The Complexity of a Flat Groupoid

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Abstract. Grothendieck proved that any finite epimorphism of noetherian schemes factors into a finite sequence of effective epimorphisms. We define the complexity of a flat groupoid $R \rightrightarrows X$ with finite stabilizer to be the length of the canonical sequence of the finite map $R \to X \times_{X/R} X$, where X/R is the Keel–Mori geometric quotient. For groupoids of complexity at most 1, we prove a theorem of descent along the quotient $X \to X/R$ and a theorem on the existence of the quotient of a groupoid by a normal subgroupoid. We expect that the complexity could play an important role in the finer study of quotients by groupoids.

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1 Introduction

Motivation. Let X be a scheme endowed with an action of a group scheme G such that there exists a quotient $\pi : X \to Y = X/G$. Consider the category $\mathcal{C}(X)$ of vector bundles on X. In this paper, we give new examples where one can characterize the G-linearized bundles on X that descend to bundles on Y, and similarly for other fibered categories $\mathcal{C}$. More precisely, let $\mathcal{C}(G, X)$ be the category of vector bundles endowed with a G-linearization. Let $\mathcal{C}(G, X)'$ be the subcategory of G-linearized bundles for which the action of the stabilizers of geometric points is trivial. It is not hard to see that for any vector bundle

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$\mathcal{G} \in \mathcal{C}(Y)$, the pullback $\mathcal{F} = \pi^*\mathcal{G}$ is naturally an object of $\mathcal{C}(G, X)'$. The question is:

Let $G \times X \to X$ be a group scheme action as above, with quotient $\pi : X \to Y = X/G$. When is the pullback $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(G, X)'$ an equivalence?

The correct framework for this type of question is that of algebraic spaces (which generalize schemes) and groupoids (which generalize group actions). That this is so was demonstrated twenty years ago by Keel and Mori who settled the question of existence of quotients for actions with finite stabilizer in the paper [KM97]. The main point is that groupoids allow reduction and dévissage in a much more flexible way than group actions. Moreover, groupoids include examples of interest like foliations in characteristic p, and inseparable equivalence relations as in work of Rudakov and Shafarevich [RS76] and Ekedahl [Ek88], which we will return to in the end of this introduction. We emphasize that our results are equally interesting in the restricted case of group actions. So in the sequel we let

(1) $R \rightrightarrows X$ be a flat locally finitely presented groupoid of algebraic spaces,

(2) $\mathcal{C} \to \mathrm{AlgSp}$ be a category fibered over the category of algebraic spaces,

(3) $\mathcal{C}(R, X)$ be the category of objects of $\mathcal{C}(X)$ equipped with R-linearizations (see 4.1 for a precise definition), and

(4) $\mathcal{C}(R, X)' \subset \mathcal{C}(R, X)$ be the full subcategory of objects with trivial geometric stabilizer actions.

Since R-linearized objects on X are the same as objects on the algebraic stack $\mathscr{X} = [X/R]$, the language of stacks is an alternative which is also used on that matter.

Known results. When $X \to Y$ is a *tame* quotient, which means that the geometric stabilizers of $R \rightrightarrows X$ are linearly reductive finite group schemes, and $\mathcal{C}$ is either the category of line bundles, or finite étale covers, or torsors under a fixed linearly reductive finite group scheme, Olsson showed that $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(G, X)'$ is an equivalence [Ol12, Props. 6.1, 6.2, 6.4]. When $X \to Y$ is a *good* quotient and $\mathcal{C}$ is the category of vector bundles, Alper showed that π^* is an equivalence [Al13, Thm. 10.3]. Results for good quotients and other categories $\mathcal{C}$ will be presented in an upcoming paper by the second author.

The complexity. We wish to find examples that go beyond these cases, e.g., wild actions in characteristic p. In this new setting the map $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(G, X)'$ fails to be an isomorphism in general; e.g. if $\mathcal{C}$ is the category of line bundles and $X = \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ with trivial action of $G = \mathbb{Z}/p\mathbb{Z}$, the G-line bundle L generated by a section x with action $x \mapsto (1+\epsilon)x$ is not trivial. For this, we introduce a new invariant of flat groupoids which we call the *complexity*. (This is not to be confused with the complexity as defined by Vinberg [Vi86] in another context, namely the minimal codimension of a Borel orbit in a variety acted on by a connected reductive group.) We fix our attention on the morphism $j_Y : R \to X \times_Y X$ which is finite and surjective when the groupoid has finite inertia. The complexity of the groupoid is controlled by the epimorphicity properties of this map. In order to quantify this, we use a result of Grothendieck to the effect that a finite epimorphism of noetherian schemes factors as a finite sequence of *effective* epimorphisms. We prove in 2.3.2 that there is a canonical such sequence, and we define the complexity of $R \rightrightarrows X$ as the length of the canonical sequence of j_Y.

MAIN NEW RESULTS. The complexity is equal to 0 when j_Y is an isomorphism, which means that the groupoid acts freely; in this case most questions involving $R \rightrightarrows X$ are easily answered. The next case in difficulty is the case of complexity 1. In order to obtain results in this case, we introduce the stabilizer Σ of $R \rightrightarrows X$, which is the preimage of the diagonal under $R \to X \times X$. It refines the information given by the collection of stabilizers of geometric points in that it accounts for higher ramification. We let $\mathcal{C}(R,X)^\Sigma \subset \mathcal{C}(R,X)'$ be the subcategory of R-linearized objects for which the action of Σ is trivial. In our main result we have to assume that the quotient map is flat; the payoff is that we can handle very general categories $\mathcal{C}$.

THEOREM 4.2.3. *Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid space with finite stabilizer $\Sigma \to X$ and complexity at most 1. Assume that the quotient $\pi : X \to Y = X/R$ is flat (resp. flat and locally of finite presentation). Let $\mathcal{C} \to \mathrm{AlgSp}$ be a stack in categories for the fpqc topology (resp. for the fppf topology).*

(1) *If the sheaves of homomorphisms $\mathcal{H}om_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ have diagonals which are representable by algebraic spaces, then the pullback functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R,X)^\Sigma$ is fully faithful.*

(2) *If the sheaves of isomorphisms $\mathcal{I}som_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ are representable by algebraic spaces, then the pullback functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R,X)^\Sigma$ is essentially surjective.*

In particular if $\mathcal{C}$ is a stack in groupoids with representable diagonal, the functor π^* is an equivalence.

This applies to stacks whose diagonal has some representability properties. The next theorem applies to a stack which does not enjoy such a property.

THEOREM 4.2.5. *Let $\mathcal{C} \to \mathrm{AlgSp}$ be the fppf stack in categories whose objects over X are flat morphisms of algebraic spaces $X' \to X$. Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid space with finite stabilizer $\Sigma \to X$ and complexity at most 1. Assume that the quotient $\pi : X \to Y = X/R$ is flat and locally of finite presentation. Then the functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R,X)^\Sigma$ is an equivalence.*

We give examples of groupoids satisfying the assumptions of these theorems in section 3.3. These include groupoids acting on smooth schemes in such a way that the stabilizers are symmetric groups acting by permutation of local coordinates. Other examples are given by groupoids acting on curves in positive characteristic; this is especially interesting in characteristic 2. The two theorems above can fail when π is not flat and the stabilizer groups are not tame, see section 4.5. We do not know if the assumption that the complexity is at most one is necessary.

We give an application to the existence of quotients of groupoids by normal subgroupoids. This is interesting when applying dévissage arguments, as for instance in [KM97, § 7]. This question is also natural from the point of view of understanding the internal structure of the category of groupoids. The basic observation is this: if $R \rightrightarrows X$ is a groupoid $P \subset R \rightrightarrows X$ is a normal flat subgroupoid, the actions of P on R by precomposition and postcomposition are free, but the simultaneous action of $P \times P$ is *not* free. For groupoids $R = G \times X$ given by group actions, it is nevertheless easy to make G/H act on X/H, providing a quotient groupoid $G/H \times X/H \rightrightarrows X/H$. However for general groupoids, constructing a composition law on the quotient $P\backslash R/P$ making it a groupoid acting on X/P is much more complicated. In section 4.3 we review some cases where this is possible. For subgroupoids of complexity 1 with flat quotient, we obtain a satisfying answer.

Theorem 4.3.1. *Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid of algebraic spaces. Let $P \rightrightarrows X$ be a flat, locally finitely presented normal subgroupoid of R with finite stabilizer $\Sigma_P \to X$ and complexity at most 1. Assume that the quotient $X \to Y = X/P$ is flat and locally finitely presented. Then there is a quotient groupoid $Q \rightrightarrows Y$ which is flat and locally finitely presented, with $Q = P\backslash R/P$. Moreover, the morphisms $R \to Q$ and $R \times_X R \to Q \times_Y Q$ are flat and locally finitely presented.*

Directions of further work. The natural question now is to extend these results to the case of groupoids of complexity 2. This would most likely shed some light on the case of arbitrary complexity. For the moment, we have no idea of what the correct substitute for $\mathcal{C}(R, X)^\Sigma$ should be in the general context.

The application we envision for these results is to the study of finite flat covers of algebraic varieties, typically over a field k of characteristic p. More precisely, we expect our theorems to be useful for understanding how purely inseparable morphisms of algebraic k-varieties $f : V \to W$ can be factorized. An important instance is when f is an iterate of the Frobenius morphism of V. We note that when V is smooth, f will be flat. Thus the assumption of flatness of the quotient map in our results is not too annoying; we give some more comment on this point in Remark 4.2.4.

Organization of the article. As we said already, we work in the setting of groupoids in algebraic spaces. (The relevance of this choice in questions of quotients in Algebraic Geometry is well explained in the paper [Li05] which we recommend as an excellent contextual reading.) This leads us to start in section 2 with some preparations on finite epimorphisms of spaces. In particular, we give sufficient conditions for an epimorphism of algebraic spaces to be effective, and we prove a precise form of Grothendieck's factorization of finite epimorphisms into finite effective epimorphisms. In section 3 we recall the basic vocabulary of groupoids, we define the complexity, and we present several examples. Finally in section 4 we prove the main results of the paper, presented above.

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2 Finite epimorphisms

This section of preliminary nature contains material on finite epimorphisms of algebraic spaces. The notion of epimorphism turns out to be a little more subtle in the category of algebraic spaces than its counterpart in the category of schemes, due to the lack of the locally ringed space description. The same is true for the notion of effective epimorphism. In order to have a better understanding of the situation, we will give some manageable conditions that ensure that a map of algebraic spaces is an epimorphism, or an effective epimorphism. The main result is Theorem 2.2.5, but for the convenience of the reader we will indicate here its main consequence needed in the sequel. We occasionally write *qcqs* for *quasi-compact and quasi-separated*. Recall the following two statements in the easy scheme case:

2.0.1 Proposition. *Let $f : S' \to S$ be a qcqs surjective morphism of schemes. Write $\mathcal{A}(S') = f_*\mathcal{O}_{S'}$. Then the following are equivalent:*

(1) *f is schematically dominant, that is, $\mathcal{A}(S) \to \mathcal{A}(S')$ is injective;*

(2) *f is an epimorphism in the category of schemes.*

2.0.2 Proposition. *Let $f : S' \to S$ be a qcqs submersive morphism of schemes. Write $S'' = S' \times_S S'$. Then the following are equivalent:*

(1) *$\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact;*

(2) *f is an effective epimorphism in the category of schemes.*

The main results we shall need are the following:

2.0.3 Proposition. *(Lemma 2.1.5) Let $f : S' \to S$ be a qcqs morphism of algebraic spaces which is submersive after every étale base change on S. Then the following are equivalent:*

(1) *f is schematically dominant, that is, $\mathcal{A}(S) \to \mathcal{A}(S')$ is injective;*

(2) *f is an epimorphism in the category of algebraic spaces.*

2.0.4 Proposition. *(Lemma 2.2.3 + Corollary 2.2.8) Let $f : S' \to S$ be an integral morphism of algebraic spaces. Then the following are equivalent:*

(1) *$\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact;*

(2) *f is an effective epimorphism in the category of algebraic spaces.*

Under these equivalent conditions, f is a uniform effective epimorphism.

Finally we prove Grothendieck's factorization of a finite epimorphism into a finite sequence of finite effective epimorphisms, Proposition 2.3.2, placing ourselves in a slightly more general context and giving some useful complements.

2.1 Epimorphisms

First we recall an easy characterization of epimorphisms of schemes.

2.1.1 Lemma. *Let $f : S' \to S$ be a morphism of schemes. The following conditions are equivalent:*

(1) *f is an epimorphism (of schemes).*

(2) *f does not factor through an open or closed subscheme $Z \subsetneq S$.*

(3) *f does not factor through a subscheme $Z \subsetneq S$.*

Proof : (1) $\Rightarrow$ (2). Assume that f factors through a subscheme $Z \subsetneq S$ which is either open or closed. Let $X = S \amalg_Z S$ be the ringed space obtained by gluing two copies of S along their common copy of Z. If Z is open then X is a scheme by ordinary topological gluing, and if Z is closed then X is a scheme by Ferrand [Fe03, Thm. 7.1] or [SP, Tag 0B7M]. Let $u, v : S \to X$ be the canonical maps. We have $u \neq v$ and $uf = vf$, so f is not an epimorphism.

(2) $\Rightarrow$ (3) Immediate because a subscheme is a closed subscheme of an open subscheme.

(3) $\Rightarrow$ (1). Let X be a scheme and let $u, v : S \to X$ be morphisms such that $uf = vf$. Let Z be the preimage of the diagonal $\Delta : X \to X \times X$ by the map $(u, v) : S \to X \times X$. Since Δ is an immersion, then Z is a subscheme of S. Since f factors through Z, by (3) it follows that $Z = S$. This shows that (u, v) factors through the diagonal, that is $u = v$. □

Recall that an algebraic space is called *locally separated* if its diagonal is an immersion. Clearly the lemma and its proof show that an epimorphism of schemes is also an epimorphism in the category of locally separated algebraic spaces. However, it may fail to be an epimorphism in the category of all algebraic spaces, even if it is surjective and schematically dominant. Here is a counter-example.

2.1.2 Example. Let k be a field of characteristic $\neq 2$. Consider the scheme

$$S = \mathrm{Spec}(k[x,y]/(x^2-y^2))$$

with closed subscheme $Y = V(x-y)$ and open complement $U = D(x-y) = S \setminus Y$. Let $S' = Y \amalg U$. Then the canonical map $f : S' \to S$ is a surjection to a reduced scheme, hence an epimorphism of schemes by the lemma above. The map $j : S' \to S \subset \mathbb{A}^1_k \times \mathbb{A}^1_k$ defines an étale equivalence relation on $\mathbb{A}^1_k$. We let $\pi : \mathbb{A}^1_k \to X$ be the quotient algebraic space. By construction, the pullback of the diagonal $X \subset X \times X$ to $\mathbb{A}^1_k \times \mathbb{A}^1_k$ is S'. Let $u, v : S \to \mathbb{A}^1_k \to X$ be the maps induced by the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : S \to \mathbb{A}^1_k$. These maps are distinct, since otherwise (u,v) would factor through the diagonal of X, which would mean that $(p_1,p_2) : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ factors through S', which it does not. However $uf = vf$, hence f is not an epimorphism of algebraic spaces.

In the applications that we have in mind, it is cumbersome to check that the algebraic spaces involved satisfy some separation condition. Because of this, we spend some effort on obtaining criteria for epimorphisms in the category of all algebraic spaces. In order to put 2.1.1 in perspective, it is useful to have the construction of gluing along closed subschemes available for algebraic spaces. This is originally due to Raoult [Ra74]. Variants appear in [Ar70, Thm. 6.1], [Ry11, Thm. A.4], [CLO12, Thm. 2.2.2], [TT16, Thm. 5.3.1]. In all these sources, the hypotheses allow one of the maps f, g of the gluing diagram to be finite or at least affine and usually some noetherian-like assumptions are present. It is known to most people that these assumptions are not essential at least when both maps f, g are closed immersions; we give a statement with the main input for the proof coming from [SP].

2.1.3 Lemma. *Let $i_1 : Y \hookrightarrow X_1$ and $i_2 : Y \hookrightarrow X_2$ be closed immersions of algebraic spaces. Then, there exists a pushout $W = X_1 \amalg_Y X_2$ in the category of algebraic spaces:*

$$\begin{tikzcd} Y \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \arrow[d, "b"] \\ X_1 \arrow[r, "a"] & W. \end{tikzcd}$$

Moreover, the diagram is a cartesian square; the maps a, b are closed immersions; the pushout is topological, i.e., its underlying topological space is $|X_1| \amalg_{|Y|} |X_2|$; and there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow a_*\mathcal{O}_{X_1} \oplus b_*\mathcal{O}_{X_2} \longrightarrow c_*\mathcal{O}_Y \longrightarrow 0$$

of sheaves on the small étale site of W.

Proof : We will reduce to the known case of schemes. For this we will use the following classical extension result for étale maps: if U, E, E' are disjoint

unions of affine schemes (henceforth to be called *sums of affines* for brevity) and $E \hookrightarrow U$ is a closed immersion, and $E' \to E$ is an étale morphism, then there exists a sum of affines U' and an étale morphism $U' \to U$ such that $E' \simeq U' \times_U E$. The proof can be found for example in [SP, Tag 04D1]. Note that if $E' \to E$ is surjective, we may choose $U' \to U$ surjective by adding to U' the sum of affines in a Zariski covering of $U \setminus E$.
For each $i = 1, 2$ let $\pi_i : U_i \to X_i$ be an étale surjective map where U_i is a sum of affines. Let $E_i = U_i \times_{X_i} Y$. Then $E_1 \times_Y E_2$ is étale surjective over E_1 and E_2. Let E' be the sum of affines given by a Zariski covering of $E_1 \times_Y E_2$. By the fact quoted above, for each $i = 1, 2$ there exists $U_i' \to U_i$ étale surjective whose restriction to E_i is isomorphic to E'. In this way, replacing U_i by U_i' we see that we can assume that $E_1 \simeq E_2$. Now for $i = 1, 2$ let $R_i = U_i \times_{X_i} U_i$ with its two projections $s_i, t_i : R_i \to U_i$. Let F_i be the preimage of Y in R_i. Since $\pi_i s_i = \pi_i t_i$, this is isomorphic to the preimage of E_i under any of the maps s_i or t_i. The isomorphism $E_1 \simeq E_2$ induces a compatible isomorphism $F_1 \simeq F_2$; in the sequel we view these isomorphisms as identifications so we write $E = E_1 = E_2$ and $F = F_1 = F_2$.
By the scheme case the pushouts $\mathscr{U} := U_1 \amalg_E U_2$ and $\mathscr{R} := R_1 \amalg_F R_2$ make sense as schemes. Using the pushout property for $\mathscr{R}$ we see that the maps $s \amalg s, t \amalg t : R_1 \amalg R_2 \to U_1 \amalg U_2$ induce maps which for simplicity we again denote $s, t : \mathscr{R} \to \mathscr{U}$. They are clearly surjective. We claim that moreover they are étale. This is a local property and is proved in [SP, Tag 08KQ]. Let $W = \mathscr{U}/\mathscr{R}$ be the quotient algebraic space. Checking that W is the pushout is formal, and obtaining the additional properties is easy by taking an atlas. □

We obtain at least a necessary condition.

2.1.4 Lemma. *An epimorphism of algebraic spaces does not factor through a locally closed subspace $Z \subsetneq S$.*

Proof : Same proof as 2.1.1 using Lemma 2.1.3 instead of [Fe03, Thm. 7.1]. □

We now present two simple examples of epimorphisms of algebraic spaces. The first one improves [Ry10, Prop. 7.2] where it is assumed that f is a submersion after every base change.

2.1.5 Lemma. *Let $f : S' \to S$ be a morphism of algebraic spaces which is schematically dominant, and submersive after every étale base change on S. Then f is an epimorphism of algebraic spaces, and remains an epimorphism after every étale base change.*

Proof : The assumptions are stable by étale base change, hence it is enough to prove that f is an epimorphism. Let X be an algebraic space and let $u, v : S \to X$ be morphisms such that $uf = vf$. Let Z be the preimage of the diagonal $\Delta : X \to X \times X$ by the map $(u, v) : S \to X \times X$. Since Δ is a representable monomorphism of spaces which is locally of finite type, see [SP, Tag 02X4], the map $g : Z \to S$ has the same properties. By the assumption on u, v the map f factors through Z. This shows that g is a submersive monomorphism, hence a homeomorphism. By the assumption on f, this remains true after every étale base change on S. Then [EGAIV.4, Cor. 18.12.4], whose proof uses only étale base changes, shows that g is finite. Thus g is a closed immersion which is schematically dominant, hence an isomorphism. Hence $u = v$, and f is an epimorphism of spaces. □

2.1.6 Lemma. *Let $S = \mathrm{Spec}(A)$ be a noetherian local scheme and let $S_n = \mathrm{Spec}(A/m^{n+1})$ be the n-th thickening of the closed point. Then $f : \coprod_{n\geqslant 0} S_n \to S$ is an epimorphism of algebraic spaces.*

Proof : Since f factors through the maximal-adic completion of S which is fpqc over S, it is enough to assume that S is complete. Let $u, v : S \to X$ be such that $uf = vf$, and Z as in the proof of 2.1.5. Since S is henselian we can write $Z = Z_0 \amalg Z_1$ where Z_0 is finite over S and contains the unique closed point above the closed point of S. By assumption $Z_0 \to S$ is an isomorphism over every S_n. Using Nakayama, we find that $Z_0 \to S$ is a closed immersion. Since S is noetherian, this implies that $Z_0 \to S$ is an isomorphism. □

2.1.7 Remarks. The noetherian assumption is of course crucial, since otherwise we may e.g. have $m = m^n$ for all $n \geqslant 1$.

2.2 Effective epimorphisms

2.2.1 Definition. We say that $f : S' \to S$ is an *effective epimorphism of algebraic spaces* if the diagram $S' \times_S S' \rightrightarrows S' \to S$ is exact, that is, if for all algebraic spaces X we have an exact diagram of sets:

$$\mathrm{Hom}(S, X) \to \mathrm{Hom}(S', X) \rightrightarrows \mathrm{Hom}(S' \times_S S', X).$$

Another way to say it is that S is the categorical quotient of S' by the groupoid $S' \times_S S' \rightrightarrows S'$.

2.2.2 Example. An fpqc covering of algebraic spaces is an effective epimorphism of algebraic spaces [SP, Tag 04P2].

If $f : X \to S$ is a morphism, we write $\mathcal{A}_S(X) = f_*\mathcal{O}_X$ or simply $\mathcal{A}(X) = f_*\mathcal{O}_X$ if the base S is clear from context. For instance $\mathcal{A}(S) = \mathcal{O}_S$. Also let us write $S'' = S' \times_S S'$.

2.2.3 Lemma. *Let $f : S' \to S$ be a quasi-compact and quasi-separated morphism of algebraic spaces. Assume that f is an effective epimorphism. Then the sequence $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact.*

Proof : Let us simplify the notations by setting $\mathcal{A}^* = \mathcal{A}(S^*)$ for $* \in \{\varnothing, ', ''\}$. Let $\mathcal{I}$ be the kernel of $\mathcal{A} \to \mathcal{A}'$ and let $\mathcal{B}$ be the kernel of the pair of arrows $\mathcal{A}' \rightrightarrows \mathcal{A}''$. We must prove that $\mathcal{A} \to \mathcal{B}$ is an isomorphism. Since f is quasi-compact and quasi-separated, the sheaves $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ are quasi-coherent hence the sheaves $\mathcal{I}$, $\mathcal{B}$ are also quasi-coherent. According to Lemma 2.1.4 we have $\mathcal{I} = 0$. Let us write $T = \mathrm{Spec}_S(\mathcal{B})$. We have injective sheaf morphisms $\mathcal{O}_S = \mathcal{A} \to \mathcal{B} \to \mathcal{A}'$ and corresponding scheme morphisms $g : S' \to T$, $h : T \to S$ satisfying $f = hg$. Let $p_1, p_2 : S'' \to S'$ be the projections. Since $gp_1 = gp_2$ and f is effective, there is a morphism $e : S \to T$ such that $g = ef = ehg$. As the sheaf map $g^\sharp : \mathcal{B} \to \mathcal{A}'$ is injective, this implies that $e^\sharp : \mathcal{B} \to \mathcal{A}$ is a section of the map $h^\sharp : \mathcal{A} \to \mathcal{B}$ which therefore is an isomorphism. □

Lemma 2.2.3 shows that under the qcqs assumption, it is necessary for an effective epimorphism of algebraic spaces to give rise to an *exact* sequence of $\mathcal{O}_S$-modules $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$. For the converse, in the world of schemes things are quite simple: a submersion with the above exact sequence property is an effective epimorphism, see [SGA1, Exp. VIII, Prop. 5.1].
In the world of algebraic spaces things are a bit more subtle, and our purpose in the rest of this subsection is to strengthen slightly the submersion property so as to salvage the result. We recall that to say that $f : S' \to S$ is a morphism of effective descent for étale algebraic spaces means that for any two étale S-algebraic spaces X, Y the diagram

$$\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'')$$

is exact, and that for every étale S'-algebraic space X', every descent datum on X' with respect to $S' \to S$ is effective.

2.2.4 Lemma. *Let $f : S' \to S$ be a morphism of algebraic spaces. The property for f to be a morphism of effective descent for étale algebraic spaces is local on the source and target for the étale topology. Explicitly,*

1. *if $T \to S$ is étale surjective, $T' = T \times_S S'$, and $f_T : T' \to T$ is the pullback of f, then f is a morphism of effective descent for étale algebraic spaces if and only if f_T is so; and*
2. *if $g : S'' \to S'$ is étale surjective, then f is a morphism of effective descent for étale algebraic spaces if and only if fg is so.*

Proof : (1) In one direction, assume $f : S' \to S$ is a morphism of effective descent for étale algebraic spaces, and let $T \to S$ be an étale base change. Let $T' = T \times_S S'$ and $T'' = T' \times_T T' = T \times_S S''$. We prove that $f_T : T' \to T$ descends morphisms. Let X, Y be two étale T-algebraic spaces. We prove that the diagram

$$\mathrm{Hom}_T(X,Y) \to \mathrm{Hom}_{T'}(X',Y') \rightrightarrows \mathrm{Hom}_{T''}(X'',Y'') \qquad (\star)$$

is exact. Note that $X \to T \to S$ is étale and similarly for the other algebraic spaces. Since f descends morphisms between étale spaces, we obtain an exact diagram

$$\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'').$$

Injectivity of the first map of ($\star$) now follows from the injectivity of the maps $\mathrm{Hom}_T(X,Y) \to \mathrm{Hom}_S(X,Y)$ and $\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_S(X',Y')$. Let $u' : X' \to Y'$ be a T'-morphism such that its pullbacks under the maps $T'' \rightrightarrows T'$ coincide. The second exact sequence provides an S-morphism $u : X \to Y$. Moreover if $a : X \to T$, $b : Y \to T$ are the structure morphisms, we see that a and bu become equal when pulled back to S', hence they are equal. This shows that u is in fact a map of T-algebraic spaces. Finally we prove effective descent for objects. Let $X' \to T'$ be an étale algebraic space with a descent datum with respect to $T' \to T$. Then $X' \to T' \to S'$ is étale and moreover the descent datum can be viewed as a descent datum with respect to $S' \to S$. By the assumption on f there exists an étale morphism $X \to S$ whose pullback under $S' \to S$ is X'. Moreover the map $X' \to T'$ descends to an S-map $X \to T$ and the construction of X is finished.
The other direction is a special case of [Gi64, Thm. 10.8] but for the convenience of the reader we give the argument here. Let $T \to S$ be étale surjective and assume that the base change $f_T : T' \to T$ is of effective descent for étale algebraic spaces. We prove descent of morphisms for f. Let X,Y be étale spaces over S, let X',Y' be the pullbacks to S', and let $u' : X' \to Y'$ be an S'-morphism whose pullbacks via the two maps $S' \times_S S' \rightrightarrows S'$ coincide. Then the map u'_T obtained by the base change $T' \to S'$ has coinciding pullbacks via the two maps $T' \times_T T' \rightrightarrows T'$. Since f_T descends morphisms, u'_T descends to a T-map $u_T : X_T \to Y_T$. Let us introduce some notation:

$$\begin{tikzcd}
T' \times_{S'} T' \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d, "f_{T\times_S T}"] & T' \arrow[r] \arrow[d, "f_T"] & S' \arrow[d, "f"] \\
T \times_S T \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & T \arrow[r] & S.
\end{tikzcd}$$

From the first part, we know that $f_{T\times_S T}$ is a morphism of (effective) descent. From the equality $q_1^* u'_T = q_2^* u'_T$ we thus deduce that $p_1^* u_T = p_2^* u_T$. By descent along the étale map $T \to S$, we obtain a unique S-map $u : X \to Y$ that descends u'. Now we prove effective descent for objects. Let $X' \to S'$ be an étale morphism equipped with a descent datum for S'/S. The pullback $X'_T \to T'$ has a descent datum for T'/T. By assumption it descends to $X_T \to T$. The canonical isomorphism $q_1^* X'_T \to q_2^* X'_T$ descends to an isomorphism $\psi\colon p_1^* X_T \to p_2^* X_T$ since $f_{T\times_S T}$ is a morphism of descent. Using that $f_{T\times_S T\times_S T}$ is a morphism of descent, one checks that ψ is a descent datum on X_T for the étale covering $T \to S$ and by effective descent, it descends to a unique $X \to S$ as desired.
(2) This is a special case of [Gi64, Props. 10.10 and 10.11]. $\square$

The next theorem is our main result on effective epimorphisms of algebraic spaces. In the world of schemes, a qcqs submersion such that $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact is an effective epimorphism. In the world of algebraic spaces, we reinforce these conditions slightly in order to suitably allow étale localization and descent.

2.2.5 Theorem. *Let $f : S' \to S$ be a morphism of algebraic spaces. Assume that:*

(1) *f is a qcqs submersion and remains so after every étale base change,*

(2) *the diagram of $\mathcal{O}_S$-modules $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact,*

(3) *f is a morphism of effective descent for étale algebraic spaces.*

Then f is an effective epimorphism of algebraic spaces and remains so after any étale base change.

Proof : By Lemma 2.2.4, all three assumptions are stable by étale base change on S. Therefore it is sufficient to prove that f is an effective epimorphism of algebraic spaces, i.e., for all algebraic spaces X, the diagram $X(S) \to X(S') \rightrightarrows X(S'')$ is exact. Note that after Lemma 2.1.5 we know that f is an epimorphism after every étale base change, which settles injectivity on the left. It remains to prove that if $\alpha' : S' \to X$ satisfies $\alpha' \operatorname{pr}_1 = \alpha' \operatorname{pr}_2$ then there exists $\alpha : S \to X$ such that $\alpha' = \alpha f$.

We prove that the question is Zariski-local on X. Let (X_i) be a covering of X by open subspaces and let $S'_i = (\alpha')^{-1}(X_i)$. Then S'_i is saturated, that is $S'_i = f^{-1}(f(S'_i))$. Since f is a submersion by (1), then S'_i descends to an open subspace $S_i \subset S$. If for each i there exists $\alpha_i : S_i \to X_i \subset X$ such that $\alpha'_{|S'_i} = \alpha_i f_{|S'_i}$ then by uniqueness the morphisms α_i glue to give a solution $\alpha : S \to X$.

We prove that the question has a positive answer when X is a scheme. Indeed, we can cover X by open affine subschemes and then by the preceding step we can reduce to the case where $X = \operatorname{Spec}(A)$ is affine. Since $\operatorname{Hom}(T, \operatorname{Spec}(A)) = \operatorname{Hom}(A, \Gamma(T, \mathcal{O}_T))$ for all algebraic spaces T (see [SP, Tag 05Z0]), the question reduces to a construction of ring homomorphisms and then the conclusion comes from assumption (2).

Now let X be an arbitrary algebraic space. Let $\pi : Y \to X$ be an étale surjective morphism where Y is a scheme. Let $U' = Y \times_X S'$ which is étale surjective over S', and $U'' = Y \times_X S''$. The assumption $\alpha' \operatorname{pr}_1 = \alpha' \operatorname{pr}_2$ implies that U' carries a descent datum. By assumption (3) it descends to an étale algebraic space $U \to S$. Also let $\beta' : U' \to Y$ be the pullback of α'. Let $R = U \times_S U$

and $R' = U' \times_{S'} U'$.

$$\begin{tikzcd}
 & R' \arrow[r] \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \\
U'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "c"] & U' \arrow[r] \arrow[d, "d"] \arrow[rr, bend right, "\beta'"'] & U \arrow[r, dashed, "\beta"] \arrow[d] & Y \arrow[d, "\pi"] \\
S'' \arrow[r, shift left] \arrow[r, shift right] & S' \arrow[r] \arrow[rr, bend right, "\alpha'"'] & S & X
\end{tikzcd}$$

We know $\beta' \operatorname{pr}_1 = \beta' \operatorname{pr}_2 : U'' \to Y$. Since $U' \to U$ satisfies again all the assumptions (1)–(3) and the statement holds when the test space Y is a scheme, we obtain a morphism $\beta : U \to Y$. We claim that $\pi\beta : U \to X$ is R-invariant. Since $R' \to R$ is an étale pullback of $f : S' \to S$, it is an epimorphism. Hence it is enough to prove that the compositions $R' \to R \rightrightarrows U \to X$ are equal. This follows because they equal to $\alpha' ds' = \alpha' dt'$. Thus $\pi\beta$ induces a morphism $\alpha : S \to X$ and we are done. □

Collecting some results on morphisms of effective descent for étale maps in the literature, we find the following special cases.

2.2.6 Corollary. *Let $f : S' \to S$ be a surjective morphism of algebraic spaces which is either :*

(i) *integral,*

(ii) *proper,*

(iii) *universally open and locally of finite presentation,*

(iv) *universally submersive and of finite presentation with S locally noetherian.*

Then if the sequence of modules $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact, the map f is an effective epimorphism of algebraic spaces and remains so after any flat base change.

Proof : In each case the assumptions are stable under base change, except possibly in case (iv). To deal with this, we use the notion of a *subtrusive* morphism from [Ry10] and we replace (iv) with the more general (iv)' : universally subtrusive and of finite presentation. That this is indeed more general than (iv) follows from [Ry10, Cor. 2.10], with the advantage that (iv)' is stable under base change. It follows that it is enough to prove that f is an effective epimorphism of algebraic spaces. For this we apply Theorem 2.2.5. In each case conditions (1) and (2) hold and it remains to see that f is of effective descent for étale algebraic spaces. Since by Lemma 2.2.4 this property is étale-local on source and target, by taking étale atlases of S and S' one reduces to the case where f is a map of schemes. Then the claim is [SGA4.2, Exp. VIII, Thm. 9.4] in cases (i)–(ii) and [Ry13, Thm. A.2] in cases (iii)–(iv)'. □

2.2.7 REMARK. Assume that f satisfies one of the conditions (i)–(iv). Then the property "f is an effective epimorphism" is fpqc-local on S because exactness of a sequence of quasi-coherent modules is an fpqc-local condition.

For ease of future reference, we single out the following particular case of 2.2.6. Recall that an (effective) epimorphism is *uniform* if it remains an (effective) epimorphism after all flat base changes.

2.2.8 COROLLARY. *Let $f : S' \to S$ be an integral morphism of algebraic spaces such that the sequence $\mathcal{A}(S) \to \mathcal{A}(S') \rightrightarrows \mathcal{A}(S'')$ is exact. Then f is a uniform effective epimorphism of algebraic spaces.* □

2.2.9 EXAMPLES. Here are some sufficient conditions for a morphism f : $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ defined by a finite ring extension $A \subset A'$ to be an effective epimorphism.

(1) f is faithfully flat (faithfully flat descent).

(2) f is the quotient of a flat groupoid (by the quotient property).

(3) f is unramified with fiber-degree at most 2. Indeed, by the structure of unramified morphisms, étale-locally on the target the morphism f has the form $\mathrm{Spec}(A/I) \amalg \mathrm{Spec}(A/J) \to \mathrm{Spec}(A)$. Hence we may assume that $A' = A/I \times A/J$ with $I \cap J = 0$, so that $A' \otimes_A A' = (A/I) \times (A/I+J) \times (A/I+J) \times (A/J)$. To say that $a' = (a_1, a_2) \in A'$ has equal images in $A' \otimes_A A'$ means that $a_1 \equiv a_2 \mod I + J$, hence $a_1 + i = a_2 + j$ for some $i \in I$, $j \in J$. Thus $a' \in A$.

(4) Levelt [Le65] contains some more examples. For instance if $A \subset A'$ is a local inclusion of local rings with trivial residue field extension and no intermediate subring then f is effective [Le65, Chap. IV, Lem. 4]. If for some maximal ideal $m \subset A$ we have $A'/A \simeq A/m$ as A-modules, then f is effective [Le65, Chap. IV, Lem. 7].

(5) f is weakly normal, e.g., A and A' are integral domains, f is generically étale and A is weakly normal [Ry10, Lem. B.5].

Here is a non-example showing that $d = 2$ is required in (3) above.

2.2.10 EXAMPLE. Let $A = k[x, y]/\big(xy(y-x)\big)$ and $A' = A/(x) \times A/(y) \times A/(y-x)$. Then $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is finite and unramified of fiber-degree at most 3 but not an effective epimorphism. Indeed, $A' \otimes_A A' = A' \times k^6$ is reduced so the equalizer of the two maps $A' \to A' \otimes_A A'$ is the weak subintegral closure [Ry10, Lem. B.5] which is isomorphic to $B = k[u, v, w]/(u, v)(u, w)(v, w)$. Explicitly, we have injective maps $A \to B$ and $B \to A'$ where $x \mapsto u + v$, $y \mapsto u + w$ and $u \mapsto (0, 0, x)$, $v \mapsto (0, x, 0)$, $w \mapsto (y, 0, 0)$.

2.3 The canonical factorization

The main result of this section gives a canonical factorization of a finite epimorphism as a composition of finitely many finite effective epimorphisms. It is first stated in [Gr59, A.2.b] and then used to study the functor of subgroups of multiplicative type of a group scheme [SGA3.2, Exp. XV, just before Lem. 3.7] and the relative representability of the Picard functor [SGA6, Exp. XII, Lem. 2.6]. A proof appears in the latter reference. With an eye towards the study of groupoids of higher complexity, we provide additional properties of the canonical factorization : uniqueness, compatibility with flat base change, and minimality of its length. For the convenience of the reader, we provide complete proofs.

2.3.1 Definitions. Let $f : T \to S$ be an epimorphism of algebraic spaces.

(1) An *f-sequence* is a sequence $T = T_0 \to T_1 \to T_2 \to \dots$ of epimorphisms of S-spaces such that for each $i \geqslant 0$, if $T_i \to T_{i+1}$ is an isomorphism then $T_{i+1} \to T_{i+2}$ is an isomorphism.

(2) The *length* of an f-sequence as above is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that $T_n \to T_{n+1}$ is an isomorphism, i.e., the number of non-isomorphic arrows of the sequence. If an f-sequence has finite length n and $T_n \to S$ is an isomorphism, we say that it is *finite and separated* or that it is a *factorization*.

(3) Assume that f is affine. The *canonical sequence* of f is the f-sequence $T = T_0 \to T_1 \to T_2 \to \dots$ given by $T_i = \mathrm{Spec}_S(\mathcal{A}_i)$ where $\mathcal{A}_0 := f_*\mathcal{O}_T$ and $\mathcal{A}_{i+1} := \ker(\mathcal{A}_i \rightrightarrows \mathcal{A}_i \otimes_{\mathcal{O}_S} \mathcal{A}_i)$ for all $i \geqslant 0$.

2.3.2 Proposition. *Let $f : T \to S$ be an integral epimorphism of algebraic spaces.*

(1) *The canonical sequence $T = T_0 \to T_1 \to T_2 \to \dots$ is characterized by the properties :*

(i) *for each i, the morphism $T_i \to S$ is integral and the morphism $T_i \to T_{i+1}$ is an integral effective epimorphism;*

(ii) *for each i the canonical morphism $T_i \times_{T_{i+1}} T_i \to T_i \times_S T_i$ is an isomorphism.*

(2) *The formation of the canonical sequence is compatible with flat base change and local for the flat topology on S. More precisely, let $S' \to S$ be a faithfully flat morphism of schemes. Let $\mathscr{T} = (T_0 \to T_1 \to T_2 \to \dots)$ be a sequence of morphisms of S-schemes and let $\mathscr{T}' = (T'_0 \to T'_1 \to T'_2 \to \dots)$ be the sequence obtained by the base change $S' \to S$. Then $\mathscr{T}$ is the canonical sequence of $T \to S$ if and only if $\mathscr{T}'$ is the canonical sequence of $T' \to S'$.*

(3) *The canonical sequence has length 0 if and only if f is an isomorphism, and length at most 1 if and only if f is an effective epimorphism.*

(4) *The canonical sequence is terminal among f-sequences, affine over S, whose factors are effective epimorphisms, i.e., for each such sequence $T = T'_0 \to T'_1 \to T'_2 \to \dots$ there are maps $T'_i \to T_i$ making a commutative diagram:*

$$\begin{tikzcd} T'_0 \arrow[r] \arrow[d, "\mathrm{id}_T"'] & T'_1 \arrow[r] \arrow[d] & T'_2 \arrow[r] \arrow[d] & \dots \\ T_0 \arrow[r] & T_1 \arrow[r] & T_2 \arrow[r] & \dots \end{tikzcd}$$

(5) *When S is noetherian and f is finite, the morphisms $T_i \to T_{i+1}$ are finite and the canonical sequence is finite and separated, i.e., a factorization. It has minimal length among all finite separated f-sequences whose factors are effective epimorphisms.*

All claims except (5) are actually quite formal.

Proof : (1) Write $\mathcal{A} := \mathcal{O}_S$ and $\mathcal{A}_0 := f_*\mathcal{O}_T$. Since $\mathcal{A}_i \subset \mathcal{A}_0$, the morphisms $T_i = \operatorname{Spec}_S \mathcal{A}_i \to S$ and $T_i \to T_{i+1}$ are integral. The surjective morphism $\mathcal{A}_i \otimes_{\mathcal{A}} \mathcal{A}_i \to \mathcal{A}_i \otimes_{\mathcal{A}_{i+1}} \mathcal{A}_i$ has kernel generated by the local sections $a \otimes 1 - 1 \otimes a$ for local sections $a \in \mathcal{A}_{i+1}$. By the definition of $\mathcal{A}_{i+1}$, it follows that this map is an isomorphism hence (ii) is satisfied. Therefore we have an exact diagram $\mathcal{A}_{i+1} \to \mathcal{A}_i \rightrightarrows \mathcal{A}_i \otimes_{\mathcal{A}_{i+1}} \mathcal{A}_i$. By Corollary 2.2.8, this means that $T_i \to T_{i+1}$ is an effective epimorphism, hence (i) is satisfied. Conversely, if a factorization $T = T'_0 \to T'_1 \to T'_2 \to \dots$ satisfies (ii) then $\mathcal{A}'_i \otimes_{\mathcal{A}} \mathcal{A}'_i \to \mathcal{A}'_i \otimes_{\mathcal{A}'_{i+1}} \mathcal{A}'_i$ is an isomorphism, and if moreover (i) is satisfied then $\mathcal{A}'_{i+1} = \ker(\mathcal{A}_i \rightrightarrows \mathcal{A}_i \otimes_{\mathcal{O}_S} \mathcal{A}_i)$. Thus we see that the given sequence is the canonical one.

(2) This follows because the formation of kernels of morphisms of quasi-coherent sheaves commutes with flat base change and is local for the flat topology on the base.

(3) This follows from the definitions.

(4) By induction, assume that there is a diagram of length i:

$$\begin{tikzcd} T'_0 \arrow[r] \arrow[d, "\mathrm{id}_T"'] & T'_1 \arrow[r] \arrow[d] & \dots \arrow[r] & T'_{i-1} \arrow[r] \arrow[d] & T'_i \arrow[d] \\ T_0 \arrow[r] & T_1 \arrow[r] & \dots \arrow[r] & T_{i-1} \arrow[r] & T_i. \end{tikzcd}$$

Then because $T'_i \to T'_{i+1}$ is effective, we have a containment $\mathcal{A}_{i+1} = \ker(\mathcal{A}_i \rightrightarrows \mathcal{A}_i \otimes_{\mathcal{A}} \mathcal{A}_i) \subset \ker(\mathcal{A}'_i \rightrightarrows \mathcal{A}'_i \otimes_{\mathcal{A}'_{i+1}} \mathcal{A}'_i) = \mathcal{A}'_{i+1}$. This gives a map $T'_{i+1} \to T_{i+1}$ and a diagram of length $i+1$.

(5) First, assume that the canonical sequence has finite length, so there exists $n \geqslant 0$ such that $\mathcal{A}_{n+1} = \mathcal{A}_n$. Then we have an isomorphism $\mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{A}_n \to \mathcal{A}_n \otimes_{\mathcal{A}_{n+1}} \mathcal{A}_n \simeq \mathcal{A}_n$. This means that $T_n \to S$ is a monomorphism. Being dominant and finite, it must be an isomorphism hence the sequence is separated. Now we prove that the canonical sequence has finite length. Since S

is noetherian, this property is étale-local on S. Moreover the formation of $\mathcal{A}_i$ commutes with restriction to an open subscheme and with passage to the stalks on étale local rings. If for some point $s \in S$ the sequence of stalks $(\mathcal{A}_{i,s})_{i\geqslant 0}$ is stationary, then the isomorphism $\mathcal{A}_s \to \mathcal{A}_{n,s}$ extends in a neighborhood of s. Thus we may assume that S is local with closed point s. In particular, we may assume that S (local or not) has finite dimension d. We now argue by induction on d. If $d = 0$, the rings $\mathcal{A}_i$ have finite length and the sequence $\mathcal{A}_i$ is stationary. If $d > 0$, the open $U = S \setminus \{s\}$ has dimension $< d$ so by induction the sequence $\mathcal{A}_i$ is stationary after restriction to U. By the same argument as before, we then know that for all big enough i the morphism $\mathcal{A} \to \mathcal{A}_i$ is an isomorphism away from s. It follows that the quotient $\mathcal{O}_S$-module $\mathcal{A}_i/\mathcal{A}$ has finite length. Thus $\mathcal{A}_i/\mathcal{A}$ is stationary, and hence also $\mathcal{A}_i$.
Now consider a finite separated sequence $T = T'_0 \to T'_1 \to \cdots \to T'_m = S$ of length m whose factors are effective epimorphisms. We have a diagram:

$$\begin{tikzcd}
T'_0 \arrow[r] \arrow[d, "\mathrm{id}_T"'] & T'_1 \arrow[r] \arrow[d] & \dots \arrow[r] & T'_{m-1} \arrow[r] \arrow[d] & T'_m \arrow[r, equal] \arrow[d] & S \arrow[d, equal] \\
T_0 \arrow[r] & T_1 \arrow[r] & \dots \arrow[r] & T_{m-1} \arrow[r] & T_m \arrow[r] & S.
\end{tikzcd}$$

We obtain $\mathcal{O}_S \subset \mathcal{A}_m \subset \mathcal{A}'_m = \mathcal{O}_S$. Thus $T_m \to S$ is an isomorphism, so the canonical sequence has length at most m. $\square$

2.3.3 Example. Let k be a field and S the affine cuspidal k-curve with equation $y^3 = x^4$. We shall see that the canonical sequence of the normalization map $f : T \to S$ has length $n = 2$, as follows :

$$T \longrightarrow T_1 \longrightarrow S$$

$\mathbb{A}^1_k$	spatial singularity $y^2 = xz$ $z^2 = x^2y$ $yz = x^3$	planar cuspidal singularity $y^3 = x^4$

We have $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ with $A = k[x,y]/(y^3 - x^4)$ and $B = k[t]$, the morphism $A \to B$ being given by $x = t^3$ and $y = t^4$. In other words $A \simeq k[t^3, t^4] \hookrightarrow k[t]$. We can write :

$$B \otimes_A B = \frac{k[t_1, t_2]}{(t_1^3 - t_2^3, t_1^4 - t_2^4)}$$

and the two arrows $B \rightrightarrows B \otimes_A B$ map t to t_1 and t_2 respectively. The ring $B_1 = \ker(B \rightrightarrows B \otimes_A B)$ contains A as well as the element t^5, since $t_1^5 = t_1 t_2^4 = t_1 t_2 t_1^3 = t_1^4 t_2 = t_2^5$. Therefore B contains $k[t^3, t^4, t^5]$. If we notice that the annihilator of $t_1 - t_2$ in $B \otimes_A B$ is generated by $t_1^2 + t_1t_2 + t_2^2$ and $(t_1+t_2)(t_1^2+t_2^2)$, we see that B_1 does not contain elements of the form $at+bt^2$. This proves that $B_1 = k[t^3,t^4,t^5]$. Letting $z = t^5$ we get the presentation :

$$B_1 = \frac{k[x,y,z]}{(y^2 - xz, z^2 - x^2y, yz - x^3)}.$$

In particular B_1 is a free $k[x]$-module with basis $\{1, y, z\}$. We now prove that $A = \ker(B_1 \rightrightarrows B_1 \otimes_A B_1)$. We write :

$$B_1 \otimes_A B_1 = k[x,y,z_1,z_2]/I$$

with $I = (y^2 - xz_1, z_1^2 - x^2y, yz_1 - x^3, x(z_1 - z_2), y(z_1 - z_2), z_1^2 - z_2^2)$. The two arrows $B_1 \rightrightarrows B_1 \otimes_A B_1$ map z to z_1 and z_2 respectively. Let $P = a(x) + b(x)y + c(x)z$ be an element of B_1 such that $P(x,y,z_1) = P(x,y,z_2)$, i.e., $c(x)z_1 = c(x)z_2$. In view of the structure of the annihilator of $z_1 - z_2$ inside $B_1 \otimes_A B_1$, this implies that x divides $c(x)$, hence $P \in k[x,y,xz] = k[x,y] = A$, as announced.

3 The category of groupoids

In this section we briefly recall some definitions and notations on groupoids (§ 3.1) and we define the complexity of a flat groupoid with finite stabilizer whose $j_Y : R \to X \times_Y X$ map is schematically dominant (§ 3.2).

3.1 The vocabulary of groupoids

Good references for this material are Keel–Mori [KM97] and Rydh [Ry13]. We fix a base algebraic space S, and products are fibered over S. We use the vocabulary of the functor of points: a *T-point* of an algebraic space X over S is a morphism $x : T \to X$ with values in some S-scheme T. We often write $x \in X(T)$.

3.1.1 Groupoids. We work with *groupoids in S-algebraic spaces*, also called *groupoid spaces* or simply *groupoids*. A groupoid is given by five morphisms of algebraic S-spaces $s,t : R \to X$, $c : R \times_{s,X,t} R \to X$, $e : X \to R$, $i : R \to R$ subject to the conditions that $X(T)$ is the set of objects and $R(T)$ is the set of arrows of a small category, functorially in T. The maps s,t,c,e,i are called *source*, *target*, *composition*, *unit* (or *identity*), and *inversion*. The points of $R \times_{s,X,t} R$ are called *pairs of composable arrows*. Usually we denote a groupoid simply by $s,t : R \rightrightarrows X$ and we call j the map $j = (t,s) : R \to X \times X$. Typically a T-point of X will be denoted x while a T-point of R will be denoted with a Greek letter like α. We sometimes write 1_x or simply 1 instead of $e(x)$. We occasionally write $\alpha : x \to y$ if $x = s(\alpha)$ and $y = t(\alpha)$. With our choices of c and j, note that it is more natural to picture T-points of R as arrows $y \xleftarrow{\alpha} x$ going from right to left.

3.1.2 Actions. For instance, an S-group space G acting on an algebraic space X gives rise to a groupoid $s,t : G \times X \rightrightarrows X$ where s is the second projection and t is the action. In the general setting one may shape one's intuition by thinking of a groupoid as a space R acting on a space X. If $\alpha : x \to y$ is an arrow, there is a corresponding action-like notation $\alpha(x) := t(\alpha) = y$. In these terms, the action is trivial if and only if $s = t$ and the maps c, e, i make $R \to X$ into an X-group space.

3.1.3 Stabilizers. If $R \rightrightarrows X$ is a groupoid, then its *stabilizer* is the X-group space $\mathrm{Stab}_R = j^{-1}(\Delta_X)$ where $\Delta_X \subset X \times X$ is the diagonal. This is the largest subgroupoid of R which is a group space, or also, the largest subgroupoid acting trivially.

3.1.4 Subgroupoids. A *subgroupoid* is a sub-algebraic space $P \subset R$ that is stable under composition and inversion, and contains the unit section $e(X)$. (Topologists call this a *wide subgroupoid* because they also allow subgroupoids $P \rightrightarrows Y$ whose base is an arbitrary possibly empty subspace $Y \subset X$. By sub-algebraic space, we here mean a subfunctor that is an algebraic space, that is, a monomorphism $P \to R$ of algebraic spaces.) A subgroupoid is called *normal* if for any $\alpha \in P(T)$ and $\varphi \in R(T)$ we have $\varphi\alpha\varphi^{-1} \in P(T)$ whenever composability holds. In detail, if $\varphi : x \to y$, then composability means that $\alpha \in \mathrm{Stab}_{P,x}(T)$ and then we have $\varphi\alpha\varphi^{-1} \in \mathrm{Stab}_{P,y}(T)$. In particular the condition that P be normal in R depends only on the stabilizer Stab_P. Any subgroupoid containing Stab_R is normal; in particular if Stab_R is trivial then all subgroupoids are normal.

3.1.5 Morphisms, kernels. A *morphism of groupoids* from $R \rightrightarrows X$ to $R' \rightrightarrows X'$ is a morphism of S-spaces $f : R \to R'$ such that $f(\alpha\beta) = f(\alpha)f(\beta)$ for all composable arrows $\alpha, \beta \in R(T)$. We also use the notation $f : (R, X) \to (R', X')$. Such a morphism f has various automatic compatibilities with the maps s, t, e, i. For instance, f maps identities to identities. Moreover there is an induced morphism on objects $s' \circ f \circ e = t' \circ f \circ e : X \to X'$ which we also write f for simplicity. Thus, notationally for an arrow $\alpha : x \to y$ in R we obtain an arrow $f(\alpha) : f(x) \to f(y)$ in R'. The *kernel* of a morphism $f : R \to R'$ is the preimage of the unit section $e' : X' \to R'$. It is a normal subgroupoid of R.

3.1.6 Invariant morphisms. Let $R \rightrightarrows X$ be a groupoid and let P be a subgroupoid. Then P acts on R in various natural ways. The action by precomposition is a groupoid $R \times_{(s,t)} P \rightrightarrows R$, and the action by postcomposition is a groupoid $P \times_{(s,t)} R \rightrightarrows R$. The stabilizers of both actions are trivial. The simultaneous action, to be called *by pre-post-composition*, is a groupoid $P \times_{(s,t)} R \times_{(s,t)} P \rightrightarrows R$. We have an isomorphism $\mathrm{Stab}_{P \times_{(s,t)} R \times_{(s,t)} P} \xrightarrow{\sim} \mathrm{Stab}_P \times_{(s,t)} R$ given by $(\varphi, \alpha, \psi) \longmapsto (\varphi, \alpha)$. This implies that the morphism of groupoids $f : P \times_{(s,t)} R \times_{(s,t)} P \longrightarrow R$, $f(\varphi, \alpha, \psi) = \varphi$ whose underlying morphism on objects is $f = t : R \to X$ is fixed point reflecting, in the sense of [KM97, 2.2]. Now let us consider moreover a morphism of groupoids

$f : R \to R'$. Then the following four assertions are rewordings of one and the same property : (i) $P \subset \ker(f)$, (ii) f is invariant by the left P-action on R, (iii) f is invariant by the right P-action on R, (iv) f is invariant by the pre-post-composition P-action on R. If this property holds, we say that f is *P-invariant.*

3.1.7 Quotients. Let $R \rightrightarrows X$ be a groupoid and $P \subset R$ a subgroupoid. A *categorical quotient* of R by P is a morphism of groupoids $\pi : R \to Q$ which is P-invariant and is universal among invariant morphisms $R \to R'$.

In Definition 3.1.7 we simplify the discussion by restricting to categorical quotients; other notions of quotients are recalled in 3.2 below. To shed light on the definition, note that by the universal property there is a morphism $P \to \ker(\pi)$ but contrary to what happens in the category of groups, it is not at all clear if this is an isomorphism (and we do not think it is the case in general). We will not pursue this question in this article.

3.2 The complexity

Whereas we introduced basic notions internal to the category of *groupoids*, in order to define the complexity we come back to the categories of schemes and algebraic spaces. Recall that if $s, t : R \rightrightarrows X$ is a groupoid space, then a morphism $f : X \to X'$ is called *R-invariant* if $fs = ft$. We will not repeat here the various definitions related to quotients because they receive a clear presentation in [KM97, § 1] and [Ry13, § 2]. We content ourselves with saying that a morphism $X \to Y$ is a *categorical quotient* if it is initial among R-invariant morphisms $X \to X'$, a *geometric quotient* if it is a submersion and $\mathcal{O}_Y$ is identified with the sheaf of R-invariant sections of $\mathcal{O}_X$, and a quotient of one of these types is *uniform* it its formation commutes with flat base change. We recall the statement of the fundamental Keel–Mori theorem from [KM97], [Ry13] as well as the case with trivial stabilizer from [Ar74].

3.2.1 Theorem. *Let S be an algebraic space and let $R \rightrightarrows X$ be a flat, locally finitely presented S-groupoid space with finite stabilizer.*

(1) *There is a uniform geometric and categorical quotient $X \to X/R = Y$ such that the map $j_Y : R \to X \times_Y X$ is finite and surjective. Moreover $X \to Y$ is universally open.*

(2) *The space $Y \to S$ is separated (resp. quasi-separated) if and only if $j_S : R \to X \times_S X$ is finite (resp. quasi-compact). It is locally of finite type if S is locally noetherian and $X \to S$ is locally of finite type.*

(3) *If the stabilizer is trivial, then Y is the fppf quotient sheaf of X by R, $X \to Y$ is flat locally finitely presented, j_Y is an isomorphism, and the formation of Y commutes with arbitrary base changes $Y' \to Y$.*

When $R \rightrightarrows X$ is finite and locally free, it is known moreover that $X \to Y$ is integral.

3.2.2 Remarks. (1) The map $j_Y : R \to X \times_Y X$ need not be schematically dominant, in particular it need not be an epimorphism. Here is an example. Let $X = \mathrm{Spec}(k[x]/(x^2))$ with action of $\mu_n = \mathrm{Spec}(k[z]/(z^n - 1))$ by multiplication then $Y = X/R = \mathrm{Spec}(k)$. We have $X \times_Y X = \mathrm{Spec}(k[x_1, x_2]/(x_1^2, x_2^2))$. The morphism $j_Y : R \to X \times_Y X$ is given by the map of k-algebras $k[x_1, x_2]/(x_1^2, x_2^2) \to k[x, z]/(x^2, z^n - 1)$ such that $x_1 \mapsto x$ and $x_2 \mapsto zx$. The element $x_1 x_2$ is not zero and it is mapped to $zx^2 = 0$.

(2) The map $X \to X/R$ need not be of finite type even when $R \rightrightarrows X$ is finite locally free. For example if $X = \mathrm{Spec}(k[t_1, t_2, \dots])$ with action of μ_n by $z.t_i = zt_i$ then X/R is the spectrum of the ring of polynomials all whose homogeneous components have degree a multiple of n.

In the rest of the text, we will focus on flat groupoids such that the morphism $j_Y : R \to X \times_Y X$ is an epimorphism. This occurs for instance when $X \to Y$ is flat and there is a schematically dense open subscheme $X_0 \subset X$ where the action is free. One way to measure further the good behavior of these groupoids is furnished by Proposition 2.3.2 and leads to the following notion.

3.2.3 Definition. Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid space with finite stabilizer. We say that $R \rightrightarrows X$ *has complexity* n if the map $j_Y : R \to X \times_Y X$ is an epimorphism and the length of its canonical sequence is n.

3.2.4 Remarks. (1) The groupoid $R \rightrightarrows X$ has complexity 0 if and only if it is free. It has complexity at most 1 if and only if j_Y is an effective epimorphism.

(2) If j_Y is an epimorphism, then, by Proposition 2.3.2(5), a sufficient condition for a groupoid to have finite complexity is that X is of finite type over a fixed noetherian base scheme.

(3) Levelt's results [Le65], see Example 2.2.9(4), hint that finite locally free groupoids with isolated fixed points of stabilizer degree at most 2 (e.g., an action of a group scheme of order 2 with isolated fixed points) should have complexity at most 1. We shall see examples of this in the next section.

3.3 Examples

Because the formation of the canonical sequence is local on the base for the flat topology (Proposition 2.3.2(2)), the computation of the complexity can be done locally. It follows that computations in this section provide results also for groupoids which are group actions only locally for the flat topology, or locally after passage to a completed local ring. This applies for instance to quotients of surfaces by p-closed vector fields, studied by many people in the last 40 years (Rudakov–Shafarevich, Russell, Ekedahl, Katsura–Takeda, Hirokado...).
We start with examples valid in any characteristic.

3.3.1 Proposition. *Let $X = \mathbb{A}^n_S$ be affine n-space over a scheme S. Let G be the symmetric group on n letters, acting by permutation of the coordinates* *of X. Then the quotient map $\pi : X \to Y = X/G$ is finite locally free of rank $n!$ and the groupoid $G \times X \rightrightarrows X$ has complexity 1.*

Proof : First we set the notations. We may assume $S = \mathrm{Spec}(R)$ affine. Then $X = \mathrm{Spec}(B)$ where $B = R[x_1, \dots, x_n]$ is a polynomial ring in n variables, and $Y = \mathrm{Spec}(A)$ where $A = B^G$ is the ring of invariants. Let $S_k(X_1, \dots, X_n)$ be the symmetric function of degree k in $X_1, \dots, X_n$ and $s_k = S_k(x_1, \dots, x_n) \in A$. By the Main Theorem on symmetric functions, we have $A = R[s_1, \dots, s_n]$ which is a ring of polynomials in the variables s_i, moreover

$$B \simeq \frac{A[x_1, \dots, x_n]}{(S_1(x_i) - s_1, \dots, S_n(x_i) - s_n)}$$

and therefore

$$B \otimes_A B \simeq \frac{B[X_1, \dots, X_n]}{(S_1(X_i) - s_1, \dots, S_n(X_i) - s_n)}$$

is B-free of rank $n!$ with basis the set of monomials

$$\mathscr{B} = \{X_1^{d_1} \dots X_n^{d_n} ; 0 \leqslant d_i < i, \forall i\}.$$

The map $j : G \times X \to X \times_Y X$ corresponds to the map of B-algebras which is given by evaluation on $(x_1, \dots, x_n)$ and its permutations:

$$\begin{aligned} \mathrm{ev} : \frac{B[X_1, \dots, X_n]}{(S_1(X_i) - s_1, \dots, S_n(X_i) - s_n)} &\longrightarrow \prod_{\sigma \in \mathfrak{S}_n} B \\ P &\longmapsto (P(x_{\sigma(1)}, \dots, x_{\sigma(n)}))_{\sigma \in \mathfrak{S}_n}. \end{aligned}$$

The stabilizer $\Sigma \to X$ of the groupoid has function ring:

$$B[\Sigma] = \prod_{\tau \in \mathfrak{S}_n} \frac{B}{(x_1 - x_{\tau(1)}, \dots, x_n - x_{\tau(n)})}.$$

The two maps $\mathrm{pr}_2, d : \Sigma \times_X (G \times X) \rightrightarrows G \times X$ correspond to the maps of B-algebras

$$\alpha, \beta : \prod_{\sigma \in \mathfrak{S}_n} B \longrightarrow \prod_{\sigma, \tau \in \mathfrak{S}_n} \frac{B}{(x_1 - x_{\tau(1)}, \dots, x_n - x_{\tau(n)})}$$

defined by $\alpha(Q)_{\sigma,\tau} = Q_\sigma$ and $\beta(Q)_{\sigma,\tau} = Q_{\tau\sigma}$ for all $Q = (Q_\sigma)_{\sigma \in \mathfrak{S}_n}$.
Since the action of G on X is not free, the complexity of the groupoid is not 0. Hence what remains to be proved is that ev is injective and $\mathrm{im}(\mathrm{ev}) = \ker(\alpha - \beta)$. In order to describe the image of ev let us introduce some more notation. Let E be the set of pairs of integers (i, j) with $1 \leqslant i < j \leqslant n$. Let $V = V(x_1, \dots, x_n) = \prod_{(i,j) \in E} (x_j - x_i)$ be the Vandermonde determinant of the x_i. To each subset $F \subset E$ we attach a monomial $\mu(F) = \prod_{(i,j) \in F} X_j$. For example

if $n=4$ and $F=\{(1,3),(2,4),(3,4)\}$ then $\mu(F)=X_3X_4^2$. Obviously the map $\mu:\mathscr{P}(E)\to\mathscr{B}$ is surjective and if $M=\mu(F)$ then $\deg(M)=\operatorname{card}(F)$. Now for each basis monomial $M\in\mathscr{B}$ we define a B-linear form $\varphi_M:\prod_{\sigma\in\mathfrak{S}_n}B\to B$ by

$$Q=(Q_\sigma)_{\sigma\in\mathfrak{S}_n}\longmapsto\varphi_M(Q)=\sum_\sigma\varepsilon(\sigma)\Big(\sum_{\substack{F\subset E\\ \mu(F)=M}}\ \prod_{(i,j)\in E-F}x_{\sigma(i)}\Big)Q_\sigma.$$

(Here $\varepsilon(\sigma)$ is the sign of the permutation σ.) We let $\varphi:\prod_{\sigma\in\mathfrak{S}_n}B\to\prod_{M\in\mathscr{B}}B$ be the map with components φ_M and we use the same letter to denote the map with values in $\prod_{M\in\mathscr{B}}B/VB$ obtained by reduction mod V. We claim that the following sequence is exact:

$$0\longrightarrow\frac{B[X_1,\dots,X_n]}{(S_1(X_i)-s_1,\dots,S_n(X_i)-s_n)}\xrightarrow{\mathrm{ev}}\prod_{\sigma\in\mathfrak{S}_n}B\xrightarrow{\varphi}\prod_{M\in\mathscr{B}}B/VB.$$

In order to prove this we introduce suitable Lagrange interpolation polynomials which allow us to invert the map ev after the base change $B\to B[1/V]$. Precisely, we set:

$$L_\sigma(X_1,\dots,X_n)=\frac{\varepsilon(\sigma)}{V}\prod_{(i,j)\in E}(X_j-x_{\sigma(i)}).$$

We have $\deg_{X_i}(L_\sigma)<i$ for all $i=1,\dots,n$. Thus, after inverting V, the polynomial L_σ lies in the submodule $\oplus_{M\in\mathscr{B}}R\cdot M\subset B[X_1,\dots,X_n]$ which as we said earlier maps isomorphically onto $B[X_1,\dots,X_n]/(S_1(X_i)-s_1,\dots,S_n(X_i)-s_n)$. Moreover one sees that $L_\sigma(x_{\tau(1)},\dots,x_{\tau(n)})=\delta_{\sigma,\tau}$ (Kronecker δ). From these remarks follows that the inverse to $\mathrm{ev}\otimes\mathrm{id}_{B[1/V]}$ is given by interpolation, that is:

$$\mathrm{int}(Q)=\sum_{\sigma\in\mathfrak{S}_n}Q_\sigma L_\sigma.$$

From this, since V is a nonzerodivisor in B, the injectivity of ev follows. By expanding one finds:

$$\begin{aligned}\mathrm{int}(Q)&=\frac{1}{V}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)Q_\sigma\prod_{(i,j)\in E}(X_j-x_{\sigma(i)})\\&=\frac{1}{V}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)Q_\sigma\sum_{F\subset E}(-1)^{\operatorname{card}(E-F)}\cdot\prod_{(i,j)\in E-F}x_{\sigma(i)}\cdot\mu(F)\\&=\frac{1}{V}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)Q_\sigma\sum_{M\in\mathscr{B}}\sum_{\substack{F\subset E\\ \mu(F)=M}}(-1)^{\frac{n(n-1)}{2}-\deg(M)}\prod_{(i,j)\in E-F}x_{\sigma(i)}\cdot M\\&=\frac{1}{V}\sum_{M\in\mathscr{B}}(-1)^{\frac{n(n-1)}{2}-\deg(M)}\varphi_M(Q)\cdot M.\end{aligned}$$

Since $Q = (\mathrm{ev} \otimes \mathrm{id}_{B[1/V]})(\mathrm{int}(Q))$, we see that Q lies in the image of ev if and only if the components of $\mathrm{int}(Q)$ on the basis vectors $M \in \mathscr{B}$ lie in B. This means precisely that $\varphi_M(Q)$ is divisible by V for all $M \in \mathscr{B}$, which proves the exactness of the sequence.
We can now conclude. It is clear that $\mathrm{im}(\mathrm{ev}) \subset \ker(\alpha - \beta)$. In order to prove the reverse inclusion let $Q = (Q_\sigma)_{\sigma \in \mathfrak{S}_n}$ lie in the equalizer of α and β, that is:

$$Q_{\tau\sigma} \equiv Q_\sigma \mod (x_1 - x_{\tau(1)}, \dots, x_n - x_{\tau(n)}), \quad \text{for all } \sigma, \tau \in \mathfrak{S}_n.$$

We want to prove that $\varphi_M(Q)$ is divisible by V for all $M \in \mathscr{B}$. It is enough to prove that $\varphi_M(Q)$ is divisible by $x_v - x_u$ for all $(u,v) \in E$. Consider the transposition $\tau = (u,v)$. Then $\mathfrak{S}_n$ is partitioned into $n!/2$ pairs $\{\sigma, \tau\sigma\}$ and it is enough to prove that for each σ the sum

$$\varepsilon(\sigma)\Bigg(\sum_{\substack{F \subset E \\ \mu(F)=M}} \prod_{(i,j)\in E-F} x_{\sigma(i)}\Bigg) Q_\sigma + \varepsilon(\tau\sigma)\Bigg(\sum_{\substack{F \subset E \\ \mu(F)=M}} \prod_{(i,j)\in E-F} x_{\tau\sigma(i)}\Bigg) Q_{\tau\sigma}$$

is divisible by $x_v - x_u$. This is clear, because modulo $x_v - x_u$ we have $Q_{\tau\sigma} \equiv Q_\sigma$ by the assumption on Q and $x_{\tau\sigma(i)} \equiv x_{\sigma(i)}$ by the definition of τ. □

3.3.2 Remark. More generally, we can ask if the complexity is at most 1 for a finite constant group G acting on a smooth scheme X in such a way that the pointwise stabilizers G_x are generated by reflections, in the sense that there is a system of local coordinates such that G_x is generated by linear automorphisms of order 2.

Here is another example in arbitrary characteristic.

3.3.3 Lemma. *Let R be a ring. Let $n \geqslant 2$ be an integer. Let $X = \mathbb{A}^1_R$ be the affine line over R, with the action of $G = \mu_{n,R}$ given by $G \times X \to X$, $(z,x) \mapsto zx$. Then the groupoid $G \times X \rightrightarrows X$ has complexity 1 if $n = 2$ and at least 2 otherwise. If $n = 3$, the complexity is equal to 2.*

Proof : We have $X = \mathrm{Spec}(B)$ and $Y = X/G = \mathrm{Spec}(A)$ with $B = R[x]$, $A = R[y]$ and $y = x^n$. Let $C_\infty = B \otimes_A B = B[X]/(X^n - x^n)$ and $C_0 = B[z]/(z^n - 1)$. The question is about the finite morphism of B-algebras $\rho : C_\infty \to C_0$ with $\rho(X) = zx$. Note that ρ identifies C_∞ with the sub-B-algebra of $B[z]/(z^n - 1)$ generated by zx. We have $C_0 \otimes_{C_\infty} C_0 = B[z_1, z_2]/(z_1^n - 1, z_2^n - 1, x(z_1 - z_2))$ with the maps $\alpha, \beta : C_0 \to C_0 \otimes_{C_\infty} C_0$ given by $\alpha(z) = z_1$ and $\beta(z) = z_2$. Let $C_1 \subset C_0$ be the equalizer of these maps, this is the sub-B-algebra generated by the elements $y_i := z^i x$ for $i = 1, \dots, n-1$. If $n = 2$ we have $C_\infty = C_1$, so the complexity is 1. If $n \geqslant 3$ we have $z^2 x \in C_1 \backslash C_\infty$ and the complexity is at least 2. In general C_1 has a fairly complicated structure. We leave it to the reader to check that for $n = 3$ we have $C_1 = B[y_1, y_2]/(y_1^3 - x^3, y_1 y_2 - x^2, y_2^2 - x y_1)$ and that the map $C_\infty \to C_1$ is effective. □

Finally an example in characteristic p.

3.3.4 Lemma. *Let R be a ring of characteristic $p > 0$. Let $X = \mathbb{A}^1_R$ be the affine line over R, with the action of $G = \alpha_{p,R}$ given by $G \times X \to X$, $(a, x) \mapsto \frac{x}{1+ax}$. Then the groupoid $G \times X \rightrightarrows X$ has complexity 1 if $p = 2$ and at least 2 otherwise.*

Proof : We have $X = \mathrm{Spec}(B)$ and $Y = X/G = \mathrm{Spec}(A)$ with $B = R[x]$, $A = R[y]$ and $y = x^p$. The question is about exactness of the sequence of B-algebras:

$$\frac{B[X]}{X^p - x^p} \xrightarrow{\ \rho\ } \frac{B[a]}{a^p} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \frac{B[a_1, a_2]}{a_1^p, a_2^p, x^2(a_1 - a_2)}$$

with $\rho(X) = \frac{x}{1+ax}$, $\alpha(a) = a_1$, $\beta(a) = a_2$. In order to find the image of ρ we compute in the localizations with respect to x. Since ρ is injective we write X for $\rho(X)$. From $X = \frac{x}{1+ax}$ we get $a = X^{-1} - x^{-1}$ so if $Q(a) = \sum_{i=0}^{p-1} Q_i a^i$ is the image of some P under ρ then we have:

$$\begin{aligned} P(X) &= Q(X^{-1} - x^{-1}) \\ &= \sum_{i=0}^{p-1} (-1)^i x^{-i} Q_i + \sum_{j=1}^{p-1} \left(\sum_{i=j}^{p-1} (-1)^{i-j} \binom{i}{j} x^{-p-i+j} Q_i \right) X^{p-j}. \end{aligned}$$

We find that the image of ρ is the set of Q such that x^{p-1} divides $\sum_{i=1}^{p-1} (-1)^i x^{p-1-i} Q_i$ and x^{2p-1-j} divides $\sum_{i=j}^{p-1} (-1)^i \binom{i}{j} x^{p-1-i} Q_i$ for all $j = 1, \dots, p-1$. This may be rewritten as the set of Q such that x^{i+1} divides Q_i for all $i = 1, \dots, p-1$ (say $Q_i = x^{i+1} R_i$ for some $R_i \in B$) and x^{p-1-j} divides $\sum_{i=j}^{p-1} (-1)^i \binom{i}{j} R_i$ for all $j = 1, \dots, p-1$. On the other hand, the equalizer of α and β is the set of Q such that x^2 divides Q_i for all $i = 1, \dots, p-1$. These sets are equal if and only if $p = 2$. □

4 Main theorems

After the work of the previous sections, we are ready to give an answer to the descent question from the introduction, for groupoids of complexity at most 1. It applies to the objects of a stack whose isomorphism sheaves are representable: see Theorem 4.2.3.

4.1 Equivariant objects

4.1.1 Definition. Let $s, t : R \rightrightarrows X$ be a groupoid and $c, \mathrm{pr}_1, \mathrm{pr}_2 : R \times_{s,X,t} R \to R$ the composition and projections. Let $\mathcal{C} \to \mathrm{AlgSp}$ be a category fibered over the category of algebraic spaces and let $\mathcal{F} \in \mathcal{C}(X)$ be an object. An *R-linearization* on $\mathcal{F}$ is an isomorphism $\phi : s^*\mathcal{F} \xrightarrow{\sim} t^*\mathcal{F}$ satisfying the cocycle condition $c^*\phi = (\mathrm{pr}_1^* \phi) \circ (\mathrm{pr}_2^* \phi)$, meaning that the following triangle is commutative :

$$
\begin{tikzcd}
(s\,\mathrm{pr}_2)^*\mathcal{F} = (sc)^*\mathcal{F} \arrow[rr, "c^*\phi"] \arrow[dr, "\mathrm{pr}_2^*\phi"'] & & (tc)^*\mathcal{F} = (t\,\mathrm{pr}_1)^*\mathcal{F} \\
& (t\,\mathrm{pr}_2)^*\mathcal{F} = (s\,\mathrm{pr}_1)^*\mathcal{F}. \arrow[ur, "\mathrm{pr}_1^*\phi"'] &
\end{tikzcd}
$$

An *R-equivariant object of $\mathcal{C}$ over X* is an object $\mathcal{F} \in \mathcal{C}(X)$ together with an R-linearization. We write $\mathcal{C}(R, X)$ for the category of R-equivariant objects.

4.1.2 Example. Let $R \rightrightarrows X$ be a groupoid as above and let $\pi : X \to Y$ be an R-invariant morphism, i.e., $\pi s = \pi t$. Then for any object $\mathcal{G} \in \mathcal{C}(Y)$, the pullback $\mathcal{F} = \pi^*\mathcal{G}$ is endowed with a canonical R-linearization $\phi : s^*\mathcal{F} = s^*\pi^*\mathcal{G} \simeq (\pi s)^*\mathcal{G} = (\pi t)^*\mathcal{G} \simeq t^*\pi^*\mathcal{G} = t^*\mathcal{F}$.

We recall the notion of a *square*, which is closely related to that of R-equivariant object.

4.1.3 Definition. A morphism of groupoids $f : (R', X') \to (R, X)$ is called a *square* or *cartesian* when the commutative diagram

$$
\begin{tikzcd}
R' \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f"] \\
R \arrow[r] & X
\end{tikzcd}
$$

is cartesian, if we take for horizontal maps either both source maps, or both target maps.

To illustrate these definitions, take for $\mathcal{C}$ the category of algebraic spaces over algebraic spaces. For $(X' \to X) \in \mathcal{C}(X)$, the following lemma makes it clear that an R-linearization on X' is the same as a lift of the R-action to X'.

4.1.4 Lemma. *Let $s, t : R \rightrightarrows X$ be a groupoid. Let $(f : X' \to X, \phi : s^*X' \xrightarrow{\sim} t^*X')$ be an R-equivariant X-space. Complete X' to a quintuple (R', X', s', t', c') as follows :*

(1) *$R' = s^*X' = R \times_{s,X,f} X'$ whose T-points are pairs (α, x') with $\alpha \in R(T)$ and $x' \in X'(T)$,*

(2) *$s' = \mathrm{pr}_2 : R' \to X'$,*

(3) *$t' = \mathrm{pr}_2 \circ \phi : R \times_{s,X,f} X' \longrightarrow R \times_{t,X,f} X' \longrightarrow X'$,*

(4) *$c' : R' \times_{s',X',t'} R' \longrightarrow R'$ defined on T-points by $c'\big((\alpha, x'), (\beta, y')\big) = (\alpha\beta, y')$.*

Then (R', X', s', t', c') is a groupoid and the morphism $(R', X') \to (R, X)$ is a square morphism of groupoids.
Conversely, a square morphism of groupoids $(R', X') \to (R, X)$ gives an R-equivariant X-space $(X' \to X, s^*X' \xrightarrow{\sim} R' \xrightarrow{\sim} t^*X')$.

Proof : This is [SP, Tag 0APC]. □

4.2 Descent along the quotient

Let $s, t : R \rightrightarrows X$ be a flat locally finitely presented groupoid. In this section we are interested in the problem of descending objects of a category $\mathcal{C}$ fibered over the category of algebraic spaces along the quotient map $\pi : X \to X/R = Y$. We know that for any object $\mathcal{G} \in \mathcal{C}(Y)$, the pullback $\mathcal{F} = \pi^*\mathcal{G}$ is endowed with a canonical R-linearization (example 4.1.2). Conversely, if $\mathcal{F} \in \mathcal{C}(X)$ then the datum of an R-linearization allows to descend $\mathcal{F}$ to an object based on $[X/R]$, the quotient *as an algebraic stack*, but is not enough to descend $\mathcal{F}$ to an object of $\mathcal{C}(Y)$ in general. Let $\mathcal{C}(R, X)$ be the category of R-equivariant objects $(\mathcal{F}, \phi)$. Descent Theory as formulated by Grothendieck seeks to characterize the essential image of the pullback functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R, X)$. When $\mathcal{C}$ is the category of étale morphisms of spaces, and without additional conditions on $R \rightrightarrows X \to Y$, Keel and Mori [KM97, Lem. 6.3], Kollár [Ko97, § 2], Rydh [Ry13, § 3] obtain such a characterization in terms of fixed-point reflecting R-equivariant objects. In a different direction, we shall prove that if $R \rightrightarrows X$ has complexity at most 1 and flat quotient $X \to Y$, there is a nice description of the image of π^* for very general stacks $\mathcal{C}$.

4.2.1 Definition. Let $\Sigma = \mathrm{Stab}_R$ be the stabilizer of the groupoid, let $a : \Sigma \to R$ be the inclusion, and put $b = sa = ta$. We denote by $\mathcal{C}(R, X)^\Sigma$ the full subcategory of $\mathcal{C}(R, X)$ consisting of R-equivariant objects $(\mathcal{F}, \phi)$ such that the action of Σ is trivial, meaning that the following map is the identity:

$$b^*\mathcal{F} \simeq a^*s^*\mathcal{F} \xrightarrow{a^*\phi} a^*t^*\mathcal{F} \simeq b^*\mathcal{F}.$$

To dispel the dryness of the formalism of groupoids, we emphasize that if $\mathcal{C}$ is the category of schemes or algebraic spaces, and if the groupoid is given by the action of a group G, then a $G \times X$-linearization on some $X' \in \mathcal{C}(X)$ is equivalent to a lift of the action of G to X' and the action of Σ is trivial in the above sense if and only if it is trivial in the usual sense.

4.2.2 Lemma. *The functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R, X)$ takes values in $\mathcal{C}(R, X)^\Sigma$.*

Proof : We have to show that the canonical R-linearization of a pullback $\mathcal{F} = \pi^*\mathcal{G}$ becomes trivial when restricted to Σ. Recall from [Gr59, A.1] or [SP, Tag 003N], that in a fibered category, there are isomorphisms $(fg)^* \xrightarrow{\sim} g^*f^*$

between pullback functors, and commutative squares giving compatibility for triple compositions :

$$\begin{tikzcd}
(fgh)^* \arrow[r] \arrow[d] & (gh)^*f^* \arrow[d] \\
h^*(fg)^* \arrow[r] & h^*g^*f^*.
\end{tikzcd}$$

We write the two squares picturing such compatibility for the two compositions $\pi sa : \Sigma \to Y$ and $\pi ta : \Sigma \to Y$, taking advantage of the fact that $\pi s = \pi t$ in order to glue them on one side:

$$\begin{tikzcd}
(sa)^*\pi^* \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & (\pi sa)^* = (\pi ta)^* \arrow[l] \arrow[r] \arrow[d] & (ta)^*\pi^* \arrow[d] \\
a^*s^*\pi^* \arrow[rr, bend right, "a^*\phi"'] & a^*(\pi s)^* = a^*(\pi t)^* \arrow[l] \arrow[r] & a^*t^*\pi^*.
\end{tikzcd}$$

Since $sa = ta$ we see that the top row is the identity. The commutativity of the exterior diagram is exactly the claim we want to prove. ☐

4.2.3 Theorem. *Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid space with finite stabilizer $\Sigma \to X$ and complexity at most 1. Assume that the quotient $\pi : X \to Y = X/R$ is flat (resp. flat and locally of finite presentation). Let $\mathcal{C} \to \mathrm{AlgSp}$ be a stack in categories for the fpqc topology (resp. for the fppf topology).*

(1) *If the sheaves of homomorphisms $\mathcal{H}om_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ have diagonals which are representable by algebraic spaces, then the pullback functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R,X)^{\Sigma}$ is fully faithful.*

(2) *If the sheaves of isomorphisms $\mathcal{I}som_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ are representable by algebraic spaces, then the pullback functor $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R,X)^{\Sigma}$ is essentially surjective.*

In particular if $\mathcal{C}$ is a stack in groupoids with representable diagonal, the functor π^* is an equivalence.

In Section 3.3 many examples were given that satisfy the assumptions of the Theorem.

4.2.4 REMARK. This result is not really an alternative to faithfully flat descent, but rather a refinement of it. Indeed, faithfully flat descent *does* provide an answer to the question of the image of π^*: it is the particular case of our theorem for the flat groupoid $R_1 := X \times_Y X \rightrightarrows X$ whose stabilizer is trivial. The category $\mathcal{C}(R_1, X)$ comprises objects with descent data, the latter being isomorphisms on products $X \times_Y X$ with conditions on triple products $X \times_Y X \times_Y X$. However, it is often the case in concrete geometric situations that there is a natural action of a group or groupoid $R \neq R_1$ such that it is much easier to handle R-equivariant objects. In these situations, the functor of points of the quotient $Y = X/R$ is usually hard to describe, as well as the square and the cube of X over Y, making $\mathcal{C}(R_1, X)$ less convenient.

PROOF : The assumptions on $\mathcal{C}$ and π imply that effective descent along π holds in $\mathcal{C}$; in the fpqc case note that π is an fpqc covering since it is open (3.2.1) and faithfully flat, see e.g. Vistoli [Vi05, Prop. 2.35]. Since the map $j_Y : R \to X \times_Y X$ will come up repeatedly, we write simply $j := j_Y$.

(1) Let $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}(Y)$ and let $(\mathcal{F}_1, \phi_1), (\mathcal{F}_2, \phi_2) \in \mathcal{C}(R, X)^\Sigma$ be their pullbacks. We must prove that the map:

$$\mathrm{Hom}_{\mathcal{C}(Y)}(\mathcal{G}_1, \mathcal{G}_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}(R,X)^\Sigma}\big((\mathcal{F}_1, \phi_1), (\mathcal{F}_2, \phi_2)\big)$$

is bijective. Injectivity is a consequence of the fact that $\pi : X \to Y$ is a covering for the topology for which $\mathcal{C}$ is a stack, and the fact that $\mathrm{Hom}_{\mathcal{C}(Y)}(\mathcal{G}_1, \mathcal{G}_2)$ is a separated presheaf. For surjectivity let $f : (\mathcal{F}_1, \phi_1) \to (\mathcal{F}_2, \phi_2)$ be a morphism. Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the projections. By descent it is enough to prove that $\pi_1^* f = \pi_2^* f$. By construction ϕ_i is the identity of $q^*\mathscr{G}_i$ for $i = 1, 2$, where $q = \pi s = \pi t$. Therefore $s^* f = t^* f$. Write $H := \mathcal{H}om_{\mathcal{C}(Y)}(\mathcal{G}_1, \mathcal{G}_2)$. We have a commutative diagram:

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d] & X \times_Y X \arrow[d, "d"] \\ H \arrow[r, "\Delta"] & H \times_Y H \end{tikzcd}$$

where $d := (\pi_1^* f, \pi_2^* f)$. Since the diagonal Δ is assumed to be representable, the fiber product

$$P := H \times_{(\Delta, d)} X \times_Y X$$

is representable and the map j factors through a map $k : P \to X \times_Y X$. Since the groupoid has complexity at most 1, the map j is an effective epimorphism. It follows by formal arguments that k has the same property. Since k is a pullback of the diagonal, it is also a monomorphism. Thus, k is an isomorphism, and therefore $\pi_1^* f = \pi_2^* f$.

(2) Let $(\mathcal{F}, \phi) \in \mathcal{C}(R, X)^\Sigma$ be an R-equivariant object. Given that $R \times_{X \times_Y X} R$ is isomorphic to $\Sigma \times_{(s,t)} R$ via the map $(\varphi, \psi) \mapsto (\varphi\psi^{-1}, \psi)$, the exact sequence

for the effective epimorphism j is:

$$\Sigma \times_{(s,t)} R \underset{\mathrm{pr}_2}{\overset{d}{\rightrightarrows}} R \xrightarrow{j} X \times_Y X.$$

Here d is the composition $\Sigma \times_{(s,t)} R \xrightarrow{a\times\mathrm{id}} R \times_{(s,t)} R \xrightarrow{c} R$. It follows that for all $X \times_Y X$-algebraic spaces I, we have an exact diagram of sets:

$$\mathrm{Hom}(X \times_Y X, I) \xrightarrow{j^*} \mathrm{Hom}(R, I) \underset{\mathrm{pr}_2^*}{\overset{d^*}{\rightrightarrows}} \mathrm{Hom}(\Sigma \times_{(s,t)} R, I).$$

Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the projections, and let $I = \mathcal{I}som_{X\times_Y X}(\pi_2^*\mathcal{F}, \pi_1^*\mathcal{F})$. This is an algebraic space by assumption, so from the above we obtain an exact diagram of sets:

$$\mathrm{Isom}_{X\times_Y X}(\pi_2^*\mathcal{F}, \pi_1^*\mathcal{F}) \xrightarrow{j^*} \mathrm{Isom}_R(s^*\mathcal{F}, t^*\mathcal{F}) \underset{\mathrm{pr}_2^*}{\overset{d^*}{\rightrightarrows}} \mathrm{Isom}_{\Sigma\times R}(\mathrm{pr}_2^*\, s^*\mathcal{F}, \mathrm{pr}_2^*\, t^*\mathcal{F}).$$

Here we use the identifications $d^*s^*\mathcal{F} \simeq (sd)^*\mathcal{F} = (s\,\mathrm{pr}_2)^*\mathcal{F} \simeq \mathrm{pr}_2^*\, s^*\mathcal{F}$ which need no further comment, and the similar identifications with s replaced by t which require the observation that $td = t\,\mathrm{pr}_2$ since source and target agree on the stabilizer. Now consider the cocycle condition $c^*\phi = \mathrm{pr}_1^*\,\phi \circ \mathrm{pr}_2^*\,\phi$ on $R \times_{(s,t)} R$ satisfied by the R-linearization $\phi : s^*\mathcal{F} \to t^*\mathcal{F}$. Then after pullback along $a \times \mathrm{id} : \Sigma \times_{(s,t)} R \to R \times_{(s,t)} R$, and since the stabilizer acts trivially on $\mathcal{F}$, this becomes:

$$d^*\phi = (a\,\mathrm{pr}_1)^*\phi \circ \mathrm{pr}_2^*\,\phi = \mathrm{pr}_2^*\,\phi.$$

Therefore by exactness of the diagram of Isom sets, ϕ descends to an isomorphism $\psi : \pi_2^*\mathcal{F} \xrightarrow{\sim} \pi_1^*\mathcal{F}$. To conclude, we use descent along the map $\pi : X \to Y$. For ψ to be a descent datum with respect to $X \to Y$, it need only satisfy the usual gluing condition:

$$(\star) \quad \pi_{13}^*\psi = \pi_{12}^*\psi \circ \pi_{23}^*\psi$$

where $\pi_{ij} : X \times_Y X \times_Y X \to X \times_Y X$ are the projections. In order to prove that this indeed holds, we consider the commutative diagram:

$$\begin{tikzcd}
R \times_{s,X,t} R \arrow[r, "j\times j"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2, "\mathrm{pr}_1,\mathrm{pr}_2,c"'] & X \times_Y X \times_Y X \arrow[d, shift left=2, "\pi_{12},\pi_{23},\pi_{13}"] \arrow[d] \arrow[d, shift right=2] \\
R \arrow[r, "j"] & X \times_Y X.
\end{tikzcd}$$

On pulling back the relation $(\star)$ by $j \times j$ we obtain the relation $c^*\phi = (\mathrm{pr}_1^*\,\phi) \circ (\mathrm{pr}_2^*\,\phi)$ which holds by assumption. Since $X \to Y$ is flat, the morphism $j \times j$ is finite, surjective and schematically dominant, hence an epimorphism. Therefore Condition ($\star$) holds, hence by descent $\mathcal{F}$ is the pullback of an object $\mathcal{G} \in \mathcal{C}(Y)$. $\square$

4.2.5 Theorem. *Let* $\mathcal{C} \to \mathrm{AlgSp}$ *be one of the following stacks in categories:*

(1) $\mathcal{C}_1 = \mathrm{Flat}$*, the fppf stack whose objects over* X *are flat morphisms of algebraic spaces* $X' \to X$*.*

(2) $\mathcal{C}_2 = \mathrm{Flat}^{\mathrm{qa}}$*, the fpqc stack whose objects over* X *are quasi-affine flat morphisms of algebraic spaces* $X' \to X$*.*

Let $R \rightrightarrows X$ *be a flat, locally finitely presented groupoid space with finite stabilizer* $\Sigma \to X$ *and complexity at most* 1*. Assume that the quotient* $\pi : X \to Y = X/R$ *is flat and locally of finite presentation if* $\mathcal{C} = \mathcal{C}_1$*, and flat if* $\mathcal{C} = \mathcal{C}_2$*. Then the functor* $\pi^* : \mathcal{C}(Y) \to \mathcal{C}(R, X)^\Sigma$ *is an equivalence.*

Recall that an object $(X' \to X) \in \mathcal{C}(X)$ is equivalent to a flat square morphism of groupoids $(R', X') \to (R, X)$ (Lemma 4.1.4) and when $(X' \to X) = \pi^*(Y' \to Y)$, then $Y' = X'/R'$. The essential surjectivity of π^* can thus be rephrased as: the natural morphism $X' \to (X'/R') \times_Y X$ is an isomorphism, and $X'/R' \to Y$ is flat.

Proof : Here the conditions on the representability of the diagonal of $\mathcal{C}$ fail to hold, so we need different arguments. In order to prove full faithfulness let W_1, W_2 be objects of $\mathcal{C}(Y)$ and (V_1, ϕ_1), (V_2, ϕ_2) the pullbacks to X. We prove bijectivity of the map:

$$\mathrm{Hom}_{\mathcal{C}(Y)}(W_1, W_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}(R,X)^\Sigma}\big((V_1, \phi_1), (V_2, \phi_2)\big).$$

Since $\mathrm{Hom}_Y(W_1, W_2)$ is a sheaf in the fpqc topology (this does not use flatness of $W_i \to Y$), injectivity goes as in 4.2.3. For the surjectivity part let $f : (V_1, \phi_1) \to (V_2, \phi_2)$ be a morphism, so $s^*f = t^*f$. Since $\mathrm{Hom}_Y(W_1, W_2)$ is a sheaf, it is enough to prove that $\pi_1^* f = \pi_2^* f$. We have commutative diagrams:

$$\begin{tikzcd}
V_1 \times_X R \arrow[r, "j^*\pi_i^* f"] \arrow[d] & V_2 \times_X R \arrow[d] \\
V_1 \times_X (X \times_Y X) \arrow[r, "\pi_i^* f"] & V_2 \times_X (X \times_Y X)
\end{tikzcd}$$

for $i = 1, 2$. From $s^*f = t^*f$ it follows that $j^*\pi_1^* f = j^*\pi_2^* f$. Since the left vertical map is the pullback of j along the flat map $V_1 \to X$, it is an epimorphism. It then follows that $\pi_1^* f = \pi_2^* f$.
In order to show essential surjectivity let $(V, \phi) \in \mathcal{C}(R, X)^\Sigma$. Let $V_i = \pi_i^* V$ be the pullbacks of $V \to X$ along the projections $\pi_1, \pi_2 : X \times_Y X \to X$. When

pulling back j along the flat morphism $h\colon V_1 \to X \times_Y X$, it remains an effective epimorphism.

$$\begin{tikzcd} h^*(\Sigma \times_{(s,t)} R) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\square"] & j^*V_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_1 \arrow[d, "h"] \\ \Sigma \times_{(s,t)} R \arrow[r, shift left] \arrow[r, shift right] & R \arrow[r, "j"] & X \times_Y X. \end{tikzcd}$$

The morphism $j^*V_1 = s^*V \xrightarrow{\phi} t^*V = j^*V_2 \longrightarrow V_2$ is $h^*(\Sigma \times_{(s,t)} R)$-invariant, so by effectivity we obtain a unique morphism $\psi : V_1 \to V_2$. Similarly we obtain a unique morphism $\chi : V_2 \to V_1$. We claim that ψ and χ are inverse isomorphisms. Since j is a uniform epimorphism, in order to prove that the $X \times_Y X$-morphism $\psi \circ \chi$ is the identity it is enough to do it after pullback along j. In this case it is clear since $j^*\psi = \phi$ and $j^*\chi = \phi^{-1}$. Similarly we prove that $\chi \circ \psi$ is the identity. One shows as in the end of the proof of 4.2.3 that the isomorphism $\psi : \pi_1^*V \xrightarrow{\sim} \pi_2^*V$ is a descent datum for V with respect to $\pi : X \to Y$. The assumptions of the theorem imply that effective descent along π holds in $\mathcal{C}$ so V descends to a unique flat morphism $W \to Y$. $\square$

4.3 Quotient by a subgroupoid

In this section we come to the quotient question from the introduction, i.e., the construction of a quotient of a groupoid by a normal subgroupoid. Let us first review some known cases where this construction is possible.

(1) If $R \rightrightarrows X$ is given by the action of a group space G and $P \rightrightarrows X$ is given by a flat normal subgroup H. In this case the quotient groupoid $Q \rightrightarrows Y$ is the action of G/H on X/H. More generally the quotient exists when $R \rightrightarrows X$ is a local group action (i.e., it is given by a group action, fppf locally on X/R) and P is a flat local normal subgroup action.

(2) If $R \rightrightarrows X$ is finite locally free and P is a normal open and closed subgroupoid; this is the Bootstrap Theorem of [KM97, 7.8].

(3) If P is included in the stabilizer; this is the process of rigidification of [ACV03, §5.1] and [AOV08, §A].

With suitable flatness assumptions, we shall provide another case in a different direction: the quotient exists when P has complexity at most 1. We emphasize that the existence of the quotients $Y = X/P$ and $Q = P\backslash R/P$ appearing in the statement is granted by 3.2.1.

4.3.1 Theorem. *Let $R \rightrightarrows X$ be a flat, locally finitely presented groupoid of algebraic spaces. Let $P \rightrightarrows X$ be a flat, locally finitely presented normal subgroupoid of R with finite stabilizer $\Sigma_P \to X$ and complexity at most 1. Assume that the quotient $X \to Y = X/P$ is flat and locally finitely presented.*

Then there is a quotient groupoid $Q \rightrightarrows Y$ which is flat and locally finitely presented, with $Q = P\backslash R/P$. Moreover, the morphisms $R \to Q$ and $R \times_X R \to Q \times_Y Q$ are flat and locally finitely presented.

The rest of this subsection is devoted to the proof. We denote by $s, t : R \rightrightarrows X$ and $\sigma, \tau : P \rightrightarrows X$ the source and target maps of the groupoids, and by $\rho : R \to Q$ and $\pi : X \to Y$ the quotient maps.

Step 1. There exist flat locally finitely presented maps $\bar{s}, \bar{t} : Q \rightrightarrows Y$ and commutative squares:

$$\begin{tikzcd} R \arrow[r, "s{,}t"] \arrow[d, "\rho"'] & X \arrow[d, "\pi"] \\ Q \arrow[r, dotted, "\bar{s}{,}\bar{t}"] & Y \end{tikzcd}$$

and ρ is flat. To prove this we start with the action of P on R by postcomposition. This action is free so there is a flat, locally finitely presented quotient morphism $\rho_{\rm post} : R \to P\backslash R$ where $P\backslash R$ is an algebraic space. Since $s : R \to X$ is invariant by the action of P, there is an induced faithfully flat locally finitely presented morphism $s' : P\backslash R \to X$. The map $R \times_{(s,\sigma)} P \to R$, $(\alpha, \varphi) \mapsto \alpha\varphi^{-1}$ is equivariant for the action of P on the R-factors by postcomposition. Using that the formation of the quotient $\rho_{\rm post} : R \to P\backslash R$ commutes with the flat base change $\sigma : P \to X$, we deduce that there is an induced map $(P\backslash R) \times_{(s',\sigma)} P \to P\backslash R$. In this way we obtain a P-linearization on the X-object $P\backslash R$, as follows:

$$\begin{array}{rcl} \sigma^*(P\backslash R) = (P\backslash R) \times_{(s',\sigma)} P & \xrightarrow{\sim} & \tau^*(P\backslash R) = (P\backslash R) \times_{(s',\tau)} P. \\ (\alpha, \varphi) & \longmapsto & (\alpha\varphi^{-1}, \varphi) \end{array}$$

We claim that because P is normal, the restriction of this P-linearization to the stabilizer Σ_P is trivial. In order to check this, we take advantage of the fact that the space $P\backslash R$ is equal to the fppf quotient sheaf so locally $(P\backslash R)(T) = P(T)\backslash R(T)$. If $\varphi \in \Sigma_P(T)$ and $\alpha \in R(T)$, we have $\psi := \alpha\varphi^{-1}\alpha^{-1} \in \Sigma_P(T)$ and hence $\alpha\varphi^{-1} = \psi\alpha$ in $R(T)$ which is equal to α in $(P\backslash R)(T)$. This proves our claim. It follows from case (1) of Theorem 4.2.5 that $s' : P\backslash R \to X$ descends to a faithfully flat locally finitely presented map $\bar{s} : Q_1 \to Y$.
Similarly, considering the action of P on R by precomposition, we obtain a flat, locally finitely presented quotient morphism $\rho_{\rm pre} : R \to R/P$, and a flat locally finitely presented morphism $t' : R/P \to X$ induced by t. The latter supports a P-linearization with trivial stabilizer action and descends to a faithfully flat locally finitely presented map $\bar{t} : Q_2 \to Y$.
Since the formation of the quotient $X \to Y$ commutes with flat base change, we see that Q_1 is the quotient of $P\backslash R$ by P acting by postcomposition and that Q_2 is the quotient of R/P by P acting by precomposition. Both quotients are isomorphic since they enjoy the same universal property as $Q = P\backslash R/P$. So $Q = Q_1 = Q_2$ canonically and we obtain maps $\bar{s}, \bar{t} : Q \rightrightarrows Y$. In this way

we obtain also that $\rho : R \to Q$ is flat, being the composition of the flat map $\rho_{\text{post}} : R \to P\backslash R$ and of the morphism $P\backslash R \to Q$ which is a base change of the flat map $X \to Y$. We have thus produced the commutative diagrams

$$\begin{tikzcd} R \arrow[r, "\rho_{\text{post}}"] \arrow[dr, "\rho"'] & P\backslash R \arrow[r, "s'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ & Q \arrow[r, "\bar{s}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\rho_{\text{pre}}"] \arrow[dr, "\rho"'] & R/P \arrow[r, "t'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ & Q \arrow[r, "\bar{t}"'] & Y \end{tikzcd}$$

in which all maps are flat.

Step 2. There exists a flat locally finitely presented map $\bar{c} : Q \times_Y Q \to Q$ and a commutative square:

$$\begin{tikzcd} R \times_X R \arrow[r, "c"] \arrow[d, "\rho\times\rho"'] & R \arrow[d, "\rho"] \\ Q \times_Y Q \arrow[r, dotted, "\bar{c}"] & Q \end{tikzcd}$$

where $\rho \times \rho$ is flat. To prove this, note that there are three commuting actions of R on $R \times_X R$: pre-composition $(\alpha, \beta, \gamma) : (\alpha, \beta) \to (\alpha, \beta\gamma)$, post-composition $(\gamma, \alpha, \beta) : (\alpha, \beta) \to (\gamma\alpha, \beta)$ and middle-composition $(\alpha, \gamma, \beta) : (\alpha, \beta) \to (\alpha\gamma, \gamma^{-1}\beta)$. The joint action of any two of these are free. The composition c is equivariant with pre- and post-composition and invariant under middle-composition.
Taking the quotient by post-composition under P, we obtain a flat morphism $c' : (P\backslash R) \times_{(s',t)} R \to P\backslash R$. Since $s' : P\backslash R \to X$ is the pull-back of $\bar{s} : Q \to Y$, we can identify the source of c' with $Q \times_{\bar{s},\pi t} R$. Middle-composition then becomes post-composition on the second factor so c' factors into two flat maps

$$\begin{tikzcd} Q \times_{\bar{s},\pi t} R \arrow[r] \arrow[rr, bend left, "c'"] & Q \times_{\bar{s},\bar{t}\pi'} P\backslash R \arrow[r, "c''"'] & P\backslash R. \end{tikzcd}$$

The map c'' fits into the diagram

$$\begin{tikzcd} Q \times_{\bar{s},\bar{t}\pi'} P\backslash R \arrow[r, "c''"] \arrow[d] \arrow[rr, bend left, "s'\,\mathrm{pr}_2"] & P\backslash R \arrow[r, "s'"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ Q \times_Y Q \arrow[r, dotted, "\bar{c}"] \arrow[rr, bend right, "\bar{s}\,\mathrm{pr}_2"'] & Q \arrow[r, "\bar{s}"] & Y \end{tikzcd}$$

where the outer square also is cartesian, so c'' descends to a flat map $\bar{c}$ as indicated in the diagram (Theorem 4.2.5). The map $\rho \times \rho : R \times_X R \to Q \times_Y Q$

is flat, being the composition of the flat map $R \times_X R \to Q \times_{\bar{s},\bar{t}\pi'} P\backslash R$ (quotient map of the free middle-post-composition) and the pull-back of the flat map π.

Step 3. Conclusion. It is easy to construct the maps $\bar{e} : Y \to Q$ and $\bar{\imath} : Q \to Q$ fitting in commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\pi"'] & R \arrow[d, "\rho"] \\ Y \arrow[r, dotted, "\bar{e}"] & Q \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "i"] \arrow[d, "\rho"'] & R \arrow[d, "\rho"] \\ Q \arrow[r, dotted, "\bar{\imath}"] & Q. \end{tikzcd}$$

We skip the details. From the fact that $\rho : R \to Q$ and $\rho \times \rho : R \times_X R \to Q \times_Y Q$ are epimorphisms of algebraic spaces, it follows formally that the maps $\bar{s}, \bar{t}, \bar{c}, \bar{e}, \bar{\imath}$ are unique, that they give $Q \rightrightarrows Y$ the structure of a groupoid, and that the map $\rho : R \to Q$ is a morphism of groupoids. Finally we can prove that the groupoid $Q \rightrightarrows Y$ is a quotient of $R \rightrightarrows X$ by P. Let $f : (R, X) \to (R', X')$ be a morphism of groupoids such that $P \subset \ker(f)$. Then the map $f : R \to R'$ is invariant by the pre-post-composition of P on R, hence it factors through a map $Q \to R'$. Similarly the map $f : X \to X'$ is invariant by the action of P hence it factors through a map $Y \to X'$. That $(Q, Y) \to (R', X')$ is a morphism of groupoids follows again from the fact that ρ and $\rho \times \rho$ are epimorphisms.

4.4 Stacky interpretations

Let $R \rightrightarrows X$ be a flat locally finitely presented groupoid in algebraic spaces and let $\mathcal{C} \to \mathrm{AlgSp}$ be a stack for the fppf topology. Then the category $\mathcal{C}(R, X)$ of R-equivariant object is equivalent with the category of morphisms $[X/R] \to \mathcal{C}$ between stacks. A morphism $\varphi : [X/R] \to \mathcal{C}$ corresponds to an object with trivial Σ-action if and only if the following equivalent conditions hold

(1) For every algebraic space T, object $x \in [X/R](T)$, and automorphism $\tau \in \mathrm{Aut}(x)$, the image $\varphi(\tau)$ is the identity on $\varphi(x)$.

(2) The induced morphism of inertia stacks $I\varphi : I[X/R] \to I\mathcal{C}$ is trivial, i.e., factors through $\mathcal{C}$.

(3) The morphism φ factors, up to equivalence, through the fppf-sheafification $[X/R] \to \pi_0([X/R])$.

If $R \rightrightarrows X$ has finite inertia, then the coarse space $[X/R] \to X/R$ factors through the fppf sheaf quotient $\pi_0[X/R] = (X/R)_{\mathrm{fppf}}$ and $\pi_0[X/R] \to X/R$ is an isomorphism if the action is free. Theorem 4.2.3 thus says that the functor

$$\mathrm{Hom}(X/R, \mathcal{C}) \to \mathrm{Hom}(\pi_0[X/R], \mathcal{C})$$

is an equivalence of categories if $R \rightrightarrows X$ has complexity at most 1, $X \to X/R$ is flat, and under certain assumptions on $\mathcal{C}$, e.g., if $\mathcal{C}$ is a stack in groupoids with representable diagonal.

In the setting of Theorem 4.2.5, the category $\mathcal{C}(R,X)$ is equivalent to the category of flat morphisms of algebraic stacks $\mathcal{X}' \to \mathcal{X} = [X/R]$ that are representable by algebraic spaces. The subcategory $\mathcal{C}(R,X)^\Sigma$ consists of stabilizer-preserving morphisms, i.e., those such that the induced morphism of inertia stacks $I\mathcal{X}' \to (I\mathcal{X}) \times_{\mathcal{X}} \mathcal{X}'$ is an isomorphism. Theorem 4.2.5 thus says that the category of flat morphisms $Y' \to Y = X/R$ is equivalent to the category of flat stabilizer-preserving representable morphisms of algebraic stacks $\mathcal{X}' \to \mathcal{X}$.

4.4.1 Remark. It can be proved that case (2) of Theorem 4.2.5 holds for arbitrary flat morphisms $X' \to X$. Indeed, let $\mathcal{X}' \to \mathcal{X}$ be the corresponding stabilizer-preserving representable morphism of algebraic stacks. Then $\mathcal{X}'$ also has finite stabilizer and a coarse moduli space $Y' = X'/R'$. It is enough to show that the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is cartesian. This can be checked étale-locally on Y' and Y, so we can assume that Y and Y' are affine. After further étale localization on Y, we can assume that $\mathcal{X} = [X/R]$ where $X \to \mathcal{X}$ is finite: this follows from the proof of the Keel–Mori theorem [Ry13, Thm. 6.12]. Since $\mathcal{X}' \to \mathcal{X}$ is representable, we obtain a finite presentation $X' \to \mathcal{X}'$ where $X' = X \times_{\mathcal{X}} \mathcal{X}'$. It follows that X' and X are affine since $X' \to Y'$ and $X \to Y$ are affine [Ry13, Thm. 5.3]. Thus $X' \to X$ is affine and case (2) of Theorem 4.2.5 applies.

Finally, Theorem 4.3.1 can be described as follows using stacks. We have a locally finitely presented flat morphism $[X/P] \to [X/R]$. This gives rise to a groupoid

$$[X/P] \times_{[X/R]} [X/P] \rightrightarrows [X/P]$$

with quotient $[X/R]$. That $P \subset R$ is a normal subgroupoid implies that the morphisms of the groupoid above are stabilizer-preserving. We can also make the identification $[X/P] \times_{[X/R]} [X/P] = [P\backslash R/P]$. By Theorem 4.2.5, we thus obtain a cartesian diagram

$$\begin{tikzcd} {[P\backslash R/P]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/P]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/R]} \arrow[d] \\ Q = P\backslash R/P \arrow[r, shift left] \arrow[r, shift right] & X/P \arrow[r] & {[(X/P)/Q]} \end{tikzcd}$$

where the horizontal morphisms are flat and locally of finite presentation and the vertical morphisms are (relative) coarse moduli spaces.

4.5 A non-flat counter-example

We give an example that shows that Theorems 4.2.3 and 4.2.5 do not hold when $\pi : X \to Y$ is not flat. The counter-example satisfies:

(1) X is an affine 1-dimensional scheme in characteristic p with an action of $G = \mathbb{Z}/p\mathbb{Z}$ but $\pi : X \to Y = X/G$ is not flat.

(2) There is a torsion equivariant line bundle $\mathscr{L} \in \mathrm{Pic}_p^G(X)^\Sigma$ that does not come from $\mathrm{Pic}(Y)$. In particular, Theorem 4.2.3 fails for the algebraic stacks $\mathcal{C} = \mathrm{Pic} = B\mathbb{G}_m$ and $\mathcal{C} = \mathrm{Pic}_p = B\mu_p$.

(3) There is a smooth morphism $X' \to X$ that is not the pull-back of a smooth morphism $Y' \to Y$. In particular, Theorem 4.2.5 fails even for smooth morphisms.

Let k be a field of characteristic p. Let $X = \operatorname{Spec} k[\epsilon, x]/(\epsilon^2)$ and let $\mathbb{Z}/p\mathbb{Z}$ act via $t.(\epsilon, x) = (\epsilon, x + t\epsilon)$. Then $Y = \operatorname{Spec} k[\epsilon, x^p, \epsilon x, \epsilon x^2, \dots, \epsilon x^{p-1}]$.
Consider the following $\mathbb{Z}/p\mathbb{Z}$-equivariant line bundle $\mathscr{L}$ on X: as a line bundle it is trivial $\mathscr{L} = \mathscr{O}_X \cdot e$ and it has the action $t.e = (1 + t\epsilon)e$.
The stabilizer acts trivially on this line bundle. Indeed, the stabilizer Σ of X is given by the closed subscheme $t\epsilon = 0$ of $(\mathbb{Z}/p\mathbb{Z}) \times X = \operatorname{Spec} k[t, \epsilon, x]/(t^p - t, \epsilon^2)$.
The line bundle is not in the image of $\pi^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}^G(X)$. Indeed, since π^* has the right adjoint $(\pi_* -)^G$, it is enough to verify that the counit $\pi^*(\pi_*\mathscr{L})^G \to \mathscr{L}$ is not an isomorphism. But an easy calculation gives that $\pi^*(\pi_*\mathscr{L})^G = (\epsilon) \cdot \mathscr{L} \subsetneq \mathscr{L}$.
In terms of algebraic stacks, the line bundle $\mathscr{L}$ corresponds to the morphism

$$[X/(\mathbb{Z}/p\mathbb{Z})] \to B(\mathbb{Z}/p\mathbb{Z})_S \xrightarrow{B\varphi} B\mu_p \to B\mathbb{G}_m,$$

where $S = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ and $\varphi\colon (\mathbb{Z}/p\mathbb{Z})_S \to \mu_p$ is the group homomorphism given by $t \mapsto (1+\epsilon)^t = 1 + t\epsilon$. Here the map between inertia stacks $I[X/(\mathbb{Z}/p\mathbb{Z})] \to IB(\mathbb{Z}/p\mathbb{Z})_S \to IB\mu_p$ is induced by

$$\begin{array}{ccccc} k[\lambda]/(\lambda^p - 1) & \longrightarrow & k[\epsilon, t]/(\epsilon^2, t^p - t) & \longrightarrow & k[\epsilon, x, t]/(\epsilon^2, t^p - t, t\epsilon) \\ \lambda & \longmapsto & 1 + t\epsilon & \longmapsto & 1 \end{array}$$

so it factors through $B\mu_p$.
The line bundle corresponds to the smooth stabilizer-preserving G-equivariant morphism $X' = \operatorname{Spec} k[\epsilon, x, y]/(\epsilon^2)$ where the G-action is $t.(\epsilon, x, y) = (\epsilon, x + t\epsilon, y + t\epsilon y)$. This is not the pull-back of the morphism $Y' = X'/G \to Y = X/G$. Indeed, a similar calculation as for the line bundle gives that $Y' = \operatorname{Spec} k[\epsilon, x^p, y^p, \epsilon x^i y^j]$.

References

[ACV03] D. Abramovich, A. Corti, A. Vistoli, *Twisted bundles and admissible covers*, Special issue in honor of Steven L. Kleiman, Comm. Algebra 31 (2003), no. 8, 3547–3618.

[Al13] J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402.

[Ar70] M. Artin, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. (2) 91 (1970), 88–135.

[Ar74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. 27 (1974), 165–189.

[AOV08] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 4, 1057–1091.

[CLO12] B. Conrad, M. Lieblich, M. Olsson, *Nagata compactification for algebraic spaces*, J. Inst. Math. Jussieu 11 (2012), no. 4, 747–814.

[EGAIV.3] A. Grothendieck, with J. Dieudonné, *Éléments de géométrie algébrique IV*, Étude locale des schémas et des morphismes de schémas III, Publ. Math. IHES no. 28, 1966.

[EGAIV.4] A. Grothendieck, with J. Dieudonné, *Éléments de géométrie algébrique IV*, Étude locale des schémas et des morphismes de schémas IV, Publ. Math. IHES no. 32, 1967.

[Ek88] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Publ. Math. IHÉS no. 67 (1988), 97–144.

[Fe03] D. Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France 131 (2003), no. 4, 553–585.

[Gi64] J. Giraud, *Méthode de la descente*, Bull. Soc. Math. France Mém. 2 (1964).

[Gr59] A. Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique I. Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki, Vol. 5, Exp. No. 190, 299–327, SMF, 1995.

[KM97] S. Keel, S. Mori, *Quotients by groupoids*, Ann. Math. (2) 145, No.1, 193-213 (1997).

[Ko97] J. Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) 145 (1997), no. 1, 33–79.

[Le65] A. H. M. Levelt, *Sur la pro-représentabilité de certains foncteurs en géométrie algébrique*, Notes from Mathematisch Instituut, Katholieke Universiteit, Nijmegen (The Netherlands), 1965. Available at `http://www.math.ru.nl/~ahml/prorep.pdf`.

[Li05] M. Lieblich, *Groupoids and quotients in algebraic geometry*, Snowbird lectures in algebraic geometry, 119–136, Contemp. Math., 388, Amer. Math. Soc., 2005.

[Ol12] M. Olsson, *Integral models for moduli spaces of G-torsors*, Ann. Inst. Fourier (Grenoble) 62 (2012), no. 4, 1483–1549.

[Ra74] J.-C. Raoult, *Compactification des espaces algébriques*, C. R. Acad. Sci. Paris Sér. A 278 (1974), 867–869.

[RS76] A. N. Rudakov, I. Shafarevich, *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 6, 1269–1307.

[Ry10] D. Rydh, *Submersions and effective descent of étale morphisms*, Bull. Soc. Math. France 138(2) (2010), 181–230.

[Ry11] D. Rydh, *Compactification of tame Deligne–Mumford stacks*, preprint, 2011.

[Ry13] D. Rydh, *Existence and properties of geometric quotients*, J. Algebraic Geom. 22 (2013), no. 4, 629–669.

[SGA1] A. Grothendieck, *Revêtements étales et groupe fondamental*, Documents Mathématiques 3, Société Mathématique de France, 2003.

[SGA3.2] M. Demazure, A. Grothendieck, *Schémas en groupes*, Tome II. Groupes de type multiplicatif, et structure des schémas en groupes généraux. Lecture Notes in Mathematics 152, Springer, 1970.

[SGA4.2] M. Artin, A. Grothendieck, J. L. Verdier, Théorie des topos et cohomologie étale des schémas, Tome 2, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Mathematics 270. Springer-Verlag, 1972.

[SGA6] P. Berthelot, A. Grothendieck, L. Illusie, *Théorie des intersections et théorème de Riemann–Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, 1971.

[SP] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2018.

[TT16] M. Temkin, I. Tyomkin, *Ferrand pushouts for algebraic spaces*, Eur. J. Math. 2 (2016), no. 4, 960–983.

[Vi86] E. B. Vinberg, *Complexity of actions of reductive groups*, Funktsional. Anal. i Prilozhen. 20 (1986), no. 1, 1–13. English translation: Functional Anal. Appl. 20 (1986), no. 1, 1–11.

[Vi05] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, 1–104, Math. Surveys Monogr. 123, AMS, 2005.

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