[r, "\varphi"] \arrow[d, "v"] & S' \arrow[d, "u"] \arrow[drr, dotted, bend left, "h"] & & \\
\mathscr{R} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathscr{X} \arrow[r, "f"] \arrow[rrr, bend right, "g"'] & \mathscr{Y} & & \mathscr{Z}.
\end{tikzcd}$$

By composition of $\beta : g \circ s \Rightarrow g \circ t$ with w and using the 2-isomorphisms $\mathrm{pr}_i \circ w \Rightarrow v \circ \mathrm{pr}_i$ with $i = 1, 2$, we obtain a 2-isomorphism $\beta * w : g \circ v \circ s' \Rightarrow g \circ v \circ t'$. But by definition of $\mathscr{Y}$, the fppf S'-groupoid $\mathscr{R}' \rightrightarrows \mathscr{X}'$ is free and transitive i.e. it is the groupoid defined by the fibres of the fppf morphism $\mathscr{X}' \to S'$. According to part (1) of the theorem, it follows that there is a unique pair (h', γ') composed of a morphism $h' : S' \to \mathscr{Z}$ and a 2-isomorphism $\gamma' : g \circ v \Rightarrow h \circ \varphi$. For variable S', the collection of pairs (h', γ') defines a pair $h : \mathscr{Y} \to \mathscr{Z}$, $\gamma : h \circ f \Rightarrow g$ that gives a factorisation of (g, β). This finishes the proof of (2).

3 Étale quotients of a stack and connected components

The end of the first part will be devoted to a couple of results dealing with a quotient problem, or more precisely a problem of representability for covariant functor: given a flat, finitely presented S-scheme X, we wish to investigate representability for the functor defined on the category of *étale* S-schemes by $F(Y) = \mathrm{Hom}_S(X, Y)$. As a matter of fact, we introduce the object that represents F rather via its functor of points describing the connected components of X above S. There is an interesting variant with irreducible components. Besides, since we envision applications to the connected components of some classifying stacks of curves, we start off with an S-algebraic stack $\mathscr{X}$ rather than a scheme. This being said, the interest of our results is the same for schemes and stacks. In the main, they all belong to the article *Composantes connexes et irréductibles en familles* [8].

3.1 Open connected components. Let S be a scheme or an algebraic space, and let $\mathscr{X}$ be a flat finitely presented S-algebraic stack. We call *open connected component* $\mathscr{X}/S$ (o.c.c. for short) a open substack $\mathscr{C} \subset \mathscr{X}$, flat and finitely presented over S, such that for all géometric points $\bar{s} \in S$, the fibre $\mathscr{C}_{\bar{s}} \subset \mathscr{X}_{\bar{s}}$ is a connected component. We denote by $\pi_0(\mathscr{X}/S)$ the functor on the category of S-schemes defined by:

$$\pi_0(\mathscr{X}/S)(T) = \big\{ \text{ o.c.c. de } \mathscr{X} \times_S T/T \ \big\}.$$

Similarly we can define the notion of *open irreducible component* (o.i.c. for short) and the associated functor $\mathrm{Irr}(\mathscr{X}/S)$. By their definition $\pi_0(\mathscr{X}/S)$ and $\mathrm{Irr}(\mathscr{X}/S)$ are fppf sheaves over S with open quasi-compact diagonal. Moreover as functors they are formally étale, that is to say with vanishing deformation theory. Also noteworthy is the fact that $\pi_0(\mathscr{X}/S)$ is always representable by an étale quasi-compact algebraic space *provided* we restrict to a finite stratification S^* of S by locally closed subschemes (see [8], lemmas 2.1.2, 2.1.3).

In order to prove that $\pi_0(\mathscr{X}/S)$ is representable, it is crucial to be able to construct o.c.c.'s. The simplest case where this is possible is the case where $\mathscr{X} \to S$ has geometrically reduced fibres, for then the classical result about the open connected component along a section generalizes (see [8] 2.2.1). Interestingly, there is a similar result for irreducible components: one shows that there is a largest open substack $\mathscr{U} \subset \mathscr{X}$ whose geometric points belong to a single irreducible component of their fibre, called *unicomponent locus*, and one constructs the open irreducible component along a section included in the unicomponent locus (see [8] 2.2.4). Using Artin's representability criteria and the theorem on quotients in 2.2, we prove:

Theorem. *Let $\mathscr{X}$ be a flat, finitely presented S-algebraic stack with geometrically reduced fibres.*

(1) *The functors $\pi_0(\mathscr{X}/S)$ and $\mathrm{Irr}(\mathscr{X}/S)$ are representable by étale, quasi-compact S-algebraic spaces.*

(2) *Let $\mathscr{R} \subset \mathscr{X} \times \mathscr{X}$ be the equivalence relation defined as the subcategory such that any two points $u, v : T \to \mathscr{X}$ are equivalent if and only if for all geometric points $t : \mathrm{Spec}(\Omega) \to T$, the points $u(t)$ and $v(t)$ lie in the same connected component of $\mathscr{X}_\Omega$. This relation is representable by the o.c.c. of $\mathscr{X} \times \mathscr{X}$ along the diagonal section. Moreover, there exists a morphism $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ allowing an identification of $\mathscr{X}$ with the universal o.c.c. and an identification of $\pi_0(\mathscr{X}/S)$ with the quotient $\mathscr{X}/\mathscr{R}$.*

(3) *Let $\mathscr{S} \subset \mathscr{U} \times \mathscr{U}$ be the equivalence relation on the unicomponent locus which is the full subcategory such that the points $u, v : T \to \mathscr{U}$ are equivalent if and only if for all geometric points $t : \mathrm{Spec}(\Omega) \to T$, the points $u(t)$ and $v(t)$ lie in the same open irreducible component of $\mathscr{U}_\Omega$. This relation is representable by the o.i.c. of $\mathscr{U} \times \mathscr{U}$ along the diagonal section. Morever, there exists a morphism $\mathscr{U} \to \mathrm{Irr}(\mathscr{X}/S)$ allowing an identification of $\mathscr{U}$ with the universal o.i.c. and an identification of $\mathrm{Irr}(\mathscr{X}/S)$ with the quotient $\mathscr{U}/\mathscr{S}$.*

This is [8] 2.5.2. The description $\pi_0(\mathscr{X}/S) = \mathscr{X}/\mathscr{R}$ permits to prove that $\pi_0(\mathscr{X}/S)$ is functorial in a very strong sense: each S-rational (resp. S-birational) map $f : \mathscr{X} \dashrightarrow \mathscr{Y}$ induces a morphism (resp. an isomorphism) $\pi_0(f) : \pi_0(\mathscr{X}/S) \to \pi_0(\mathscr{Y}/S)$.

3.2 Open connected components in the proper case. When $\mathscr{X} \to S$ is proper, flat and finitely presented, we can say a bit more about $\pi_0(\mathscr{X}/S)$. For all S-algebraic stacks $\mathscr{X}$ and all points $s \in S$, let us write $n_{\mathscr{X}}(s)$ the number of connected components of the geometric fibre $\mathscr{X}_{\bar{s}}$. Case (ii) in the statement below was indicated to me by Laurent Fargues ; this additional case of representability complements the results of [8].

Theorem. *Let S be a scheme and $\mathscr{X}$ a proper, flat, finitely presented S-algebraic stack. Suppose that one of the following assumptions is satisfied:* (i) *$\mathscr{X} \to S$ has reduced geometris fibres, or* (ii) *the function $n_{\mathscr{X}}$ is upper semi-continuous. Then, we have:*

(1) *The functor $\pi_0(\mathscr{X}/S)$ is representable by a finite étale S-scheme.*

(2) *The description of $\pi_0(\mathscr{X}/S)$ as the quotient $\mathscr{X}/\mathscr{R}$ like in theorem 3.1 is valid without modification.*

(3) *Let us denote by* $\mathscr{X} \to \mathrm{St}(\mathscr{X}/S) \to S$ *the Stein factorisation. Then we have a factorisation* $\mathrm{St}(\mathscr{X}/S) \to \pi_0(\mathscr{X}/S) \to S$*, where* $\mathrm{St}(\mathscr{X}/S) \to \pi_0(\mathscr{X}/S)$ *is a universal homeomorphism, and an isomorphism in case* (i).

Preuve : In case (i), everything is proven in [8] 2.5.2 and 3.2.5. Hence we focus on case (ii).

(1) We repeat the proof of point (i) in Theorem 2.5.2 of [8], using Artin's criteria. Let us set $F = \pi_0(\mathscr{X}/S)$. Like in *loc. cit.* we are reduced to showing that for each complete noetherian local ring (R, m, k), the map $F(R) \to F(k)$ is bijective. Let (R', m', k') be a finite étale local extension of R such that all o.c.c.'s of $\mathscr{X}_{\bar{k}}$ are defined over k'. Let $\mathscr{Z}_{1,k'}, \dots \mathscr{Z}_{n,k'}$ be the o.c.c.'s de $\mathscr{X}_{k'}$. According to [EGA] III, prop. 5.5.1, there exist open and closed substacks $\mathscr{Z}_1, \dots, \mathscr{Z}_n$ of $\mathscr{X}_{R'}$ such that $\mathscr{Z}_i \otimes k' = \mathscr{Z}_{i,k'}$ for all i. (The proof of this result of [EGA] uses Grothendieck's théorème d'existence des faisceaux, which holds for algebraic stacks by Olsson [Ol], thm. 1.4.) As $n_{\mathscr{X}}$ is upper semi-continuous, we deduce that it is in fact constant on $\mathrm{Spec}(R')$. Thus $F(R') \to F(k')$ is a bijection. It restricts to a bijection $F(R) \to F(k)$ for the o.c.c.'s defined over R. This completes the proof that $\pi_0(\mathscr{X}/S)$ is representable by an algebraic space. Furthermore we see using Lemma 3.2.2 of [8] that the o.c.c.'s of $\mathscr{X}$ are proper over the base and hence closed in $\mathscr{X}$. It follows that $\pi_0(\mathscr{X}/S)$ is separated, hence a scheme by [Kn] II.6.17, hence finite since $n_{\mathscr{X}}$ is locally constant.

(2) The proof of point (ii) of theorem 2.5.2 of [8] applies word for word.

(3) Since $\pi_0(\mathscr{X}/S)$ is affine over S, the morphisms $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ factors through the affine hull of $\mathscr{X} \to S$ i.e. through $\mathrm{St}(\mathscr{X}/S)$. It is clear that the morphism $\mathrm{St}(\mathscr{X}/S) \to \pi_0(\mathscr{X}/S)$ is finite, surjective and radicial, and therefore a universal homeomorphism. $\square$

3.3 Closed connected components. We continue with the flat finitely presented S-algebraic stack $\mathscr{X}$ (but flatness will not be essential here). If $\mathscr{X} \to S$ fails to fulfill one of the conditions (i)-(ii) of 3.2, that is to say if it fails to have geometrically reduced fibres *and* to be proper with constant function $n_{\mathscr{X}}$, then the sheaf $\pi_0(\mathscr{X}/S)$ is not representable in general. Still, in the proper case a relevant alternative is to consider *closed connected components* (c.c.c. for short), which by definition are closed substacks $\mathscr{C} \subset \mathscr{X}$, flat and finitely presented over S, such that for all geometric points $\bar{s} \in S$, the fibre $\mathscr{C}_{\bar{s}} \subset \mathscr{X}_{\bar{s}}$ is supported by a connected component. We say that a c.c.c. is *reduced* if its fibres are geometrically reduced. We write $\pi_0(\mathscr{X}/S)^{\mathfrak{f}}$ the functor of c.c.c.'s and $\pi_0(\mathscr{X}/S)^{\mathfrak{r}}$ the subfunctor of reduced c.c.c.'s. We show:

Theorem. *Let* $\mathscr{X}$ *be a proper finitely presented* S*-algebraic stack. Then, the functor* $\pi_0(\mathscr{X}/S)^{\mathfrak{f}}$ *is representable by a formal locally finitely presented separated* S*-algebraic space. The functor* $\pi_0(\mathscr{X}/S)^{\mathfrak{r}}$ *is representable by a formal quasi-finite separated* S*-scheme.*

This result is [8] 3.2.1. For the proof, one relies on the "Hilbert scheme" of $\mathscr{X}/S$ – which really is a Hilbert space. Away from the proper case, representability of these functors is probably uncommon. Here is one favourable situation: if $\mathscr{X}$ is a quasi-finite S-algebraic space, there is a canonical isomorphism $\pi_0(\mathscr{X}/S)^{\mathfrak{r}} = \mathscr{X}$. Indeed, if $\mathscr{C}$ is a reduced c.c.c. of $\mathscr{X}/S$, then its geometric fibres are reduced points and it follows that $\mathscr{C} \to S$ is étale of degree 1, hence a section de $\mathscr{X}/S$. This is used in 3.4 below.

3.4 Application to the connected-étale sequence of group schemes. Let $\mathcal{E}t/S$ be the category of étale S-group algebraic spaces. For all flat finitely presented S-group schemes (or S-group spaces) G, we define a covariant functor $F : \mathcal{E}t/S \to \mathrm{Ens}$ by the formula:

$$F(H) = \mathrm{Hom}_{S\text{-}\mathrm{Gr}}(G, H).$$

If this functor is representable, we denote the representing algebraic space by $G^{\text{ét}}$ and we call it the *biggest étale quotient* of G. It is a classical fact that if S is the spectrum of a henselian local ring and G is proper or affine, then there is a biggest étale quotient: it is just the spectrum of the biggest étale subalgebra of the ring of global functions of G. The kernel $G^0 = \ker(G \to G^{\text{ét}})$ is the connected component of the unit of the special fibre, open and closed in G, and we have the *connected-étale exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 1.$$

Keeping in mind the example of the group scheme μ_p over the ring of p-adic integers $\mathbb{Z}_p$, having $\mu_p^{\text{ét}} = 1$, we observe that the formation of these objects does not commute with a change of henselian (possibly finite flat) base ring, and that G^0 may differ from the connected component along the unit section (i.e. the union of the connected components of the units in the fibres).

Our results above allow to obtain such an exact sequence over an arbitrary base S. For this we have to add more assumptions, but we are rewarded by better properties. More precisely, let G be a plat finitely presented S-group scheme (or S-group space), and assume that one of the following assumptions is satisfied:

(i) G is smooth over S,

(ii) G is proper over S and the function n_G is upper semi-continuous.

Then the results above show that the connected component of G along the unit section is a normal open subgroup space $G^0 \subset G$ and the functor $G^{\text{ét}} := \pi_0(G/S)$ is representable. Moreover $G^{\text{ét}}$ is an S-group space since π_0 commutes with products. Hence we have the connected-étale sequence $1 \to G^0 \to G \to G^{\text{ét}} \to 1$ all whose terms have formation commuting with base change on S.

Note that over the base scheme $S = \mathrm{Spec}(\mathbb{Z}_p)$, the group scheme $G = \mu_{p,S}$ satisfies neither (i) nor (ii). For this group, the scheme of reduced c.c.c. is $\pi_0(G/S)^{\mathfrak{r}} = G$. This fits into a natural exact sequence $1 \to \ker \to G \to \pi_0(G/S)^{\mathfrak{r}} \to 1$, but it is not on the "good" side of G in view of the connected-étale sequence defined previously over a henselian base...

3.5 Application to moduli spaces of curves. The main motivation for my work on families of components was to answer some questions asked by Pierre Lochak on a concrete example. In the moduli space of curves of genus g, let $\mathscr{M}_g(G) \subset \mathscr{M}_g$ be the locus of curves that admit an action of the (fixed) finite group G. Now ask, how do the irreducible components of the fibres of $\mathscr{M}_g(G)$ vary at the different primes? Does their number vary? Does a given component have a sort of "ring of definition", a finite extension of $\mathbb{Z}$?

Over the open set $S = D(30|G|) \subset \mathrm{Spec}(\mathbb{Z})$, we may answer these questions quite easily using functoriality of π_0 and the stable compactification of the Hurwitz stack of curves with G action, as presented in Section 7. The factor 30 owes its presence to the existence of curves with extra automorphisms: for instance the factor 2 is related to hyperelliptic curves (we refer to the proof of [8], prop. 3.4.1 for more details on this point). More specifically, let us denote by $\mathscr{H}_{g,G}$ (resp. $\bar{\mathscr{H}}_{g,G}$) the classifying stack of smooth (resp. stable) projective genus g curves $C \to T$ with a faithful (stable) action $\rho : G_T \hookrightarrow \mathrm{Aut}_T(C)$. The stack $\bar{\mathscr{H}}_{g,G}$ is a smooth S-compactification of $\mathscr{H}_{g,G}$. The group $\mathrm{Aut}(G)$ acts upon these stacks by precomposition on ρ, and we write

$$\mathscr{H}'_{g,G} = \mathscr{H}_{g,G} / \mathrm{Aut}(G), \qquad \bar{\mathscr{H}}'_{g,G} = \bar{\mathscr{H}}_{g,G} / \mathrm{Aut}(G)$$

the quotient stacks. Due to the properties of the quotient by finite étale groups, these S-stacks are smooth (see Theorem 2.2 and Remark (ii) after it) and $\bar{\mathscr{H}}'_{g,G}$ is proper. For a generic curve C in $\mathscr{M}_g(G)$, we have $\mathrm{Aut}(C) = G$ (except for a little problem with extra automorphisms as we said before, but this is easily fixed). It follows that two actions of G differ by an element of $\mathrm{Aut}(G)$. In this way one proves that $\mathscr{H}'_{g,G} \to \mathscr{M}_g(G)$ is birational. Using smoothness of $\mathscr{H}'_{g,G}$ and functoriality for the S-birational morphisms

$$\mathscr{M}_g(G) \xleftarrow{\alpha} \mathscr{H}'_{g,G} \xrightarrow{\beta} \bar{\mathscr{H}}'_{g,G},$$

we end up with

$$\begin{aligned} \mathrm{Irr}(\mathscr{M}_g(G)/S) &= \mathrm{Irr}(\mathscr{H}'_{g,G}/S) \ \text{ by functoriality for } \alpha, \\ &= \pi_0(\mathscr{H}'_{g,G}/S) \ \text{ by smoothness}, \\ &= \pi_0(\bar{\mathscr{H}}'_{g,G}/S) \ \text{ by functoriality for } \beta. \end{aligned}$$

Theorem 3.2 shows that the latter functor is a finite étale S-scheme.

If $\mathscr{C}_\Omega$ is an irreducible component of $\mathscr{M}_g(G)_\Omega$ over an algebraically closed field Ω, it determines a geometric point $\mathrm{Spec}(\Omega) \to \mathrm{Irr}(\mathscr{M}_g(G)/S)$. The connected component of this point is a finite étale connected S-scheme, which is the spectrum of a finite extension of $\mathbb{Z}[1/(30|G|)]$. This ring deserves the name of a "ring of definition" for $\mathscr{C}_\Omega$.

Part II

Models of group schemes

In this part, we present some results on integral models of finite group schemes defined over the fraction field of a discrete valuation ring $R = (R, K, k, \pi)$. The situation of interest is when the order of the group is a multiple of the residue characteristic $p = \mathrm{char}(k)$. When such a group is given together with an action on the generic fibre of an R-scheme X, we look for models acting on X.

For example, if R contains a primitive p-th root of unity ζ, let us consider the action of the group $G_K = (\mathbb{Z}/p\mathbb{Z})_K$ on the generic fibre of the affine line $X = \mathbb{A}^1_R$ given by the homotheties ζ^i. This action extends to X in two natural ways: the first is an action of $G = (\mathbb{Z}/p\mathbb{Z})_R$ and the second is an action of $G' = \mu_{p,R}$. On the special fibre, the action of G_k is trivial whereas the action of G'_k is faithful. These two group actions show the typical choice that we have to make. Here, it is of course the model G' that we shall prefer.

In order to treat situations where no properness assumption is available, we introduced the use of the notion of *purity*. It seems to be the optimal condition to put on schemes that are acted upon in order to obtain the existence of finite flat models acting faithfully, like the group G' above. This is discussed in Section 5. As a preliminary, we give in Section 4 some definitions and elementary properties of pure schemes. Finally in Section 6 we investigate the models of the group scheme of p^n-th roots of unity, one of the most important groups in Arithmetic and Geometry.

4 Pure schemes

The notion of *purity* of morphisms of schemes was introduced in his doctoral thesis by M. Raynaud in the search for criteria of cohomological flatness under weaker assumptions than properness (see [R], e.g. II.2.11 and III.1.4). By definition, a morphism locally of finite type $f : X \to S$ is *pure* if and only if it is so after restriction to the henselisations of all points $s \in S$, so it is enough to set:

Definition. Let S be a local henselian scheme. We say that a morphism locally of finite type $f : X \to S$ is *pure* if the closure of any point $x \in X$ which is an associated point in its fibre X_s ($s = f(x)$) meets the closed fibre.

The definition extends to algebraic stacks ([8], Annexe B). A morphism which is proper, or fppf with geometrically irreducible fibres without embedded components, is pure. Purity was also used by Raynaud and Gruson as a key ingredient to describe in geometric terms the relations between flatness and projectivity: a finitely presented algebra which is flat and pure is a projective module (see [RG], 3.3.5). This fact is crucial to prove the existence of certain scheme-theoretic closures over a discrete valuation ring in the work *Effective models of group schemes* [7] which is further described in Section 5.

In several occasions, I have had to prove that such-and-such classical property of proper morphisms extends to pure morphisms. Here are three examples. The first one is the property of faithfulness of the formal completion functor.

Proposition. *Let A be a noetherian adic ring, I an ideal of definition, $S = \mathrm{Spec}(A)$, $S_0 = \mathrm{Spec}(A/I)$. Let $X \to S$, $Y \to S$ be morphisms of finite type with X pure and Y separated, and let $\widehat{S}$, $\widehat{X}$, $\widehat{Y}$ be the completions of S, X, Y along S_0, $X_0 = X \times_S S_0$, $Y_0 = Y \times_S S_0$ respectively. Then the formal completion map*

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_{\widehat{S}}(\widehat{X}, \widehat{Y}) \quad , \quad f \mapsto \widehat{f}$$

is injective.

In the proper case this is proven in [EGA], III_1, 5.4.1. In the pure case, this is proven in [7], 2.1.9 (the result is stated there in the case where A is discrete valuation ring but this assumption is not used in the proof). Of course, in the proper case the difficult point is the surjection: it requires to use Grothendieck's existence theorem. There is no analogue in the pure case. The second example concerns some properties of the fibres of morphisms:

Theorem. *Let $\mathscr{X} \to \mathscr{S}$ be a finitely presented, flat and pure morphism of algebraic stacks. Let $n \geqslant 1$ be an integer. Then the following sets are open in $|\mathscr{S}|$:*

(i) *the set of those $s \in |\mathscr{S}|$ such that $\mathscr{X}_s$ is geometrically reduced,*

(ii) *the set of those $s \in |\mathscr{S}|$ such that the geometric fibre $\mathscr{X}_{\bar{s}}$ is reduced with at most n connected components,*

(iii) *the set of those $s \in |\mathscr{S}|$ such that the geometric fibre $\mathscr{X}_{\bar{s}}$ is reduced with at most n irreducible components.*

The reader will find some conventions about the fibres of morphisms of algebraic stacks in [8], annexe A.1. In the proper case this statement is in [EGA] IV_3 12.2 and in the pure case this is in [7], th. 2.2.1 and [8], th. B.4. The third example is about the representability of the Weil restriction functor of closed subschemes:

Theorem. *Let $h : X \to S$ be a finitely presented, flat and pure morphism of schemes and $Z \hookrightarrow X$ a closed immersion. Then the Weil restriction h_*Z is representable by a closed subscheme of S.*

In the proper case, this is a consequence of Grothendieck's and Artin's theorems on representability of the Hilbert functor (see for example [Ar], § 6). In the pure case, this is proven in Appendix B.3 of the article *Moduli of Galois p-covers in mixed characteristics* [9]. This result has a big number of classical corollaries to representability of various equalizers, kernels, centralizers, normalizers, etc.

5 Effective models of finite flat group schemes

Let $R = (R, K, k, \pi)$ be a discrete valuation ring with residue characteristic p. It is known that each covering $f_K : Y_K \to X_K$ between projective, smooth, geometrically connected K-curves extends after a finite extension of K to to a morphism between semi-stable curves $f : X \to Y$. If f_K is Galois with group G, we can choose X stable, endowed with an action of G, and $Y = X/G$. When p divides the order of G, we can not avoid the apparition of inseparability in some components of the special fibre. One of our aims is to understand this inseparability. When G is cyclic of order p, an easy local computation shows that on the components where it is inseparable, the morphism f_k is generically a torsor under one of the group schemes μ_p or α_p. In his thesis [He], Henrio characterized the special fibres $f_k : X_k \to Y_k$ in terms of the combinatorial data given by these torsors (encoded by logarithmic or exact differential forms on the components), called a *Hurwitz tree*.

When the p-adic valuation of $|G|$ is at least 2, the situation is far more complicated. Let us consider the generic point ξ of an irreducible component of X_k. It is not clear a priori that there should exist a natural group scheme G' acting on a neighbourhood of ξ (we shall show that this is true), and that the morphism f_k is locally a torsor under this group (this however is not always true). If we remove from X the irreducible components of X_k whose generic points are not in the G-orbit of ξ, we get an open X' that although not proper is *pure* over R. In order to show that there exists a group G' acting faithfully on X', called the *effective model*, we established the following general result (see the article *Effective models of group schemes* [7], th. A or th. 4.3.5).

Theorem. *Let X be an R-scheme which is finitely presented, separated, flat and pure. Let G be a finite flat group scheme acting on X, faithfully on the generic fibre. Then, the scheme-theoretic image of G in $\mathrm{Aut}_R(X)$ is representable by a finite flat group scheme G'.*

The strength of this result lies in the fact that when X is not proper, the sheaf $\mathrm{Aut}_R(X)$ is not representable by a scheme or an algebraic space a priori. However there is a well-defined notion of scheme-theoretic closure, which gives meaning to the statement. We also point out that in [7] we establish generalisations (e.g. to more general finite type group schemes G, when X is affine) and variants (for formal schemes) that will not be mentioned here.

One of the main arguments of the proof is an amalgame property of finite flat subschemes of a pure scheme (see [7], 3.2.5 and 3.2.6).

Theorem. *Let R be a henselian discrete valuation ring. Let X be an R-scheme locally of finite type, flat and pure. Then, the family of closed subschemes $Z_\lambda \subset X$ finite flat over R is R-universally schematically dense. Moreover, for all diagrammes in solid arrows*

$$\begin{tikzcd}
\amalg Z_{\lambda,K} \arrow[r] \arrow[d] & X_K \arrow[d] \arrow[ddr] & \\
\amalg Z_\lambda \arrow[r] \arrow[drr] & X \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

with Y separated over R, there exists a unique morphism $X \to Y$ making the full diagramme commutative.

Once we have this amalgame theorem, we can give an idea of the proof of existence of the effective model in the previous theorem. For each closed subscheme $Z_\lambda \subset X$ which is G-stable and finite flat over R, the existence of an effective model G'_λ for the action on Z_λ is easy. By a noetherian argument, one proves that the G'_λ are dominated by one of them, that we shall denote G'. The only thing left to prove is that G' acts on X, and this is a consequence of the amalgame property with X, Y and Z_λ equal to $G \times X$, X and $G' \times Z_\lambda$ respectively.

Let us come back to the original problem of describing the reduction of a Galois cover $f : X \to Y$ in the neighbourhood of a generic point ξ of the special fibre. Once we have the existence of a model $G \to G'$ acting faithfully in a neighbourhood of the orbit of ξ, we want to describe this action: is it free at the point ξ? Does it have a relation with some sort of "ramification" of the morphism f_k? Can one construct Hurwitz trees analogue to Henrio's? The first bits of information available are given by examples for the cyclic group of order p^2. Let us mention three points.

(1) In the case of equal characteristic, one finds in [7], 5.2.4 an example of a Galois étale cover with group $\mathbb{Z}/p^2\mathbb{Z}$ of the affine line $\mathbb{A}^1_K$, over a valued field of characteristic p, whose reduction is finite flat of degree p^2 above the affine line $\mathbb{A}^1_k$, endowed with a faithful action of the group $G'_k = (\alpha_p)^2$. On the special fibre, each point has a stabilizer of order p, which shows that there is no torsor structure even at the generic point.

(2) In the case of unequal characteristics, D. Tossici goes much further in his thesis. First of all, he classifies the models of the group scheme $\mu_{p^2,K}$ ([To2]). Then, over a base ring R containing a primitive p^2-th root of unity, he considers extensions of a torsor $Y_K \to X_K$ under the group $G = \mathbb{Z}/p^2\mathbb{Z}$ into a finite G-invariant morphism $h : Y \to X$. Assuming X, Y affine normal flat with integral fibres over R and $\mathrm{Pic}(X_K)[p^2] = 0$, for each $h : Y \to X$ he defines four integral invariants j, γ_1, γ_2, $\kappa \in \mathbb{Z}$ that determine explicitly the effective model G' of the reduction of the action: γ_1 and γ_2 give the effective models of the intermediary quotients of degree p; the pair (κ, γ_2) describes G'; finally j is related to the particular cover Y_K (see [To1], 6.2.1 for a more precise statement). Then he goes on to characterize the case where the reduction is a torsor by the equality $\kappa = \gamma_1$, and provides examples where this is not the case ([To1], 6.2.11).

(3) A construction of Hurwitz trees of an arbitrary finite group G was given by L. H. Brewis and S. Wewers in the paper [BW]. This allows them to find new obstructions the problem of lifting actions from characteristic p to characteristic 0. However, these trees contain no differential data analogous to Henrio's logarithmic or exact forms, that is to say no information on the degeneration of the group. It would be interesting to add to these trees some information on the local effective models of the group, and to use them in order to study the lifting problem.

6 Models and Kummer-type exact sequences

The study of degenerations of Galois coverings, like in the previous section, puts in the forefront the integral models of the Galois group. In the present section, our main aim is to explain our approach to study the finite flat models of the group scheme μ_{p^n} of p^n-th roots of unity over a complete discrete valuation ring of unequal characteristics $(0,p)$.

First and foremost, let us insist on our point of view. Our interest for a given finite flat group scheme $G \to S$ comes to a large extent on the interest for the coverings of which it is the structure group. Thus when it is possible, we wish to have a good knowledge of the group together with an "explicit" description of the coverings, typical inspiration being given by Kummer theory for the group μ_n and by Artin-Schreier-Witt theory for the group $\mathbb{Z}/p^n\mathbb{Z}$ in characteristic p. So given a finite flat group scheme $G \to S$, we wish to see it as the kernel of an isogeny between smooth affine groups. In this connection, we emphasize that in the commutative case, the resolution of a finite flat group scheme by abelian schemes given by Raynaud's embedding theorem [BBM] 3.1.1, or the "standard smooth resolution" by affine smooth groups of Bégueri [Be] 2.2.1, do not seem to be well-suited to the computations we contemplate doing.

6.1 Kummer-type sequences for group schemes of order p. Let S be a scheme and $G \to S$ a finite locally free group scheme of order p. Under a fairly weak assumption, we can put G inside an exact sequence alike to the ones described before. In order to give a precise statement of this result, we use the notion of *full set of sections* taken from Katz-Mazur [KM] 1.8 or to [9] 1.1 to which we refer. Let us say that a morphism of group schemes $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ is a *generator* if the $\gamma(i)$, $0 \leqslant i \leqslant p-1$, form a full set of sections. For example, the trivial morphism $(\mathbb{Z}/p\mathbb{Z})_S \to \mu_{p,S}$ is a generator if S has characteristic p. Let us say that a morphism $\kappa : G \to \mu_{p,S}$ is a *cogenerator* if the Cartier dual $(\mathbb{Z}/p\mathbb{Z})_S \to G^\vee$ is a generator.

Theorem. *Let S be a scheme and $G \to S$ a finite locally free group scheme of order p. Let $\kappa : G \to \mu_{p,S}$ be a cogenerator. Then κ can be canonically inserted into a commutative diagramme with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\kappa"] & \mathscr{G} \arrow[r, "\varphi_\kappa"] \arrow[d] & \mathscr{G}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu_{p,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r, "p"] & \mathbb{G}_{m,S} \arrow[r] & 0
\end{tikzcd}$$

where $\varphi_\kappa : \mathscr{G} \to \mathscr{G}'$ is an isogeny between affine smooth one-dimensional S-group schemes with geometrically connected fibres.

This result is the topic of Appendix A in the article *Moduli of Galois p-covers in mixed characteristics* [9]. The proof proceeds by constructing a family of group schemes of order p denoted $H^M_{\lambda,\mu}$ defined as kernels of certain isogenies, and giving a correspondence between the triples of parameters (M,λ,μ) and the triples (L,a,b) of the Tate-Oort classification, where $G = G^L_{a,b}$. The groups $\mathscr{G}$ and $\mathscr{G}'$ are in fact very explicit: they are global variants of the groups $\mathscr{G}^{(\lambda)}$ of Sekiguchi, Oort and Suwa [SOS]. Finally, we notice that the condition of existence of a cogenerator is weak (it is always fulfilled after a finite locally free extension of the base of degree $p-1$) but necessary: for example, if $p>3$ and S is the spectrum of the field $\mathbb{Q}(\zeta_{p-1})$ of $(p-1)$-th roots of unity, then the group $\mathbb{Z}/p\mathbb{Z}$ can *not* be embedded into an affine smooth 1-dimensional group scheme. (Note that $\mathbb{Q}(\zeta_{p-1})$ is nothing but the ring $\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$ where Λ is the Tate-Oort ring, see [TO].)

6.2 Kummer-type sequences for the models of μ_{p^n}. Let R be a discrete valuation ring of unequal characteristics $(0,p)$ and fraction field K. In the article *Models of group schemes of roots of unity* [11] in collaboration with A. Mézard and D. Tossici, for all integers $n\geqslant 1$ we construct a family of R-models of the group scheme $\mu_{p^n,K}$ of p^n-th roots of unity, called *Kumer group schemes*. These are *defined* as kernels of some isogenies between affine smooth n-dimensional group schemes. We believe that all models of $\mu_{p^n,K}$ are Kummer, at least when R is complete with perfect residue field. It is not easy to extract a single statement that would give a fair summary of this work; we prefer to describe briefly the three main steps of our work.

First step. First of all, we translate in concrete terms, in the case of models of $\mu_{p^n,K}$, the Breuil-Kisin classification (stemming from [Br] and [Ki]) of finite flat commutative p-groups over R in terms of semilinear modules. This classification holds for a perfect residue field k and a complete totally ramified extension of the ring of Witt vectors $R/W(k)$. The result is a classification of models in terms of certain matrices with entries in the polynomial ring $k[u]$, which is this context must be seen as a subring of the discrete valuation ring $k[[u]]$. This is Theorem 4.2.2 in [11].

Second step. Then, we construct the Kummer group schemes. For this we recall the construction due to Sekiguchi and Suwa of *filtered groups*, that are affine smooth iterated extensions of the groups

$$\mathcal{G}^{\lambda}=\mathrm{Spec}(R[x,(1+\lambda x)^{-1}])=\ker(\mathbb{G}_{m,R}\to i_*\mathbb{G}_{m,R/\lambda}),$$

where i stands for the closed immersion $\mathrm{Spec}(R/\lambda)\hookrightarrow\mathrm{Spec}(R)$ and the kernel is taken in the category of fppf sheaves on the small fppf site of $\mathrm{Spec}(R)$. (The groups $\mathcal{G}^{\lambda}$ are natural because they are exactly the smooth models with connected fibres of $\mathbb{G}_{m,K}$.) After a description of the construction of these extensions in terms of certain generalized Artin-Hasse exponentials in sections 4 and 5 of [SeSu], Sekiguchi and Suwa specialize the parameters of their construction in sections 8 and 9 of [SeSu] in order to reach their goal: the unification of the Kummer and Artin-Schreier-Witt exact sequences. In contrast, for the construction of Kummer groups we do *not* specialize the parameters; thus we have to give the congruence conditions on these parameters that ensure that the morphisms produced in this way are isogenies, i.e. have *finite flat* kernel.

We obtain some finite flat models of $\mu_{p^n,K}$, that we call *Kummer groups*, parameterized by matrices with entries in the ring $W(R)$ of Witt vectors with coefficients in R. This is Theorem 7.2.1 in [11]. We remark that the construction of Kummer groups does not require the discrete valuation ring R to be complete with perfect residue field; in fact the process may be carried on over an almost general base, as we explain in the article *Sekiguchi-Suwa Theory revisited* [10] (see notably Th. 5.2.7).

Third step. Finally, we compare Breuil-Kisin modules and Kummer models. The comparison is done through the corresponding matrices on either side. The main difficulty comes from the fact that the matrices on the Breuil-Kisin module side have entries in a ring of characteristic p whereas the matrices on the Kummer group side have entries in a ring of characteristic 0. One further difficulty is that the matrices on the Breuil-Kisin side are in *bijection* with the models of $\mu_{p^n,K}$ whereas the matrices on the other side only *surject* to the Kummer groups. Put differently, we do not know how to construct a convenient "distinguished" matrix defining a Kummer group. However, we propose a map that is a candidate to settle a correspondence between the sets of matrices on either side. More precisely, we exhibit a mapping $k[[u]] \to R$, $c \mapsto c^*$ (non-additive, non-multiplicative) and to each matrix $A = (a_{ij})$ with entries in $k[[u]]$, we attach the matrix $A^* = ([a^*_{ij}])$ with entries in $W(R)$ obtained by taking the Teichmüller representatives of the a^*_{ij}. Our desired goal is to see that *A is the matrice of a Breuil-Kisin module if and only if A^* is the matrix of a Kummer group.* We check this for $n = 2$ (see [11] 8.2.6), and also for $n = 3$ (see [11] 8.3.5) under a supplementary assumption on the valuations of certains differents, which amounts to restricting to some components of the k-variety that plays the role of a moduli space for the models of $\mu_{p^n,K}$.

We conjecture that each model of $\mu_{p^n,K}$ is a Kummer group. Our initial hope was to use Breuil-Kisin modules so as to obtain this result. At present, we rather believe that our ideas may allow us to prove this conjecture independently and *a posteriori* to compute the Breuil-Kisin module of a model of $\mu_{p^n,K}$ given as a Kummer group. That would be one of the first non-trivial examples of computation of a Breuil-Kisin module.

Part III

Moduli spaces of coverings

Let us recall from the introduction that the donkey has its motivation: the carrot, and we have ours: the wish to understand the reduction at p of the proper moduli spaces $\bar{\mathscr{H}} = \bar{\mathscr{H}}_{g,G}$ of Galois covers of curves. It is interesting also to widen our horizons and recall why it is desirable in general to have a *proper* moduli space, that is, to have a *compactification* for the algebraic stack $\mathscr{H}_{g,G}$ classifying covers between smooth curves. Over a base field, having a proper variety (or a proper stack) allows to do intersection theory. Over the ring of integers of a local field, it allows to study the question of a (smooth) variety: indeed, if the situation has no other particularity like a group law, the constraint of properness ensures that a model is unique, hence meaningful and useful. Finally, over a more general base, compactifying allows to study some global or local invariants of the fibres, like in the proof of the irreducibility of the moduli space of curves by Deligne and Mumford. In this part of the report, these three occurrences of properness will be considered – and in fact have already been considered in 3.5.

It is essential to remark that in the course of the study of coverings and their moduli spaces, the representations of the group G are ubiquitous:

- via differential geometry objects like tangent spaces, spaces of differential forms, spaces of jets and ramification theory,

- via algebraic topology objects coming in particular from sheaf cohomology, since the cohomology of the cotangent complex controls the local structure of moduli spaces.

This explains the importance of the representation theory of G. At all characteristics prime to its order, the group G is semisimple and its representation theory is essentially the same (see Serre [Se], 15.5). It follows that over the ring $\mathbb{Z}[1/|G|]$, the geometry of Hurwitz spaces is essentially the same at all residue characteristics, which by the way does not mean that this is always easy to verify. In this case, we have a nice compactification using covers of stable curves. Our results in this case are summarized in Section 7. In contrast, at all characteristics $p > 0$ that divide the order of G, the existence of good compactifications remains open; if one manages to construct one, its geometry and arithmetic will necessarily be very different from what happens in characteristic 0. Our results concerning this case are presented in Section 8.

7 Galois group whose order is invertible on the base

In the text *Champs de Hurwitz* [4] in collaboration with J. Bertin, we present a complete study of Hurwitz spaces classifying covers of algebraic curves under the following assumption: the order of the Galois groups of all coverings or of their Galois closures is *invertible* in the base fields. This topic was taken up by other authors including Ekedahl [Ek] or Abramovich, Corti and Vistoli [ACV] who see it as a particular case of moduli spaces of stable maps. Our work includes the construction of these stacks, their compactification, the combinatorial description of their boundary, some significant examples including curves with level structure, and some intersection computations. Here we can hardly give more than a broad overview of the text, and a selection of some of the main results. For clarity of the presentation, we will single out three stages in the text [4]: the definitions about families of coverings; the construction of Hurwitz stacks and various examples; the enumerative geometry of Hurwitz stacks.

7.1 Definitions about families of coverings. In Sections 1 to 5, we present various definitions and basic results on coverings of smooth or nodal curves, their ramification, their families, their deformations. Over a base field k, the typical object of study is an algebraic curve C which is *stable* in the sense of Deligne and Mumford [DeMu], endowed with an *admissible* action of a finite group G, which means that any element $g \in G$ that fixes a node without switching the branches acts on the tangent spaces by mutually inverse characters. Overall these sections put together things that are known, except maybe for two of them:

(1) the fact that a finite big enough number of m-canonical G-representations $H^0(C, \omega_{C/k}^{\otimes m})$ suffices to determine the ramification of the l'action: we call this the "inversion" of the Chevalley-Weil formula ([4], thm. 3.2.2 and prop. 4.2.5),

(2) the precise study of the collision of ramification points in a family and the role of automorphisms that switch the branches ([4], thm. 4.2.2).

7.2 Construction of Hurwitz stacks and examples. In Sections 6 to 9, we construct various Hurwitz stacks. This is the heart of our work. Let us fix an integer g, a finite group G and a ramification datum ξ (a notion which is defined in [4] 2.2.1). We show that the stack $\bar{\mathscr{H}}_{g,G,\xi}$ whose objects are families of stable curves of genus g endowed with a faithful admissible action of G with ramification ξ is an algebraic Deligne-Mumford stack, proper and smooth over $\mathbb{Z}[1/|G|]$, containing the stack $\mathscr{H}_{g,G,\xi}$ of smooth curves as an open relatively dense substack ([4], thm. 6.3.1). It has a coarse mioduli space which is projective and normal. If it is convenient, one can also consider the ramification divisor as a marking in order to ensure that it stays in the smooth locus. The construction of numerous known moduli spaces reduce to that of $\bar{\mathscr{H}}_{g,G,\xi}$:

(1) *non-Galois coverings*: for a covering of smooth curves $C \to D$, the triple $m = (G, H, \xi)$ composed of the Galois groups $G = \mathrm{Gal}(Z/D)$, $H = \mathrm{Gal}(Z/C)$ and the ramification

datum of G acting on Z, is independent of the choice of a Galois closure $Z \to C$. We call it the *monodromy type*. We set

$$\mathrm{Aut}(m) = \{\theta \in \mathrm{Aut}(G),\, \theta(H) = H \text{ et } \theta(\xi) = \xi\}$$

and $\Delta(m) = \mathrm{Aut}(m)/H$, where the ramification datum $\theta(\xi)$ has an obvious definition that the reader may find after Proposition 2.2.3 in [4]. We show that the stack $\mathscr{H}_{g,g',m}$ classifying coverings between smooth curves of genera g and g' with fixed monodromy equal to m is isomorphic to the quotient stack $\mathscr{H}_{g,G,\xi}/\Delta(m)$ ([4], thm. 6.6.6). Thus it has a natural smooth compactification $\bar{\mathscr{H}}_{g,g',m} := \bar{\mathscr{H}}_{g,G,\xi}/\Delta(m)$.

(2) *level structures*: we show that the stack ${}_G\mathscr{M}_g^0$ classifying smooth curves endowed with a Teichmüller structure of level G (as defined in Deligne and Mumford [DeMu], 5.7, 5.8) is isomorphic to the rigidified stack $\mathscr{H}_{h,G,0} /\!\!/ Z(G)$, where h is defined by the equality $h-1 = |G|(g-1)$ and $Z(G)$ is the centre of G. (The process of rigidification denoted by the symbol "$/\!\!/$" is described in [ACV], 5.1.1). Thus it has a natural smooth compactification ${}_G\bar{\mathscr{M}}_g := \bar{\mathscr{H}}_{h,G,0} /\!\!/ Z(G)$ ([4], thm. 8.2.2 and rem. 8.2.3). This compactification has a proper birational morphism (which is not representable in general) to the normalization ${}_G\mathscr{M}_g$ (which is not smooth in general) defined by Deligne and Mumford.

(3) *curves with symmetries*: the substack $\mathscr{M}_g(G) \subset \mathscr{M}_g$ composed of curves admitting an action of the group G (studied among others by González-Díez and Harvey [GDH]) is a closed substack, which is singular and reducible in general. We show that its normalization is isomorphic to a disjoint sum of stacks of the form $\mathscr{H}_{g,G,\xi}/\mathrm{Aut}(G)$, so that in particular it is smooth (see a more precise description in [8] 3.4.1). Thus the closure $\bar{\mathscr{M}}_g(G)$ of $\mathscr{M}_g(G)$ in $\bar{\mathscr{M}}_g$ has a natural proper desingularization, a disjoint sum of stacks of the form $\bar{\mathscr{H}}_{g,G,\xi}/\mathrm{Aut}(G)$. The functor of irreducible components of $\mathscr{M}_g(G)$ is a finite étale scheme over $\mathbb{Z}[1/(30|G|)]$, as we saw in 3.5.

7.3 Enumerative geometry of Hurwitz stacks. In Section 10, following the strategy initiated by Mumford [Mu], we study the tautological classes in the Chow ring of the Hurwitz stack $\bar{\mathscr{H}}_{g,G,\xi}$ and the tautological relations between them. Here a crucial role is played by the correspondence

$$\bar{\mathscr{M}}_{g',b} \longleftarrow \bar{\mathscr{H}}_{g,G,\xi} \longrightarrow \bar{\mathscr{M}}_{g,r}$$

where the left arrow maps a curves with action (C,G) to the quotient curve C/G marked by the branch points, and the right arrow maps (C,G) to the curve C marked by the ramification points. Because of the action of the group G, we obtain plenty of tautological classes by considering isotypical components of preimages of the tautological bundles living on $\bar{\mathscr{M}}_{g,r}$. Let us state simply two results.

(1) In the Picard group, the Riemann-Hurwitz relation gives an expression of the canonical sheaf of the universal curve $\bar{\mathscr{C}}_{g,G,\xi}$ in terms of the preimage of the canonical sheaf of the universal curve $\bar{\mathscr{C}}_{g',b}$ and of certain classes coming from the ramification. Taking powers of this relation in the Chow ring, we get higher Riemann-Hurwitz formulas that give expressions of κ classes (direct images of Chern classes of the Hodge bundle on the universal curve) coming from $\bar{\mathscr{M}}_{g,r}$ in terms of those coming $\bar{\mathscr{M}}_{g',b}$ and classes coming from the normal sheaf along the ramification ([4], thm. 10.3.4).

(2) Applying the Grothendieck-Riemann-Roch theorem, we compute the Chern character of the bundle defined by the branch locus of the coverings on the universal curve of genus g' (or better on a modification of the universal curve where the branch divisor becomes Cartier). The result gives this character as a function of the ψ classes (defined by the morphisms of evaluation at the marked sections) and the κ classes. We refer to [4], thm 10.4.1 for a precise statement.

8 Galois group whose order is not invertible

8.1 An example : Potts curves. In genus $g \geqslant 2$, very few examples are available where stacks of Galois coverings of curves are studied thoroughly at the characteristics that divide the order of the Galois group. It may therefore be useful to recall that in my Ph.D. thesis I worked out the example of some Galois smooth coverings of genus $p-1$ of the projective line, with dihedral Galois group $G = \mathbb{D}_p$, called *Potts curves*. It is shown in [1] that:

(1) the stack of Potts curves is a Deligne-Mumford stack of dimension 1,

(2) the formation of its coarse moduli space commutes with passage to the fibre at p,

(3) the fibre at p is an étale gerbe over its moduli space.

8.2 The stack of p-torsors. Let $R = (R, K, k, \pi)$ be a discrete valuation ring. As we said already in Section 5, a torsor $Y_K \to X_K$ under the group $G = \mathbb{Z}/p\mathbb{Z}$, let us say in characteristic 0 to fix ideas, may be extended on a non-empty open set over R to a torsor $Y \to X$ under one of the three groups of order p, namely $\mathbb{Z}/p\mathbb{Z}$, μ_p or α_p. However if for instance X_K and Y_K are proper and we insist to extend them into proper R-schemes, then in general we can at best extend the torsor structure in a neighbourhood of the generic points of the special fibre. A necessary condition is to find a *flat* R-morphism $Y \to X$, and this is not always possible.

Let us now start out from a torsor $Y_K \to X_K$ whose base and total space are smooth curves. In order to extend it, we can choose the stable model Y of Y_K and the quotient $X = Y/G$ (after a finite extension of the base if needed). This morphism is not flat a priori, but a crucial observation due to D. Abramovich is that we can modify the curve X into a *twisted curve* $\mathscr{X} \to X$ *in a unique way* so that $Y \to X$ lifts to a *flat* G-invariant morphism. (We refer to [AOV], Section 2 for the definition of a twisted curve; let us simply say that this is a one-dimensional Artin stack with finite linearly reductive stabilizers.) Then we show that the scheme-theoretic image of the morphism $G_{\mathscr{X}} \to \mathrm{Aut}_{\mathscr{X}}(Y)$ is a finite flat $\mathscr{X}$-group scheme $\mathscr{G}$ of degree p, and that $Y \to \mathscr{X}$ is a torsor under $\mathscr{G}$. Moreover one checks that the morphism $\gamma : G_{\mathscr{X}} \to \mathscr{G}$ is a *generator* in the sense that the p sections $\gamma(i)$, $0 \leqslant i \leqslant p-1$, form a full set of sections of $\mathscr{G}$ (see [KM] 1.8 and Section 6.1).

If we include marked points, we are led to the following definition. We fix integers $g, h, n \geqslant 0$ such that $2g - 2 + n > 0$. Over a base scheme S, we call *stable n-marked p-torsor* a triple composed of

(i) an n-marked twisted curve $(\mathscr{X}, \{\Sigma_i\}_{1 \leqslant i \leqslant n})$ of genus h,

(ii) a stable marked curve with étale markings $(Y, \{P_i\}_{1\leqslant i\leqslant n})$,

(iii) a finite locally free $\mathscr{X}$-group scheme $\mathscr{G}$ of degree p endowed with a generator $\gamma : (\mathbb{Z}/p\mathbb{Z})_{\mathscr{X}} \to \mathscr{G}$,

such that $Y \to \mathscr{X}$ is a $\mathscr{G}$-torsor and $P_i = \Sigma_i \times_{\mathscr{X}} Y$ for all i. The category of n-marked stable p-torsors (of genera g and h) is a $\mathbb{Z}$-stack denoted $ST_{p,g,h,n}$. In a stable p-torsor, the P_i have degree 1 or p above S. Those that have degree 1 (i.e. those that are sections) correspond to ramification points of the morphism $Y \to \mathscr{X} \to X$ where $\mathscr{X} \to X$ is the coarse moduli space, and their number m is determined by the Riemann-Hurwitz formula $2g-2 = p(2h-2)+m(p-1)$. The others correspond to marked points, in number $n-m$. In the article [9], in collaboration with D. Abramovich, we show the following result.

Theorem. *The stack $ST_{p,g,h,n}$ of n-marked stable p-torsors is a Deligne-Mumford stack which is proper over* $\mathrm{Spec}(\mathbb{Z})$.

The considerations that we gave as an introduction yielded a sketch of proof of the valuative criterion for properness for $ST_{p,g,h,n}$, for the K-points that factor through the locus of *smooth* p-torsors. This is however not enough to prove properness because these may not be dense, and more work is needed in our paper [9].

8.3 The question of reduction at p of the Hurwitz stack. Over the ring $\mathbb{Z}[1/p]$, let us compare the stack $ST_{p,g,h,n} \otimes \mathbb{Z}[1/p]$ with the Hurwitz stacks of Section 7. In the case of the group $G = \mathbb{Z}/p\mathbb{Z}$, the ramification datum reduces to the integer m equal to the number of ramification points. It is not hard to see that the stack $ST_{p,g,h,n} \otimes \mathbb{Z}[1/p]$ is isomorphic to the Hurwitz stack $\bar{\mathscr{H}}_{g,G,m,d}$ of stable genus g curves endowed with an admissible action of G ramified at m points, with $d = n-m$ marked points. In order to answer the question of reduction at p of the Hurwitz stacks $\bar{\mathscr{H}}_{g,G,m,d}$, what we have to do now is to study with more details the special fibre of $ST_{p,g,h,n}$. The deformation theory splits into three steps: first deform the twisted curve $\mathscr{X}$ (this problem is unobstructed), then the group scheme $\mathscr{G} \to \mathscr{X}$, and finally the stable curve $Y \to \mathscr{X}$ as a torsor. This will be the topic of a further work; we hope for instance to be able to find in the fibre at p some nice components, those containing the smooth curves, etc.

8.4 Galois groups of order divisible by p^2. As a conclusion, we wish to point out that the case of cyclic coverings of order p is particularly simple because of the fact that the group $\mathbb{Z}/p\mathbb{Z}$ has no nontrivial subgroup. As far as the Galois group $G = \mathbb{Z}/p^2\mathbb{Z}$ is concerned (for example), things become significantly harder. It is plausible that in the situation of 8.2, the scheme-theoretic image of the morphism $G_{\mathscr{X}} \to \mathrm{Aut}_{\mathscr{X}}(Y)$ is again a finite flat $\mathscr{X}$-group scheme $\mathscr{G}$ of degree p^2. However, we have many examples showing that $Y \to \mathscr{X}$ is not a torsor under $\mathscr{G}$ in general (we gave such examples in Section 5).

Another hope that one may have is that on the special fibre, the group $\mathscr{G}_k$ might be constant at least on open sets (like what happens in degree p), that is to say of the form $G'_k \times_{\mathrm{Spec}(k)} \mathscr{X}_k$ for a certain finite k-group G'_k which is nothing else than (the special fibre of) an effective model of G as constructed in the article [7], see Section 5. The examples of Tossici's article [To1] show that this is not the case: we give a counter-example below. Thus much remains to be understood about these models, the variation of the stabilizers, the ramification...

8.5 An example. We give a counter-example in the local case – it is easy to extend it to an affine curve. We omit some computations in order to keep the text at a reasonable length. Let us take a discrete valuation ring $R = (R, K, k, \pi)$ containing a primitive p^2-th root of unity ζ and set $G = \mathbb{Z}/p^2\mathbb{Z}$. We denote by $X = \mathrm{Spec}(A)$ the localization of the affine R-line at the origin of the special fibre, that is $A = R[Z]_m$ where $m = (\pi, Z)$. We choose integers $\gamma_1, \gamma_2 \geqslant 2$ such that $v(p) > p\gamma_1 = p^2\gamma_2 > 0$, an element $f \in A$ whose reduction modulo π is invertible but not a p-th power, we set $r = \pi^{\gamma_2 - 1}Z$ and $g = 1 + r^p f$. Finally, for each $\gamma \leqslant v(p)/(p-1)$ we define $P_\gamma(T) = \pi^{-p\gamma}((1+\pi^\gamma T)^p - 1) \in R[T]$ and we consider the finite morphism $Y \to X$ described inside $\mathbb{A}^2_R \times X = \mathrm{Spec}(A[T_1, T_2])$ by the equations :

$$P_{\gamma_1}(T_1) = f \quad , \quad P_{\gamma_2}(T_2) = \frac{\frac{1+r^p f}{(1+rT_1)^p}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}.$$

One checks that Y is normal, flat over R with integral fibres. On the generic fibre, $Y_K \to X_K$ is the G-torsor with equation

$$T^{p^2} = (1 + \pi^{p\gamma_1} f)g^p$$

where $T = (1+rT_1)(1+\pi^{\gamma_2}T_2)$. The G-action $T \mapsto \zeta T$ extends to Y. Set $H(T_1) = 1 + rT_1$. According to Theorem 6.2.1 in [To1], if $p \geqslant 3$ then the effective model of G is controlled by the solutions $a \in R$ of the equation, for variable $m \geqslant \gamma_1$:

$$(\triangle)_m : \qquad aH(T_1) \equiv \pi^{m-\gamma_1} H'(T_1) \mod \pi^{\gamma_2}.$$

In order to evaluate the invariant κ of [To1] 6.1.3, let us do $m = \gamma_1$ i.e. look at the equation

$$a(1 + rT_1) \equiv r \mod \pi^{\gamma_2}.$$

If we consider the action of G over the discrete valuation ring $R_1 = R$, this equation has no solution and it follows that the associated invariant κ satisfies $\kappa_1 > \gamma_1$. Now let us call η the generic point of X_k. Over the discrete valuation ring $R_2 = \mathcal{O}_{X,\eta}$ with uniformizer π, the equation has the solution $a = r \in R_2$. It follows that $\kappa_2 = \gamma_1$. According to [To1] 6.2.1, the effective models G'_1 and G'_2 are different.

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