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Unramified F–divided objects and the étale fundamental pro-groupoid in positive characteristic

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Let $\mathscr{X}/S$ be a flat algebraic stack of finite presentation. We define a new étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$, generalizing Grothendieck's enlarged étale fundamental group from SGA 3 to the relative situation. When S is of equal positive characteristic p, we prove that $\Pi_1(\mathscr{X}/S)$ naturally arises as colimit of the system of relative Frobenius morphisms $\mathscr{X} \to \mathscr{X}^{p/S} \to \mathscr{X}^{p^2/S} \to \cdots$ in the pro-category of Deligne Mumford stacks. We give an interpretation of this result as an adjunction between Π_1 and the stack Fdiv of F–divided objects. In order to obtain these results, we study the existence and properties of relative perfection for algebras in characteristic p.

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1 Introduction

1.1 Motivation

Using Cartier's theorem on the descent of vector bundles under Frobenius, Gieseker and Katz were able to give another viewpoint on stratified vector bundles on a smooth

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variety of characteristic p. Namely, they showed that these objects are equivalent to *F–divided vector bundles*, that is, sequences $\{E_i\}_{i\geqslant 0}$ of vector bundles and isomorphisms $E_i \simeq F^* E_{i+1}$, where F is the Frobenius endomorphism of the variety; see Gieseker [9]. Since then, these have occupied an important place in the research on vector bundles in characteristic p. Looking only at the recent literature, we can mention the works of dos Santos [26; 27], Esnault and Mehta [6], Berthelot [1] and Tonini and Zhang [30; 29].

More generally, one can expect that in the study of curves, or morphisms, or torsors (etc) in characteristic p, the F–divided curves, morphisms, torsors (etc) are natural objects which are likely to play an important role. Here, for any algebraic stack $\mathscr{M}/S$, we introduce the stack $\mathrm{Fdiv}(\mathscr{M}/S)$ of F–divided objects of $\mathscr{M}$ and we seek to understand it (see Remark 2.3.5 for a warning on notation). Note that F–divided vector bundles correspond to the case where $\mathscr{M}$ is the classifying stack $B\mathrm{GL}_n$, a typical example of Artin stack with affine positive-dimensional inertia. We study the somehow opposite case where $\mathscr{M}$ is a Deligne–Mumford stack. In this case we call the objects of $\mathrm{Fdiv}(\mathscr{M}/S)$ *unramified F–divided objects*. Roughly speaking, our main result says that unramified F–divided objects defined over geometrically reduced bases are quasi-isotrivial. In order to achieve this, we establish various results on the perfection of algebras, and on the coperfection of algebraic spaces and stacks, which have independent interest. These will be presented in Sections 1.3 and 1.4 after a short topological detour.

1.2 Fundamental group(oid) in topology

Our main result below features a certain pro-étale stack which is an algebro-geometric version of the fundamental groupoid. We wish to clarify the link between the present algebraic constructions and the classical topological notions. For this we give a brief recollection of the topological notion of *fundamental groupoid*. A classical invariant associated to a pointed topological space (X, x) is its fundamental group $\pi_1(X, x)$, consisting of homotopy classes of loops based at x. There are at least two issues worth discussing: First, in some contexts one would prefer not to have to carry around the extra datum of a basepoint; however, there is in general no *canonical choice* of identification $\pi_1(X, x) \to \pi(X, y)$ between two fundamental groups based at different points; therefore a notion of basepoint-free fundamental group would not be well defined. Secondly, if X is not path-connected, $\pi_1(X, x)$ will not see any information coming from a path-connected component not containing x.

An elegant way out is offered by the notion of *fundamental groupoid*. This is defined to be the category $\Pi(X)$ whose set of objects is the set underlying X and whose arrows

$x \to y$ are homotopy classes of paths from x to y. It is easy to check that compositions of arrows and identities are well defined, and that moreover every arrow has a unique inverse; hence the term *groupoid*. The notion does not rely on the choice of a base point. Taking the set of isomorphism classes of objects of $\Pi(X)$ retrieves the set $\pi_0(X)$ of path-connected components; while for every $x \in X$, the group $\mathrm{Aut}_{\Pi(X)}(x)$ retrieves the fundamental group based at x. In this regard, the fundamental groupoid is an invariant which is richer than the group. Also, the category of locally constant sheaves on X is naturally equivalent to the category of *functors* from $\Pi(X)$ to sets. For example, letting BG denote the category with one object and automorphism group G, the fundamental groupoid of the circle $\mathbb{S}^1_{\mathbb{R}}$ is (noncanonically) equivalent to the groupoid $B\mathbb{Z}$.

In this paper we introduce an incarnation of the fundamental groupoid in algebraic geometry; it generalizes the notion of enlarged étale fundamental group appearing in SGA 3_{II} [14, section 6], and it works for families $X \to S$ over a general base.

1.3 Étale fundamental pro-groupoid and unramified F–divided objects

Let S be an algebraic space and $\mathscr{X} \to S$ a flat, finitely presented algebraic stack. We construct a 2–pro-object of the 2–category of étale algebraic stacks, called the *étale fundamental pro-groupoid*, with a map $\mathscr{X} \to \Pi_1(\mathscr{X}/S)$. If $\dim(S) = 0$ or if $\mathscr{X} \to S$ is *separable* (ie it has geometrically reduced fibres), then $\Pi_1(\mathscr{X}/S)$ has a coarse moduli space which is the space of connected components $\pi_0(\mathscr{X}/S)$ of Romagny [23], seen as a constant 2–pro-object. When S is the spectrum of a field k and $\mathscr{X}$ is geometrically connected, the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ recovers known objects:

- If k is separably closed and $x \in \mathscr{X}(k)$ is a rational point, $\Pi_1(\mathscr{X}/k)$ is the classifying stack of Grothendieck's enlarged fundamental group $\pi_1^{\mathrm{SGA3}}(\mathscr{X}, x)$ (Proposition 5.4.2).
- In general, the pro-finite quotient of $\Pi_1(\mathscr{X}/k)$ defines a projective system in the 2–category of stacks whose limit is the étale fundamental gerbe $\Pi^{\text{ét}}_{\mathscr{X}/k}$ of Borne and Vistoli (Proposition 5.4.3).

Let us now assume that S has characteristic p. Then, from the fact that étale morphisms are perfect, it follows that the natural map $\mathrm{Fdiv}(\Pi_1(\mathscr{X}/S)/S) \to \Pi_1(\mathscr{X}/S)$ is an isomorphism. This is all we need to state our main result.

Theorem A *Let S be a quasicompact, quasiseparated algebraic space of characteristic p and $\mathscr{X} \to S$ a flat, finitely presented algebraic stack. Assume that either*

$\dim(S) = 0$ *or* $\mathscr{X} \to S$ *is separable. Let* $\mathscr{M} \to S$ *be a Deligne–Mumford stack. Then, by applying* Fdiv *and precomposing with* $\mathscr{X} \to \Pi_1(\mathscr{X}/S)$, *we obtain an equivalence*

$$\mathsf{Hom}(\Pi_1(\mathscr{X}/S), \mathscr{M}) \xrightarrow{\sim} \mathsf{Hom}(\mathscr{X}, \mathsf{Fdiv}(\mathscr{M}/S))$$

between the categories of morphisms of pro-Deligne–Mumford stacks (with $\mathscr{M}$ *seen as a constant 2–pro-object) on the source, and morphisms of stacks on the target. This equivalence is functorial in* $\mathscr{X}$ *and* $\mathscr{M}$.

See Theorem 5.6.2. Intuitively, this means that any F–divided object of $\mathscr{M}$ over the base $\mathscr{X}$ becomes constant after étale surjective base changes on S and on $\mathscr{X}$, ie is quasi-isotrivial in a suitable sense. Here is a simple illustration. Let us assume that X is a connected, simply connected variety over a separably closed field k. Then Theorem A implies that all F–divided families $C \to X$ of stable n–pointed curves of genus g with $2g - 2 + n > 0$ are constant. The same assertion with vector bundles replacing curves is the analogue of Gieseker's conjecture, proved by Esnault and Mehta [6]. However, Esnault and Mehta's situation and ours are different in nature. In fact, in loc. cit. as well as in our work, the approach has two comparable steps. First, one uses the fact that objects are described by a morphism from a suitable fundamental group(oid) scheme Π (the étale fundamental pro-groupoid for us and the stratified fundamental group scheme in [6]). Second, one proves that, under the given assumptions, the group scheme Π vanishes. The crucial difference is that in our setting the first step is the difficult part of the argument and the second step is almost trivial, while for Esnault and Mehta the first step is easy and the second step is where all the effort lies.

If contemplated with a focus on $\mathscr{X}$, Theorem A gives information on its coperfection. The viewpoint being substantially different, it is worth giving the corresponding version of the statement. For this we denote by $\mathscr{X}^{p^i/S}$ the i^{th} Frobenius twist of $\mathscr{X}/S$ and

$$F_i : \mathscr{X}^{p^i/S} \to \mathscr{X}^{p^{i+1}/S}$$

the relative Frobenius morphism.

Theorem A′ *Let* S *be a quasicompact quasiseparated algebraic space of characteristic* p.

(1) *Let* $X \to S$ *be a flat, finitely presented morphism of algebraic spaces. Assume that either* $\dim(S) = 0$ *or* $X \to S$ *is separable. The inductive system of relative Frobenii*

$$X \xrightarrow{F_0} X^{p/S} \xrightarrow{F_1} X^{p^2/S} \xrightarrow{F_2} \cdots$$

admits a colimit in the category of algebraic spaces over S. This colimit is the algebraic space of connected components $\pi_0(X/S)$; it is a coperfection of $X \to S$.

(2) *Let $\mathscr{X} \to S$ be a flat, finitely presented algebraic stack. Assume that either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable. The inductive system of relative Frobenii*

$$\mathscr{X} \xrightarrow{F_0} \mathscr{X}^{p/S} \xrightarrow{F_1} \mathscr{X}^{p^2/S} \xrightarrow{F_2} \cdots$$

admits a 2–colimit in the 2–category of pro-Deligne–Mumford stacks over S. This 2–colimit is the pro-étale stack $\Pi_1(\mathscr{X}/S)$; it is a 2–coperfection of $\mathscr{X}/S$ in the 2–category of pro-Deligne–Mumford stacks.

See Remarks 5.1.2 and 5.6.3. Statement (2) is equivalent to Theorem A, as explained in Remark 2.3.3. Note that (2) includes (1) as a special case, because $\Pi_1(\mathscr{X}/S)$ has coarse moduli space $\pi_0(\mathscr{X}/S)$. We include (1) for emphasis and also because the proof actually proceeds by deducing (2) from (1).

Theorem A′ seems to suggest that taking coperfection in the higher category of pro-Deligne–Mumford n–stacks would eventually recover the whole relative étale homotopy type of $X \to S$. We plan to investigate this eventuality in a future article.

1.4 Perfection of algebras; largest étale subalgebras

Within the category of algebras, the situation is somehow more subtle. Given a characteristic p ring R and an algebra $R \to A$, denote by

$$F_i : A^{p^{i+1}/R} \to A^{p^i/R}$$

the relative Frobenius of $A^{p^i/R}$, the i^{th} Frobenius twist of A. Define the *preperfection* of A/R:

$$A^{p^\infty/R} = \lim(\cdots \xrightarrow{F_2} A^{p^2/R} \xrightarrow{F_1} A^{p/R} \xrightarrow{F_0} A).$$

The name is explained by a surprising fact: the algebra $A^{p^\infty/R}$ is not perfect in general, even if $R \to A$ is flat, finitely presented and separable. We give an example of this with R equal to the local ring of a nodal curve singularity (see Lemma 4.5.2). In our example the double preperfection is perfect but we do not know if iterated preperfections should converge to a perfect algebra in general. In the affine case $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, we write $\pi_0(A/R)$ instead of $\pi_0(X/S)$. The following is an immediate consequence of Theorem A′:

Theorem A″ *Let R be a ring of characteristic p and $R \to A$ flat and finitely presented. Assume that either* $\dim(R) = 0$ *or $R \to A$ is separable. There is an isomorphism of R–algebras*

$$\mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

See Theorem 4.3.2. Here $\mathcal{O}(-)$ is the functor of global functions. Given the bad properties of the rings under consideration, this could not really be anticipated: indeed, in general $\mathcal{O}(\pi_0(A/R))$ is not étale and $A^{p^\infty/R}$ is not perfect. Although we present Theorem A″ as a corollary to Theorem A′, the structure of the proof is actually to first establish the former and then deduce the geometric statement for spaces and stacks (Theorem A′).

This begs for a further study of perfection of algebras. Our general expectation is that, for algebras of finite type, there should exist a largest étale subalgebra and this should be (at least close to) the perfection of $R \to A$. In striving to materialize this picture, we study étale hulls in more detail. We take inspiration from recent work of Ferrand [8] on the separated étale hull $\pi^s(X/S)$. We prove the following result, which is not special to characteristic p:

Theorem B *Let S be a noetherian, geometrically unibranch algebraic space without embedded points. Let $f : X \to S$ be a faithfully flat, finitely presented morphism of algebraic spaces.*

1. *The category of factorizations $X \to E \to S$ such that $X \to E$ is a schematically dominant morphism of algebraic spaces and $E \to S$ is étale and affine is a lattice; that is, any two objects have a supremum and an infimum (for the obvious relation of domination). Moreover, it has a largest element $\pi^a(X/S)$.*
2. *The functor $X \mapsto \pi^a(X/S)$ is left adjoint to the inclusion of the category of étale, affine S–schemes into the category of faithfully flat, finitely presented S–algebraic spaces.*

See Theorem 3.2.7 and Corollary 3.2.9. The largest element $\pi^a(X/S)$ is the relative spectrum of a sheaf of $\mathcal{O}_S$–algebras which is the largest étale subalgebra of $f_*\mathcal{O}_X$. It is called the *étale affine hull* of $X \to S$. When S is artinian or $X \to S$ is separable, the functor $\pi_0(X/S)$ is an étale algebraic space and we have morphisms

$$X \to \pi_0(X/S) \to \pi^s(X/S) \to \pi^a(X/S).$$

We can take advantage of this to analyze perfection of algebras in characteristic p. When $S=\mathrm{Spec}(R)$ and $X=\mathrm{Spec}(A)$, the largest étale subalgebra is written $A^{\text{ét}/R}\subset A$; that is, $\pi^a(A/R)=\mathrm{Spec}(A^{\text{ét}/R})$. We then obtain the following positive results:

Theorem C *Let $R\to A$ be a flat, finite-type morphism of noetherian rings of characteristic p.*

(1) *Assume that either*

  (i) *R is artinian, or*

  (ii) *R is one-dimensional, reduced and geometrically unibranch, and $R\to A$ is separable.*

  Then there are natural isomorphisms

$$A^{\text{ét}/R}\xrightarrow{\sim}\mathcal{O}(\pi_0(A/R))\xrightarrow{\sim}A^{p^\infty/R}.$$

(2) *Assume that R is regular and F–finite. Then the natural map gives rise to an isomorphism*

$$\mathcal{O}(\pi^s(A/R))\xrightarrow{\sim}A^{p^\infty/R}.$$

In all these cases, the R–algebra $A^{p^\infty/R}$ is perfect.

See Theorem 4.2.1 and Corollaries 4.3.3 and 4.4.3.

1.5 "Covers from atlases" and groupoidification

To conclude this introduction, we would like to bring the reader's attention to a trick which is at the heart of most of our constructions, but whose depth may be hidden by the technical developments necessary to its practical use. This trick can be called "covers from atlases". To explain it, let $U\to X$ be a smooth atlas of an algebraic stack, or a Zariski cover of a scheme, and let $R=U\times_X U$. Under the correct assumptions, applying the functor π_0 allows one to form a diagram, with a bottom line composed of algebraic spaces étale over the base,

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d, dashed] \\ \pi_0(R) \arrow[r, shift left] \arrow[r, shift right] & \pi_0(U) \arrow[r, dashed] & ? \end{tikzcd}$$

Here $R\rightrightarrows U$ is a groupoid but $\pi_0(R)\rightrightarrows\pi_0(U)$ is *not* anymore. Fortunately, it turns out that there is a smallest groupoid $\pi_0(R)^{\mathrm{gpd}}\rightrightarrows\pi_0(U)$ mapping out of it and called its *groupoidification*. One can then complete the diagram by setting $?=[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$, which typically is a nontrivial étale gerbe over $\pi_0(X)$. In summation, our procedure associates to any smooth finitely presented atlas $U \to X$ a natural map $X \to \mathscr{E}$ towards an étale stack. In the special case where $U \to X$ is an étale Galois cover of connected schemes with Galois group G, the map $X \to \mathscr{E}$ we find is none other than the classifying map $X \to BG$ of the cover.

That such a groupoidification exists is largely due to the fact that categorical constructions with étale spaces are almost as easy as those with sets. The natural framework to study groupoidification is homotopical algebra; in Appendix A, it is defined and constructed as a left adjoint to the inclusion of étale groupoids into 2-étale spaces (aka 2–truncated simplicial étale spaces).

Overview of the paper and notation

Each section starts with a small description of contents, where the reader will find more detail. In Section 2, we give definitions and basic facts on perfect stacks, perfection and coperfection. In Section 3, which makes no assumption on the characteristic, we give complements on the functor π_0 and we prove Theorem B. In Section 4, we study the commutative algebra of perfection, proving the results summarized in Theorem C. In Section 5, we introduce the étale fundamental pro-groupoid and prove Theorems A and A$'$, first for algebraic spaces and then for algebraic stacks. Finally, in Appendix A we provide a proof that any étale 2–space possesses a groupoidification.

As already said, throughout the paper we say that $\mathscr{X} \to S$ is *separable* if it has geometrically reduced fibres; note that this differs from the convention of SGA 1 [16], where flatness is additionally required. All sheaves and stacks are considered for the fppf topology unless explicitly stated otherwise. We write Hom, Hom and $\mathscr{H}om$ for sets, categories, and sheaves and stacks, respectively, of morphisms.

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2 Perfection and coperfection

Throughout this section, we let S be an algebraic space of characteristic p. Our purpose is to make some preliminary remarks on perfection and coperfection: definitions and formal properties (Sections 2.1 and 2.2), description in the 2–category of stacks (Section 2.3), and structure of perfect algebraic stacks (Section 2.4).

There is unfortunately no uniform use of the word "perfection" in the literature. Our convention is to call *perfection* and *coperfection* the right adjoint and the left adjoint, respectively, to the inclusion of the full subcategory of perfect objects in the ambient category. This choice is prompted by the fact that, in most cases of existence, the construction of perfections uses limits while the construction of coperfections uses colimits. For example, this is the way one can form the perfection A^{pf} and the coperfection A^{copf} of an $\mathbb{F}_p$–algebra A with absolute Frobenius F_A:

$$A^{\mathrm{pf}} = \lim(\cdots \to A \xrightarrow{F_A} A \xrightarrow{F_A} A), \quad A^{\mathrm{copf}} = \operatorname{colim}(A \xrightarrow{F_A} A \xrightarrow{F_A} A \to \cdots).$$

We emphasize that our interest is in perfection of algebras, and coperfection of algebraic spaces and stacks. This means that our setting is *relative* (over a possibly imperfect base) and *geometric* (with schemes, spaces and stacks). Both features introduce difficulties; we do not know if perfection of algebras and coperfection of algebraic spaces and stacks exist in full generality.

2.1 Categorical definitions

2.1.1 Frobenius Let $f : X \to S$ be a fibred category over S and let $X^{p/S} = X \times_{S,F_S} S$ be its Frobenius twist. The *absolute Frobenius* is the functor $F_X : X \to X$ defined by $F_T^* : X(T) \to X(T)$ for all schemes T of characteristic p, where X is thought of as a fibred category over $\mathbf{Sch}_{\mathbb{F}_p}$. The *relative Frobenius* is the functor $F_{X/S} := (F_X, f) : X \to X^{p/S}$. Note that F_X is not a morphism of fibred categories over S, while $F_{X/S}$ is.

2.1.2 Perfection and coperfection We say that $X \to S$ is *perfect* if $F_{X/S}$ is an isomorphism of fibred categories. Let C be a fibred 2–category over S whose objects are fibred categories over S. We write $\mathsf{Hom}_C(X,Y)$ for the categories of morphisms in C and $\mathrm{Hom}_C(X,Y)$ for the object sets of the latter. The objects $X\in C$ which are perfect form a full 2–subcategory $\mathbf{Perf}(C)$ whose inclusion we denote by $i\colon \mathbf{Perf}(C)\to C$. Now let $X\in C$ be any object. We say that an object $X^{\mathrm{pf}}\in\mathbf{Perf}(C)$ together with a map $iX^{\mathrm{pf}}\to X$ is a *2–perfection of* X if, for all $P\in\mathbf{Perf}(C)$, the induced functor $\mathsf{Hom}_{\mathbf{Perf}(C)}(P,X^{\mathrm{pf}})\to\mathsf{Hom}_C(iP,X)$ is an equivalence. Similarly, a *2–coperfection of* X is an object $X^{\mathrm{copf}}\in\mathbf{Perf}(C)$ together with a map $X\to iX^{\mathrm{copf}}$ such that for all $P\in\mathbf{Perf}(C)$ the induced functor $\mathsf{Hom}_{\mathbf{Perf}(C)}(X^{\mathrm{copf}},P)\to\mathsf{Hom}_C(X,iP)$ is an equivalence. We often simply say *perfection* and *coperfection* for simplicity. Hence, if all objects have perfections (resp. coperfections), then the functor $X\mapsto X^{\mathrm{pf}}$ (resp. $X\mapsto X^{\mathrm{copf}}$) is right (resp. left) adjoint to the inclusion i. Note that, if a given X of interest may be seen as an object of different fibred 2–categories C and C', then its hypothetical perfections in C and C' differ in general, and similarly for its hypothetical coperfections.

2.1.3 Cofibred setting While algebraic spaces and stacks and the 2–categories that contain them fall under the scope of the "fibred" categorical setting, algebras and the categories that contain them live in the "cofibred" categorical setting. The cofibred analogues of the notions just presented exist with the obvious modifications; notably, for a cofibred category $A\to S$, the relative Frobenius is a functor $F_{A/S}\colon A^{p/S}\to A$. In this setting, perfection (resp. coperfection) is again defined as the right (resp. left) adjoint of the inclusion of perfect objects.

2.1.4 Base change and composition If $f\colon X\to Y$ is a morphism of fibred categories over S, we can define $X^{p/Y}:=X\times_{Y,F_Y}Y$ and the relative Frobenius $F_{X/Y}:=(F_X,f)\colon X\to X^{p/Y}$. We say that f is (*relatively*) *perfect* if $F_{X/Y}$ is an isomorphism. The formation of $F_{X/Y}$ commutes with base change on Y; that is, for all $Z\to Y$, $F_{X\times_Y Z/Z}\colon X\times_Y Z\to(X\times_Y Z)^{p/Z}=X^{p/Y}\times_Y Z$ coincides with $F_{X/Y}\times_Y\mathrm{id}_Z$.

If $g\colon Y\to Z$ is another morphism of fibred categories over S, we have $F_{X/Z}=(f^{p/Z})^*F_{Y/Z}\circ F_{X/Y}$ as one can see from the diagram with cartesian squares

$$\begin{tikzcd}
X \arrow[r, "F_{X/Y}"] \arrow[rr, bend left, "F_{X/Z}"] & X^{p/Y} \arrow[r, "(f^{p/Z})^*F_{Y/Z}"] \arrow[d] \arrow[dr, phantom, "\square"] & X^{p/Z} \arrow[r] \arrow[d, "f^{p/Z}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& Y \arrow[r, "F_{Y/Z}"'] & Y^{p/Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
& & Z \arrow[r, "F_Z"'] & Z
\end{tikzcd}$$

Using these remarks, one checks the following facts:

(i) Perfectness is stable by base change and, if X and Y are algebraic stacks, then perfectness of f is local on Y for the fppf topology.

(ii) Perfectness is stable by composition.

(iii) Morphisms between fibred categories perfect over Z are perfect.

(iv) If $X \to Y$, $X \to Z$ are perfect and $f^{p/Z}$ descends isomorphisms (eg f is a perfect and faithfully flat quasicompact morphism of algebraic stacks), then $Y \to Z$ is perfect.

(v) If $X \to Y$ is represented by a perfect morphism of algebraic spaces, then it is perfect.

Finally, if $X \to S$ is an algebraic stack, we point out two features of the absolute Frobenius and relative Frobenius. Firstly, they are universal homeomorphisms, and, secondly, they are in general not representable (by algebraic spaces). For example, if $X = B\mu_p$ is the classifying stack of the scheme of p^{th} roots of unity, all the maps $\mathrm{Aut}(x) \to \mathrm{Aut}(F_{X/S}(x))$ are trivial. This follows easily using the fact that, if we denote by $I_X = X \times_{\Delta, X\times_S X, \Delta} X$ the inertia stack, then the Frobenius of inertia

$$F_{I_X/S} \colon I_X \to (I_X)^{p/S} = I_{X^{p/S}}$$

is canonically isomorphic to the map induced by $F_{X/S} \colon X \to X^{p/S}$ on the level of inertia.

2.2 Base restriction

For the sake of simplicity, let us come back to algebraic spaces. Let $f \colon S' \to S$ be a morphism of algebraic spaces. The *base restriction along* f is the functor $X' \mapsto f_! X'$ that sends an S'–algebraic space X' to the S–algebraic space $X' \to S' \to S$. It is left adjoint to the pullback f^* and should not be confused with the Weil restriction functor f_*, which is right adjoint to f^*. We will need to use the fact that coperfection commutes with base restriction. This is a consequence of the simple categorical fact that, if two functors commute and have left adjoints, then the left adjoints commute. Here is a precise statement in our context.

2.2.1 Lemma *Let X, T and S be $\mathbb{F}_p$–algebraic spaces. Let $f \colon T \to S$ be a morphism which is relatively perfect, and $X \to T$ a morphism which admits a coperfection X^{copf}. Then $f_!(X^{\mathrm{copf}})$ is a coperfection for $f_! X$. Thus, we obtain an isomorphism*

$$f_!(X^{\mathrm{copf}}) \xrightarrow{\sim} (f_! X)^{\mathrm{copf}}.$$

Proof Let $\mathbf{Sp}_S$ be the category of S–algebraic spaces and $i_S\colon \mathbf{Perf}_S \to \mathbf{Sp}_S$ the inclusion of perfect objects. Since $f:T\to S$ is relatively perfect and relatively perfect morphisms are stable by composition, the functor $f_!$ maps $\mathbf{Perf}_T$ into $\mathbf{Perf}_S$, that is, it commutes with i_S and i_T. Similarly, f^* maps $\mathbf{Perf}_S$ into $\mathbf{Perf}_T$. For each $Y\in\mathbf{Perf}_S$, we have canonical bijections

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sp}_S}(f_!X, i_SY) &= \mathrm{Hom}_{\mathbf{Sp}_T}(X, f^*i_SY)\\ &= \mathrm{Hom}_{\mathbf{Sp}_T}(X, i_Tf^*Y)\\ &= \mathrm{Hom}_{\mathbf{Perf}_T}(X^{\mathrm{copf}}, f^*Y)\\ &= \mathrm{Hom}_{\mathbf{Perf}_S}(f_!X^{\mathrm{copf}}, Y).\end{aligned}$$

This shows that $f_!X^{\mathrm{copf}}$ is the coperfection of $f_!X$. □

The same result holds, with the same proof, for pairs of commuting adjoints in similar situations. For example, it holds for the inclusion of quasicompact étale algebraic spaces in the category of faithfully flat, finitely presented, separable algebraic stacks; there, the left adjoint "étalification" functor is given by the functor of connected components π_0, which we will review in Section 3.

2.3 The case of stacks; F–divided objects

In this section we describe concretely the perfection and coperfection of fppf stacks over S and highlight some properties. As we said in the introduction, all sheaves and stacks are considered for the fppf topology, so most of the time we omit the adjective.

2.3.1 Coperfection of stacks Let $\mathscr{X}$ be a stack over S. We let

$$\mathscr{X}^{\mathrm{copf}/S} = \mathrm{colim}(\mathscr{X} \xrightarrow{F_0} \mathscr{X}^{p/S} \xrightarrow{F_1} \mathscr{X}^{p^2/S} \to \cdots)$$

be the colimit in the 2–category of stacks. The inductive system being filtered, the prestack colimit satisfies the stack property for coverings of affine schemes $\mathrm{Spec}(A')\to\mathrm{Spec}(A)$, and its Zariski stackification is an fppf stack, and hence is the fppf stackification. One checks the following facts:

(i) $\mathscr{X}^{\mathrm{copf}/S}$ is perfect and is a coperfection of $\mathscr{X}$ in the 2–category of S–stacks.

(ii) The formation of $\mathscr{X}^{\mathrm{copf}/S}$ commutes with all base changes $S'\to S$ and is functorial in $\mathscr{X}$.

(iii) $\mathscr{X}^{\mathrm{copf}/S}$ is locally of finite presentation (that is, limit-preserving) if $\mathscr{X}$ is.

(iv) If $\mathscr{X}$ is an algebraic stack, then $\mathscr{X}^{\mathrm{copf}}$ is far from algebraic in general. For example if $\mathscr{X}$ is the affine line over $\mathbb{F}_p$ then for an $\mathbb{F}_p$–algebra A, the set $\mathscr{X}^{\mathrm{copf}}(A)$ is equal to $A^{\mathrm{copf}/\mathbb{F}_p}$, the absolute coperfection of A. In particular, for $A = \mathbb{F}_p[\![t]\!]$ the set $\mathscr{X}^{\mathrm{copf}}(A) = \mathbb{F}_p[\![t^{p^{-\infty}}]\!]$ is much bigger than $\lim \mathscr{X}^{\mathrm{copf}}(A/t^n) = \mathbb{F}_p$. This violates the effectivity of formal objects, a necessary condition for algebraicity; see the Stacks project [28, Tag 07X8].

2.3.2 Perfection of stacks; *F*–divided objects Let $\mathscr{M}$ be a stack over S. For each $i \geqslant 0$, let $F^i_{S,*}$ be the Weil restriction along the i^{th} absolute Frobenius of S, and

$$G_i : F^{i+1}_{S,*}\mathscr{M} \to F^i_{S,*}\mathscr{M}$$

the morphism which maps a T–valued object $x \in \mathscr{M}(T^{p^{i+1}/S}) = (F^{i+1}_{S,*}\mathscr{M})(T)$ to the pullback

$$G_i(x) := F^*_{T^{p^i/S}/S}x$$

under the Frobenius $F = F_{T^{p^i/S}/S} : T^{p^i/S} \to T^{p^{i+1}/S}$. Then we define

$$\mathscr{M}^{\mathrm{pf}/S} = \lim(\cdots \to F^2_{S,*}\mathscr{M} \xrightarrow{G_1} F_{S,*}\mathscr{M} \xrightarrow{G_0} \mathscr{M}),$$

the limit being taken in the 2–category of stacks. One has the following facts:

(i) $\mathscr{M}^{\mathrm{pf}/S}$ is perfect and is a perfection of $\mathscr{X}$ in the 2–category of S–stacks.

(ii) The formation of $\mathscr{M}^{\mathrm{pf}/S}$ commutes with all *perfect* base changes $S' \to S$ and is functorial in $\mathscr{M}$.

(iii) $\mathscr{M}^{\mathrm{pf}/S}$ is not locally of finite presentation in general, even if $\mathscr{M}$ is.

(iv) Assume that $F_S : S \to S$ is finite locally free. If $\mathscr{M}$ is a scheme, it is proven in Kato [19, Proposition 1.4] that $\mathscr{M}^{\mathrm{pf}/S}$ is a scheme and the morphism $\mathscr{M}^{\mathrm{pf}} \to \mathscr{M}$ is affine. We extend this to Deligne–Mumford stacks in Corollary 2.4.2 below. For Artin stacks, one can prove by the same arguments that the diagonal of $\mathscr{M}^{\mathrm{pf}/S}$ is representable by algebraic spaces, but in general it is not locally of finite type and $\mathscr{M}^{\mathrm{pf}/S}$ is not algebraic. For instance, in the case of $\mathscr{M} = B\mathbb{G}_m$ over $S = \mathrm{Spec}(\mathbb{F}_p)$, the diagonal is a torsor under $\mu_{p^\infty} = \lim \mu_{p^i}$. Finally, if F_S is not finite locally free, then already the diagonal may fail to be representable.

(v) If $\mathscr{M}' \to \mathscr{M}$ is perfect, the natural morphism $\mathscr{M}'^{\mathrm{pf}/S} \to \mathscr{M}^{\mathrm{pf}/S} \times_{\mathscr{M}} \mathscr{M}'$ is an isomorphism of stacks.

(vi) Perfection preserves fibre products.

2.3.3 Remark For arbitrary S–stacks $\mathscr{X}$ and $\mathscr{M}$, we have canonical equivalences

$$\mathsf{Hom}(\mathscr{X},\mathscr{M}^{\mathrm{pf}})=\mathsf{Hom}(\mathscr{X}^{\mathrm{copf}},\mathscr{M}^{\mathrm{pf}})=\mathsf{Hom}(\mathscr{X}^{\mathrm{copf}},\mathscr{M}).$$

(These can be promoted to isomorphisms of stacks on the perfect-étale site of S introduced in Section 5.1.) This equality is what explains the dual interpretation of our result embodied by Theorems A and A′. Indeed, assume we have a satisfactory understanding of the above object as a bifunctor in $(\mathscr{X},\mathscr{M})$. Then, letting $\mathscr{X}$ vary, we obtain a description of the perfection of $\mathscr{M}$, while, letting $\mathscr{M}$ vary, we obtain a description of the coperfection of $\mathscr{X}$. Going still further, since $T^{\mathrm{copf}}=\operatorname{colim} T^{p^i/S}$, we have

$$\begin{aligned}\mathscr{M}^{\mathrm{pf}}(T)&=\mathsf{Hom}(\operatorname{colim} T^{p^i/S},\mathscr{M})=\lim \mathsf{Hom}(T^{p^i/S},\mathscr{M})\\&=\lim \mathsf{Hom}(T,F^i_{S,*}\mathscr{M})=\mathsf{Hom}(T,\lim F^i_{S,*}\mathscr{M})=(\lim F^i_{S,*}\mathscr{M})(T).\end{aligned}$$

This shows that, once we know coperfection in the 2–category of stacks, the construction of the perfection is forced upon us.

The points of the stack $\mathscr{M}^{\mathrm{pf}/S}$ are exactly the F–divided objects of $\mathscr{M}$. We want to give the latter an existence of their own, independent of the adjointness property.

2.3.4 Definition We denote by $\mathrm{Fdiv}_S(\mathscr{M})$ the stack described as follows:

(1) An *F–divided object of $\mathscr{M}$* over an S–scheme T is a collection of pairs $(x_i,\sigma_i)_{i\geqslant 0}$ where $x_i\in\mathscr{M}(T^{p^i/S})$ and $\sigma_i\colon x_i\to F^*x_{i+1}$ is an isomorphism; here $F=F_{T^{p^i/S}/S}\colon T^{p^i/S}\to T^{p^{i+1}/S}$ is Frobenius.

(2) A morphism between $(x_i,\sigma_i)_{i\geqslant 0}$ and $(y_i,\tau_i)_{i\geqslant 0}$ is a collection of morphisms $u_i\colon x_i\to y_i$ such that $\tau_i\circ u_i=F^*u_{i+1}\circ\sigma_i$ for all $i\geqslant 0$.

To make things clear, $\mathrm{Fdiv}_S(\mathscr{M})$ and $\mathscr{M}^{\mathrm{pf}/S}$ are really two names for the same object.

2.3.5 Remark In most of the existing literature, eg [26; 29], $\mathrm{Fdiv}(\mathcal{Z})$ is used for the category of F–divided vector bundles on $\mathcal{Z}$. Tonini and Zhang [29, Definition 6.20] extend the notation to the effect that $\mathrm{Fdiv}(\mathcal{Z},\mathcal{Y})$ denotes the category of F–divided objects of a stack $\mathcal{Y}$ over the base $\mathcal{Z}$. Our emphasis is on the stack where divided objects take their values rather than the base that supports them. We are therefore led to drop $\mathcal{Z}$ from the notation, so that our $\mathrm{Fdiv}(\mathscr{M})$ is Tonini and Zhang's $\mathrm{Fdiv}(-,\mathscr{M})$. We warn the reader that, as a result, the notation $\mathrm{Fdiv}(\mathscr{M})$ does not have the same meaning in both works. Writing **Vect** for the stack of vector bundles, the following table gives a summary of the correspondence of notation:

here	in [29]
$\mathrm{Fdiv}(\mathscr{M})$	$\mathrm{Fdiv}(-,\mathscr{M})$
$\mathscr{M}^{\mathrm{pf}/S}$	$\mathrm{Fdiv}(-,\mathscr{M})$
$\mathrm{Fdiv}(\mathbf{Vect})(T)$	$\mathrm{Fdiv}(T)$
$\mathscr{X}^{\mathrm{copf}/S}$	$\mathscr{X}^{(\infty,S)}$

We end this subsection with a lemma which is a consequence of the fact that the diagonal of the perfection is the perfection of the diagonal. This will be useful in Section 5.

2.3.6 Lemma *Let S be an algebraic space of characteristic p and $\mathscr{Y}\to S$ a perfect stack. Let $\mathscr{M}$ be a stack and $f\colon \mathrm{Fdiv}_S(\mathscr{M})\to\mathscr{M}$ the perfection morphism. Let $x,y\colon\mathscr{Y}\to\mathrm{Fdiv}_S(\mathscr{M})$ be two morphisms, and write $x_0,y_0\colon\mathscr{Y}\to\mathscr{M}$ for the compositions fx and fy. Then there is an isomorphism of stacks*

$$\mathscr{H}om(x,y)\xrightarrow{\sim}\mathrm{Fdiv}_S(\mathscr{H}om(x_0,y_0))$$

identifying the morphism

$$\mathscr{H}om(x,y)\to\mathscr{H}om(x_0,y_0)$$

with the S–perfection morphism.

Proof As $\mathscr{M}^{\mathrm{pf}/S}=\mathrm{Fdiv}_S(\mathscr{M})$ is defined as a limit, the formation of Fdiv_S commutes with products, and the natural equivalence

$$\mathrm{Fdiv}_S(\mathscr{M})\times_S\mathrm{Fdiv}_S(\mathscr{M})\xrightarrow{\sim}\mathrm{Fdiv}_S(\mathscr{M}\times_S\mathscr{M})$$

identifies the diagonal $\Delta_{\mathrm{Fdiv}_S(\mathscr{M})}$ with $\mathrm{Fdiv}_S(\Delta_{\mathscr{M}})$. We have a 2–cartesian diagram of stacks on S,

$$\begin{tikzcd}
\mathscr{H}om(x,y) \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "{(x,y)}"] \\
\mathrm{Fdiv}_S(\mathscr{M}) \arrow[r, "\Delta"] & \mathrm{Fdiv}_S(\mathscr{M})\times_S\mathrm{Fdiv}_S(\mathscr{M})
\end{tikzcd}$$

Because $\mathscr{Y} \to S$ is perfect, the morphism $\mathrm{Fdiv}_S(\mathscr{Y}) \to \mathscr{Y}$ is an isomorphism of stacks. Applying Fdiv_S to the 2–cartesian diagram

$$\begin{tikzcd} \mathscr{H}om(x_0, y_0) \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "{(x_0,y_0)}"] \\ \mathscr{M} \arrow[r, "\Delta"] & \mathscr{M} \times_S \mathscr{M} \end{tikzcd}$$

we obtain the desired isomorphism $\mathscr{H}om(x, y) \xrightarrow{\sim} \mathrm{Fdiv}_S(\mathscr{H}om(x_0, y_0))$. □

2.4 Perfect and étale algebraic stacks

As in [28, Tag 0CIK], we call a morphism of algebraic stacks $\mathscr{X} \to \mathscr{Y}$ *étale* if it is relatively Deligne–Mumford and there exist smooth surjective morphisms $V \to \mathscr{Y}$ and $U \to \mathscr{X} \times_{\mathscr{Y}} V$ such that $U \to V$ is an étale map of algebraic spaces (or schemes). In the following lemma we discuss the relations between perfect and étale morphisms $\mathscr{X} \to \mathscr{Y}$. For simplicity we stick to the case where $\mathscr{Y}$ is an algebraic space, but using a smooth atlas by a scheme allows us to extend the results immediately to the case of an arbitrary $\mathscr{Y}$.

2.4.1 Lemma *Let $\mathscr{X}$ be an algebraic stack over S. Consider the following conditions*:

(1) *$\mathscr{X}$ is étale over S.*
(2) *$\mathscr{X}$ is an étale gerbe over an étale S–algebraic space.*
(3) *$\mathscr{X}$ is an étale gerbe over a perfect S–algebraic space.*
(4) *$\mathscr{X}$ is perfect over S.*
(5) *There exists an étale*, *surjective morphism $U \to \mathscr{X}$ from a perfect S–algebraic space.*
(6) *$\mathscr{X}$ is formally étale over S.*

Then we have the implications (*loc. f.p. means locally finitely presented*)

$$\begin{tikzcd} (1) \arrow[r, Leftrightarrow] & (2) \arrow[r, Rightarrow] & (3) \arrow[r, Rightarrow] & (4) \arrow[l, Rightarrow, bend right, "\text{if } \Delta_{\mathscr{X}/S} \text{ loc. f.p.}"'] \arrow[r, Leftrightarrow] & (5) \arrow[r, Rightarrow] & (6) \arrow[lllll, Rightarrow, bend left, "\text{if } \mathscr{X}/S \text{ loc. f.p.}"] \end{tikzcd}$$

In particular, *all perfect algebraic stacks are Deligne–Mumford*, *and all perfect*, *locally finitely presented algebraic stacks are étale.*

One can obtain examples of perfect algebraic stacks that do not satisfy (3) starting from a positive-dimensional scheme X over a perfect field k with a nonfree, nontrivial action of a finite group G and letting $\mathscr{X} = [X^{\mathrm{pf}/k}/G]$. For instance, one can take $X = \mathbb{A}^1_k$ with the standard action of μ_p.

Proof We use the facts collected in Section 2.1.4 without explicit mention.

(1) $\Longrightarrow$ (2) If $\mathscr{X}$ is étale then its diagonal is étale; see [28, Tag 0CJ1]. It follows that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is étale and therefore there is an algebraic space X and an étale gerbe morphism $\mathscr{X} \to X$; see [28, Tag 06QJ]. Since $\mathscr{X} \to S$ is étale and surjective, it follows that also $X \to S$ is étale.

(2) $\Longrightarrow$ (1) This is clear.

(1) $\Longrightarrow$ (4) Let $U \to \mathscr{X}$ be an étale atlas by a scheme, and let $R = U \times_{\mathscr{X}} U$. Then R and U are algebraic spaces which are étale, and hence perfect over S. We have a diagram of presentations

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F_{R/S}"'] & U \arrow[r] \arrow[d, "F_{U/S}"] & \mathscr{X} \arrow[d, "F_{\mathscr{X}/S}"] \\ R^{p/S} \arrow[r, shift left] \arrow[r, shift right] & U^{p/S} \arrow[r] & \mathscr{X}^{p/S} \end{tikzcd}$$

Then $F_{R/S}$ and $F_{U/S}$ are isomorphisms, so also $F_{\mathscr{X}/S}$ is an isomorphism.

(2) $\Longrightarrow$ (3) This follows from the implication (1) $\Longrightarrow$ (4).

(3) $\Longrightarrow$ (4) According to the already-proven implication (1) $\Longrightarrow$ (4), $\mathscr{X}$ is perfect.

(4) $\Longrightarrow$ (6) Let $i\colon S_0 \hookrightarrow S_1$ be a closed immersion of affine schemes with square-zero ideal sheaf. We want to prove that the functor $\mathscr{X}(S_1) \to \mathscr{X}(S_0)$ is an equivalence. Base changing $\mathscr{X} \to S$ along $S_1 \to S$, we can assume that $S_1 = S$. We know that $F_S\colon S \to S$ factors through a morphism $f\colon S \to S_0$ in such a way that $i \circ f = F_S$ and $f \circ i = F_{S_0}$. Applying $\mathscr{X}$, we obtain the composition $\mathscr{X}(S) \to \mathscr{X}(S_0) \to \mathscr{X}^{p/S}(S)$, which, by the perfectness assumption, is an equivalence. To conclude, it is enough to prove that $\mathscr{X}(S) \to \mathscr{X}(S_0)$ is full and essentially surjective. Fullness follows if we show that $\mathscr{X}(S_0) \to \mathscr{X}^{p/S}(S)$ is faithful. As the composition $\mathscr{X}(S_0) \to \mathscr{X}^{p/S}(S) \to \mathscr{X}^{p/S}(S_0)$ is an equivalence, the first map is indeed faithful. Now, for essential surjectivity, let $a_0\colon S_0 \to \mathscr{X}$ be an S_0–valued point; thus $s \circ a_0 = i$, where $s\colon \mathscr{X} \to S$ is the structure map. By perfectness again, the square in the following diagram is

2–cartesian:

$$\begin{tikzcd} S \arrow[dr, dashed, "a"] \arrow[drr, bend left, "a_0\circ f"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & \mathscr{X} \arrow[r, "F_{\mathscr{X}}"] \arrow[d, "s"] & \mathscr{X} \arrow[d, "s"] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

We deduce the existence of a filling arrow $a\colon S \to \mathscr{X}$. Moreover,

$$F_{\mathscr{X}} \circ a \circ i \simeq a_0 \circ f \circ i = a_0 \circ F_{S_0} \simeq F_{\mathscr{X}} \circ a_0, \quad s \circ a \circ i = i = s \circ a_0,$$

which, by 2–uniqueness, implies that $a \circ i \simeq a_0$. This means that a is the desired lifting.

(4) $\Longrightarrow$ (5) If $\mathscr{X} \to S$ is perfect, then so is $\mathscr{X} \times_S \mathscr{X} \to S$ and hence also the diagonal $\Delta_{\mathscr{X}/S}$. By the implication (4) $\Longrightarrow$ (6) we see that $\Delta_{\mathscr{X}/S}$ is formally unramified. Being locally of finite type [28, Tag 04XS], it is unramified in the sense of [22; 28]. It follows that $\mathscr{X}$ is Deligne–Mumford [28, Tag 06N3]. Let $U \to \mathscr{X}$ be an étale surjective morphism from an algebraic space; then $U \to \mathscr{X}$ is perfect and it follows that $U \to S$ is perfect.

(5) $\Longrightarrow$ (4) By Section 2.1.4(iv), if U is perfect and $U \to \mathscr{X}$ is étale surjective then $\mathscr{X}$ is perfect.

(4) $\Longrightarrow$ (3) Assuming $\Delta_{\mathscr{X}/S}$ locally of finite presentation. In this case, $\Delta_{\mathscr{X}/S}$ is formally étale, and hence étale. It follows that the inertia stack $I_{\mathscr{X}} \to \mathscr{X}$ is étale. Similarly as before, there is an algebraic space X and an étale gerbe morphism $\mathscr{X} \to X$.

(6) $\Longrightarrow$ (1) assuming $\mathscr{X} \to S$ locally of finite presentation. Since $\mathscr{X} \to S$ is assumed formally étale, it is formally unramified. It follows automatically that the diagonal $\Delta_{\mathscr{X}/S}$ is formally unramified, thus $\mathscr{X}$ is a Deligne–Mumford stack. Let $U \to \mathscr{X}$ be an étale atlas from a scheme. The morphism of algebraic spaces $U \to S$ is formally étale and locally of finite presentation, and hence étale. Then $\mathscr{X} \to S$ is étale as well. □

These remarks have consequences for Deligne–Mumford stacks, because these have étale — and hence perfect — atlases.

2.4.2 Corollary *Assume that $F_S\colon S \to S$ is finite locally free. If $\mathscr{M}$ is a Deligne–Mumford stack, then the stack perfection $\mathscr{M}^{\mathrm{pf}/S}$ is a Deligne–Mumford stack and the morphism $\mathscr{M}^{\mathrm{pf}/S} \to \mathscr{M}$ is affine.*

Proof Let $U \to \mathscr{M}$ be an étale atlas from a scheme U. Then $U \to \mathscr{M}$ is perfect and it follows from Section 2.3.2(v) that the following square is cartesian:

$$\begin{array}{ccc} U^{\mathrm{pf}} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathscr{M}^{\mathrm{pf}} & \longrightarrow & \mathscr{M} \end{array}$$

Thus $U^{\mathrm{pf}} \to \mathscr{M}^{\mathrm{pf}}$ is representable, surjective and étale. It follows from [28, Tag 06DC] that $\mathscr{M}^{\mathrm{pf}}$ is a Deligne–Mumford stack. Since moreover $U^{\mathrm{pf}} \to U$ is an affine morphism of schemes (see Section 2.3.2(iv)), by descent, also $\mathscr{M}^{\mathrm{pf}/S} \to \mathscr{M}$ is affine. □

3 Étale hulls and connected components

In this section, we provide some complements on the functor π_0 introduced in [23] and some of its variants. Although these results hold for algebraic stacks, we restrict most of the time to algebraic spaces because this simplifies the treatment a little and is enough for our needs.

There are two viewpoints on the functor π_0, and we consider both. Firstly π_0 is a left adjoint to the inclusion of the category of étale finitely presented spaces in the category of flat, finitely presented, separable spaces. In the study of such "étalification" functors, Ferrand [7; 8] recently highlighted the importance of the category of factorizations $X \to E \to S$, where the second arrow is étale. He proved that, when the base S has finitely many irreducible components, there is a left adjoint π^s to the inclusion of étale, separated spaces into all flat, finitely presented spaces. In Section 3.2 we prove that the category of factorizations as well as some interesting subcategories satisfy topological invariance (in the sense of SGA 4_2 [15, théorème 1.1]). Then we prove that, when S is noetherian, geometrically unibranch and without embedded points, there is a left adjoint π^a to the inclusion of étale, affine spaces into all flat, finitely presented spaces. In Section 3.3 we compare π^a with the affine hull of π_0.

Secondly, π_0 is the functor of connected components of a relative space. In Section 3.4 we describe ways to compute $\pi_0(X/S)$ by using an atlas of X, or completing along a closed fibre of $X \to S$.

We sometimes impose some finiteness or regularity assumptions on the base S, but nothing on the characteristics; it is only in later sections that we specialize to characteristic p.

3.1 A summary of properties of the functor π_0

We recall the main properties of the functor $\pi_0(\mathscr{X}/S)$ which to $T \to S$ associates the set of open relative connected components of $\mathscr{X}_T \to T$.

3.1.1 Lemma *Let S be an algebraic space and $\mathscr{X} \to S$ a flat, finitely presented algebraic stack.*

(1) *The functor $\pi_0(\mathscr{X}/S)$ is a sheaf on the big étale site of S, locally of finite presentation, formally étale, quasicompact, with representable, open quasicompact diagonal. Its formation commutes with base change on S. Its restriction to the small étale site is a constructible sheaf.*

If moreover $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable, the following hold:

(2) *The sheaf $\pi_0(\mathscr{X}/S)$ is an étale, finitely presented algebraic space. It can be constructed as the quotient of $\mathscr{X}$ by the open equivalence relation $\mathscr{R} \subset \mathscr{X} \times_S \mathscr{X}$ defined by the relative connected component of the diagonal of $\mathscr{X}$.*

(3) *The morphism $\mathscr{X} \to \pi_0(\mathscr{X}/S) = \mathscr{X}/\mathscr{R}$ is faithfully flat, finitely presented and realizes $\mathscr{X}$ as the universal connected component inside $\mathscr{X} \times_S \pi_0(\mathscr{X}/S)$.*

(4) *The functor $\pi_0(-/S)$ is left adjoint to the inclusion of the category of étale finitely presented S–algebraic spaces in the category of flat, finitely presented, separable S–algebraic stacks.*

(5) *The functor $\pi_0(-/S)$ commutes with products.*

Proof This is essentially all proven in [23], to which we refer:

(1) This is found in the end of définition 2.1.1 and lemmes 2.1.2 and 2.1.3.

(2)–(4) When $\mathscr{X} \to S$ is separable, these assertions are found in théorème 2.5.2 and corollaire 2.6.2 (note that quasiseparation of $\pi_0(\mathscr{X}/S)$ was forgotten in that statement but is in lemme 2.1.2). When $\dim(S) = 0$, representability follows from lemme 2.1.3 and the remaining assertions follow easily using the topological nature of $\pi_0(\mathscr{X}/S)$.

(5) This is because the natural morphism $\pi_0(\mathscr{X} \times_S \mathscr{Y}/S) \to \pi_0(\mathscr{X}/S) \times_S \pi_0(\mathscr{Y}/S)$ is an isomorphism in the geometric fibres. All spaces in place being étale, this settles the question. □

3.2 Étale affine hulls and largest étale subalgebras

Let us briefly recall what is known on étale hulls, also called étalification functors. Consider the diagram of fully faithful subcategories of the category of S–spaces (“fp” stands for finitely presented)

$$\begin{tikzcd}
\mathbf{\acute{E}tAff}_S \arrow[r] & \mathbf{\acute{E}tSep}_S \arrow[r] & \mathbf{\acute{E}t}_S \arrow[r] & \mathbf{Spb}_S \arrow[r] \arrow[l, bend right, "\pi_0"'] & \mathbf{Flat}_S \arrow[llll, bend right, "\pi^a\ (S\text{ unibranch})"'] \arrow[lll, bend right, "\pi^s"'] \arrow[ll, bend right, "\exists?\pi"'] \\
\text{étale affine} & \text{étale fp separated} & \text{étale fp} & \text{flat fp separable} & \text{flat fp}
\end{tikzcd}$$

Here are some positive facts on the existence of these adjoints:

(i) π^s is constructed in [7] when S has finitely many irreducible components (and more generally in [8] when S is locally connected). It has no known moduli description. It has functoriality and base change properties available only in restricted cases. The morphism $X \to \pi^s(X/S)$ is surjective but its geometric fibres are usually not connected.

(ii) π^a is constructed is the present subsection when S is noetherian, geometrically unibranch, without embedded points. It shares the same features as those just listed for π^s, except that $X \to \pi^a(X/S)$ is schematically dominant [12, 11.10.2; 28, Tag 0CMH] but maybe not surjective.

Here are some negative facts:

(iii) π is not known to exist in general (note however that it exists when S is zero-dimensional, in which case $\pi = \pi_0$).

(iv) π_0 *does* extend naturally to a functor $\mathbf{Flat}_S \to \mathbf{\acute{E}t}_S$ but this is not a left adjoint to the inclusion $i : \mathbf{\acute{E}t}_S \to \mathbf{Flat}_S$. Indeed, Lemma 3.1.1(1) implies that, for all $X \in \mathbf{Flat}_S$, the functor $\pi_0(X/S)$ defines a constructible sheaf on $S_{\text{ét}}$, and hence an étale quasicompact algebraic space (which does *not* represent the functor $\pi_0(X/S)$ on all S–schemes!). Moreover, for each étale $E \to S$ there is a map $\mathrm{Hom}(X, E) \to \mathrm{Hom}(\pi_0(X/S), E)$. However, in general there is no map in the other direction; in particular, there is no morphism $X \to \pi_0(X/S)$ and this prevents π_0 from being an adjoint of i. For instance, let S be the spectrum of a discrete valuation ring R with fraction field K and let $X = \mathrm{Spec}(R[x]/(x^2 - \pi x))$. Then $\pi_0(X/S) \simeq \mathrm{Spec}(K) \sqcup \mathrm{Spec}(K)$ and the map $\pi_0(X/S) \to S$ is not even surjective.

We now start our investigations on π^a. To start with, we recall the definition of the category of factorizations from [8, 2.1.1]. In order to make Theorem 3.2.7 possible, we modify the definition slightly by relaxing the assumption of surjectivity.

3.2.1 Definition Let $X \to S$ be a morphism of algebraic spaces. The *category of factorizations* is the category $\mathsf{E}(X/S)$ whose objects are the factorizations $X \to E \to S$ such that $E \to S$ is étale, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} & E_1 \arrow[d] \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & \downarrow & S \\ & E_2 \arrow[ur] & \end{tikzcd}$$

The category $\mathsf{E}^{\mathrm{surj}}(X/S)$ (resp. $\mathsf{E}^{\mathrm{dom}}(X/S)$) is the full subcategory of factorizations such that $X \to E$ is surjective (resp. schematically dominant). The category $\mathsf{E}^{\mathrm{sep}}(X/S)$ (resp. $\mathsf{E}^{\mathrm{aff}}(X/S)$) is the full subcategory of factorizations such that $E \to S$ is separated (resp. affine). We write $\mathsf{E}^{\mathrm{dom,aff}}(X/S) = \mathsf{E}^{\mathrm{aff}}(X/S) \cap \mathsf{E}^{\mathrm{dom}}(X/S)$ and similarly for other intersections.

We will often denote a factorization $X \to E \to S$ simply by using the letter E. We draw the attention of the reader to the fact that, for the subcategories $\mathsf{E}^{\sharp}(X/S)$ defined above, the property "$\sharp$" applies either to $E \to S$ or to $X \to S$, depending on the case.

In the following lemma, we use the notions of associated and embedded points. For a review of these notions in the context of algebraic spaces and stacks, we refer to [28, Tag 0CTV] or [23, définition A.2.4].

3.2.2 Lemma *Let $X \to S$ be a morphism of algebraic spaces. Let $f : S' \to S$ be a morphism of spaces which is integral, radicial and surjective. Let $X' = X \times_S S'$.*

(1) *The pullback functor $f^* : \mathsf{E}(X/S) \to \mathsf{E}(X'/S')$ is an equivalence which preserves the subcategories $\mathsf{E}^{\mathrm{sep}}$, $\mathsf{E}^{\mathrm{aff}}$ and $\mathsf{E}^{\mathrm{surj}}$.*

(2) *If moreover S and S' are locally noetherian, f induces a bijection $\mathrm{Emb}(S') \to \mathrm{Emb}(S)$ of embedded points, and $X \to S$ is faithfully flat, then f^* preserves also the subcategory $\mathsf{E}^{\mathrm{dom}}$.*

Proof First, we recall basic facts on the topological invariance of the étale site. Let $f : S' \to S$ be a morphism of algebraic spaces which is integral, radicial and surjective. Then the pullback functor f^* induces an equivalence between the category of étale S–spaces and the category of étale S'–spaces; see [15] for schemes and [28, Tag 05ZG] for spaces. This equivalence preserves affine objects; see [28, Tag 07VW].

(1) We prove that f^* is essentially surjective. Let $X' \to E' \to S'$ be a factorization. By topological invariance of the étale site, there exists an essentially unique $E \to S$ such that $E' \simeq E \times_S S'$. In order to descend $u' \colon X' \to E'$ to a morphism $u \colon X \to E$, by descent of morphisms to an étale scheme along universal submersions [16, Exposé IX, proposition 3.2] it is enough to prove that $\mathrm{pr}_1^* u' = \mathrm{pr}_2^* u'$, where $\mathrm{pr}_1, \mathrm{pr}_2 \colon S' \times_S S' \to S'$ are the projections. By [16, Exposé IX, proposition 3.1] it is enough to find a surjective morphism $g \colon S''' \to S' \times_S S'$ such that the two maps agree after base change along g. We can take $S''' = S'$ and g the diagonal map. This proves essential surjectivity; we leave full faithfulness to the reader. We now prove that f^* preserves the indicated subcategories. Since the diagonal of $E \to S$ is a closed immersion if and only if the diagonal of $E' \to S'$ is a closed immersion, we see that f^* preserves $\mathsf{E}^{\mathrm{sep}}(X/S)$. The fact that f^* preserves $\mathsf{E}^{\mathrm{aff}}$ was recalled above. Finally, f^* preserves $\mathsf{E}^{\mathrm{surj}}$ because f is a universal homeomorphism.

(2) Here the morphisms $X \to E$ in the factorizations are automatically flat. Thus such a morphism is schematically dominant if and only if its image contains the set of associated points $\mathrm{Ass}(E)$ [28, Tags 0CUP and 0CTX]. Since $\mathrm{Ass}(E) = \bigcup_{s \in \mathrm{Ass}(S)} E_s$ by EGA IV_2 [11, 3.3.1], we see that $X \to E$ is schematically dominant if and only if the image of $X \to E$ contains all fibres E_s with $s \in \mathrm{Ass}(S)$. But f induces a bijection of the nonembedded associated points since it is a homeomorphism, and a bijection on embedded points by assumption. Hence it is equivalent to say that the image of $X' \to E'$ contains all fibres $E'_{s'}$ with $s' \in \mathrm{Ass}(S')$. ☐

3.2.3 Suprema and infima We say that E_1 and E_2 have a *supremum* if the category of factorizations E mapping to E_1 and E_2 has a terminal element. We say that E_1 and E_2 have an *infimum* if the category of factorizations E receiving maps from E_1 and E_2 has an initial element. In pictures,

$$
\begin{tikzcd}
 & & & E_1 \arrow[dr] & \\
X \arrow[r] & E \arrow[r, dotted] \arrow[urr, bend left] \arrow[drr, bend right] & \sup(E_1, E_2) \arrow[ur] \arrow[dr] & & S \\
 & & & E_2 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & E_1 \arrow[dr] \arrow[drr, bend left] & & & \\
X \arrow[ur] \arrow[dr] & & \inf(E_1, E_2) \arrow[r, dotted] & E \arrow[r] & S \\
 & E_2 \arrow[ur] \arrow[urr, bend right] & & &
\end{tikzcd}
$$

Note that, in the three categories $\mathsf{E}^{\mathrm{surj}}(X/S)$, $\mathsf{E}^{\mathrm{dom,sep}}(X/S)$ and $\mathsf{E}^{\mathrm{dom,aff}}(X/S)$, if there is a morphism between E_1 and E_2 then it is unique. In other words, these categories really are posets.

3.2.4 Corollary *Let* $\mathsf{E}^\sharp(X/S) \subset \mathsf{E}(X/S)$ *be any subcategory with*

$$\sharp \in \{\varnothing, \mathrm{sep}, \mathrm{aff}, \mathrm{surj}, \mathrm{dom}\}.$$

Let $f : S' \to S$ *be a morphism of spaces which is integral, radicial and surjective. When* $\sharp = \mathrm{dom}$, *assume moreover that* f *and* X *satisfy the assumptions of Lemma 3.2.2*(2). *Then the following hold*:

(1) $\mathsf{E}^\sharp(X/S)$ *has an initial element if and only if* $\mathsf{E}^\sharp(X'/S')$ *has one.*

(2) *Let* E_1 *and* E_2 *be factorizations in* $\mathsf{E}^\sharp(X/S)$ *and* E_1' *and* E_2' *their images in* $\mathsf{E}^\sharp(X'/S')$. *Then* E_1 *and* E_2 *have a supremum* (*resp. an infimum*) *if and only if* E_1' *and* E_2' *have a supremum* (*resp. an infimum*).

Proof Suprema and infima are defined in terms of morphisms and are therefore preserved by the equivalences $f^* : \mathsf{E}^\sharp(X/S) \to \mathsf{E}^\sharp(X'/S')$. □

We arrive at the main existence result of this subsection. We prepare the proof with two lemmas.

3.2.5 Lemma *Let* $f : X \to S$ *be a morphism of algebraic spaces which is faithfully flat, quasicompact, separated and schematically birational. Then* f *is an isomorphism.*

Proof Let $U \subset S$ and $V \subset X$ be schematically dense opens mapped isomorphically via f. It is enough to prove the claim after fpqc base change on S. Hence we can base change by f itself and assume that f has a section $\epsilon : S \to X$ such that $\epsilon^{-1}(V)$ is schematically dense in X. Since f is separated, this is a closed immersion, so the comorphism $\epsilon^\sharp : \mathcal{O}_X \to \epsilon_*\mathcal{O}_S$ is surjective. Restricting to U and V, we see that $\epsilon^\sharp$ is also injective. It follows that ϵ is an isomorphism and f is its inverse. □

3.2.6 Lemma *Let* S *be a separated noetherian scheme and* $U \subset S$ *a nonempty dense open. Then the set of opens* V *containing* U *and such that* $V \to S$ *is affine is finite and has a smallest element for inclusion.*

Proof If V is such an open, the complement $S \smallsetminus V$ is included in $S \smallsetminus U$ and has pure codimension 1 in S by EGA IV_4 [13, 21.12.7]. This proves that $S \smallsetminus V$ is a union of codimension 1 irreducible components of $S \smallsetminus U$. Since these are finite in number, we see the set of interest is finite. Since S is separated, the intersection of all its elements is again S–affine and is the smallest element. □

3.2.7 Theorem *Let $f : X \to S$ be a faithfully flat, finitely presented morphism of algebraic spaces. Assume that S is noetherian, geometrically unibranch and without embedded points. Then the category $\mathsf{E}^{\mathrm{dom,aff}}(X/S)$ is a lattice; that is, any two objects have a supremum and an infimum. Moreover, $\mathsf{E}^{\mathrm{dom,aff}}(X/S)$ has a largest element.*

A similar statement holds in the category $\mathsf{E}^{\mathrm{surj,sep}}(X/S)$, where existence of suprema and maximum are due to Ferrand [8, théorème 2.3.4].

Proof Throughout the proof we write $\mathsf{E} = \mathsf{E}^{\mathrm{dom,aff}}(X/S)$. Note that, for each factorization $X \to E \to S$, the morphism $X \to E$ is flat and finitely presented because it is the composition of the open quasicompact immmersion $X \to X \times_S E$ and the projection $X \times_S E \to E$.

We start with the proof that any two factorizations $E_1, E_2 \in \mathsf{E}$ have a supremum. By topological invariance of the étale site (Lemma 3.2.2(2)), we can assume that S is reduced. Let E be the schematic image of the morphism $X \to E_1 \times_S E_2$. As a closed subscheme of $E_1 \times_S E_2$, it is affine and unramified over S. To prove that $E \in \mathsf{E}$, it remains to prove that it is flat. For this we may pass to an étale atlas of S and assume that S is a scheme. By the theorem on unramified morphisms over unibranch schemes [13, 18.10.1], it is enough to prove that, for each $e \in E$ with image $s \in S$, the map of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{E,e}$ is injective. To argue this, let $\eta_1, \dots, \eta_n$ be the generic points of the irreducible components of S containing s and let $\mathcal{O}_{E,\eta_i}$ be the semilocal rings of the fibres of $E \to S$ at η_i. We have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{S,s} \arrow[r] \arrow[d, hook] & \mathcal{O}_{E,e} \arrow[d] \\ \prod_{i=1}^n \mathcal{O}_{S,\eta_i} \arrow[r, hook] & \prod_{i=1}^n \mathcal{O}_{E,\eta_i} \end{tikzcd}$$

The left vertical map is injective. The horizontal map on the bottom is injective because the rings $\mathcal{O}_{S,\eta_i}$ are fields and $\mathcal{O}_{E,\eta_i} \neq 0$. Therefore $\mathcal{O}_{S,s} \to \mathcal{O}_{E,e}$ is injective and this concludes the argument.

Now we prove that there is a largest element. For each $E \in \mathsf{E}$, the image of $X \to E$ is an open subscheme $U \subset E$ that is étale, separated and quasicompact over S, which we call the "image" of the factorization E. It is determined by the scheme $R := X \times_E X = X \times_U X$ which is the graph in $X \times_S X$ of an open and closed equivalence relation; indeed, we recover U as the quotient algebraic space X/R. Because S is noetherian, the same is true for $X \times_S X$ and hence there are finitely many open and closed equivalence relations and finitely many "images" U. By the existence of suprema in E, the poset of "images" forms a directed finite set, hence it has a largest element.

We fix $E \in \mathsf{E}$ whose "image" U is largest. It is now enough to prove that the directed set of maps $u\colon F \to E$ in E has a largest element $u^{\max}\colon E^{\max} \to E$. Since E is a directed set, $E^{\max}$ will automatically be a largest element for it, concluding the proof.

Given a map $u\colon E' \to E$, we observe that there is an induced isomorphism $U' \simeq U$ between the "images". Moreover, $U \subset E$ and $U' \subset E'$ are schematically dense in E. It follows that the induced étale surjective separated morphism from E' onto its image $u(E') \subset E$ is birational, and hence an isomorphism by Lemma 3.2.5. That is, u is an open immersion. Since E' is affine over S, then so is $u(E')$; hence, Lemma 3.2.6 applied to the open $U \subset E$ implies that the directed set of maps $F \to E$ stabilizes, so eventually an $E^{\max}$ is achieved.

Finally, we construct an infimum for E_1 and E_2. Let E_0 be the pushout of the diagram $E_1 \leftarrow X \rightarrow E_2$, that is, the quotient of $E_1 \sqcup E_2$ by the étale equivalence relation that identifies the image of $X \to E_1$ and the image of $X \to E_2$. Let E be the largest element of the category $\mathsf{E}^{\mathrm{dom,aff}}(E_0/S)$. This is the infimum of E_1 and E_2. □

3.2.8 Definition With the notation and assumptions of Theorem 3.2.7, the largest element of the poset $\mathsf{E}^{\mathrm{dom,aff}}(X/S)$ is called the *étale affine hull of X/S* and denoted by $\pi^a(X/S)$. Its $\mathcal{O}_S$–sheaf of functions is called the *largest (quasicoherent) étale $\mathcal{O}_S$–subalgebra of $f_*\mathcal{O}_X$*.

3.2.9 Corollary *Let S be a noetherian geometrically unibranch scheme without embedded points.*

(1) *For each morphism $u\colon X \to Y$ between faithfully flat, finitely presented S–algebraic spaces, there is an induced morphism of étale affine hulls $\pi^a(X/S) \to \pi^a(Y/S)$.*

(2) *The functor π^a is left adjoint to the inclusion of the category of étale, affine S–schemes into the category of faithfully flat, finitely presented S–algebraic spaces.*

Proof (1) By topological invariance of the étale site (Lemma 3.2.2), we can assume that S is reduced. Let E be the schematic image of $X \to Y \to \pi^a(Y/S)$. It follows from the theorem on unramified morphisms over unibranch schemes [13, 18.10.1] that $E \to S$ is étale. By the definition of $\pi^a(X/S)$, we obtain a morphism $\pi^a(X/S) \to \pi^a(Y/S)$.

(2) Let $u\colon X \to E$ be an S–morphism from a faithfully flat, finitely presented space to an étale, affine scheme. By (1) there is an induced morphism $\pi^a(X/S) \to \pi^a(E/S)$. Since $E \to \pi^a(E/S)$ is an isomorphism, we obtain a morphism $\pi^a(X/S) \to E$. □

3.3 Properties of the étale affine hull

Like for most algebro-geometric constructions, the étale affine hull is easier to handle when one can count on good formal properties, like compatibility with localization, completion, various other base changes, formation of products, and so on. To start with, the nonconstructive existence proof given in the previous subsection can hardly be circumvented as long as one misses these properties. In this short subsection we initiate the investigation of these properties, which will certainly deserve further effort in the future. We note that, for the related étale separated hull, a sample of base change results is given in [7, sections 6–7].

We focus on the study of affine hulls (that is, spectra of sheaves of functions) of étale schemes and spaces, and its application to base change. Following the rule indicated in the introduction of this section, here we make no assumption on the residue characteristics; some of our results will be complemented in Section 4.4 in the characteristic p setting.

3.3.1 On Ferrand's results We occasionally need results from Ferrand's preprint [7], which contains some material cut down in the final version [8]. We also need to cite results in greater generality than stated in [7; 8]. Namely, Ferrand's results are stated for schemes but are valid for algebraic spaces, and his statements in [7, sections 6, 7 and 8; 8, sections 5 and 6] have the assumption that the base scheme is normal but are valid more generally if it is merely geometrically unibranch. To reach these generalizations, no modification of the proofs is required. Indeed, the use of schemes rather than algebraic spaces is essentially a convenience, and the fact that the correct assumption on the base scheme is geometric unibranchness is noted by Ferrand in [7, section 6.1] or [8, section 4.3.2]; see also [8, section 4.4.3]:

> La normalité elle-même n'intervient dans la suite que par les propriétés topologiques suivantes ....

Recall that a local ring is $\mathbb{Q}$*–factorial* if it is normal and its divisor class group is torsion (equivalently all Weil divisors of the spectrum are $\mathbb{Q}$–Cartier). A scheme is *locally* $\mathbb{Q}$*–factorial* if its local rings are $\mathbb{Q}$–factorial. A regular scheme is locally $\mathbb{Q}$–factorial.

3.3.2 Lemma *Let S be a locally noetherian algebraic space, all of whose étale atlases are locally $\mathbb{Q}$–factorial. Let $U \to S$ be an étale, separated, quasicompact morphism of schemes. Then the following conditions are equivalent:*

(1) *There exists an open immersion $U \hookrightarrow E$ with $E \to S$ étale and affine.*

(2) *The affine hull $U^{\mathrm{aff}} \to S$ is étale.*

Proof (1) $\Longrightarrow$ (2) The claim is étale-local on S so we may assume that S is an affine scheme. Let $Z = E \setminus U$ and write $Z = D \cup T$ as a union of a divisor D and a closed subscheme T of codimension at least 2. Since E is locally $\mathbb{Q}$–factorial by the assumption on S, the scheme $E \setminus D$ is affine, being the complement of a divisor; see Brenner [3]. Moreover, S is normal, and hence S_2 is too, and hence also E because it is étale over it. Since $\operatorname{codim}(T) \geqslant 2$, it follows that the restriction map $\mathcal{O}(E \setminus D) \to \mathcal{O}(E \setminus Z) = \mathcal{O}(U)$ is an isomorphism. This shows that the affine hull of U is $E \setminus D$, which is an open of E, and hence étale over S.

(2) $\Longrightarrow$ (1) Since $U \to S$ is quasifinite and separated, it is quasiaffine [13, 18.12.12], and hence $U \to U^{\mathrm{aff}}$ is an open immersion. Hence, we may take $E = U^{\mathrm{aff}}$ to have (1) satisfied. □

For a factorization $E \in \mathsf{E}^{\mathrm{dom,aff}}(X/S)$, let us write $\mathrm{im}(E)$ for the image of the map $X \to E$, called the "image" of the factorization. If $U = \mathrm{im}(E)$, we have $U \in \mathsf{E}^{\mathrm{surj,sep}}(X/S)$. The proof of Theorem 3.2.7 showed how useful it is to study all the factorizations E with the same "image", and Lemma 3.3.2 shows us how to find the unique largest E with "image" U. Here is an application for base change for π^a:

3.3.3 Lemma *Let S and S' be integral, locally noetherian algebraic spaces all of whose étale atlases are locally $\mathbb{Q}$–factorial. Let $X \to S$ be a flat, separable, finitely presented morphism of schemes.*

(1) *Let $\mathsf{E}_*^{\mathrm{dom,aff}} \subset \mathsf{E}^{\mathrm{dom,aff}}$ be the set of factorizations that are largest among those with a given "image". This inclusion has a retraction given by the map $E \mapsto \mathrm{im}(E)^{\mathrm{aff}}$ taking a factorization to the affine hull of its image, and we have an injection $\mathrm{im}\colon \mathsf{E}_*^{\mathrm{dom,aff}}(X/S) \hookrightarrow \mathsf{E}^{\mathrm{surj,sep}}(X/S)$.*

(2) *Let $S' \to S$ be a faithfully flat, quasicompact morphism. Then the natural base change morphism $\pi^a(X \times_S S'/S') \to \pi^a(X/S) \times_S S'$ is an isomorphism.*

Proof (1) Let $E \in \mathsf{E}^{\mathrm{dom,aff}}$ and U its "image". It follows from Lemma 3.3.2 that $U^{\mathrm{aff}} \to S$ is étale, whence the claim.

(2) Replacing X by $\pi_0(X/S)$, we can assume that $X \to S$ is étale. Let $X' = X \times_S S'$. Ferrand [7, Proposition 7.3.1] states that the natural base change morphism $\pi^s(X'/S') \to \pi^s(X/S) \times_S S'$ is an isomorphism (read Section 3.3.1 for the applicability of [7]). The proof in loc. cit. proceeds by showing that the pullback map $\mathsf{E}^{\mathrm{surj,sep}}(X/S) \to \mathsf{E}^{\mathrm{surj,sep}}(X'/S')$ is a bijection. Moreover, by fpqc descent and flat base change for the affine hull, for $U \in \mathsf{E}^{\mathrm{surj,sep}}(X/S)$ the conditions "$U^{\mathrm{aff}} \to S$ is étale" and "$(U \times_S S')^{\mathrm{aff}} \to S'$ is étale" are equivalent. In view of (1), this proves that our bijection preserves the largest affine dominant factorizations $\mathsf{E}_*^{\mathrm{dom,aff}} \subset \mathsf{E}^{\mathrm{surj,sep}}$. Our claim follows. □

Finally, we relate π^a to the affine hull of π_0 when the base has dimension at most 1.

3.3.4 Lemma *Let $X \to S$ be a morphism of algebraic spaces which is flat, separable, and finitely presented. Assume that S is reduced, geometrically unibranch, locally noetherian. Then the natural map $\pi_0(X/S) \to \pi^s(X/S)$ induces an isomorphism of affine hulls.*

Proof The space $\pi_0(X/S)$ exists by Lemma 3.1.1. The space $\pi^s(X/S)$ exists by [8, théorème 2.3.4]. To prove the lemma, we may assume that S is affine. Also we may replace X by $\pi_0(X/S)$, hence assume that $X \to S$ is étale (in particular smooth). By [8, proposition 6.1.2] the map $X \to \pi^s(X/S)$ is initial among maps to separated schemes. Since $X^{\mathrm{aff}} \to S$ is separated, we obtain a factorization $X \to \pi^s(X/S) \to X^{\mathrm{aff}}$. Taking global sections, the map $\mathcal{O}(X^{\mathrm{aff}}) \to \mathcal{O}(\pi^s(X/S)) \to \mathcal{O}(X) = \mathcal{O}(X^{\mathrm{aff}})$ is the identity; since $X \to \pi^s(X/S)$ is faithfully flat and hence dominant, we see that X has the same affine hull as $\pi^s(X/S)$. Since $X = \pi_0(X/S)$, we are done. □

3.3.5 Proposition *Let S be a reduced noetherian scheme of dimension 1. Let $X \to S$ be a morphism of algebraic spaces which is flat, separable and finitely presented.*

(1) *If S is geometrically unibranch, the affine hull $\pi_0(X/S)^{\mathrm{aff}}$ is étale, ie the natural map $\pi_0(X/S)^{\mathrm{aff}} \to \pi^a(X/S)$ is an isomorphism.*
(2) *If S is excellent, the affine hull $\pi_0(X/S)^{\mathrm{aff}}$ is quasifinite.*

Proof (1) It is enough to prove that $\pi_0(X/S)^{\mathrm{aff}} \to S$ is étale. For this we may assume that S is affine and, by Lemma 3.3.4, we can replace X by $\pi^s(X/S)$. Hence, we may assume that $X \to S$ is étale and separated. By Zariski's main theorem, X is quasiaffine hence $X \to X^{\mathrm{aff}}$ is a dominant open immersion. The closed complement $X^{\mathrm{aff}} \smallsetminus X$ has codimension at least 2 by [13, 21.12.6], and this proves that $X^{\mathrm{aff}} = X$ is étale.

(2) Quasifiniteness of $\pi_0(X/S)^{\mathrm{aff}}$ may be checked étale locally on S. So we let s be a geometric point of S and $(S', s') \to (S, s)$ an étale neighbourhood such that:

(i) The irreducible components $S_1, \dots, S_n$ of S' are geometrically unibranch.

(ii) $S_i \cap S_j = \{s'\}$ for every $i \neq j$.

(iii) The fibre $\pi_0(X/S)_{s'}$ is a disjoint union of copies of $\mathrm{Spec}(k(s'))$.

(iv) $S' \smallsetminus \{s'\}$ is regular.

(v) $S' = \mathrm{Spec}(R')$ is affine.

The reason why condition (i) can be fulfilled is the following: the étale local ring of S at s has irreducible components that are geometrically unibranch; we may spread out to an étale neighbourhood $(S', s') \to (S, s)$ having the same property, since, by excellence, the regular — and hence the geometrically unibranch — locus is open dense.

Write $\pi = \pi_0(X/S)$. Then $\pi = \pi' \sqcup \pi^*$, where π^* is the union of those connected components that do not meet the fibre $\pi_{s'}$. Then π^* lives over $S' \smallsetminus \{s'\}$, which, by condition (ii), is geometrically unibranch. By Proposition 3.3.5(1), the map $(\pi^*)^{\mathrm{aff}} \to S$ is étale, and in particular quasifinite. It remains to check that π'^{aff} is quasifinite.

Up to restricting S by a further étale neighbourhood of s, we may assume that the isomorphism $\bigsqcup_{i=1}^n \mathrm{Spec}(k(s)) \to \pi'_s$ extends to an open immersion $\alpha\colon \bigsqcup_{i=1}^n S \hookrightarrow \pi'$. We claim that α has dense image. Indeed, let Z be an irreducible component of π'. Then Z maps to some irreducible component S_i of S. By assumption, S_i is geometrically unibranch, so, by [13, théorème 18.10.1], $Z \to S_i$ is étale. In particular, $Z \to \pi'_{S_i}$ is an étale, closed immersion; that is, Z is a connected component of π'_{S_i}. Thanks to condition (ii), Z is also a connected component of π', and therefore meets the closed fibre. In particular, it meets the image of α. This proves the claim.

The morphism α is dominant and induces an injective R–algebra morphism $\mathcal{O}(\pi') \hookrightarrow R^n$. It follows that $\mathcal{O}(\pi')$ is finite as an R–module. In particular, $\pi'^{\mathrm{aff}} \to S$ is finite. □

3.4 Computing π_0

In this subsection, we collect various techniques to compute the space of connected components $\pi_0(\mathscr{X}/S)$, notably by using an atlas of $\mathscr{X}$, or completing along a closed fibre of $\mathscr{X} \to S$. These are used crucially in Sections 4 and 5. We start with an elementary result related to factorizations in the sense of Definition 3.2.1, which holds irrespective of representability of $\pi_0(\mathscr{X}/S)$ by an algebraic space.

3.4.1 Lemma *Let $\mathscr{X} \xrightarrow{h} \mathscr{E} \xrightarrow{f} S$ be morphisms of algebraic stacks of finite presentation.*

(1) *If $\mathscr{E} \to S$ is an étale algebraic space, there is a morphism of S–functors*

$$f_!\pi_0(\mathscr{X}/\mathscr{E}) \to \pi_0(f_!\mathscr{X}/S)$$

which is an isomorphism when $\mathscr{X} \to \mathscr{E}$ is universally open.

(2) *If $\mathscr{X} \to \mathscr{E}$ is a universal submersion with connected geometric fibres and either $\mathscr{X} \to \mathscr{E}$ or $\mathscr{E} \to S$ is flat, there is an isomorphism*

$$\pi_0(\mathscr{X}/S) \xrightarrow{\sim} \pi_0(\mathscr{E}/S).$$

Proof (1) If $\mathscr{X} \to \mathscr{E}$ is flat, finitely presented and separable, this follows from Lemma 2.2.1 and the comments after it. However, here we assume much less. The morphism in the statement is constructed as follows. For each S–scheme T, a point of $f_!\pi_0(\mathscr{X}/\mathscr{E})$ with values in T is a pair composed of an S–morphism $u\colon T \to \mathscr{E}$ and a T–relative connected component $\mathscr{C}' \subset \mathscr{X} \times_{\mathscr{E}} T$. Since $\mathscr{E} \to S$ is étale, the map $\mathscr{X} \times_{\mathscr{E}} T \to \mathscr{X} \times_S T$ is an open immersion globally and a closed immersion in the fibres, showing that $\mathscr{C} := \mathscr{C}'$ is a T–relative connected component of $\mathscr{X} \times_S T$, ie a T–valued point of $\pi_0(f_!\mathscr{X}/S)$. Let us describe the inverse morphism, assuming $\mathscr{X} \to \mathscr{E}$ universally open. Let $\mathscr{C} \subset \mathscr{X} \times_S T$ be a T–relative connected component. By the assumption on $\mathscr{X} \to \mathscr{E}$, the image $\mathscr{D}$ of $\mathscr{C}$ in $\mathscr{E} \times_S T$ is open, and hence étale over T with nonempty geometrically connected T–fibres. By the implication (c) $\Longrightarrow$ (a) of [13, 17.9.1], which extends easily to algebraic spaces, it follows that $\mathscr{D} \to T$ is an isomorphism. Using its inverse, we obtain a morphism $T \to \mathscr{D} \to \mathscr{E}$ and the pair $(T \to \mathscr{E}, \mathscr{C})$ is a T–point of $f_!\pi_0(\mathscr{X}/\mathscr{E})$. These constructions are inverse to each other.

(2) We will use an elementary fact from topology: if L, M are topological spaces and $h\colon L \to M$ is a submersion with connected fibres, then the sets of connected components of L and M are in bijection by the maps $C \mapsto h(C)$ and $D \mapsto h^{-1}(D)$. We use this to define $\pi_0(\mathscr{X}/S) \to \pi_0(\mathscr{E}/S)$. Let $C \subset \mathscr{X}$ be a relative connected component. In the fibre of a geometric point $\bar{s} \to S$, the image $D_{\bar{s}}$ of $C_{\bar{s}}$ in $\mathscr{E}_{\bar{s}}$ is a connected component. Consider the union $D := \bigcup D_{\bar{s}}$. Since its preimage in $\mathscr{X}$ is equal to C, which is open, and $\mathscr{X} \to \mathscr{E}$ is a submersion, then D is an open substack of $\mathscr{E}$. Moreover, D is of finite presentation over S. Thus, if $\mathscr{X} \to \mathscr{E}$ is flat then $D \to S$ is flat by the "critère de platitude par fibres", and if $\mathscr{E}$ is flat over S then D is also, being an open of $\mathscr{E}$. Hence, it is a relative connected component. This defines $\pi_0(\mathscr{X}/S) \to \pi_0(\mathscr{E}/S)$, which is an isomorphism with inverse $D \mapsto h^{-1}(D)$. □

We continue with a description of $\pi_0(\mathscr{X}/S)$ in terms of an atlas. This takes the form of a pushout property which is a consequence of the right exactness of the functor π_0, and will have an important refinement in the context of stacks in the later Lemma 5.5.2.

3.4.2 Lemma *Let $\mathscr{X} \to S$ be a flat, finitely presented, separable algebraic stack. Let $U \to \mathscr{X}$ be a faithfully flat, finitely presented, separable morphism from an algebraic space.*

(1) *Let $R \rightrightarrows U$ be the groupoid presentation defined by $U \to \mathscr{X}$, so that $\mathscr{X}$ is identified with the quotient stack $[U/R]$. Then $\pi_0(\mathscr{X}/S)$ is the coequalizer of the pair of maps $\pi_0(R/S) \rightrightarrows \pi_0(U/S)$ in the category of algebraic spaces.*

(2) *For every fppf sheaf Y and maps $\mathscr{X} \to Y$, $\pi_0(U/S) \to Y$ making the solid diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \arrow[dr, bend left] & \\ \pi_0(U/S) \arrow[r] \arrow[rr, bend right] & \pi_0(\mathscr{X}/S) \arrow[r, dashed] & Y \end{tikzcd}$$

commute, there exists a unique dashed arrow $\pi_0(\mathscr{X}/S) \to Y$ making the diagram commute.

We warn the reader that, even if $\mathscr{X}$ is an algebraic space, the map

$$\pi_0(R/S) \to \pi_0(U/S) \times_S \pi_0(U/S)$$

may fail to be injective; eg R may be disconnected in a connected U.

Proof Throughout, we write $\pi_0(\mathscr{X})$ instead of $\pi_0(\mathscr{X}/S)$ and we omit S from fibred products.

(1) The pair of maps $\pi_0(R) \rightrightarrows \pi_0(U)$ are the datum of a 2–étale algebraic space (Section A.1); we denote by $\pi_0(R)^{\mathrm{gpd}} \rightrightarrows \pi_0(U)$ the associated groupoid in étale algebraic spaces (Section A.6), and by $\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}$ the étale algebraic space obtained as quotient of the étale equivalence relation given by the image of $\pi_0(R)^{\mathrm{gpd}} \to \pi_0(U) \times \pi_0(U)$.

By Corollary A.7.1, the quotient algebraic space $\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}$ is a coequalizer for the pair of maps $\pi_0(R) \rightrightarrows \pi_0(U)$ in the category of algebraic spaces. This gives us a unique map $\pi_0(U)/\pi_0(R)^{\mathrm{gpd}} \to \pi_0(\mathscr{X})$ of étale spaces; on the other hand, the morphism of groupoids $(R \rightrightarrows U) \to (\pi_0(R)^{\mathrm{gpd}} \rightrightarrows \pi_0(U))$ induces a morphism of quotient stacks $\mathscr{X} \to [\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$, and by composition with the coarse moduli space a map $\mathscr{X} \to \pi_0(U)/\pi_0(R)^{\mathrm{gpd}}$ to the quotient étale algebraic space. By the universal property of $\pi_0(\mathscr{X})$ we have a map $\pi_0(\mathscr{X}) \to \pi_0(U)/\pi_0(R)^{\mathrm{gpd}}$. By the uniqueness part of the universal properties of the two spaces, the two maps constructed are inverse to each other.

(2) Let Y be an fppf sheaf and let $a\colon \mathscr{X} \to Y$, $b\colon \pi_0(U) \to Y$ be maps that coincide on U. Denote by $u\colon U \to \mathscr{X}$ the chosen atlas and $s, t\colon R \to U$ the projections. Let σ and τ be the maps $\pi_0(s), \pi_0(t)\colon \pi_0(R) \to \pi_0(U)$. The map $R \to \pi_0(R)$ is flat hence an epimorphism of sheaves, so from $aus = aut$ we deduce $b\sigma = b\tau$. Then (1) implies that b factors through a unique map $\pi_0(\mathscr{X}) \to Y$. □

3.4.3 Completion We finish this subsection with a description of $\pi_0(X/S)$ over a complete local base, which will be crucial for the proof of Theorem 4.3.2. To start with, we point out:

3.4.4 Lemma *A quasifinite algebraic space over an artinian ring is a finite scheme.*

Proof See eg [28, Tag 06LZ]. □

Let S be the spectrum of a complete noetherian local ring R with maximal ideal $\mathfrak{m}$. For each $n \geqslant 0$, let $S_n = \mathrm{Spec}(R/\mathfrak{m}^{n+1})$. By [13, 18.5.15], restriction to S_0 yields an equivalence $\mathbf{F\acute{E}t}/S \simeq \mathbf{F\acute{E}t}/S_0$ between the categories of finite étale algebras. In particular, given $X \to S$ flat of finite type and separable, there exists a unique finite étale scheme $\widehat{\pi}/S$ restricting to $\pi_0(X \times_S S_n/S_n)$ over each S_n. Alternatively, one can see $\widehat{\pi}$ as the algebraization of the formal completion of $\pi_0(X/S)$, which explains the choice of notation $\widehat{\pi}$. As $\widehat{\pi}$ is finite over S, it is a product of complete local rings. By [28, Tag 0AQH], there is a natural morphism of S–algebraic spaces

$$\psi\colon \widehat{\pi} \to \pi_0(X/S), \tag{1}$$

which restricts to an isomorphism over each S_n.

3.4.5 Proposition *Let R be a complete noetherian ring and A a flat separable R–algebra of finite type. Write $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and s for the closed point* *of S, and let $V = S \setminus \{s\}$. The commutative diagram of S–algebraic spaces*

$$\begin{tikzcd} \widehat{\pi}_V \arrow[r, hook] \arrow[d, "\psi_V"] & \widehat{\pi} \arrow[d, "\psi"] \\ \pi_V \arrow[r, hook] & \pi_0(X/S) \end{tikzcd}$$

is a pushout in the category of fppf sheaves over S.

Proof In the proof we write $\pi := \pi_0(X/S)$. In order to prove the claim, it suffices to show that any diagram of solid arrows

$$\begin{tikzcd} \widehat{\pi}_V \arrow[r] \arrow[d, "\psi_V"] & \widehat{\pi} \arrow[d, "\psi"] \arrow[ddr, bend left, "a"] & \\ \pi_V \arrow[r] \arrow[drr, bend right, "b"'] & \pi \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

where Z is an S–sheaf admits a unique dashed arrow making the diagram commute.

First of all, notice that $\psi\colon \widehat{\pi} \to \pi$ is étale; writing $U = \pi_V \sqcup \widehat{\pi}$, it follows that $U \to \pi$ is faithfully flat of finite presentation, and hence it is a coequalizer for $U \times_\pi U \to U$. Therefore, in order to obtain a unique dashed arrow, it suffices to check that $a \circ p_1 = a \circ p_2$, where p_1 and p_2 are the projections $\widehat{\pi} \times_\pi \widehat{\pi} \to \widehat{\pi}$.

The S–scheme $\widehat{\pi}$ is finite étale, and hence the map $\psi\colon \widehat{\pi} \to \pi$ is separated and quasifinite, and so is also the base change $p_1\colon \widehat{\pi} \times_\pi \widehat{\pi} \to \widehat{\pi}$. Moreover, we know that $\widehat{\pi}$ is a finite disjoint union of spectra of completed local rings; by the classification of separated quasifinite schemes over henselian local rings, $\widehat{\pi} \times_\pi \widehat{\pi}$ decomposes into a disjoint union $P^f \sqcup P'$ such that $p_1\colon P^f \to \widehat{\pi}$ is finite (and étale), and $P' = P'_V$ has empty closed fibre. One obtains a similar decomposition for the map p_2; let us say $\widehat{\pi} \times_\pi \widehat{\pi} = Q^f \sqcup Q'$. However, the compositions $\widehat{\pi} \times_\pi \widehat{\pi} \xrightarrow{p_i} \widehat{\pi} \to \pi \to S$ are the same map for $i = 1, 2$, and are both quasifinite, separated; so both P^f and Q^f are equal to the finite part of the composition, and we find $P^f = Q^f$.

The restriction of ψ to the closed fibre, $\psi_s\colon \widehat{\pi}_s \to \pi_s$, is an isomorphism by construction of $\widehat{\pi}$, and therefore so is $P^f_s = (\widehat{\pi} \times_\pi \widehat{\pi})_s \xrightarrow{p_1} \widehat{\pi}_s$. The restriction $P^f \to \widehat{\pi}$ of the projection is therefore an isomorphism as well.

Consider the diagram of solid arrows

$$\begin{tikzcd} \widehat{\pi} \sqcup P' \arrow[r, "p_2"] \arrow[d, "p_1"] & \widehat{\pi} \arrow[d, "a"] \\ \widehat{\pi} \arrow[r, "a"] & Z \end{tikzcd}$$

where we have identified P^f with $\widehat{\pi}$. We want to show that it is commutative.

For $i = 1, 2$, the morphism p_i is the identity on $\widehat{\pi}$, so we really only need to show that $a \circ p_1$ agrees with $a \circ p_2$ on P'. As P' is contained in $(\widehat{\pi} \times_\pi \widehat{\pi})_V$, we have $a \circ (p_1)_V = b \circ \psi_V \circ (p_1)_V = b \circ \psi_V \circ (p_2)_V = a \circ (p_2)_V$ and the proof is complete. □

4 Perfection of algebras

The commutative algebra developed in this section has independent interest but is also fruitfully introduced with an eye towards the geometric applications of the next section. Let $X \to S$ be a flat, finitely presented morphism of algebraic spaces of characteristic p. In order to study the coperfection of X in the category of S–algebraic spaces, we will use the étale algebraic spaces $\pi_0(X/S)$ and $\pi^a(X/S)$ (assuming they exist). Since étale implies relatively perfect, the morphism $X \to \pi_0(X/S)$ extends to the direct Frobenius system and we have a diagram

$$(X \xrightarrow{F_0} X^{p/S} \xrightarrow{F_1} \cdots) \to \pi_0(X/S) \to \pi^a(X/S).$$

The present section is devoted to the case where $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. The main question is whether there exists a *perfection functor*, right adjoint to the inclusion of perfect R–algebras into all R–algebras. In such generality we do not know if such perfection exists. At least an obvious approximation should be the *preperfection*

$$A^{p^\infty/R} := \lim A^{p^i/R} = \lim(\cdots \to A^{p^2/R} \xrightarrow{F_A} A^{p/R} \xrightarrow{F_A} A).$$

The above diagram of spaces provides a diagram of algebras

$$A^{\text{ét}/R} \to \mathcal{O}(\pi_0(A/R)) \to A^{p^\infty/R},$$

where $A^{\text{ét}/R} = \mathcal{O}(\pi^a(A/R))$ is the largest étale subalgebra of A; see Definition 3.2.8. Our goal is roughly to find as many situations as possible where both maps above are isomorphisms.

We start in Section 4.1 with preliminary material on base change in the formation of the preperfection. Then we prove that both maps above are indeed isomorphisms when R is artinian and $R \to A$ is of finite type (see Section 4.2) or R is regular of dimension $\leqslant 2$ and $R \to A$ is of finite type and separable (see Section 4.4). Over a general ring, only the map $\mathcal{O}(\pi_0(A/R)) \to A^{p^\infty/R}$ is an isomorphism (see Section 4.3). This is already remarkable, given the poor properties of both algebras: in general, $\mathcal{O}(\pi_0(A/R))$ is not étale and $A^{p^\infty/R}$ is not perfect, even when $R \to A$ is flat, of finite type and separable. One may expect that, after iterating the preperfection functor $(-)^{p^\infty/R}$ a finite (sufficiently high) number of times, one reaches a perfect R–algebra. With the hope that this might be true, we establish in Section 4.4 some finiteness properties of $A^{p^\infty/R}$. We conclude the section with counterexamples.

4.1 Base change in preperfection

For each morphism of $\mathbb{F}_p$–algebras $R \to A$ and each base change morphism $R \to R'$, we have a natural base change map for preperfection,

$$\phi = \phi_{R,R',A} \colon A^{p^\infty/R} \otimes_R R' \to (A \otimes_R R')^{p^\infty/R'}.$$

It is important to understand this map for at least two reasons. The first is that the study of $A^{p^\infty/R}$ with the usual tools (localization, completion on $R, \dots$) involves many base changes. The second is that the base change map along Frobenius $F \colon R \to R$ controls the success or failure of $A^{p^\infty/R}$ to be perfect; we elaborate on this in Remarks 4.1.4. Before stating the first lemma devoted to properties of ϕ, we recall a result of T Dumitrescu:

4.1.1 Theorem [5, Theorem 3] *Let $R \to A$ be a morphism of noetherian commutative rings. Let $F_{A/R} \colon A^{p/R} \to A$ be the relative Frobenius morphism. Then the following are equivalent*:

(i) *$R \to A$ is flat and separable.*

(ii) *$F_{A/R}$ is injective and its cokernel is a flat R–module.* □

4.1.2 Remark If we do not assume that R and A are noetherian but $R \to A$ is of finite presentation, then (i) $\Longrightarrow$ (ii) is true. To see this, write R as the filtered union of its finitely generated subrings R_i. By [12, 11.2.7], there exists i such that A is the base change of a flat, finitely presented R_i–algebra A_i. By [12, 12.1.1(vii)], the locus of points $\mathfrak{p} \in \operatorname{Spec}(R_i)$ where the fibre $A_\mathfrak{p}$ is separable is an open U_i. Since $\operatorname{Spec}(R) \to \operatorname{Spec}(R_i)$ factors through U_i, using [12, 8.3.2] we see that, for some $j \geqslant i$, the map $\operatorname{Spec}(R_j) \to \operatorname{Spec}(R_i)$ will have image in U_i. Applying [12, 11.2.7] again, we find $j \geqslant i$ such that A is the base change of a flat, separable, finitely presented R_j–algebra A_j. By the noetherian case, it follows that F_{A_j/R_j} is injective with R_j–flat cokernel. By base change, $F_{A/R}$ is injective with R–flat cokernel.

4.1.3 Lemma *The base change map $\phi_{R,R',A}\colon A^{p^\infty/R} \otimes_R R' \to (A \otimes_R R')^{p^\infty/R'}$ is:*

1. *An isomorphism if $R \to R'$ is finite locally free.*
2. *Injective in each of the following cases:*
   1. (i) *R' is a projective R–module.*
   2. (ii) *$R \to R'$ is flat and $R \to A$ is flat, separable and finitely presented.*
   3. (iii) *$R' = \mathrm{colim}(R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots)$ is the absolute coperfection of a ring R such R is a projective R–module via $F\colon R \to R$.*

Proof Note that since $(A \otimes_R R') \otimes_{R',F^i} R' = A^{p^i/R} \otimes_R R'$, the map $\phi_{R,R',A}$ is just a special case for the R–module $M := R'$ of the map $\phi_{R,M,A}$ that appears as the upper horizontal row in the commutative diagram

$$\begin{tikzcd}
(\lim A^{p^i/R}) \otimes_R M \arrow[r, "\phi_{R,M,A}"] \arrow[d] & \lim(A^{p^i/R} \otimes_R M) \arrow[d] \\
\left(\prod_{i\geqslant 0} A^{p^i/R}\right) \otimes_R M \arrow[r, "\psi_{R,M,A}"] & \prod_{i\geqslant 0}(A^{p^i/R} \otimes_R M)
\end{tikzcd}$$

In the sequel we assume that M is flat, so the left-hand vertical map is injective. If M is free (resp. free of finite rank), then $\psi_{R,M,A}$ is injective (resp. an isomorphism). It follows that also $\phi_{R,M,A}$ is injective (resp. an isomorphism). If M is projective, one reaches the same conclusions by embedding it in a free module (resp. a free module of finite rank) and using the facts that $\phi_{R,M,A}$ and $\psi_{R,M,A}$ are additive in M. This settles cases (1) and (2i).

In case (2ii), by Dumitrescu's Theorem 4.1.1, all the maps $A^{p^{i+1}/R} \to A^{p^i/R}$ are injective; it follows that $\lim A^{p^i/R} \to A^{p^j/R}$ is injective for each fixed j. By flatness of $R \to R'$, the tensored map $(\lim A^{p^i/R}) \otimes_R R' \to A^{p^j/R} \otimes_R R'$ is injective. Therefore $\phi_{R,R',A}$ is also injective.

In case (2iii) we can write the coperfection as $R' = \mathrm{colim}\, R^{p^{-j}}$. Since the absolute Frobenius of R is projective, it is in fact faithfully flat. It follows that the maps $R^{p^{-j}} \to R^{p^{-(j+1)}}$ are faithfully flat, hence universally injective. Thus, for each i and j the map

$$A^{p^i/R} \otimes R^{p^{-j}} \to A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective. Then, for each i,

$$A^{p^i/R} \otimes R^{p^{-j}} \to \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective. Taking limits,

$$\lim_i (A^{p^i/R} \otimes R^{p^{-j}}) \to \lim_i \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}}$$

is injective, which implies that

$$\operatorname*{colim}_j \lim_i (A^{p^i/R} \otimes R^{p^{-j}}) \to \lim_i \operatorname*{colim}_j A^{p^i/R} \otimes R^{p^{-(j+1)}} = \lim_i A^{p^i/R} \otimes R'$$

is injective. Since also by (2i) the map

$$\big(\lim_i A^{p^i/R}\big) \otimes R' = \operatorname*{colim}_j \big(\lim_i A^{p^i/R}\big) \otimes R^{p^{-j}} \to \operatorname*{colim}_j \lim_i (A^{p^i/R} \otimes R^{p^{-j}})$$

is injective, by composition we obtain the result. □

4.1.4 Remarks (1) Let $R \to A$ be a map of rings of characteristic $p > 0$. When inquiring whether the preperfection $A^{p^\infty/R}$ is perfect, we are led to ask if the Frobenius of the preperfection ($F \circ (-)^{p^\infty}$ below) is an isomorphism. In general it is not; an example is given in Lemma 4.5.2. In contrast, the preperfection of the Frobenius ($(-)^{p^\infty} \circ F$ below), that is, the morphism obtained by taking limits in the morphism of systems $\{F_{A^{p^i/R}}\}\colon \{A^{p^i/R} \otimes_{R,F} R\} \to \{A^{p^i/R}\}$, *is* an isomorphism: it is essentially a shift by one in the indices, which is invisible in the infinite system. In fact, "Frobenius of the preperfection" and "the preperfection of the Frobenius" are the two edges of a commutative triangle whose third edge, the base change map in preperfection, serves to compare them:

$$\begin{tikzcd} & A^{p^\infty/R} & \\ A^{p^\infty/R} \otimes_{R,F} R \arrow[ur, "F\circ(-)^{p^\infty}"] \arrow[rr, "\phi_{R,R,A}"'] & & (A \otimes_{R,F} R)^{p^\infty/R} \arrow[ul, "(-)^{p^\infty}\circ F"'] \end{tikzcd}$$

Since $(-)^{p^\infty} \circ F$ is an isomorphism, we see that $A^{p^\infty/R}$ is a perfect R–algebra if and only if the base change map $\phi_{R,R,A}$ is an isomorphism. According to Lemma 4.1.3(1), this happens when Frobenius is finite locally free, eg when R is regular and F–finite; see Kunz [20].

(2) In case (2ii), it will be a consequence of Theorem 4.3.2 that the base change map is in fact an isomorphism. Indeed, one just has to recall that the formation of π_0

commutes with arbitrary base change (Lemma 3.1.1(1)) and the formation of the ring of global sections commutes with flat base change.

(3) Here is an example where the base change map is not surjective. Let k be a field of characteristic p and k' an infinite-dimensional field extension. Let

$$A = k[\epsilon_0, \epsilon_1, \dots]/(\epsilon_0^p, \epsilon_{i+1}^p - \epsilon_i)$$

and $A' = A \otimes_k k'$. Let $t_0, t_1, \dots$ be an infinite family of elements of k' that is k–linearly independent. Let $x_i' = \epsilon_0 t_i + \epsilon_1 t_{i-1} + \dots + \epsilon_i t_0 \in (A')^{p^i/k'}$. Then $F_{A'/k'}(x_{i+1}') = x_i'$, so $x' = (x_i')$ is an element of $(A')^{p^\infty/k'}$, which obviously does not come from $A^{p^\infty/k} \otimes k'$.

4.2 Perfection over artinian rings

In this subsection we consider the case where R is an artinian ring. For such a ring, Theorem 3.2.7 implies that any flat, finitely generated algebra $R \to A$ has a largest étale subalgebra $A^{\text{ét}}$. Below we prove that the natural map $A^{\text{ét}} \to A^{p^\infty}$ to the preperfection is an isomorphism. In particular, the preperfection is perfect, and hence a perfection. We point out that in this special situation the separability of $R \to A$ is not needed.

4.2.1 Theorem *Let R be an artinian local ring of characteristic p, and let A be a flat R–algebra of finite type. Then $\pi_0(A)$ is finite étale and the maps $A^{\text{ét}} \to \mathcal{O}(\pi_0(A)) \to A^{p^\infty}$ are isomorphisms.*

Proof It follows from Lemmas 3.1.1 and 3.4.4 that $\pi_0(A)$ is a finite étale scheme. In particular, it is affine and the map $A^{\text{ét}} \to \mathcal{O}(\pi_0(A))$ is an isomorphism. It remains to prove that $A^{\text{ét}} \to A^{p^\infty}$ is an isomorphism. The proof of this is in five steps.

Step 1 We reduce to the case where $R = k$ is a field. Let $\mathfrak{m}$ (resp. k) be the maximal ideal (resp. residue field). Let $F \colon R \to R$ be the absolute Frobenius and e an integer such that $\mathfrak{m} = \ker F^e$. Then F^e induces a ring map $\alpha \colon k \to R$, which we use to view R as a k–algebra. We compute the perfection of A using the cofinal system of indices $e\mathbb{N} \subset \mathbb{N}$. For each $i \geqslant 0$, the morphism $F^{ei} \colon R \to R$ has a factorization

$$R \twoheadrightarrow k \xrightarrow{F^{e(i-1)}} k \xrightarrow{\alpha} R.$$

Writing $A_0 = A \otimes_R k$, it follows that $A^{p^{ei}/R} = A_0^{p^{e(i-1)}/k} \otimes_k R$. Passing to the limit and using Lemma 4.1.3(1), we deduce an isomorphism

$$\lambda \colon A_0^{p^\infty/k} \otimes_k R \xrightarrow{\sim} A^{p^\infty/R}.$$

On the other hand, the e–fold absolute Frobenius $F_A^e\colon A_0^{\text{ét}/k} \to A^{\text{ét}/R}$ extends the map $\alpha\colon k \to R$, providing an isomorphism, which is a witness of topological invariance (Lemma 3.2.2),

$$\mu\colon A_0^{\text{ét}/k} \otimes_{k,\alpha} R \xrightarrow{\sim} A^{\text{ét}/R}.$$

Since λ and μ fit together in a commutative square, the reduction step follows.

Step 2 We reduce to the case where k is algebraically closed. Let k' be an algebraic closure of k, and $A' := A \otimes_k k'$. We have injections

$$A^{\text{ét}/k} \otimes_k k' \hookrightarrow A^{p^\infty/k} \otimes_k k' \hookrightarrow (A')^{p^\infty/k'},$$

where the first is deduced from $A^{\text{ét}/k} \hookrightarrow A^{p^\infty/k}$ and the second comes from case (2i) of Lemma 4.1.3. It is classical that $A^{\text{ét}/k} \otimes_k k' = (A')^{\text{ét}/k'}$; see Waterhouse [31, Theorem 6.5]. It follows that, if $(A')^{\text{ét}/k'} \to (A')^{p^\infty/k'}$ is an isomorphism, then $A^{\text{ét}/k} \otimes_k k' \hookrightarrow A^{p^\infty/k} \otimes_k k'$ is an isomorphism and hence $A^{\text{ét}/k} \to A^{p^\infty/k}$ is an isomorphism.

Step 3 We reduce to the case where A is reduced. Let A_{red} be the reduced quotient. On the side of separable closure, since $A^{\text{ét}/k}$ does not meet the nilradical $\text{Nil}(A)$ and all separable elements of A_{red} lift to A, we have an isomorphism $A^{\text{ét}/k} \xrightarrow{\sim} (A_{\text{red}})^{\text{ét}/k}$. On the side of preperfection, we use the isomorphisms $A^{p^i/k} \xrightarrow{\sim} A$, $a \otimes \lambda \mapsto a\lambda^{p^{-i}}$, to obtain an isomorphism of rings $A^{p^\infty/k} \xrightarrow{\sim} A^{p^\infty/\mathbb{F}_p}$, and similarly for A_{red}. Since $\text{Nil}(A)$ is finitely generated, there is $e \geqslant 0$ such that $\text{Nil}(A) = \ker F^e$, where $F\colon A \to A$ is the absolute Frobenius. Then the computation of the perfection can be carried out along the cofinal system of indices $e\mathbb{N} \subset \mathbb{N}$, showing that the projection $A^{p^\infty/\mathbb{F}_p} \to (A_{\text{red}})^{p^\infty/\mathbb{F}_p}$ is an isomorphism. Contemplating the commutative diagram

$$\begin{tikzcd}
A^{\text{ét}/k} \arrow[r] \arrow[d, "\simeq"] & A^{p^\infty/k} \arrow[r, "\simeq"] & A^{p^\infty/\mathbb{F}_p} \arrow[d, "\simeq"] \\
(A_{\text{red}})^{\text{ét}/k} \arrow[r] & (A_{\text{red}})^{p^\infty/k} \arrow[r, "\simeq"] & (A_{\text{red}})^{p^\infty/\mathbb{F}_p}
\end{tikzcd}$$

we see that, if $(A_{\text{red}})^{\text{ét}/k} \to (A_{\text{red}})^{p^\infty/k}$ is an isomorphism, then $A^{\text{ét}/k} \to A^{p^\infty/k}$ is also.

Step 4 We reduce to the case where A has connected spectrum. This is straightforward, because, if $A = A_1 \times \cdots \times A_d$ is the decomposition of A as a product of rings with connected spectrum, we have $\bigl(\prod A_i\bigr)^{\text{ét}/k} \simeq \prod A_i^{\text{ét}/k}$ and $\bigl(\prod A_i\bigr)^{p^\infty/k} \simeq \prod A_i^{p^\infty/k}$.

Step 5 We conclude that $A^{\text{ét}/k} \to A^{p^\infty/k}$ is surjective. Let x be an element of the ring

$$A^{p^\infty/k} \simeq A^{p^\infty/\mathbb{F}_p} = \bigcap_{n \geqslant 0} A^{p^n},$$

with $x = x_n^{p^n}$ and $x_n \in A$ for each n. By noetherianity, the increasing sequence of ideals (x_i) stabilizes at some N. It follows that $y := x_N$ satisfies $(y) = (y^p)$; in particular, $(y) = (y^2)$ is an idempotent ideal. Since $X = \operatorname{Spec}(A)$ is connected, we must have $(y) = (0)$ or $(y) = A$. Hence $y = 0$ or y is a unit; therefore, $x = 0$ or x is a unit in A. If $x = 0$ we are done; henceforth assume that x is a unit. Let A_i for $i = 1, \dots, n$ be the quotients of A by the minimal primes. Again by connectedness, the injection $A \hookrightarrow A_1 \times \dots \times A_n$ induces a morphism of groups of units modulo constants $A^\times/k^\times \hookrightarrow (A_1^\times/k^\times) \times \dots \times (A_n^\times/k^\times)$ which is *injective*. It is a classical result of Rosenlicht [25, Lemma to Proposition 3] that each $A_i^\times/k^\times$ is a finitely generated free abelian group; hence, the same is true for $A^\times/k^\times$. In particular, the class of x in this group cannot be infinitely p–divisible, so $x \in k^\times$ and this proves the claim. $\square$

4.2.2 Remark If the base is a field k of characteristic p, we have, more generally,

$$\pi_0(X/k) \simeq \operatorname{Spec}(\mathcal{O}(X)^{\text{ét}}) \simeq \operatorname{Spec}(\mathcal{O}(X)^{p^\infty})$$

for all algebraic stacks of finite type X/k. Indeed the first isomorphism follows since $\pi_0(X/k)$ is affine and the second isomorphism follows from [29, Theorem 6.23(2)].

4.3 Preperfection over arbitrary rings

The aim of this section is to generalize the statement that $\mathcal{O}(\pi_0(A)) \to A^{p^\infty}$ is an isomorphism to the case of a general base ring R, in the case of *separable* algebras. The proof proceeds by thickening from an artinian base to a complete local base, then a Zariski-local base and then to a general base by induction on the dimension.

4.3.1 Lemma *Let R be a complete noetherian local ring and A a flat separable R–algebra of finite type. Write $\widehat{A}$ for the completion of A with respect to the maximal ideal of R, and write $\widehat{\pi}$ for the finite étale R–scheme built from $\pi_0(A/R)$ as in the situation of Section 3.4.3. Then the natural map $\mathcal{O}(\widehat{\pi}) \to (\widehat{A})^{p^\infty/R}$ is an isomorphism.*

Proof Let $\mathfrak{m}$ be the maximal ideal of R. Write $B = \mathcal{O}(\widehat{\pi})$. For every $n \geqslant 0$, let $R_n = R/\mathfrak{m}^{n+1}$, $A_n = A \otimes_R R_n$ and $B_n = B \otimes_R R_n$. As $B_n = \mathcal{O}(\pi_0(A_n/R_n))$, for every n we have an inclusion $B_n \hookrightarrow A_n$. Taking the limit over n, and noticing that B

is finite over R and hence complete, we obtain an inclusion $B \hookrightarrow \widehat{A}$. As B is also étale over R, it is in fact contained in $(\widehat{A})^{p^\infty/R}$.

On the other hand, a section to the inclusion $B \hookrightarrow (\widehat{A})^{p^\infty}$ is given by the map

$$(\widehat{A})^{p^\infty} = \lim_i (\widehat{A})^{p^i} = \lim_i (\lim_n A_n)^{p^i} \to \lim_i \lim_n (A_n^{p^i}) = \lim_n \lim_i (A_n^{p^i}) = \lim_n (A_n)^{p^\infty}$$
$$= \lim_n B_n = B.$$

Here, the second-to-last equality comes from Theorem 4.2.1. Notice that we have suppressed parentheses in the expression $A_n^{p^\infty}$ since $(A_n)^{p^\infty} = A_n^{p^\infty}$ anyway. To complete the proof it suffices to show that $\widehat{A}^{p^\infty} \to B$ is injective, or that $(\lim_n A_n)^{p^i} \to \lim_n (A_n^{p^i})$ is injective. The latter is the completion morphism

$$(\widehat{A})^{p^i} \to \widehat{(\widehat{A})^{p^i}}. \tag{2}$$

Here we have used that $A_n^{p^i} = \widehat{A}^{p^i} \otimes_R R_n$ [28, Tag 05GG].

We claim that the map $R \to \widehat{A}$ satisfies condition (i) of Theorem 4.1.1. As A is noetherian, $\widehat{A}$ is noetherian and $A \to \widehat{A}$ is flat. It remains to show that $R \to \widehat{A}$ is separable. Since $R \to A$ is separable and $A \to \widehat{A}$ is flat, we may reduce to showing that $A \to \widehat{A}$ is separable. By [28, Tag 0BK9], we reduce to showing that, for every prime $\mathfrak{p}$ of A, the map $A_{\mathfrak{p}} \to A_{\mathfrak{p}}^{\wedge}$ to the $\mathfrak{p}$–adic completion has geometrically reduced fibres. By [28, Tag 0BJ0], this is equivalent to $A_{\mathfrak{p}}$ being Nagata. By [28, Tag 0335], A is Nagata and by [28, Tag 032U] so is $A_{\mathfrak{p}}$. This proves the claim.

We apply Theorem 4.1.1 and deduce that $(\widehat{A})^{p^i}$ is a subalgebra of $\widehat{A}$. As the latter is $\mathfrak{m}$–adically separated — that is, $\bigcap_{i=1}^n \mathfrak{m}^i \widehat{A} = 0$ — so is its subalgebra $(\widehat{A})^{p^i}$. Hence the completion morphism (2) is injective and we conclude. □

4.3.2 Theorem *Let R be a ring and A a flat R–algebra of finite presentation. Assume that either $\dim(R) = 0$ or $R \to A$ is separable. Then the natural map*

$$\phi\colon \mathcal{O}(\pi_0(A/R)) \to A^{p^\infty/R}$$

is an isomorphism.

Proof As a first step, we claim that we may reduce to R noetherian. Suppose then that the result holds for R noetherian and let $R \to A$ as in the hypotheses. As discussed in Remark 4.1.2, we may find a noetherian subring $R_0 \subset R$ and $R_0 \to A_0$ flat and separable of finite type, with $A \cong R \otimes_{R_0} A_0$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}(\pi_0(A_0/R_0)) \arrow[r, "\cong"] \arrow[d] & A_0^{p^\infty/R_0} \arrow[r] \arrow[d] & A_0 \arrow[d] \\
\mathcal{O}(\pi_0(A/R)) \arrow[r, "\alpha"] & A^{p^\infty/R} \arrow[r] & A
\end{tikzcd}$$

All maps in the diagram are injective, by flatness of $R_0 \to A_0$, by Theorem 4.1.1 and by injectivity of $\mathcal{O}(\pi_0(A/R)) \to A$ (which in turn is due to the fact that $\mathrm{Spec}(A) \to \pi_0(A/R)$ is faithfully flat). We need to show that α is surjective. Let then $a \in A^{p^\infty/R}$. Up to extending R_0, we may assume that $a \in A_0$; we only need to show that $a \in A_0^{p^\infty/R_0}$. In other words, we have reduced to showing that

$$A^{p^\infty/R} \cap A_0 = A_0^{p^\infty/R_0}.$$

Consider the commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & A_0^{p^\infty/R_0} \arrow[r] \arrow[d] & A_0 \arrow[r, "f"] \arrow[d] & Q_0 \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A^{p^\infty/R} \arrow[r] & A \arrow[r] & Q_0 \otimes_{R_0} R \arrow[r] & 0
\end{tikzcd}$$

where Q_0 is by definition the cokernel of $A_0^{p^\infty/R_0} \to A_0$ and the bottom row is obtained from the top one by tensoring with R over R_0. By flatness of $R_0 \to A_0$, the central vertical arrow is injective; and, by Theorem 4.1.1, Q_0 is R_0–flat; hence, the map g is injective and both rows are exact. Now $gf(A_0 \cap A^{p^\infty/R}) = 0$, which implies that $f(A_0 \cap A^{p^\infty/R}) = 0$ and that $A_0 \cap A^{p^\infty/R}$ is contained in $A_0^{p^\infty/R_0}$. This proves the claim and we will from now on assume that R is noetherian.

In the case $\dim(R) = 0$, the subring R_0 above is artinian and the result is provided by Theorem 4.2.1. Hence we are left with the case where $R \to A$ is separable.

As our second step, we claim that we may reduce to the case of R complete local. Indeed, let $R \to R'$ be the completion of the local ring at some prime $\mathfrak{p} \subset R$. The morphism $R \to R'$ is flat. We have a map

$$\mathcal{O}(\pi_0(A \otimes_R R'/R')) = \mathcal{O}(\pi_0(A/R)) \otimes_R R' \to A^{p^\infty} \otimes_R R' \hookrightarrow (A \otimes_R R'/R')^{p^\infty}.$$

The first equality is compatibility of global sections and flat base change, the second arrow is $\phi \otimes_R R'$, while the last arrow is injective by Lemma 4.1.3. We see that, if the composition is an isomorphism, then also the central arrow $\phi \otimes_R R'$ is an isomorphism.

As $R_{\mathfrak{p}} \to R'$ is faithfully flat, the map $\phi \otimes_R R_{\mathfrak{p}}$ is also an isomorphism. Repeating the argument for all $\mathfrak{p} \subset R$, we find that ϕ is an isomorphism. This proves the claim.

We argue by induction on the dimension d of R. The case $d = 0$ was considered previously, so assume that $d \geqslant 1$ and the result is true for base rings of dimension at most $d-1$. We may assume R local and complete with respect to its maximal ideal. Let s be the closed point of $\mathrm{Spec}(R)$, and $V = S \setminus \{s\}$. Notice that V is of dimension $d-1$. Cover V with open affines $U_i = \mathrm{Spec}(R_i)$. Consider the commutative diagram of solid arrows

$$\begin{tikzcd} & A \arrow[r, dashed] \arrow[d, dashed] \arrow[dl, dashed, bend right] & \widehat{A} \arrow[d] \\ & \prod_i A \otimes_R R_i \arrow[r] \arrow[d] & \prod_i \widehat{A} \otimes_R R_i \arrow[d] \\ 0 \arrow[r] & \prod_{i,j} A \otimes_R R_i \otimes_R R_j \arrow[r] & \prod_{i,j} \widehat{A} \otimes_R R_i \otimes_R R_j \end{tikzcd}$$

Clearly, A admits natural compatible maps towards the diagram, represented by dashed arrows in the diagram.

Next, we take the preperfection of the diagram. By Lemma 4.3.1, we have $\widehat{A}^{p^\infty} = \mathcal{O}(\widehat{\pi})$. Moreover, for every R–algebra R', there is a natural map $\mathcal{O}(\widehat{\pi} \otimes_R R') = \widehat{A}^{p^\infty} \otimes_R R' \to (\widehat{A} \otimes_R R')^{p^\infty}$. Finally, by the induction hypothesis, $(A \otimes R_i)^{p^\infty} = \mathcal{O}(\pi(X_{U_i}/U_i))$. We get a commutative diagram

$$\begin{tikzcd} & A^{p^\infty} \arrow[r, dashed] \arrow[d, dashed] \arrow[dl, dashed, bend right] & \mathcal{O}(\widehat{\pi}) \arrow[d] \\ & \prod_i \mathcal{O}(\pi_0(X_{U_i}/U_i)) \arrow[r] \arrow[d] & \prod_i \mathcal{O}(\widehat{\pi}_{U_i}) \arrow[d] \\ 0 \arrow[r] & \prod_{i,j} \mathcal{O}(\pi_0(X_{U_{ij}}/U_{ij})) \arrow[r] & \prod_{i,j} \mathcal{O}(\widehat{\pi}_{U_{ij}}) \end{tikzcd}$$

where the horizontal arrows are those induced by the natural morphism $\psi\colon \widehat{\pi} \to \pi_0(X/S)$ of Section 3.4.3. The limit of the diagram of solid arrows coincides with the limit of the subdiagram of solid arrows in the commutative diagram

$$\begin{tikzcd} A^{p^\infty} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{O}(\widehat{\pi}) \arrow[d] \\ \mathcal{O}(\pi_0(X_V/V)) \arrow[r] & \mathcal{O}(\widehat{\pi}_V) \end{tikzcd} \tag{3}$$

Taking global sections in the pushout diagram of Proposition 3.4.5, we see that $\mathcal{O}(\pi_0(X/S))$ is a fibre product for the subdiagram (3) of solid arrows. Therefore we get a natural map $\chi\colon A^{p^\infty} \to \mathcal{O}(\pi_0(X/S))$. The maps

$$A^{p^\infty} \xrightarrow{\chi} \mathcal{O}(\pi_0(X/S)) \xrightarrow{\phi} A^{p^\infty}$$

are compatible with the natural inclusions of A^{p^∞} and $\mathcal{O}(\pi_0(X/S))$ into A. Hence ϕ is injective, and, because $\phi \circ \chi$ is the identity, it is also surjective, as we wished to show. □

With the notation of Theorem 4.3.2, the algebraic space $\pi_0(X/S)$ is étale; however, its R–algebra of global sections $\mathcal{O}(\pi_0(X/S))$ may fail to be unramified (and therefore étale and perfect); see for instance Lemma 4.5.2. In particular, the preperfection $A^{p^\infty/R}$ need not be perfect.

Here is still a simple, favourable case.

4.3.3 Corollary *Let R be a reduced, noetherian, one-dimensional ring of characteristic p. Let A be a flat, separable R–algebra of finite type with preperfection $A^{p^\infty/R}$.*

(1) *If R is geometrically unibranch, we have isomorphisms*

$$A^{\text{ét}} \xrightarrow{\sim} \mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}.$$

In particular, $A^{p^\infty/R}$ is étale, hence perfect and of finite type.

(2) *If R is excellent then $A^{p^\infty/R}$ is quasifinite, and in particular of finite type.*

Proof Since $\mathcal{O}(\pi_0(A/R)) \xrightarrow{\sim} A^{p^\infty/R}$ by Theorem 4.3.2, the results follow from Proposition 3.3.5. □

4.4 Perfection over regular rings

Here we show that, when the base scheme S is noetherian, formally unramified factorizations are dominated by étale ones. In characteristic p, this sheds light on the relation between coperfections and étale hulls. This relation becomes particularly simple over regular F–finite rings.

4.4.1 Lemma *Let S be a noetherian algebraic space. Let X and Y be S–algebraic spaces with $X \to S$ faithfully flat and finitely presented and $Y \to S$ formally unramified. Then any S–morphism $X \to Y$ factors uniquely as $X \to E \to Y$ with $E \to S$ étale, $X \to E$ faithfully flat and finitely presented and $E \to Y$ a monomorphism of algebraic spaces which is formally étale and of finite type.*

Proof The uniqueness is immediate: any such factorization consists of an epimorphism of fppf sheaves $X \to E$, followed by a monomorphism $E \to Y$; hence, E is identified with the image sheaf of $X \to Y$. Now we prove the existence. It is formal to check that the diagonal of a formally unramified morphism is formally étale. Hence $\Delta_{Y/S}: Y \hookrightarrow Y \times_S Y$ is a formally étale monomorphism locally of finite type; see [28, Tag 03HK]. By pullback we obtain a morphism $j: X \times_Y X \hookrightarrow X \times_S X$ with the same properties. Since $X \times_S X$ is noetherian, the map j is automatically locally of finite presentation; see [28, Tag 06G4]. Hence it is an open immersion. It follows that $R := X \times_Y X$ is the graph of an open equivalence relation on X. Let $E := X/R$ be the quotient sheaf, which is a quasicompact S–algebraic space.

To prove that $E \to Y$ is a monomorphism, it is enough to prove that $\Delta_{E/Y}: E \to E \times_Y E$ is an isomorphism. This follows because $X \to E$ is faithfully flat and finitely presented, and hence so is $X \times_Y X \to E \times_Y E$; applying this base change to $\Delta_{E/Y}$ results in the isomorphism $R \xrightarrow{\sim} X \times_E X$.

The map $E \to Y$ is formally unramified, because it is a monomorphism; hence so is $E \to S$. Moreover, $E \to S$ is also flat and of finite presentation, because these properties descend along the fppf covering $X \to E$. It follows that $E \to S$ is étale.

Since E is noetherian and $E \to S$ is of finite type, then also $E \to Y$ is of finite type. Finally, $E \to Y$ is formally étale by [13, 17.1.4] applied to the composition $E \to Y \to S$. $\square$

4.4.2 Corollary *Let S be a noetherian scheme and $X \to S$ be a faithfully flat, finitely presented S–algebraic space.*

(1) *Assume that the category of morphisms $X \to E$ to some étale, finitely presented S–algebraic space has an initial object π, for instance $\pi = \pi_0(X/S)$ when it exists as an algebraic space (see Lemma 3.1.1(2) for sufficient conditions). Then $X \to \pi$ is also initial among morphisms to all formally unramified S–algebraic spaces. In particular, if S has characteristic p then $X \to \pi$ is a coperfection of X/S.*

(2) *The map $X \to \pi^s(X/S)$ is initial among morphisms to all separated, formally unramified S–algebraic spaces. In particular, if S has characteristic p then $X \to \pi^s(X/S)$ is initial among morphisms to all separated, relatively perfect S–algebraic spaces.*

Proof (1) Note that $X \to \pi$ is surjective, because, if π' denotes the image, then $X \to \pi'$ also satisfies the universal property of the initial object. Let $X \to Y$ be a morphism to a formally unramified S–algebraic space. By Lemma 4.4.1, there is a factorization $X \to E \to Y$ with $E \to S$ étale and finitely presented. Set-theoretically, E is the image of $X \to Y$. By the universal property of π the map $X \to E$ factors further through π. Uniqueness of the factorization follows from the fact that $X \to \pi$ is an fppf epimorphism.

Assume S has characteristic p. Since a relatively perfect morphism is formally unramified, we see that $X \to \pi$ satisfies the universal property of the coperfection for $X \to S$.

(2) Note that $X \to \pi^s(X/S)$ exists and is surjective by [8, théorème 2.3.4]. Keeping the same notation as before, if $Y \to S$ is separated then so is $E \to S$. Hence the proof carries over similarly in this case. ◻

Here is a consequence for perfection over regular F–finite base rings. A remarkable feature of this result is that no separability hypothesis is needed; this generalizes Theorem 4.3.2 because π_0 and π^s have the same rings of global functions, as we showed in Lemma 3.3.4.

4.4.3 Corollary *Let R be a regular F–finite ring of characteristic p. Let A be a flat R–algebra of finite type with preperfection $A^{p^\infty/R}$. Then we have an isomorphism $\mathcal{O}(\pi^s(A/R)) \xrightarrow{\sim} A^{p^\infty/R}$.*

Proof Write $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $A^{p^\infty} = A^{p^\infty/R}$. We have seen in Remarks 4.1.4 that in this case A^{p^∞} is R–perfect. By Corollary 4.4.2, the map $X \to \mathrm{Spec}(A^{p^\infty})$ factors through $\pi^s(X/S)$. Since $\pi^s(X/S) \to S$ is (étale and hence) perfect, for each $i \geqslant 0$ this factors further as

$$X \to X^{p^i/S} \to \pi^s(X/S) \to \mathrm{Spec}(A^{p^\infty}).$$

Taking affine hulls and passing to the limit over i provides the row of the diagram

$$\begin{tikzcd} A^{p^\infty} \arrow[r] \arrow[dr] & \mathcal{O}(\pi^s(A/R)) \arrow[r] \arrow[d, hook] & A^{p^\infty} \arrow[dl] \\ & A & \end{tikzcd}$$

The two maps of the row compose to the identity. Since $X \to \pi^s(X/S)$ is faithfully flat and hence dominant, the vertical map is injective. We deduce that the maps in the row are inverse bijections. ◻

4.5 Examples

We shall see that the coperfection of the spectrum of an algebra is in general not the spectrum of its perfection. In fact, in the flat and separable case the coperfection of an affine scheme is π_0 and may be nonseparated. Here is an example.

4.5.1 Lemma *Let R be a noetherian ring, $u \in R$ and set*

$$A = \frac{R[x,y,(x-y)^{-1}]}{(xy-u)}.$$

Let $X = \mathrm{Spec}(A) \to S = \mathrm{Spec}(R)$ be the associated map of schemes. Then $\pi_0(X/S)$ consists of two copies of S glued along $S_u := \mathrm{Spec}(R_u)$, and we have

$$A^{p^\infty} = \{(a,b) \in R \times R \mid u^n(a-b) = 0 \text{ for some } n \geqslant 0\}.$$

Proof Notice first that the map $X \to S$ is flat and separable. Consider the Zariski cover of X given by the two opens $X_x = \mathrm{Spec}(A_x)$ and $X_y = \mathrm{Spec}(A_y)$. Both X_x/S and X_y/S have geometrically connected fibres; it follows from the universal property of π_0 that the composition

$$X_x \sqcup X_y \twoheadrightarrow X \twoheadrightarrow \pi_0(X/S)$$

factors via $\pi_0(X_x/S) \sqcup \pi_0(X_y/S) = S \sqcup S$. This gives us an étale surjection $S \sqcup S \to \pi_0(X/S)$. The locus where the two maps $S \to \pi_0(X/S)$ coincide is the open in S where the two opens X_x and X_y coincide, ie S_u; this justifies the claimed presentation of $\pi_0(X/S)$. Finally, by Theorem 4.3.2, $A^{p^\infty} = \mathcal{O}(\pi_0(X/S))$, and the latter consists of the pairs $(a,b) \in R \times R$ such that $a = b$ in R_u. □

The following is the most basic example of a nonperfect preperfection, that is, an R–algebra A which is flat, separable, of finite presentation, for which the preperfection $A^{p^\infty/R}$ is not perfect. The ring R is one-dimensional; we remark that, in accordance with Proposition 3.3.5, we need to choose R with multiple branches. Since the preperfection is not perfect, it is natural to ask what happens if we take the preperfection once more. Here is the answer:

4.5.2 Lemma *Let $R = \mathbb{F}_p[\![u,v]\!]/(uv)$ and $A = R[x,y,(x-y)^{-1}]/(xy-u)$. If $p \neq 2$, we have:*

(1) *$A^{p^\infty} \simeq R[\alpha]/(u\alpha, v^2 - \alpha^2)$ mapping to A by $\alpha \mapsto v(x+y)/(x-y)$.*

(2) *$(A^{p^\infty})^{p^\infty} \simeq R$.*

Notice that the restriction of $R \to A^{p^\infty}$ to the branch $\{u=0\}$ is

$$\mathbb{F}_p[\![v]\!] \to \frac{\mathbb{F}_p[\![v]\!][\alpha]}{(v^2-\alpha^2)},$$

which is not formally étale. Therefore ϕ itself is not formally étale and in particular not relatively perfect by Lemma 2.4.1. The restriction $p \neq 2$ allows a simpler presentation of A^{p^∞} but is inessential.

Proof (1) We apply Lemma 4.5.1 to find that $A^{p^{\infty}/R}$ is the sub-R–module of $R \times R$ generated by $(1,1)$ and $(v,0)$. As $p \neq 2$, we may choose $(1,1)$ and $(v,-v)$ as generators instead. Writing $\alpha=(v,-v)$, we obtain

$$A^{p^\infty} = \frac{R[\alpha]}{(u\alpha, \alpha^2-v^2)}.$$

The map $A^{p^\infty} \to A$ is induced by the natural map $R \times R = (A_x)^{p^\infty} \times (A_y)^{p^\infty} \to A_x \times A_y$ that sends $(1,0)$ and $(0,1)$ to the idempotents $(x/(x-y),0)$ and $(0,y/(y-x))$, respectively. It follows that $\alpha=(v,-v)$ is sent to $v(x+y)/(x-y) \in A$.

(2) Let $B=A^{p^\infty}$. Notice first that any element of B can be written uniquely as $f+g\alpha$, with $f \in R$ and $g \in R/u$. Therefore, any element of $B^{(p^n)} = B \otimes_{R,F^n} R$ takes either the form $1 \otimes f$ with $f \in R$ or $\alpha \otimes g$ with $g \in R/u^{p^n}$. In fact, the map of R–modules

$$B^{(p^n)} \to R \oplus R/u^{p^n}, \qquad 1 \otimes f \mapsto (f,0), \quad \alpha \otimes g \mapsto (0,g),$$

is an isomorphism, which we will use to rewrite the preperfection diagram of B. The n^{th} map in the diagram is $B^{(p^n)} \to B^{(p^{n-1})}$ sending $1 \otimes f$ to $1 \otimes f$ and $\alpha \otimes g$ to $\alpha^p \otimes g = v^{p-1}\alpha \otimes g = \alpha \otimes v^{p^n-p^{n-1}} g$. Using the isomorphism of R–modules above, this becomes the map of R–modules

$$G_n \colon R \oplus R/u^{p^n} \to R \oplus R/u^{p^{n-1}}$$

sending (f,g) to $(f, gv^{p^n-p^{n-1}})$. Consider now the preperfection diagram

$$\cdots \xrightarrow{G_{n+1}} R \oplus R/u^{p^n} \xrightarrow{G_n} R \oplus R/u^{p^{n-1}} \xrightarrow{G_{n-1}} \cdots \xrightarrow{G_1} R \oplus R/u.$$

Let $H_n = G_1 \circ \cdots \circ G_n \colon R \oplus R/u^{p^n} \to R \oplus R/u$ and let $(\ldots, a_n, a_{n-1}, \ldots, a_0)$ be an element of the limit of the diagram. We can of course consider the limit in the category of R–modules, as it will automatically have an R–algebra structure making it into the limit in the category of R–algebras. Now, the image of $(f,g) \in R \oplus R/u^{p^n}$ via H_n is (f, gv^{p^n-1}). Hence $a_0=(f_0,g_0)$ is such that, for every $n \geqslant 1$, g_0 is in the ideal of R/u generated by v^{p^n-1}. Therefore $g_0 = 0$. One can use the same argument to show that, for every $a_n = (f_n, g_n)$, g_n vanishes. Therefore the limit is simply the limit of the diagram

$$\cdots \xrightarrow{\text{id}} R \xrightarrow{\text{id}} R \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} R.$$

This shows that $B^{p^\infty} = R$. □

5 Unramified F–divided objects and the étale fundamental pro-groupoid

In this section, we define the étale fundamental pro-groupoid $\mathscr{X} \to \Pi_1(\mathscr{X}/S)$ of a flat finitely presented algebraic stack and we prove Theorem A, namely that if moreover $\mathscr{X}/S$ is separable or $\dim(S) = 0$, and $\mathscr{M}/S$ is a Deligne–Mumford stack, there is an equivalence $\mathsf{Hom}(\Pi_1(\mathscr{X}/S), \mathscr{M}) \to \mathsf{Hom}(\mathscr{X}, \mathrm{Fdiv}(\mathscr{M}/S))$. As a first step, in Section 5.1 we build on Theorem 4.3.2 to prove this when $\mathscr{X}$ and $\mathscr{M}$ are algebraic spaces. Then, in Section 5.2, we introduce the étale fundamental pro-groupoid and its basic properties. In Section 5.3, we show how to use smooth atlases of a stack in order to construct enough étale factorizations to describe $\Pi_1(\mathscr{X}/S)$. In Section 5.4, we show how $\Pi_1(\mathscr{X}/S)$ relates with Grothendieck's enlarged fundamental group and Borne and Vistoli's fundamental gerbe, when the base is a field. Finally, in Section 5.5, we prove that such factorizations enjoy a key pushout property; this technical fact is the heart of the argument for the proof of the main theorem, which we derive in Section 5.6.

Some material on groupoid closures is needed to handle $\Pi_1(\mathscr{X}/S)$. In order to spare the reader unpleasant technicalities, this has been relegated to Appendix A.

As we observed in Remark 2.3.3, the canonical equivalence

$$\mathsf{Hom}(\mathscr{X}, \mathrm{Fdiv}(\mathscr{M})) = \mathsf{Hom}(\mathscr{X}^{\mathrm{copf}}, \mathscr{M})$$

allows an equivalent interpretation of the result in terms of the coperfection of $\mathscr{X}$. The interplay between the two viewpoints pervades the section, and the proofs.

5.1 The case of algebraic spaces

Let S be an algebraic space of characteristic p. We denote by $S_{\mathrm{perf,ét}}$ its *perfect-étale* site, which is the category of relatively perfect S–algebraic spaces endowed with the étale topology (see [19]).

Let X be a flat, finitely presented, separable S–algebraic space. The algebraic space $\pi_0(X/S)$ is relatively perfect over S. Thus the natural morphism $\mathrm{Fdiv}(\pi_0(X/S)) \to \pi_0(X/S)$ is an isomorphism, and we obtain a natural morphism

$$\rho\colon X \to \pi_0(X/S) \xrightarrow{\sim} \mathrm{Fdiv}(\pi_0(X/S)).$$

5.1.1 Theorem *Let S be an algebraic space of characteristic p and $X \to S$ a flat, finitely presented algebraic space. Assume that either* $\dim(S) = 0$ *or $X \to S$ is separable. Let $M \to S$ be an arbitrary algebraic space. Then the natural morphism given by $\alpha \mapsto \mathrm{Fdiv}(\alpha) \circ \rho$,*

$$\mathscr{H}om(\pi_0(X/S), M) \xrightarrow{\sim} \mathscr{H}om(X, \mathrm{Fdiv}(M/S)),$$

is a bifunctorial isomorphism of sheaves on $S_{\mathrm{perf,ét}}$.

We make three remarks before giving the proof.

5.1.2 Remarks (1) In terms of coperfection, this theorem says that, if $X \to S$ is a flat, finitely presented, separable morphism of algebraic $\mathbb{F}_p$–spaces, then the inductive system of relative Frobenii

$$X \xrightarrow{F_{X/S}} X^{p/S} \xrightarrow{F_{X^{p}/S}} X^{p^2/S} \to \cdots$$

admits a colimit in the category of algebraic spaces over S; the colimit is the algebraic space $\pi_0(X/S)$, and is also a coperfection of $X \to S$.

(2) Point (1) is remarkable if we consider that, for a ring R and a flat, finitely presented separable algebra $R \to A$, taking the preperfection of A, ie the limit of relative Frobenius morphisms

$$\cdots \to A^{p^2/R} \xrightarrow{F_{A^{p}/R}} A^{p/R} \xrightarrow{F_{A/R}} A,$$

does not guarantee to produce a perfect object, as illustrated in Lemma 4.5.2.

(3) In the particular case $M = \mathbb{A}^1_S$, we find $\mathcal{O}(\pi_0(X/S)) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Fdiv}(\mathbb{A}^1_S)) = \lim \mathcal{O}(X^{p^n/S})$. If $S = \mathrm{Spec}(R)$ and R is regular (in which case the absolute Frobenius of R is flat), then $\mathcal{O}(X^{p^n/S}) = \mathcal{O}(X)^{p^n/S}$ and Theorem 5.1.1 gives us

$$\mathcal{O}(\pi_0(X/S)) \xrightarrow{\sim} \mathcal{O}(X)^{p^\infty/R},$$

which generalizes Theorem 4.3.2.

Proof Throughout, we write $\pi_0(X)$ and $\mathrm{Fdiv}(M)$ instead of $\pi_0(X/S)$ and $\mathrm{Fdiv}(M/S)$. Let $\rho_0\colon X \to \pi_0(X)$ be the natural map. Since $\pi_0(X) \to S$ is perfect, we have a canonical isomorphism $\mathscr{H}om(\pi_0(X), M) = \mathscr{H}om(\pi_0(X), \mathrm{Fdiv}(M))$, so the statement to be proven is that

$$\Phi := \rho_0^* \colon \mathscr{H}\!om(\pi_0(X), \mathrm{Fdiv}(M)) \xrightarrow{\sim} \mathscr{H}\!om(X, \mathrm{Fdiv}(M))$$

is a bifunctorial isomorphism of sheaves on $S_{\mathrm{perf,ét}}$.

We start with easy observations. Obviously we can assume that S is affine. Since the formation of $\pi_0(X)$ and $\mathrm{Fdiv}(M)$ is compatible with perfect base change, it is enough to consider the sections over S and prove that we have a bijection of Hom sets. Also, the injectivity part is clear because ρ_0 is an epimorphism of sheaves.

First we reduce to the case M affine. We are free to fix a morphism $u \colon X \to M$ and prove that Φ induces a bijection between the subsets $\mathrm{Hom}_u(\pi_0(X), \mathrm{Fdiv}(M))$ and $\mathrm{Hom}_u(X, \mathrm{Fdiv}(M))$ of maps that induce u. Since X is quasicompact, the map u factors through a quasicompact open subspace $M' \subset M$, and all maps in the above Hom_u subsets factor through $\mathrm{Fdiv}(M')$. Therefore, replacing M by M' if necessary, we can assume that M is quasicompact. Let $g \colon V \to M$ be an étale surjection with V an affine scheme. The map $X \times_M V \to X$ satisfies the hypotheses of Lemma B.1. Hence there exists a quasicompact open subalgebraic space $U' \subset X \times_M V$ such that the map $U' \to X$ is surjective (and étale). We take $U \to U'$ étale surjective with U a scheme of finite presentation over S. As the composition $U \to U' \to X \times_M V \to X$ is étale, we deduce that $U \to S$ is flat and separable.

Now start from a map $f \colon X \to \mathrm{Fdiv}(M)$. Taking into account that $\mathrm{Fdiv}(V) \xrightarrow{\sim} \mathrm{Fdiv}(M) \times_M V$ — see Section 2.3.2(v) — by pullback along $V \to M$ we obtain a map $X_V \to \mathrm{Fdiv}(V)$ and by precomposition a map $f' \colon U \to \mathrm{Fdiv}(V)$. By assumption, since V is affine, the map

$$\mathrm{Hom}(\pi_0(U), \mathrm{Fdiv}(V)) \to \mathrm{Hom}(U, \mathrm{Fdiv}(V))$$

is an isomorphism; hence f' factors uniquely via $\pi_0(U)$. By the pushout property of Lemma 3.4.2, the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
\pi_0(U) \arrow[r] \arrow[dr] & \pi_0(X) \arrow[dr, dashed] & \\
& \mathrm{Fdiv}(V) \arrow[r] & \mathrm{Fdiv}(M)
\end{tikzcd}$$

can be completed by a dashed arrow and the claim is proven.

We now reduce to the case X affine. Let $U \to X$ be an étale atlas with U affine. Starting from a map $X \to \mathrm{Fdiv}(M)$, by assumption the composition $U \to X \to \mathrm{Fdiv}(M)$ factors through $\pi_0(U)$. Using once more the pushout of Lemma 3.4.2, the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
\pi_0(U) \arrow[r] \arrow[drr, bend right] & \pi_0(X) \arrow[dr, dashed] & \\
& & \mathrm{Fdiv}(M)
\end{tikzcd}$$

can be completed by a dashed arrow and this completes the proof.

To conclude when S, M and X are affine, let X^{copf} be the coperfection in the sense of sheaves as in Section 2.3.1, and compute

$$\begin{aligned}
\mathrm{Hom}(X, \mathrm{Fdiv}(M)) &= \mathrm{Hom}(X^{\mathrm{copf}}, M) && \text{(by Remark 2.3.3)} \\
&= \lim \mathrm{Hom}(X^{p^i}, M) \\
&= \lim \mathrm{Hom}(\mathcal{O}(M), \mathcal{O}(X^{p^i})) && \text{(because } M \text{ is affine)} \\
&= \mathrm{Hom}(\mathcal{O}(M), \lim \mathcal{O}(X^{p^i})) \\
&= \mathrm{Hom}(\mathcal{O}(M), \mathcal{O}(\pi_0(X))) && \text{(by Theorem 4.3.2)} \\
&= \mathrm{Hom}(\pi_0(X), M) && \text{(because } M \text{ is affine)} \\
&= \mathrm{Hom}(\pi_0(X), \mathrm{Fdiv}(M)). && \square
\end{aligned}$$

5.2 The étale fundamental pro-groupoid

In this subsection, the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ of a flat finitely presented algebraic stack $\mathscr{X}/S$ is defined as a 2–*pro-object* of the 2–category of algebraic stacks. Let us recall the definition of this concept. For more details, we refer to Descotte and Dubuc [4].

5.2.1 Definition A nonempty 2–category $\mathcal{I}$ is 2–*cofiltered* if it satisfies the following conditions:

(1) Given two objects $i, j \in \mathcal{I}$, there is an object $k \in \mathcal{I}$ and arrows $k \to i$ and $k \to j$.

(2) Given two arrows $f, g\colon j \to i$, there is an arrow $h\colon k \to j$ and a 2–isomorphism $\alpha\colon fh \to gh$.

(3) Given two 2–arrows $\alpha, \beta\colon f \to g$, where $f, g \in \mathrm{Hom}_{\mathcal{I}}(j, i)$, there is an arrow $h\colon k \to j$ such that $\alpha h = \beta h$.

Clearly, a nonempty 1–category is cofiltered if and only if it is 2–cofiltered when seen as a 2–category.

5.2.2 Definition A 2*–pro-object* of a 2–category $\mathcal{C}$ is a 2–functor $F:\mathcal{I}\to\mathcal{C}$ from an essentially small 2–cofiltered 2–category $\mathcal{I}$. The 2–category of 2–pro-objects of $\mathcal{C}$ is denoted by 2–Pro($\mathcal{C}$). The category of morphisms between two 2–pro-objects $F:\mathcal{I}\to\mathcal{C}$ and $G:\mathcal{J}\to\mathcal{C}$ is

$$\mathsf{Hom}_{2\text{–Pro}(\mathcal{C})}(F,G) := \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathsf{Hom}_{\mathcal{C}}(F(i),G(j)),$$

where lim (resp. colim) is the pseudolimit (resp. pseudocolimit) for strict 2–categories; see [4, Proposition 2.1.5]. In particular, by a *pro-*(*algebraic stack*) we mean a 2–pro-object of the 2–category **AlgStack** of algebraic stacks.

The index 2–category for defining Π_1 will be a 2–category of factorizations similar to that of Definition 3.2.1, with the difference that the étale part $\mathscr{E}\to S$ is allowed to be an algebraic stack rather than an algebraic space. More generally, it is relevant to introduce categories of factorizations $\mathscr{X}\to\mathscr{E}\to S$ through an étale n–stack $\mathscr{E}\to S$. It is natural to denote such categories $\mathsf{E}_n(\mathscr{X}/S)$; the category of Definition 3.2.1 is $\mathsf{E}_0(\mathscr{X}/S)$ while the category introduced below is $\mathsf{E}_1(\mathscr{X}/S)$. We leave the study of such categories for $n\geqslant 2$ for subsequent investigation. In fact, since from now on no other factorization category will appear in the article, for simplicity we keep the notation $\mathsf{E}(\mathscr{X}/S)$.

5.2.3 Definition Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. We define $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)=\mathsf{E}_1^{\mathrm{surj}}(\mathscr{X}/S)$ to be the following 2–category:

- Objects are factorizations $\mathscr{X}\xrightarrow{h}\mathscr{E}\to S$, where $\mathscr{E}/S$ is an étale algebraic stack and h is faithfully flat.
- 1–Arrows $(\mathscr{X}\xrightarrow{h}\mathscr{E}\to S)\to(\mathscr{X}\xrightarrow{h'}\mathscr{E}'\to S)$ are pairs (f,α), with $f:\mathscr{E}\to\mathscr{E}'$ and $\alpha: fh\to h'$ giving a 2–commutative diagram

$$\begin{tikzcd} & \mathscr{E} \arrow[dd, "f"] \arrow[dr] & \\ \mathscr{X} \arrow[ur, "h"] \arrow[dr, "h'"'] & & S \\ & \mathscr{E}' \arrow[ur] & \end{tikzcd}$$

- 2–Arrows $(f,\alpha)\to(g,\beta)$ are 2–morphisms $u: f\to g$ giving a commutative diagram

$$\begin{tikzcd} fh \arrow[r, "uh"] \arrow[dr, "\alpha"'] & gh \arrow[d, "\beta"] \\ & h' \end{tikzcd}$$

The stacks $\mathscr{E} \to S$ are quasicompact; this is automatic from the same property for $\mathscr{X} \to S$. The condition that the factorizations $\mathscr{X} \to \mathscr{E}$ are flat is redundant, as this follows from flatness of $\mathscr{X} \to S$ together with étaleness of $\mathscr{E} \to S$.

5.2.4 Lemma *Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. The 2–category $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is nonempty and 2–cofiltered. Moreover, it is equivalent to a 1–category.*

Proof The category $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is nonempty, because it contains the factorization with $\mathscr{E}$ equal to the image of $\mathscr{X}$ in S, which is open in S and hence étale over S. Next, we check the three conditions for 2–cofilteredness.

(1) Given two factorizations $h\colon \mathscr{X} \to \mathscr{E}$ and $h'\colon \mathscr{X} \to \mathscr{E}'$, there is the common refinement $\mathscr{X} \to \mathscr{E} \times_S \mathscr{E}'$ and 2–commutative diagram

$$\begin{tikzcd} \mathscr{X} \arrow[dr] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\ & \mathscr{E} \times_S \mathscr{E}' \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ & \mathscr{E}' \arrow[r] & S \end{tikzcd}$$

Take the image $\mathscr{E}''$ of $\mathscr{X} \to \mathscr{E} \times_S \mathscr{E}'$; this is an open substack of $\mathscr{E} \times_S \mathscr{E}'$. Then $\mathscr{E}''$ is again an étale S–stack and $h''\colon \mathscr{X} \to \mathscr{E}''$ is a common refinement of h and h' in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$.

(2) Given two morphisms (f,α) and (g,β),

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, dashed, bend right, "h''"'] \arrow[d, "h"] \arrow[dr, bend left, "h'"] & \\ \mathscr{E}'' \arrow[r, dashed, "(k,\gamma)"] & \mathscr{E} \arrow[r, shift left, "(f,\alpha)"] \arrow[r, shift right, "(g,\beta)"'] & \mathscr{E}' \end{tikzcd}$$

we want to find a third morphism $(k,\gamma)\colon \mathscr{E}'' \to \mathscr{E}$ and a 2–isomorphism $u\colon fk \to gk$. For this we consider the 2–fibred product

$$\begin{tikzcd} \mathscr{E}'' \arrow[r] \arrow[d, "k"] & \mathscr{E}' \arrow[d, "\Delta"] \\ \mathscr{E} \arrow[r, "(f,g)"] & \mathscr{E}' \times_S \mathscr{E}' \end{tikzcd}$$

Then u is given by definition. Moreover, the morphisms $h\colon \mathscr{X} \to \mathscr{E}$ and $h'\colon \mathscr{X} \to \mathscr{E}'$ and the 2–commutativity isomorphisms

$$fh \xrightarrow{\alpha} h' \xrightarrow{\beta^{-1}} gh$$

provide a morphism $(h,h')\colon \mathscr{X} \to \mathscr{E}''$. Finally, we replace $\mathscr{E}''$ by the image of $\mathscr{X} \to \mathscr{E}''$, so as to get an object of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$.

(3) Given two morphisms (f,α) and (g,β) and two 2–morphisms $u,v\colon (f,\alpha) \to (g,\beta)$,

$$\begin{tikzcd}
 & \mathscr{X} \arrow[dl, dashed, bend right, "h''"'] \arrow[d, "h"] \arrow[dr, bend left, "h'"] & \\
\mathscr{E}'' \arrow[r, dashed, "(k,\gamma)"] & \mathscr{E} \arrow[r, bend left, "(f,\alpha)"] \arrow[r, bend right, "(g,\beta)"'] \arrow[r, phantom, "\Downarrow u \ \Downarrow v"] & \mathscr{E}'
\end{tikzcd}$$

we want to find a third morphism $(k,\gamma)\colon \mathscr{E}'' \to \mathscr{E}$ such that $uk = vk$. For this we view f and g as $\mathscr{E}$–valued points of the stack $\mathscr{E}'$ and u and v as sections of the Isom functor $I := \mathrm{Isom}_{\mathscr{E}}(f,g) \to \mathscr{E}$, that is $u,v\colon \mathscr{E} \to \mathrm{Isom}_{\mathscr{E}}(f,g)$. Since the diagonal of $\mathscr{E}'$ is an étale morphism, the map $I \to \mathscr{E}$ is representable and étale, so its diagonal is an open immersion. We consider the fibred product

$$\begin{tikzcd}
\mathscr{E}'' \arrow[r] \arrow[d, hook] & I \arrow[d, hook, "\Delta"] \\
\mathscr{E} \arrow[r, "{(u,v)}"] & I \times_{\mathscr{E}} I
\end{tikzcd}$$

The 2–commutativity isomorphisms

$$fh \xrightarrow{\alpha} h' \xrightarrow{\beta^{-1}} gh$$

provide a morphism $\mathscr{X} \to I$. Moreover, the conditions $\beta \circ uh = \beta \circ vh = \alpha$ ensure that $(uh, vh) = (\beta^{-1}\alpha, \beta^{-1}\alpha)$, that is, we have a commutative square

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\beta^{-1}\alpha"] \arrow[d, "h"'] & I \arrow[d, hook, "\Delta"] \\
\mathscr{E} \arrow[r, "{(u,v)}"] & I \times_{\mathscr{E}} I
\end{tikzcd}$$

We deduce a morphism $h''\colon \mathscr{X} \to \mathscr{E}''$. Moreover, since we have the diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "h''"] \arrow[dr, two heads, "h"'] & \mathscr{E}'' \arrow[d, hook, "k"] \\
 & \mathscr{E}
\end{tikzcd}$$

where the map h is surjective, the vertical inclusion is in fact an isomorphism. Hence the two 2–morphisms u and v are equalized by an isomorphism $k\colon \mathscr{E}'' \to \mathscr{E}$.

In particular, we see that for any two such morphisms (f, α) and (g, β), there is at most one 2–isomorphism between them, thus $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is equivalent to a 1–category. $\square$

5.2.5 Definition Let $\mathscr{X}/S$ be a flat finitely presented algebraic stack. We define the *étale fundamental pro-groupoid* $\Pi_1(\mathscr{X}/S)$ of $\mathscr{X}$ to be the pro-algebraic stack

$$\Pi_1(\mathscr{X}/S)\colon \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S) \to \mathbf{AlgStack}_S, \quad \{\mathscr{X} \to \mathscr{E}\} \mapsto \mathscr{E}.$$

5.2.6 Remarks (1) By definition, the étale fundamental pro-groupoid is a pro-(étale stack). In what follows we will just write "pro-étale stack" but this should not be confused with a (pro-étale) stack.

(2) Strictly speaking, the index category $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is not essentially small because it includes étale stacks $\mathscr{E} \to S$ with arbitrarily unbounded inertia. This is not a serious problem (see eg the beginning of [4, Section 1]). To fix this, one may for instance fix a large enough cardinal κ and restrict to factorizations such that the cardinality of the fibres of the inertia of $\mathscr{E}$ is bounded by κ. In fact we will not really be concerned by this problem, because we will work with algebraic stacks for which we can replace $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ by the essentially small category $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$; see Lemma 5.3.4.

The pro-algebraic stack $\Pi_1(\mathscr{X}/S)$ comes with a canonical morphism $\mathscr{X} \to \Pi_1(\mathscr{X}/S)$. This object defines a 2–functor

$$\Pi_1\colon \mathbf{FlStack}_S \to 2\text{–}\operatorname{Pro}(\mathbf{\acute{E}tStack}_S)$$

from the 2–category of flat finitely presented algebraic stacks over S to the 2–category of pro-(étale stacks over S). It is tautological from its definition that the 2–functor $\Pi_1(-/S)$ is pro-left adjoint to the inclusion $\mathbf{\acute{E}tStack}_S \hookrightarrow \mathbf{FlStack}_S$. Finally, if either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable, the space of connected components $\pi_0(\mathscr{X}/S)$ is a member of the category $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$; see Lemma 3.1.1(2)–(3). It follows that there is a morphism $\Pi_1(\mathscr{X}/S) \to \pi_0(\mathscr{X}/S)$ with target the constant 2–pro-object. This morphism is easily seen to be universal for morphisms from $\Pi_1(\mathscr{X}/S)$ to an étale algebraic space; we call it the *coarse moduli space*.

5.3 Π_1 via smooth atlases

The main results of this section will hold under an extra assumption on the base space S, a slight generalization of quasicompactness, which we now introduce.

5.3.1 Definition A topological space is τ–quasicompact if it admits an open cover $\{S_i\}_{i\in I}$ by open quasicompacts such that, for every $i \in I$, there are only finitely many $j \in I$ for which $S_j \cap S_i \neq \varnothing$.

From now on S will be a τ–quasicompact algebraic space. As before we let $\mathscr{X} \to S$ be a flat, finitely presented algebraic stack and assume additionally that either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable. By Lemma B.2, there exists a smooth atlas $U \to \mathscr{X}$ with U/S finitely presented; let $R = U \times_{\mathscr{X}} U$. In the sequel, for simplicity let us write $\pi_0(U)$ for $\pi_0(U/S)$. The groupoid presentation $R \rightrightarrows U$ of $\mathscr{X}$ induces a 2–commutative diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \pi_0(R) \arrow[r, shift left] \arrow[r, shift right] & \pi_0(U) \arrow[r] & {[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]} \end{tikzcd}$$

where $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ is the quotient stack of the groupoid in étale algebraic spaces $\pi_0(R)^{\mathrm{gpd}} \rightrightarrows \pi_0(U)$ naturally associated to $\pi_0(R) \rightrightarrows \pi_0(U)$. For details on the construction of the groupoidification, see Appendix A. The quotient $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ is an étale stack over S and a 2–coequalizer for $\pi_0(R) \rightrightarrows \pi_0(U)$ in the category of algebraic stacks; see Corollary A.7.1. The map $\mathscr{X} \to [\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$, which results from $R \rightrightarrows U \to \mathscr{X}$ being a 2–coequalizer, is surjective, since so is $U \to [\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$. Hence the factorization $\mathscr{X} \to [\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ is an object of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$.

Notationally speaking, as a general rule we will write $[U/R]$ the stack coequalizer of a pair of maps of spaces $R \rightrightarrows U$ when it exists, so in particular $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ can be shortened to $[\pi_0(U)/\pi_0(R)]$.

5.3.2 Example Here is an example showing that $\pi_0(R)^{\mathrm{gpd}}$ need not be quasicompact and that therefore $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ need not be quasiseparated.

Let $S = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field, $n \geqslant 2$ an integer, and let X be a so-called Néron polygon, that is, the projective curve having irreducible components $X_i = \mathbb{P}^1_k$ for $i \in \mathbb{Z}/n\mathbb{Z}$, and with X_i meeting X_{i-1} and X_{i+1} at distinct nodes p_i and p_{i+1}. The components are therefore arranged in a circle. Notice that $H^1(X_{\text{ét}}, \mathbb{Z}) = H^1(X_{\mathrm{Zar}}, \mathbb{Z}) = \mathbb{Z}$; that is, X admits a nontrivial $\mathbb{Z}$–torsor (see [14, Examples]).

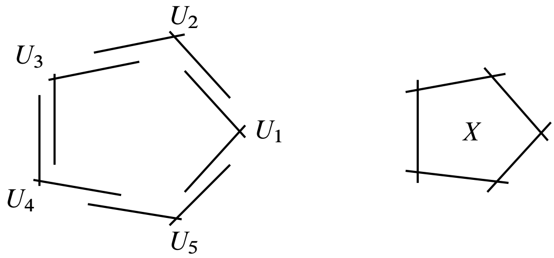
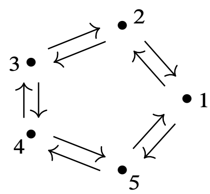


Figure 1: A cover of the Néron n–gon.

Let $U = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n \to X$ be the Zariski cover where U_i is the open subset of X given by $X_i \cup X_{i+1} \setminus \{p_i, p_{i+2}\}$. Write $R = U \times_X U \rightrightarrows U$ for the associated groupoid; see Figure 1.

The 2–étale space $\pi_0(U_\bullet)$ obtained by applying π_0 to this groupoid is the one described in Example A.7.2. It can be pictured as follows:

$\pi_0(R)$:



The example shows that the groupoidification $\pi_0(R)^{\mathrm{gpd}} \rightrightarrows \pi_0(U)$ is equivalent to the groupoid $\mathbb{Z} \rightrightarrows \{\star\}$, with composition given by addition,

$$\pi_0(R)^{\mathrm{gpd}}\colon\ \cdots \rightleftarrows \overset{5}{\bullet} \rightleftarrows \overset{1}{\bullet} \rightleftarrows \overset{2}{\bullet} \rightleftarrows \overset{3}{\bullet} \rightleftarrows \overset{4}{\bullet} \rightleftarrows \overset{5}{\bullet} \rightleftarrows \overset{1}{\bullet} \rightleftarrows \cdots.$$

In particular, $\pi_0(R)^{\mathrm{gpd}}$ is infinite and the quotient stack $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ is $B\mathbb{Z}$.

5.3.3 Definition Assume that S is quasiseparated and τ–quasicompact. Let $\mathscr{X}/S$ be a flat, finitely presented, algebraic stack. Assume that either $\mathscr{X} \to S$ is separable or $\dim(S) = 0$. We define $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ to be the full subcategory of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, which consists of objects of the form

$$\mathscr{X} \to [\pi_0(U/S)/\pi_0(R/S)^{\mathrm{gpd}}],$$

where $U \to \mathscr{X}$ is a smooth atlas with $U \to S$ finitely presented and $R := U \times_{\mathscr{X}} U$.

5.3.4 Lemma *The category $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ is essentially small. The inclusion functor $i : \mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S) \hookrightarrow \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ is initial. In particular, the full subcategory $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ is cofiltered.*

Proof The fact that $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ is essentially small (ie its isomorphism classes of objects form a set) is standard; we recall briefly the argument. First, we consider the set of open substacks $\mathscr{U} \subset \mathscr{X}$. By choosing a groupoid presentation $\mathscr{X} \simeq [U/R]$ and recalling the explicit description of the quotient stack $[U/R]$, one sees that, for all S–schemes T, the category $\mathscr{X}(T)$ is essentially small. Therefore, for each $\mathscr{U}$ we can choose a set of standard fppf coverings $\{f_j : T_j \to \mathscr{U}\}_{j=1,\dots,m}$ as in [28, Tag 021L]. By construction, any fppf atlas $U \to \mathscr{X}$ is refined by a set of open substacks $\{\mathscr{U}_i\}$ and sets of fppf coverings $\{f_{ij} : T_{ij} \to \mathscr{U}_i\}_{j=1,\dots,m_i}$, which proves the claim.

For the definition of initial functor, see [28, Tag 09WP]. Since i is fully faithful, by the dual version of SGA 4_{I}, Exposé I [17, proposition 8.1.3(c)], we only need to verify that any object of $\mathsf{E}^{\mathrm{surj}}$ can be dominated by an object of $\mathsf{E}^{\mathrm{cov}}$, according to condition (F1) in loc. cit.

Let $\{\mathscr{X} \to \mathscr{E}\} \in \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$. Choose an étale surjective map $E \to \mathscr{E}$ from an étale scheme. Then $\mathscr{X} \times_{\mathscr{E}} E \to \mathscr{X}$ is smooth; by Corollary B.3 there exists a finitely presented S–scheme U with a smooth surjective map $U \to \mathscr{X}$. In particular, $U \to S$ is automatically flat and separable.

Let $R = U \times_{\mathscr{X}} U$ and $F = E \times_{\mathscr{E}} E$. The map $U \to \mathscr{X} \times_{\mathscr{E}} E \to E$ induces a map $R \to F$. Since E and F are étale S–spaces, the two morphisms $U \to E$ and $R \to F$ factor through $\pi_0(-/S)$ of the source. Taking groupoid closures and using functoriality of stack quotients [28, Tag 04Y3], we obtain a 2–commutative diagram

$$
\begin{tikzcd}
R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
\pi_0(R/S) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \pi_0(U/S) \arrow[r] \arrow[d] & {[\pi_0(U)/\pi_0(R)]} \arrow[d] \\
F \arrow[r, shift left] \arrow[r, shift right] & E \arrow[r] & \mathscr{E}
\end{tikzcd}
$$

The right column is a morphism in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$; hence,

$$\mathrm{Hom}_{\mathsf{E}^{\mathrm{surj}}}\big(i\big([\pi_0(U)/\pi_0(R)]\big), \mathscr{E}\big) \neq \varnothing$$

and i is an initial functor. ◻

Therefore the cofiltered category $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$, seen as a 2–cofiltered 2–category, defines the same object $\Pi_1(\mathscr{X}/S)$ inside the 2–category 2– Pro(**ÉtStack**$_S$),

$$\Pi_1(\mathscr{X}/S) := \lim_{\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)} \mathscr{E} = \lim_{\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)} [\pi_0(U)/\pi_0(R)].$$

Note that the stacks $[\pi_0(U)/\pi_0(R)]$ are étale gerbes over the algebraic space

$$\pi_0(U)/\pi_0(R)^{\mathrm{gpd}} = \pi_0(\mathscr{X}/S);$$

see Lemma 3.4.2.

The expression as a limit over $\mathsf{E}^{\mathrm{cov}}(\mathscr{X}/S)$ is sometimes useful for computing Π_1:

5.3.5 Proposition *Let S be quasiseparated and τ–quasicompact, and G be a flat group scheme of finite presentation over S. Assume also that either* $\dim(S) = 0$ *or $G \to S$ is smooth. Then we have a canonical isomorphism $\Pi_1(BG/S) \simeq B(\pi_0(G)/S)$. In particular, the formation of Π_1 commutes with base change in the special case of classifying stacks of smooth group schemes (and flat group schemes when* $\dim(S) = 0$).

Proof By Lemma 5.3.4, we can compute $\Pi_1(BG/S)$ using atlases. Let $U \to BG$ be a finitely presented smooth atlas; this determines a G–torsor $P \to U$. Consider the refinement $P \to U$ of atlases

$$\begin{tikzcd} P \times_U P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & S \arrow[d] \\ P \arrow[r] & U \arrow[r] & BG \end{tikzcd}$$

Since $P \times_U P \simeq G \times_S P$, the left vertical arrow is a trivial G–torsor. Hence any smooth atlas of BG is refined by an atlas corresponding to a trivial torsor; we may therefore assume that $U \to BG$ corresponds to a trivial G–torsor. Equivalently, it means that there is a factorization $U \to S \to BG$. From the cartesian squares

$$\begin{tikzcd} U \times_S U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ U \times_S G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ U \arrow[r] & S \arrow[r] & BG \end{tikzcd}$$

we have $U \times_{BG} U \simeq U \times_S U \times_S G$. Hence the groupoid presentation of BG

$$U \times U \times G \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} U \to BG$$

gives rise to the quotient stack

$$[\pi_0(U)/\pi_0(U \times U \times G)] \simeq [\pi_0(U)/\pi_0(U) \times \pi_0(U) \times \pi_0(G)] \simeq B(\pi_0(G)/S).$$

Here, we have used the commutation of π_0 with products; see Lemma 3.1.1(5). Since these atlases of trivial torsors are initial among all smooth atlases of BG and the corresponding étale quotient stacks are initial in $\mathsf{E}^{\mathrm{cov}}(BG/S)$, we deduce the canonical isomorphism $\Pi_1(BG/S) \simeq B(\pi_0(G)/S)$. □

5.4 Comparison with other fundamental gerbes

This subsection is illustrative and can be skipped by a reader who wishes to go straight to the main theorem. In [14, section 6], Grothendieck introduces a generalization of the classical étale fundamental group. For X a scheme together with a geometric point $x \to X$, he calls "pro-groupe fondamental élargi" a certain pro-group, which we will denote by $\pi_1^{\mathrm{SGA3}}(X, x)$, given by a filtered system $(\pi_i)_{i \in I}$ of groups with surjective transition morphisms.

For any (abstract) group G, write $\mathrm{Tors}(X, G, x)$ for the set of isomorphism classes of pairs (P, σ) composed of a G–torsor $P \to X$ and a trivialization $\sigma\colon P_x \to G$. Then $\pi_1^{\mathrm{SGA3}}(X, x)$ pro-represents the functor $G \mapsto \mathrm{Tors}(X, G, x)$, ie there is an isomorphism of functors from groups to sets,

$$\mathrm{Tors}(X, -, x) \cong \operatorname*{colim}_i \mathrm{Hom}(\pi_i, -) = \mathrm{Hom}_{\mathrm{Pro}}(\pi_1^{\mathrm{SGA3}}(X, x), -).$$

Now let $\mathscr{X}/S$ be a flat, finitely presented algebraic stack and fix a section $x\colon S \to \mathscr{X}$. For simplicity, assume that $\pi_0(\mathscr{X}/S) = S$ — this allows us to avoid working with $\pi_0(\mathscr{X}/S)$–group spaces below. Let I be the poset of isomorphism classes of objects of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$; see Remark 5.2.6(2). We claim that, for each $i \in I$, the map $\mathscr{E}_i \to S$ in the corresponding factorization $\mathscr{X} \to \mathscr{E}_i \to S$ is a gerbe which is neutralized by the section $e_i\colon S \to \mathscr{X} \to \mathscr{E}_i$. Indeed, since $\mathscr{E}_i \to S$ is étale it is enough to prove that e_i is surjective. For this we may assume that S is the spectrum of an algebraically closed field k. As $\mathscr{E}_i$ is connected and étale, the choice of a rational point of an atlas produces a surjective morphism $\mathrm{Spec}(k) \to \mathscr{E}_i$. Therefore $\mathscr{E}_i$ is a gerbe and e_i is surjective.

It follows that there is a canonical isomorphism $\mathscr{E}_i \to BG_i$ with $G_i = \mathrm{Aut}_{\mathscr{E}_i}(e_i) = S \times_{e_i, \mathscr{E}_i, e_i} S$ an étale S–group algebraic space. For each morphism $\mathscr{E}_i \to \mathscr{E}_j$ in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, we obtain a natural map of group spaces $G_i = S \times_{\mathscr{E}_i} S \to S \times_{\mathscr{E}_j} S = G_j$.

5.4.1 Definition With notation as above, we define the *enlarged fundamental pro-group space* by

$$\pi_1(\mathscr{X}/S, x) := (G_i)_{i \in I}.$$

For such a pro-group sheaf $G = (G_i)$ we write $BG = (BG_i)$ for the pro-classifying stack.

5.4.2 Proposition *Let $\mathscr{X}/S$ be a flat, finitely presented algebraic stack with geometrically connected fibres and keep the notation above.*

(1) *Each choice of section $x\colon S \to \mathscr{X}$ gives a canonical isomorphism*

$$\Pi_1(\mathscr{X}/S) \simeq B(\pi_1(\mathscr{X}/S, x)).$$

(2) *If S is the spectrum of a separably closed field, we have a canonical isomorphism*

$$\pi_1(\mathscr{X}/S, x) \simeq \pi_1^{\mathrm{SGA3}}(\mathscr{X}, x).$$

Proof (1) Write $(BG, *)$ for the classifying stack BG pointed by the trivial torsor. Note that we have a bijection of sets $\mathrm{Hom}((BG_1, *_1), (BG_2, *_2)) = \mathrm{Hom}(G_1, G_2)$, which to each morphism of pointed stacks $(BG_1, *_1) \to (BG_2, *_2)$ associates the morphism of S–group spaces $G_1 = \mathrm{Aut}(*_1) \to G_2 = \mathrm{Aut}(*_2)$. Now let G be an étale group algebraic space. Then, in the category of pro-(pointed stacks), we have

$$\begin{aligned}
&\mathrm{Hom}\big((\Pi_1(\mathscr{X}/S), x), (BG, *)\big)\\
&\qquad= \operatorname*{colim}_i \mathrm{Hom}_S((BG_i, *), (BG, *)) &&\text{(by definition of } I\text{)}\\
&\qquad= \operatorname*{colim}_i \mathrm{Hom}(G_i, G)\\
&\qquad= \mathrm{Hom}(\pi_1(\mathscr{X}/S, x), G) &&\text{(by definition of } \pi_1(\mathscr{X}/S, x)\text{)}\\
&\qquad= \mathrm{Hom}\big((B\pi_1(\mathscr{X}/S, x), *), (BG, *)\big).
\end{aligned}$$

This shows that $(\Pi_1(\mathscr{X}/S), x)$ and $(B\pi_1(\mathscr{X}/S, x), *)$ pro-represent the same functor, and hence are naturally identified.

(2) If S is the spectrum of a separably closed field, an étale S–group space is simply an abstract group. In this case the functor just described is the functor $G \mapsto \pi_1(X, x; G)$ of pointed torsors as defined in [14, section 6]. The (pro-)representing objects are therefore canonically isomorphic, whence our claim. □

We now compare $\Pi_1(\mathscr{X}/k)$ with the étale fundamental gerbe of Borne and Vistoli.

5.4.3 Proposition *Let k be a field. Let $\mathscr{X}$ be an algebraic k–stack of finite presentation and geometrically connected. Let $\mathsf{E}^{\mathrm{fin}}(\mathscr{X}/k) \subset \mathsf{E}^{\mathrm{surj}}(\mathscr{X}/k)$ be the subcategory composed of factorizations through a* ***finite*** *gerbe, and $\Pi_1(\mathscr{X}/k) \twoheadrightarrow \Pi_1^{\mathrm{fin}}(\mathscr{X}/k)$ the corresponding pro-finite pro-quotient. Let $\Pi^{\text{ét}}_{\mathscr{X}/k}$ be the étale fundamental gerbe of Borne and Vistoli* [2, Section 8]. *Then:*

(1) $\Pi_1^{\text{fin}}(\mathscr{X}/k)$ *defines a projective system whose limit in the category of stacks is representable by* $\Pi^{\text{ét}}_{\mathscr{X}/k}$.

(2) *If* $\mathscr{X}$ *is geometrically unibranch, we have* $\Pi_1(\mathscr{X}/k) = \Pi_1^{\text{fin}}(\mathscr{X}/k)$.

The existence of the étale fundamental gerbe is granted by [29, Proposition 4.3].

Proof (1) Let Γ be the limit in the category of stacks of the projective system defined by $\Pi_1^{\text{fin}}(\mathscr{X}/k)$. Then Γ and $\Pi^{\text{ét}}_{\mathscr{X}/k}$ are isomorphic because both are universal for morphisms to finite gerbes: for Γ this follows from [2, Proposition 3.8] and for $\Pi^{\text{ét}}_{\mathscr{X}/k}$ this is by construction; see [2, Section 8].

(2) In order to prove that $\Pi_1(\mathscr{X}/k)$ is pro-finite, it is enough to prove that any factorization $\mathscr{X}/\mathscr{E}/k$ is dominated by a factorization $\mathscr{X}/\mathscr{F}/k$ with $\mathscr{F}/k$ a finite gerbe. Let $e\colon E \to \mathscr{E}$ be an atlas from an étale scheme; since $\mathscr{E}$ is connected and quasicompact, we may pick $E = \operatorname{Spec}(\ell)$ for $k \to \ell$ a finite Galois field extension with Galois group Γ. We write $\mathscr{Y}$ for the 2–fibre product $\mathscr{X} \times_{\mathscr{E}} \operatorname{Spec}(\ell)$.

Let us first consider the case $k = \ell$ and $\Gamma = \{0\}$. Let $G = \operatorname{Aut}(e)$ be the étale k–group scheme of automorphisms. Then we have an isomorphism of gerbes $\mathscr{E} = BG$ and the morphism $\mathscr{Y} \to \mathscr{X}$ is the G–torsor associated to $\mathscr{X} \to \mathscr{E}_\ell = BG$.

Let $\mathscr{P} \subset \mathscr{Y}$ be a connected component. We claim that $\mathscr{P} \to \mathscr{X}$ is finite. For this it is enough to choose a smooth atlas $U \to \mathscr{X}$ from a quasicompact scheme U and prove that the projection $\mathscr{P} \times_{\mathscr{X}} U \to U$ is finite. Since $\mathscr{P} \times_{\mathscr{X}} U \to \mathscr{P}$ is quasicompact, $\mathscr{P} \times_{\mathscr{X}} U$ is a union of finitely many connected components of $\mathscr{Y} \times_{\mathscr{X}} U$. Since U is geometrically unibranch again, the finiteness of $\mathscr{P} \times_{\mathscr{X}} U \to U$ follows from [14, corollaire 5.14]. Therefore, the stabilizer $H \subset G$ of $\mathscr{P}$ is finite and $\mathscr{P} \to \mathscr{X}$ is an H–torsor.

The $\mathscr{X}$–morphism $\mathscr{P} \to \mathscr{Y}$ is H–equivariant, and the induced isomorphism of G–torsors $\mathscr{P} \wedge^H G \to \mathscr{Y}$ is exactly the datum of a factorization $\mathscr{X} \to BH \to BG$.

We now go back to the general case where $k \neq \ell$ and $\Gamma \neq \{0\}$. Let $\operatorname{Res}_{\ell/k}$ be the Weil restriction functor. As the gerbe $\mathscr{E}_\ell$ is trivialized by e, by the previous part of the argument there is a factorization

$$\mathscr{X}_\ell \to \mathscr{F} \to \mathscr{E}_\ell$$

with $\mathscr{F}$ a finite étale gerbe. Applying Weil restriction and adjunction, we obtain a natural map

$$\mathscr{X} \to \mathscr{F}' := \operatorname{Res}_{\ell/k} \mathscr{F} \times_{\operatorname{Res}_{\ell/k} \mathscr{E}_\ell} \mathscr{E}.$$

The map $\mathscr{E} \to \mathrm{Res}_{\ell/k} \mathscr{E}_\ell$ is faithful because so is its base change $\mathscr{E}_\ell \to \prod_\Gamma \mathscr{E}_\ell$. Hence the map $\mathscr{F}' \to \mathrm{Res}_{\ell/k} \mathscr{F}$ is faithful as well. Since the base change via $k \to \ell$ of $\mathrm{Res}_{\ell/k} \mathscr{F}$ is $\prod_\Gamma \mathscr{F}$, the stack $\mathrm{Res}_{\ell/k} \mathscr{F}$ is a finite étale gerbe. Then $\mathscr{F}'$ is étale with quasifinite stabilizers. Replacing $\mathscr{F}'$ with the schematic image $\mathscr{F}''$ of $\mathscr{X} \to \mathscr{F}'$, we obtain a factorization $\mathscr{X} \to \mathscr{F}'' \to \mathscr{E}$ in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/k)$. As $\mathscr{F}''$ is a gerbe with quasifinite stabilizers, it is finite étale, as we wished to show. □

5.5 Pushout along the component fibration of an atlas

The key fact allowing to upgrade our result to algebraic stacks is an analogue of the pushout property from Lemma 3.4.2. We establish it in Lemma 5.5.2 below. For this, we will use a strengthening of the property that $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ is initial for morphisms from $\mathscr{X}$ to étale S–algebraic spaces. Such a statement was already proven in Corollary 4.4.2. We give here another proof for a statement without noetherian assumption on S but with a weaker conclusion.

5.5.1 Lemma *Let $\mathscr{X}/S$ be a flat, finitely presented algebraic stack. Assume that either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable. Then $\mathscr{X} \to \pi_0(\mathscr{X}/S)$ is initial for morphisms from $\mathscr{X}$ to unramified S–algebraic spaces.*

Proof Let $f : \mathscr{X} \to I$ be a morphism to an unramified S–algebraic space I. According to Lemma 3.1.1, the algebraic space $\pi_0(\mathscr{X}/S)$ is the quotient of $\mathscr{X}$ by the open equivalence relation whose graph $\mathscr{R} \subset \mathscr{X} \times_S \mathscr{X}$ is the open connected component of the diagonal. Therefore, in order to obtain a factorization $\pi_0(\mathscr{X}/S) \to I$ it is enough to prove that $f\,\mathrm{pr}_1 = f\,\mathrm{pr}_2$, where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{R} \to \mathscr{X}$ are the projections. Let $\mathscr{Z} \to \mathscr{R}$ be the equalizer of $f\,\mathrm{pr}_1$ and $f\,\mathrm{pr}_2$. Since I is unramified, $\mathscr{Z}$ is an open substack of $\mathscr{R}$. Moreover, in each fibre above a point $s \in S$, we have $\mathscr{Z}_s = \mathscr{R}_s$ because I_s is étale over the residue field $k(s)$ and $\mathscr{X}_s \to \pi_0(\mathscr{X}_s/k(s))$ is initial for maps to étale $k(s)$–spaces (note that the formation of π_0 commutes with arbitrary base change). Therefore $\mathscr{Z} = \mathscr{R}$, so $f\,\mathrm{pr}_1 = f\,\mathrm{pr}_2$ and we are done. □

5.5.2 Lemma *Let $\mathscr{X}/S$ be a flat, finitely presented algebraic stack. Assume that either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable. Let $U \to \mathscr{X}$ be a faithfully flat, finitely presented, separable atlas (eg a smooth surjective atlas of finite presentation). Let $R \rightrightarrows U$ be the corresponding groupoid presentation of $\mathscr{X}$. Consider the 2–commutative* *diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \pi_0(U) \arrow[r] & {[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]} \end{tikzcd}$$

and let $\mathscr{M} \to S$ be either

(i) *a Deligne–Mumford stack*, *or*

(ii) $\mathscr{M} = \mathrm{Fdiv}_S(\mathscr{N})$ *for some algebraic stack $\mathscr{N} \to S$.*

Then the natural functor

$$F : \mathsf{Hom}([\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}], \mathscr{M}) \to \mathsf{Hom}(\mathscr{X}, \mathscr{M}) \times_{\mathsf{Hom}(U,\mathscr{M})} \mathsf{Hom}(\pi_0(U), \mathscr{M})$$

is an equivalence of categories.

Proof Throughout the proof we write $\mathscr{Q} = [\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ for the quotient stack of the 2–étale space $\pi_0(R) \rightrightarrows \pi_0(U)$. First we explain precisely what the functor F of the statement is. The target of F is the category with objects the triples $(v : \mathscr{X} \to \mathscr{M},\ f : \pi_0(U) \to \mathscr{M}, \delta : v\pi \xrightarrow{\sim} fh)$, or in other words the 2–commutative diagrams

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "h"'] & \mathscr{X} \arrow[d, "v"] \arrow[dl, Rightarrow, "\delta"'] \\ \pi_0(U) \arrow[r, "f"'] & \mathscr{M} \end{tikzcd}$$

For $\mathscr{M} = \mathscr{Q}$, we have a canonical particular object of this category (see Section 5.3)

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "h"'] & \mathscr{X} \arrow[d, "v_0"] \arrow[dl, Rightarrow, "\gamma"'] \\ \pi_0(U) \arrow[r, "f_0"'] & \mathscr{Q} \end{tikzcd}$$

Here is how the functor F is defined. For a morphism $g : \mathscr{Q} \to \mathscr{M}$, we have

$$F(g) = (v = gv_0,\ f = gf_0, \delta = g\gamma : gv_0\pi \to gf_0h).$$

To construct a quasi-inverse for F, we will construct a functor G such that $GF = \mathrm{id}$, and an isomorphism $\epsilon : FG \xrightarrow{\sim} \mathrm{id}$. This means that, given (v, f, δ), we seek to construct functorially a morphism $g\colon \mathscr{Q}\to\mathscr{M}$ and 2–isomorphisms $a\colon gf_0\to f$ and $b\colon gv_0\to v$ filling in a 2–commutative diagram

$$\begin{tikzcd}
U \arrow[r, "\pi"] \arrow[d, "h"'] & \mathscr{X} \arrow[d, "v_0"] \arrow[ddr, bend left, "v"] \arrow[dl, Rightarrow, "\gamma"'] & \\
\pi_0(U) \arrow[r, "f_0"'] \arrow[drr, bend right, "f"'] & \mathscr{Q} \arrow[dr, dashed, "g"] \arrow[r, phantom, "b"] \arrow[d, phantom, "a"] & {} \arrow[d, phantom, "\Downarrow\delta"] \\
 & {} & \mathscr{M}
\end{tikzcd}$$

We use the usual notation for the groupoid $R\rightrightarrows U$, and we complete the picture by adding in the bottom row the 2–étale space $\pi_0(R)\rightrightarrows\pi_0(U)$:

$$\begin{tikzcd}
R\times_{s,U,t}R \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "c" description] \arrow[r, shift right=2, "\mathrm{pr}_2"'] \arrow[d, "l"'] & R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "k"'] & U \arrow[r, "\pi"] \arrow[d, "h"'] & \mathscr{X} \arrow[d, "v_0"] \arrow[dr, bend left, "v"] & \\
\pi_0(R\times_{s,U,t}R) \arrow[r, shift left=2, "p_1"] \arrow[r, "d" description] \arrow[r, shift right=2, "p_2"'] & \pi_0(R) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \pi_0(U) \arrow[r, "f_0"] \arrow[rr, bend right, "f"'] & \mathscr{Q} & \mathscr{M}
\end{tikzcd}$$

First we construct the pair (g,a) using Corollary A.7.1 on the coequalizer property of the stack quotient $\pi_0(U)\to[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}]$ on objects. Consider $x=f\sigma$ and $y=f\tau$ viewed as $\pi_0(R)$–points of $\mathscr{M}$, and $I:=\mathrm{Isom}(x,y)$ the space of isomorphisms. Let $\alpha\colon\pi s\to\pi t$ and $\alpha_0\colon f_0\sigma\xrightarrow{\sim}f_0\tau$ be the canonical 2–isomorphisms. The composition

$$f\sigma k=fhs\xrightarrow{\delta^{-1}s}v\pi s\xrightarrow{v\alpha}v\pi t\xrightarrow{\delta t}fht=f\tau k$$

is an isomorphism $\widetilde{\beta}\colon k^*x\xrightarrow{\sim}k^*y$, that is, a point $\widetilde{\beta}\colon R\to I$.

We claim that $\widetilde{\beta}$ factors uniquely via $\pi_0(R)$. We perform separately the two cases of the statement, starting by case (i), where $\mathscr{M}\to S$ is Deligne–Mumford. Then $I\to\pi_0(R)$ is unramified, and hence so is $I\to S$. Lemma 5.5.1 implies that $\widetilde{\beta}$ factors uniquely as

$$R\xrightarrow{k}\pi_0(R)\xrightarrow{\beta}I.$$

The next case is (ii), suppose $\mathscr{M}=\mathrm{Fdiv}(\mathscr{N})$. We write $x_0,y_0\colon\pi_0(R)\to\mathscr{N}$ for the compositions of x and y with $\mathrm{Fdiv}(\mathscr{N})\to\mathscr{N}$. Let $I_0:=\mathrm{Isom}(x_0,y_0)$. As $\pi_0(R)\to S$

is perfect, we may apply Lemma 2.3.6, and deduce that

$$I = \mathrm{Fdiv}(I_0).$$

Then, by Theorem 5.1.1,

$$\begin{aligned}\mathrm{Hom}_S(R, I) = \mathrm{Hom}_S(R, \mathrm{Fdiv}(I_0)) = \mathrm{Hom}_S(\pi_0(R), I_0) = \mathrm{Hom}(\pi_0(R), \mathrm{Fdiv}(I_0))\\ = \mathrm{Hom}(\pi_0(R), I).\end{aligned}$$

Therefore $\widetilde{\beta}\colon R \to I$ factors uniquely via $\pi_0(R)$. This completes the proof of the claim.

We have obtained an isomorphism $\beta\colon x \xrightarrow{\sim} y$. Now we check that $\beta d = \beta p_1 \circ \beta p_2$ holds. Consider the equality $\alpha c = \alpha\,\mathrm{pr}_1 \circ \alpha\,\mathrm{pr}_2$:

$$\begin{tikzcd}[column sep=huge]
R \times_{s,U,t} R \arrow[r, bend left=60, "\pi s\,\mathrm{pr}_2 = \pi s c"{name=U}] \arrow[r, "\pi t\,\mathrm{pr}_2 = \pi s\,\mathrm{pr}_1"{name=M}] \arrow[r, bend right=60, "\pi t\,\mathrm{pr}_1 = \pi t c"'{name=D}] & \mathscr{X}
\arrow[Rightarrow, from=U, to=M, "\alpha\,\mathrm{pr}_2"]
\arrow[Rightarrow, from=M, to=D, "\alpha\,\mathrm{pr}_1"]
\arrow[Rightarrow, from=U, to=D, shift left=6, "\alpha c"]
\end{tikzcd}$$

This gives $v\alpha c = (v\alpha\,\mathrm{pr}_1) \circ (v\alpha\,\mathrm{pr}_2)$, which, using the three relations $t\,\mathrm{pr}_1 = tc$, $s\,\mathrm{pr}_1 = t\,\mathrm{pr}_2$ and $s\,\mathrm{pr}_2 = sc$, we can write as

$$(\delta tc) \circ (v\alpha c) \circ (\delta^{-1} sc) = (\delta t\,\mathrm{pr}_1) \circ (v\alpha\,\mathrm{pr}_1) \circ (\delta^{-1} s\,\mathrm{pr}_1) \circ (\delta t\,\mathrm{pr}_2) \circ (v\alpha\,\mathrm{pr}_2) \circ (\delta^{-1} s\,\mathrm{pr}_2).$$

Now, by definition, $\widetilde{\beta} = \delta t \circ v\alpha \circ \delta^{-1} s$, so the above equality becomes $\widetilde{\beta} c = \widetilde{\beta}\,\mathrm{pr}_1 \circ \widetilde{\beta}\,\mathrm{pr}_2$, which in turn can be rewritten as $\beta d l = \beta p_1 l \circ \beta p_2 l$. Finally, because l is faithfully flat and hence an epimorphism of spaces, we obtain

$$\beta d = \beta p_1 \circ \beta p_2.$$

Then Corollary A.7.1 applies and provides a pair (g, a) and a 2–commutative diagram

$$\begin{tikzcd}
\pi_0(R) \arrow[r, "\sigma"] \arrow[d, "\tau"'] & \pi_0(U) \arrow[d, "f_0"] \arrow[ddr, bend left, "f"] \arrow[dl, phantom, ""] & \\
\pi_0(U) \arrow[r, "f_0"] \arrow[drr, bend right, "f"'] & \mathscr{Q} \arrow[dr, dashed, "g"] & \\
& & \mathscr{M}
\end{tikzcd}$$

Now we construct $b\colon gv_0 \to v$ using Corollary A.7.1 on the coequalizer property of $U \to [U/R]$ on morphisms. Define $c := \delta^{-1} \circ (bh) \circ (g\gamma)$ and consider the solid diagram

$$\begin{tikzcd}
gv_0\pi s \arrow[r, "g\gamma s"] \arrow[d, "gv_0\alpha"'] \arrow[rrr, bend left, dotted, "cs"] & gf_0hs \arrow[r, "bhs"] \arrow[d, "g\alpha_0 k"'] & fhs \arrow[r, "\delta^{-1}s"] \arrow[d, "\widetilde{\beta}"'] & v\pi s \arrow[d, "v\alpha"'] \\
gv_0\pi t \arrow[r, "g\gamma t"'] \arrow[rrr, bend right, dotted, "ct"'] & gf_0ht \arrow[r, "bht"'] & fht \arrow[r, "\delta^{-1}t"'] & v\pi t
\end{tikzcd}$$

The first square is commutative because $\alpha_0 k \circ \gamma s = \gamma t \circ v_0\alpha$ by the functoriality of quotient stacks for the morphism of 2–étale spaces $(R \rightrightarrows U) \to (\pi_0(R) \rightrightarrows \pi_0(U))$. The second square is commutative by the compatibility between α_0 and b that results from Corollary A.7.1. The third square is commutative by definition of $\widetilde{\beta}$. Therefore the outer rectangle is commutative. That is, with the notation of Corollary A.7.1, the arrow c is a morphism from $(f_1, \beta_1) = (gv_0\pi, gv_0\alpha)$ to $(f_2, \beta_2) = (v\pi, v\alpha)$ in the equalizer category

$$\mathrm{eq}(\mathrm{Hom}(U, \mathscr{M}) \rightrightarrows \mathrm{Hom}(R, \mathscr{M})).$$

The quoted corollary gives existence of a 2–isomorphism $b\colon gv_0 \to v$ such that $c = b\pi$. This concludes the proof of the lemma. $\square$

5.5.3 Remark Lemma 5.5.2 does not hold if $\mathscr{M}$ is an arbitrary Artin stack. In fact, using Proposition 5.3.5 we have $\Pi_1(B\mathbb{G}_m/S) = S$ and this implies that the lemma fails already with $\mathscr{X} = \mathscr{M} = B\mathbb{G}_m$ and $U = S$. For a maybe more geometric counterexample, let k be a field and consider the 2–commutative diagram of k–algebraic stacks

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d, "\mathcal{O}(1)"] \\
\pi_0(U) \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

Here $U = \mathbb{A}^1 \sqcup \mathbb{A}^1$, $U \to \mathbb{P}^1$ is the usual affine cover and $\alpha\colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}(1)_U$ is some isomorphism. In this case, $\pi_0(R) = \pi_0(U) \times_{\pi_0(\mathbb{P}^1)} \pi_0(U)$, and the two maps towards $\pi_0(U)$ coincide with the projections. Therefore $[\pi_0(U)/\pi_0(R)^{\mathrm{gpd}}] = \pi_0(\mathbb{P}^1) = \mathrm{Spec}(k)$. However, the morphism $\mathcal{O}(1)\colon \mathbb{P}^1 \to B\mathbb{G}_m$ does not factor via $\mathrm{Spec}(k)$ since $\mathcal{O}(1)$ is not trivial.

5.6 The case of algebraic stacks

Finally we prove our main result (Theorem A from the introduction), building on the case of algebraic spaces (Theorem 5.1.1) and the pushout along an atlas (Lemma 5.5.2).

We will use a lemma about epimorphisms of algebraic stacks. Since these may fail to be right-cancellable, as point (1) below shows, the claim in (2) must be estimated at its true value.

5.6.1 Lemma *Let $f\colon \mathscr{S}' \to \mathscr{S}$ be a morphism of algebraic stacks which is schematically dominant and submersive, and remains so after any smooth base change. Let $\mathscr{X}$ be a stack whose diagonal is representable by algebraic spaces. Let $u, v\colon \mathscr{S} \to \mathscr{X}$ be morphisms of stacks.*

1. *There exist $u, v\colon \mathscr{S} \to \mathscr{X}$ such that $uf = vf$ but u and v are not isomorphic. Moreover $\mathscr{X}$ can be chosen algebraic and f can be chosen representable, finite, étale and surjective.*
2. *Let $a, b\colon u \xrightarrow{\sim} v$ be two 2–isomorphisms. If $f^*a = f^*b$, then $a = b$.*

Proof (1) Let $f\colon S \to BG$ be the canonical atlas of the classifying stack of a finite étale nontrivial group scheme G over a scheme S. Let $a\colon BG \to S$ be the structure morphism. Let $u = \mathrm{id}_{BG}\colon BG \to BG$ and $v = fa\colon BG \to BG$. Then we have $af = \mathrm{id}_S$ and hence $vf = uf$. But u anv v are not isomorphic, because, on the automorphism sheaf of the trivial torsor, the map u induces $\mathrm{id}_G\colon G \to G$ while the map v induces the trivial map $G \to S \to G$.

(2) Replacing $\mathscr{S}'$ by a smooth atlas $S' \to \mathscr{S}'$, we can assume that $\mathscr{S}' = S'$ is a scheme. Consider the $\mathscr{S}$–stack of 2–isomorphisms $I_{\mathscr{S}} := \mathscr{I}som(u, v)$. Then $I_{\mathscr{S}}$ defines a sheaf over the lisse-étale site of $\mathscr{S}$, and we have $a = b$ if and only $a_T = b_T$ for all objects $T \to \mathscr{S}$ of that site. Fix such a T and let $I_T = I_{\mathscr{S}} \times_{\mathscr{S}} T$. Let $T' := T \times_{\mathscr{S}} S'$. Because T' dominates S', the assumption $f^*a = f^*b$ implies $a_{T'} = b_{T'}$; that is, we have two equal compositions

$$T' \to T \overset{a_T}{\underset{b_T}{\rightrightarrows}} I_T.$$

But the assumption on f implies that $T' \to T$ is an epimorphism of algebraic spaces; see Romagny, Rydh and Zalamansky [24, Lemma 2.1.5]. Hence $a_T = b_T$, as was to be shown. $\square$

Now let $\mathscr{X} \to S$ be a flat, finitely presented algebraic stack with S of characteristic p. Assume that either $\dim(S) = 0$ or $\mathscr{X} \to S$ is separable. By Lemma 2.4.1, the étale fundamental pro-groupoid $\Pi_1(\mathscr{X}/S)$ is relatively perfect over S. Therefore the natural morphism

$$\mathrm{Fdiv}(\Pi_1(\mathscr{X}/S)/S) \to \Pi_1(\mathscr{X}/S)$$

is an isomorphism and we obtain a natural morphism

$$\rho\colon \mathscr{X} \to \Pi_1(\mathscr{X}/S) \xrightarrow{\sim} \mathrm{Fdiv}(\Pi_1(\mathscr{X}/S)/S).$$

5.6.2 Theorem *Let S be a quasiseparated, τ–quasicompact algebraic space of characteristic p. Let $\mathscr{X} \to S$ be a flat, finitely presented algebraic stack. Assume that either* $\dim(S) = 0$ *or $\mathscr{X} \to S$ is separable. Let $\mathscr{M} \to S$ be a Deligne–Mumford stack. Then the functor $\alpha \mapsto \mathrm{Fdiv}(\alpha) \circ \rho$ is an equivalence*

$$\mathsf{Hom}(\Pi_1(\mathscr{X}/S), \mathscr{M}) \xrightarrow{\sim} \mathsf{Hom}(\mathscr{X}, \mathrm{Fdiv}(\mathscr{M}/S))$$

between the categories of morphisms of pro-Deligne–Mumford stacks (with $\mathscr{M}$ seen as a constant 2–pro-object) on the source and morphisms of stacks on the target. This equivalence is functorial in $\mathscr{X}$ and $\mathscr{M}$.

5.6.3 Remark In terms of coperfection, this says that the inductive system of relative Frobenii

$$\mathscr{X} \xrightarrow{F_{\mathscr{X}/S}} \mathscr{X}^{p/S} \xrightarrow{F_{\mathscr{X}^{p}/S}} \mathscr{X}^{p^2/S} \to \cdots$$

admits a colimit in the 2–category of pro-Deligne–Mumford stacks over S, which is the pro-étale stack $\Pi_1(\mathscr{X}/S)$. In particular, $\Pi_1(\mathscr{X}/S)$ is a coperfection of $\mathscr{X}/S$ in the 2–category of pro-Deligne–Mumford stacks.

Proof As in the proof of Theorem 5.1.1, we write $\Pi_1(\mathscr{X}) := \Pi_1(\mathscr{X}/S)$ and $\mathrm{Fdiv}(\mathscr{M}) := \mathrm{Fdiv}(\mathscr{M}/S)$, we let $\rho_0\colon \mathscr{X} \to \Pi_1(\mathscr{X})$ be the natural map and we want to prove that

$$\Phi = \rho_0^*\colon \mathsf{Hom}(\Pi_1(\mathscr{X}), \mathrm{Fdiv}(\mathscr{M})) \xrightarrow{\sim} \mathsf{Hom}(\mathscr{X}, \mathrm{Fdiv}(\mathscr{M}))$$

is a bifunctorial equivalence over S.

We start with essential surjectivity. Consider an object f of $\mathsf{Hom}(\mathscr{X}, \mathrm{Fdiv}(\mathscr{M}))$. Just like we did in the proof of Theorem 5.1.1, we fix $u\colon \mathscr{X} \to \mathscr{M}$. We pick an étale atlas $V \to \mathscr{M}$ with V an algebraic space; define $\mathscr{X}_V := \mathscr{X} \times_{\mathscr{M}} V$. Applying Corollary B.3, we find a smooth map $U \to \mathscr{X}_V$ from an S–scheme of finite presentation, with $U \to \mathscr{X}$ smooth surjective. Notice that, if $\mathscr{X} \to S$ is separable, then also $U \to S$ is separable.

By Section 2.3.2(v) we have $\mathrm{Fdiv}(V) \xrightarrow{\sim} \mathrm{Fdiv}(\mathscr{M}) \times_{\mathscr{M}} V$, so that f induces an object $f' \in \mathsf{Hom}(\mathscr{X}_V, \mathrm{Fdiv}(V))$ and, by precomposition, an object $g \in \mathsf{Hom}(U, \mathrm{Fdiv}(V))$.

By Theorem 5.1.1, the map g is induced by a unique morphism $\pi_0(U) \to V$, or, equivalently, a morphism $\pi_0(U) \to \mathrm{Fdiv}(V)$. Using the pushout diagram of Lemma 5.5.2(ii),

$$\begin{tikzcd}
U \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\ulcorner"] & \mathscr{X} \arrow[d] \arrow[dddr, bend left] & \\
& \Pi_1(\mathscr{X}) \arrow[d] \arrow[ddr, dashed] & \\
\pi_0(U) \arrow[r] \arrow[dr] & {[\pi_0(U)/\pi_0(R)]} \arrow[dr, dashed] & \\
& \mathrm{Fdiv}(V) \arrow[r] & \mathrm{Fdiv}(\mathscr{M})
\end{tikzcd}$$

we obtain a map $\Pi_1(\mathscr{X}) \to \mathrm{Fdiv}(\mathscr{M})$ and this shows essential surjectivity.

We pass now to full faithfulness of Φ. For f and g objects of $\mathsf{Hom}(\Pi_1(\mathscr{X}), \mathrm{Fdiv}(\mathscr{M}))$, we want to prove that the map $\mathrm{Hom}(f, g) \to \mathrm{Hom}(\Phi(f), \Phi(g))$ is bijective.

We start with surjectivity. Assume given a diagram

$$\mathscr{X} \xrightarrow{\rho_0} \Pi_1(\mathscr{X}) \underset{g}{\overset{f}{\rightrightarrows}} \mathrm{Fdiv}(\mathscr{M})$$

and an isomorphism $\alpha\colon f\rho_0 \xrightarrow{\sim} g\rho_0$. By the definition of morphisms in the pro-category and cofilteredness of $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$, the morphisms f and g as well as α are defined on some common étale stack $\mathscr{E}$ corresponding to a surjective factorization $h\colon \mathscr{X} \to \mathscr{E}$. Abusing notation slightly, we therefore assume that we have $f, g\colon \mathscr{E} \to \mathrm{Fdiv}(\mathscr{M})$ and $\alpha\colon fh \xrightarrow{\sim} gh$. Our aim is to show that there exists a refinement $\mathscr{X} \xrightarrow{h'} \mathscr{E}' \xrightarrow{l} \mathscr{E}$ in $\mathsf{E}^{\mathrm{surj}}(\mathscr{X}/S)$ and a 2–isomorphism $\beta\colon fl \xrightarrow{\sim} gl$ such that $\beta h' = \alpha$. Since $\mathscr{E} \to S$ is étale and hence perfect, we have $\mathsf{Hom}(\mathscr{E}, \mathrm{Fdiv}(\mathscr{M})) = \mathsf{Hom}(\mathscr{E}, \mathscr{M})$ canonically, and similarly for $\mathscr{E}'$. We deduce that it is enough to work with the compositions $f_0, g_0\colon \mathscr{E} \to \mathrm{Fdiv}(\mathscr{M}) \to \mathscr{M}$. Indeed, if we find $(h_0'\colon \mathscr{X} \to \mathscr{E}', \beta_0\colon fl \xrightarrow{\sim} gl)$ for (f_0, g_0), then applying Fdiv will provide (h', β) suitable for (f, g). In sum, changing again notation, we can start from

$$\mathscr{X} \xrightarrow{h} \mathscr{E} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{M}.$$

Letting $\mathscr{I} := \mathrm{Isom}_{\mathscr{E}}(f, g)$, we consider the 2–commutative diagram

$$\begin{tikzcd}
& \mathscr{I} \arrow[r] \arrow[d] & \mathscr{M} \arrow[d, "\Delta"] \\
\mathscr{X} \arrow[ur, "\alpha"] \arrow[r] & \mathscr{E} \arrow[r, "{(f,g)}"] & \mathscr{M} \times_S \mathscr{M}
\end{tikzcd}$$

The assumption that $\mathscr{M} \to S$ is Deligne–Mumford guarantees that the representable morphism $\mathscr{I} \to \mathscr{E}$ is unramified. Now let us pick an étale surjective $E \to \mathscr{E}$ with $E \to S$ an étale scheme; applying again Corollary B.3, we find a smooth map $V \to \mathscr{X} \times_{\mathscr{E}} E$ from an S–scheme V of finite presentation such that $V \to \mathscr{X}$ is surjective. If $\mathscr{X} \to S$ is separable, the S–scheme V is automatically flat and separable. Let $I = E \times_{\mathscr{E}} \mathscr{I}$. We obtain a 2–commutative diagram

$$\begin{tikzcd}
V \arrow[d] & & I \arrow[dl] \arrow[d] \\
\mathscr{X} \times_{\mathscr{E}} E \arrow[r] \arrow[d] \arrow[urr, "\alpha'"] & E \arrow[d] & \mathscr{I} \arrow[dl] \\
\mathscr{X} \arrow[r] \arrow[urr, "\alpha"] & \mathscr{E} &
\end{tikzcd}$$

with α' induced by α via pullback along $E \to \mathscr{E}$. The morphism $I \to E$ is representable and unramified; therefore I is an unramified algebraic space over S. By Lemma 5.5.1, the map $V \to I$ factors uniquely via $\pi_0(V)$. Letting $R = V \times_{\mathscr{X}} V$, we obtain by Lemma 5.5.2 a dashed arrow

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \arrow[ddr, bend left] & \\
\pi_0(V) \arrow[r] \arrow[drr, bend right] & {} \arrow[dr, dashed] & \\
& & \mathscr{I}
\end{tikzcd}$$

making the diagram 2–commute. Then $\mathscr{X} \to \mathscr{E}' := [\pi_0(V)/\pi_0(R)]$ is the required h', and the dashed arrow $\mathscr{E}' \to \mathscr{I}$ is β.

We finish with injectivity. Let $a, b\colon f \to g$ be two morphisms such that $\rho_0^* a = \rho_0^* b$. Then, as before, there is a factorization $h\colon \mathscr{X} \to \mathscr{E}$ such that f and g, and a and b, are defined on $\mathscr{E}$ and we can start from

$$\mathscr{X} \xrightarrow{h} \mathscr{E} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{M}.$$

Since $h\colon \mathscr{X} \to \mathscr{E}$ is faithfully flat and locally of finite presentation, it satisfies the assumptions of Lemma 5.6.1 and we deduce that $a = b$. This concludes the proof. $\square$

5.6.4 Remark If one wants to make the statement above an actual adjunction, some rather costly strengthenings of the assumptions are needed. First, one needs to extend the functors to the 2–pro-categories; this is no big problem. Second and more seriously, we need Fdiv to take values in (the 2–pro-category of) flat, separable algebraic stacks. This is much more binding; the natural way to ensure this is to assume that the Frobenius of S is *finite locally free* (eg S regular F–finite) and $\mathscr{M}$ is *smooth.* To sum up, let $\mathbf{SpbStack}_S$ be the 2–category of faithfully flat, finitely presented, separable algebraic stacks and $\mathbf{SmDM}_S$ the 2–category of smooth Deligne–Mumford stacks. If F_S is finite locally free, we obtain a pair of 2–adjoint functors

$$\text{2–Pro}(\mathbf{SpbStack}_S) \underset{\text{Fdiv}}{\overset{\Pi_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \text{2–Pro}(\mathbf{SmDM}_S).$$

To give a concrete illustration, we take as an example the moduli stack $\mathscr{M} = \bar{\mathscr{M}}_{g,n}$ of stable curves of genus g with n marked points, with $2g-2+n>0$.

5.6.5 Proposition *Let k be a field and let X/k be a geometrically connected scheme of finite type admitting a k–rational point $x \in X(k)$. Set $\mathscr{M} = \bar{\mathscr{M}}_{g,n}$. Let*

$$(\mathscr{C}_i \to X^{p^i/k}, \sigma_i) \in \mathrm{Fdiv}(\mathscr{M})(X)$$

be a divided curve over X. Let $C \in \mathrm{Fdiv}(\mathscr{M})(k)$ be its pullback via $x\colon \mathrm{Spec}(k) \to X$; note that $\mathrm{Fdiv}(\mathscr{M})(k) = \mathscr{M}(k)$ by taking $\mathscr{X} = \mathrm{Spec}(k)$ in Remark 5.6.3. Then there exist

- *a finite étale subgroup scheme $G \subset \mathrm{Aut}_k(C)$,*
- *a G–torsor $f : P \to X$*

such that the F–divided curve on P obtained from pullback of $(\mathscr{C}_i, \sigma_i)$ via $f : P \to X$ is isomorphic to the pullback of C via $P \to \mathrm{Spec}(k)$.

Proof By Theorem 5.6.2, the F–divided curve $(\mathscr{C}_i, \sigma_i)$ corresponds to an object of

$$\mathsf{Hom}(\Pi_1(X), \mathscr{M}) = \operatorname*{colim}_{X \twoheadrightarrow \mathscr{E}} \mathsf{Hom}(\mathscr{E}, \mathscr{M})$$

and therefore to a $g \in \mathsf{Hom}(\mathscr{E}, \mathscr{M})$ for some factorization $X \twoheadrightarrow \mathscr{E} \to \mathrm{Spec}(k)$ in $\mathsf{E}^{\mathrm{surj}}(X/k)$.

Let $\mathscr{E} \to E$ be the coarse moduli space. Then E/k is an étale algebraic space, and $X \to \mathscr{E} \to E$ is surjective; we have therefore a factorization $X \to \pi_0(X/k) \twoheadrightarrow E$. As X/k is geometrically connected, $\pi_0(X/k) = \mathrm{Spec}(k)$, and so $E = \mathrm{Spec}(k)$ as well.

The gerbe $\mathscr{E} \to E = \mathrm{Spec}(k)$ has a section induced by $x \in X(k)$; hence, $\mathscr{E}$ is equivalent to BG for some étale k–group scheme G. The morphism $BG \to \mathscr{M}$ induced by g

is the datum of a curve C/k in $\mathscr{M}(k)$ and a left G–action on C. We may therefore replace G by its image in the finite group scheme $\mathrm{Aut}_k(C)$.

Now let $P \to X$ be the G–torsor associated to $X \to BG$. The 2–commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

induces a 2–commutative diagram

$$\begin{tikzcd} \mathrm{Fdiv}(\mathscr{M})(P) & \mathrm{Fdiv}(\mathscr{M})(k) = \mathscr{M}(k) \arrow[l] \\ \mathrm{Fdiv}(\mathscr{M})(X) \arrow[u] & \mathrm{Fdiv}(\mathscr{M})(BG) = \mathscr{M}(BG) \arrow[l] \arrow[u] \end{tikzcd}$$

where the equivalences on the right are due to Theorem 5.6.2 and Proposition 5.3.5.

As we said, the F–divided curve $(\mathscr{C}_i, \sigma_i)$ is in the essential image of the lower horizontal arrow, and its image in $\mathrm{Fdiv}(\mathscr{M})(P)$ is therefore isomorphic to the pullback of a curve $C \in \mathscr{M}(k)$. □

Appendix A Groupoidification

A.1 2–Objects

We introduce the notation that we will use in this section.

Let $\mathscr{C}$ be a category. We denote by $s\mathscr{C}$ the category $\mathrm{Hom}(\mathbf{\Delta}^{\mathrm{op}}, \mathscr{C})$ of simplicial objects. We denote by 2–$\mathscr{C}$ the category $\mathrm{Hom}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant 2}, \mathscr{C})$ of truncated simplicial objects: it consists of diagrams

$$\begin{tikzcd} X_2 \arrow[r, shift left=4, "\mathrm{pr}_1"] \arrow[r, "c" description] \arrow[r, shift right=4, "\mathrm{pr}_2"'] & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

satisfying the usual list of properties of simplicial objects. Such data are denoted by $X_\bullet$ or simply X. In fact, 2–$\mathscr{C}$ is naturally a 2–category. A 2–morphism $\eta\colon f \to g$ between morphisms $f, g\colon X_\bullet \to Y_\bullet$ is a so-called *simplicial homotopy* and consists of arrows $X_0 \to Y_1$ and $X_1 \to Y_2$ satisfying certain compatibilities as in May [21, Section I.5].

A.1.1 Remark Suppose $\mathscr{C}$ is the category **Sets**. For $X_\bullet \in 2$–**Sets**, we may interpret X_0 as a set of objects, X_1 as a set of arrows and X_2 as a subset of triangles $x \to y$, $y \to z$ and $x \to z$. The map e specifies the identity arrow, the maps s and t specify source and target, and the map c tells us that $x \to z$ is a composition of $x \to y$ and $y \to z$. Notice that two composable maps $x \to y$ and $y \to z$ may have multiple compositions.

A.2 Groupoids

We will say that $X_\bullet \in 2$–$\mathscr{C}$ is a *groupoid in* $\mathscr{C}$ if it satisfies:

- **Existence and uniqueness of compositions** The arrows $\mathrm{pr}_1, \mathrm{pr}_2 \colon X_2 \to X_1$ identify X_2 with $X_1 \times_{t,X_0,s} X_2$; this formalizes the fact that every pair of composable arrows should admit a unique composition.
- **Associativity** The two maps $c \circ (c, \mathrm{id}), c \circ (\mathrm{id}, c) \colon X_1 \times_{t,X_0,s} X_1 \times_{t,X_0,s} X_1 \to X_1$ coincide.
- **Existence of an inverse** There exists an involution $\iota \colon X_1 \to X_1$ such that the diagram

$$\begin{tikzcd}
X_1 \times_{t,X_0,s} X_1 \arrow[r, shift left=6, "\mathrm{pr}_1"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "\mathrm{pr}_2"'] \arrow[d, "(\iota,\iota)\circ\sigma"'] & X_1 \arrow[l, shift left=4] \arrow[l] \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] \arrow[d, "\iota"] & X_0 \arrow[l, "e"'] \arrow[d, "\mathrm{id}_{X_0}"] \\
X_1 \times_{t,X_0,s} X_1 \arrow[r, shift left=6, "\mathrm{pr}_2"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "\mathrm{pr}_1"'] & X_1 \arrow[l, shift left=4] \arrow[l] \arrow[r, shift left=3, "t"] \arrow[r, shift right=3, "s"'] & X_0 \arrow[l, "e"']
\end{tikzcd}$$

  is a morphism in 2–$\mathscr{C}$, ie all squares having corresponding horizontal maps commute (notice the change in order of the arrows between the top and bottom row!). The leftmost vertical arrow is given by the composition of the swapping $\sigma \colon X_1 \times_{t,X_0,s} X_1 \to X_1 \times_{s,X_0,t} X_1$, sending (a,b) to (b,a), with $(\iota,\iota) \colon X_1 \times_{s,X_0,t} X_1 \to X_1 \times_{t,X_0,s} X_1$.

Groupoids in $\mathscr{C}$ form in a natural way a full sub-2–category

$$\mathbf{Gpd}(\mathscr{C}) \subset 2\text{–}\mathscr{C}.$$

When $\mathscr{C} = $ **Sets**, an $X_\bullet \in$ **Gpd**(**Sets**) is the datum of a groupoid (ie a category where all arrows are invertible). Morphisms in **Gpd**(**Sets**) give functors of groupoids, and simplicial homotopies give natural transformations.

Sometimes we will denote a groupoid just by the symbol $X_1 \rightrightarrows X_0$, and leave the map $c \colon X_2 = X_1 \times_{s,t} X_1 \to X_1$ implicit.

A.3 Functors between simplicial categories of (pre)sheaves

We fix S an algebraic space; we work with the categories $\mathbf{PSh}_S$ of presheaves on $\mathbf{Sch}_S$, or the category $\mathbf{Sh}_S$ of sheaves on the big étale site over S (we could as well work with a topology finer than the étale topology, for instance the fppf topology).

Here is a list of important 2–functors whose existence and properties are classical (see eg Goerss and Jardine [10]):

$$\mathrm{tr}_2\colon s\mathbf{PSh}_S \to 2\text{–}\mathbf{PSh}_S, \qquad \mathrm{cosk}_2\colon 2\text{–}\mathbf{PSh}_S \to s\mathbf{PSh}_S,$$
$$\pi^{\mathrm{pre}}\colon s\mathbf{PSh}_S \to \mathbf{Gpd}(\mathbf{PSh}_S), \quad N\colon \mathbf{Gpd}(\mathbf{PSh}_S) \to s\mathbf{PSh}_S.$$

They are called, from left to right and top to bottom, 2–truncation, 2–coskeleton, fundamental groupoid and nerve. They are defined objectwise by the similarly named functors for simplicial sets. Every functor in the left column is left adjoint to the one to its right; moreover N is the restriction of cosk_2 via the inclusion $\mathbf{Gpd}(\mathscr{C}) \subset 2\text{–}\mathscr{C}$. However tr_2 is *not* the composition of π^{pre} with the inclusion. The functors N and cosk_2 are both fully faithful, that is, $\pi^{\mathrm{pre}} \circ N$ and $\mathrm{tr}_2 \circ \mathrm{cosk}_2$ are isomorphic to the identity functors.

We have analogous 2–functors at the level of sheaves,

$$\mathrm{tr}_2\colon s\mathbf{Sh}_S \to 2\text{–}\mathbf{Sh}_S, \qquad \mathrm{cosk}_2\colon 2\text{–}\mathbf{Sh}_S \to s\mathbf{Sh}_S,$$
$$\pi\colon s\mathbf{Sh}_S \to \mathbf{Gpd}(\mathbf{Sh}_S), \quad N\colon \mathbf{Gpd}(\mathbf{Sh}_S) \to s\mathbf{Sh}_S.$$

They are all defined by restriction of the corresponding functor for presheaves (note that cosk_2 preserves sheaves because it is constructed as a limit), except for the fundamental groupoid π, which is given by the composition $\mathrm{sh} \circ \pi^{\mathrm{pre}}$, where $\mathrm{sh}\colon \mathbf{Gpd}(\mathbf{PSh}_S) \to \mathbf{Gpd}(\mathbf{Sh}_S)$ is the termwise sheafification. It is easily checked that these two pairs of functors form adjoint pairs.

A.4 The fundamental groupoid of a Kan complex of sheaves

Let $X_\bullet \in s\mathbf{Sh}_S$ be a simplicial sheaf on S. We say that $X_\bullet$ is a *Kan complex* if, for every n and every horn $\Lambda^i[n] \subset \Delta[n]$, the map of sheaves

$$\mathscr{H}om(\Delta[n], X_\bullet) \to \mathscr{H}om(\Lambda^i[n], X_\bullet)$$

is surjective.

For a Kan complex, the description of the fundamental groupoid $\pi(X_\bullet)$

$$\pi(X_\bullet)_2 \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\underset{-c\rightarrow}{\rightrightarrows}}} \pi(X_\bullet)_1 \underset{t}{\overset{s}{\rightrightarrows}} \pi(X_\bullet)_0$$

is rather explicit (see [10, Section I.8]):

- In degree 0 we have $\pi(X_\bullet)_0 = X_0$.
- In degree 2, since $\pi(X_\bullet)$ is a groupoid, $\pi(X_\bullet)_2 = \pi(X_\bullet)_1 \times_{t,\pi(X_\bullet)_0,s} \pi(X_\bullet)_1$.
- To describe the degree 1, consider the subsheaf
$$\mathscr{H}om(\Delta[1]^2, X_\bullet)^\star \subset \mathscr{H}om(\Delta[1]^2, X_\bullet)$$
of those maps satisfying the following condition $(*)$: for $i = 0, 1$, when precomposed with the map $(d^i, \mathrm{id})\colon \Delta[0] \times \Delta[1] \to \Delta[1] \times \Delta[1]$, the resulting element of $X_1 = \mathscr{H}om(\Delta[1], X_\bullet)$ is in the image of the map $X_0 \to X_1$, $x \mapsto \mathrm{id}_x$.

  Now consider the "homotopy" equivalence relation[1]
$$\mathscr{H}om(\Delta[1]^2, X_\bullet)^\star \rightrightarrows \mathscr{H}om(\Delta[1], X_\bullet) = X_1 \tag{4}$$
with maps induced by $(\mathrm{id}, d^0), (\mathrm{id}, d^1)\colon \Delta[1] = \Delta[1] \times \Delta[0] \rightrightarrows \Delta[1]^2$.

  Thanks to the condition $(*)$ above, each of the maps $s, t\colon X_1 \to X_0$ coequalizes the two maps in (4). By definition, $\pi(X_\bullet)_1$ is the quotient of the above equivalence relation; the maps s and t descend to maps $\pi(X_\bullet)_1 \rightrightarrows X_0$.

A.5 The key lemma

We recall the following well-known fact:

A.5.1 Lemma *Let $F\colon I \to \mathbf{Sh}_S$ be a diagram of sheaves indexed by a filtered small category. If the objects $F(i)$ are representable by étale S–algebraic spaces, then the colimit of the diagram is representable by étale S–algebraic spaces.*

Proof In fact the statement holds true for arbitrary colimits, but for filtered ones the proof is more straightforward and goes as follows. For each pair of maps of I with the same target, $a\colon i \to k$ and $b\colon j \to k$, we form the fibred product $F_{a,b} := F(i) \times_{F(a),F(k),F(b)} F(j)$. Using the fact that morphisms between étale algebraic spaces are étale, we have an étale groupoid
$$\coprod_{\substack{a\colon i\to k\\ b\colon j\to k}} F_{a,b} \to \coprod_{i,j} F(i) \times F(j) = \Big(\coprod_i F(i)\Big) \times \Big(\coprod_j F(j)\Big).$$
Its image is an open equivalence relation on $\coprod F(i)$ whose quotient is an étale algebraic space, and the desired colimit. □

[1]The fact that it is indeed an equivalence relation can be seen as follows: the symmetry is given by the obvious involution of $\Delta[1]^2$; the transitivity by the observation that $\Delta[1]^2 \times_{\mathrm{pr}_1,\Delta[1],\mathrm{pr}_2} \Delta[1]^2 \cong \Delta[1]^2$.

A.5.2 Lemma *Let $E_\bullet$ be a simplicial étale S–algebraic space. Then the fundamental groupoid $\pi(E_\bullet) \in \mathbf{Gpd}(\mathbf{Sh}_S)$ is a groupoid in étale S–algebraic spaces. Moreover, its formation commutes with arbitrary base change, up to equivalence.*

Proof Write $E = E_\bullet$. For the first part of the statement, we begin by showing that we may reduce to proving the claim for E a Kan complex. We use Kan's Ex^∞ functor for sheaves; Jardine [18, Proposition 1.17] deals with presheaves and can be used as a reference. We let $\mathrm{Ex}(E)$ be the simplicial sheaf with $\mathrm{Ex}(E)_n = \mathscr{H}om(\mathrm{sd}\,\Delta[n], E)$, where $\mathrm{sd}\,\Delta[n]$ is the subdivision of $\Delta[n]$. As $\mathrm{sd}\,\Delta[n]$ is a simplicial constant sheaf, $\mathrm{Ex}(E)$ is a simplicial étale space. The last vertex map $\mathrm{sd}\,\Delta[n] \to \Delta[n]$ induces a map $E \to \mathrm{Ex}(E)$. Iterating the construction, we obtain a diagram $E \to \mathrm{Ex}(E) \to \mathrm{Ex}^2(E) \to \cdots$ of simplicial étale spaces. We denote by $\mathrm{Ex}^\infty(E)$ the colimit in $s\mathbf{Sh}_S$; this is in fact a simplicial étale space by Lemma A.5.1 and a Kan complex. We denote by $w\colon E \to \mathrm{Ex}^\infty(E)$ the induced map.

For any $T \to S$ with T quasicompact, we have $\mathrm{Ex}^\infty(E)(T) = \mathrm{Ex}^\infty(E(T))$, by [28, Tag 0739]; in this case, $w(T)\colon E(T) \to \mathrm{Ex}^\infty(E)(T)$ is a weak equivalence of simplicial sets. It induces an equivalence of fundamental groupoids $\pi(w(T))\colon \pi(E(T)) \to \pi(\mathrm{Ex}^\infty(E)(T))$. It follows that the map of groupoids in sheaves $\pi(w)\colon \pi(E) \to \pi(\mathrm{Ex}^\infty(E))$ is an equivalence. Let us rename $X = \pi(E)$ and $Y = \pi(\mathrm{Ex}^\infty(E))$ for brevity. Assume now that Y is a groupoid in étale spaces. We claim that X is as well. The fact that $\pi(w)$ is an equivalence implies that the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \times X_0 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

is cartesian. We already know that $X_0 = \pi(E)_0 = E_0$ is an étale algebraic space. Then the fibre product $Y_1 \times_{Y_0 \times Y_0} (X_0 \times X_0)$ is an étale algebraic space as well. This proves the claim, and completes the proof of reduction to the case of E a Kan complex.

Now we assume that E is a Kan complex and use the description of $\pi(E)$ given in Section A.4. In degree zero, $\pi(E)_0 = E_0$ is an étale algebraic space. It remains to show that $\pi(E)_1$ is an étale algebraic space. Since $\Delta[1]$ is a simplicial constant sheaf, $\mathscr{H}om(\Delta[1]^2, E)$ is an étale algebraic space. Its subsheaf $\mathscr{H}om(\Delta[1]^2, E)^\star$ (as defined in Section A.4) is the fibre product $\mathscr{H}om(\Delta[1]^2, E)^\star \times_{E_1 \times E_1} E_0 \times E_0$, and hence is also an étale algebraic space. The sheaf $\pi(E)_1$ is the quotient of the equivalence relation

$$\mathscr{H}om(\Delta[1]^2, E)^\star \rightrightarrows E_1$$

and hence an étale algebraic space. This completes the proof of the first part of the statement.

For the second part of the statement, let $T \to S$ be a morphism of algebraic spaces; then $\mathrm{Ex}^\infty(E)_T$ is still a Kan complex and moreover the map $E_T \to \mathrm{Ex}^\infty(E)_T$ induces an equivalence of fundamental groupoids (by the same argument as earlier in the proof). We can therefore reduce to proving the statement for E a Kan complex. This simply relies on the fact that forming quotients by étale equivalence relations commutes with base change. □

A.6 Groupoidification of 2–étale spaces

Consider now the category $2\text{–}\mathbf{\acute{E}t}_S$ of 2–étale spaces. By Lemma A.5.2, we obtain a *groupoidification* functor

$$\mathrm{gpd}\colon 2\text{–}\mathbf{\acute{E}t}_S \to \mathbf{Gpd}(\mathbf{\acute{E}t}_S), \quad X_\bullet = (X_1 \rightrightarrows X_0) \mapsto \mathrm{gpd}(X_\bullet) := \pi(\mathrm{cosk}_2(X_\bullet)),$$

which is easily seen to be left adjoint to the inclusion $\mathbf{Gpd}(\mathbf{\acute{E}t}_S) \subset 2\text{–}\mathbf{\acute{E}t}_S$. In fact, even more is true: for $E = E_\bullet \in 2\text{–}\mathbf{\acute{E}t}_S$ the unit map

$$\mathrm{cosk}_2(E) \to N\pi\,\mathrm{cosk}_2(E)$$

induced from the adjunction of N and π, gives, upon applying the 2–truncation functor, a natural map

$$E \to \mathrm{gpd}(E) = \pi\,\mathrm{cosk}_2(E)$$

for which the following holds:

A.6.1 Corollary *The map $E \to \mathrm{gpd}(E)$ is universal for morphisms from E to groupoids in $\mathbf{Sh}_S$; more precisely, for $E \in 2\text{–}\mathbf{\acute{E}t}_S$ and $G \in \mathbf{Gpd}(\mathbf{Sh}_S)$ the natural functor*

$$\mathsf{Hom}_{\mathbf{Gpd}(\mathbf{Sh}_S)}(\mathrm{gpd}(E), G) \to \mathsf{Hom}_{2\text{–}\mathbf{Sh}_S}(E, G)$$

is an equivalence of categories.

Proof We have equivalences

$$\mathsf{Hom}(\pi\,\mathrm{cosk}_2(E), G) = \mathsf{Hom}(\mathrm{cosk}_2(E), NG) = \mathsf{Hom}(E, G),$$

where the last equivalence is due to the fact that cosk_2 is a 2–fully faithful 2–functor (see [28, Tag 0185]) and N is the restriction of cosk_2. □

A.7 Universal property towards stacks

A.7.1 Corollary *Let $X_\bullet\colon s,t\colon R \rightrightarrows U$ be a 2–étale algebraic space, with groupoidification*

$$\mathrm{gpd}(X_\bullet)\colon R^{\mathrm{gpd}} \rightrightarrows U.$$

Write $\pi\colon U \to [U/R^{\mathrm{gpd}}]$ for the associated quotient stack and $\pi\colon U \to U/R^{\mathrm{gpd}}$ for the quotient algebraic space obtained from the equivalence relation given by the image of $R^{\mathrm{gpd}} \to U \times_S U$.

For each fppf stack in groupoids $\mathscr{X}/S$ (resp. fppf sheaf), let

$$\mathsf{eq}(\mathsf{Hom}(U,\mathscr{X}) \overset{s^*}{\underset{t^*}{\rightrightarrows}} \mathsf{Hom}(R,\mathscr{X}))$$

be the "equalizer" category (resp. setoid) described as follows:

(i) *Objects are pairs (f,β) composed of a 1–morphism $f\colon U \to \mathscr{X}$ and a 2–isomorphism $\beta\colon fs \to ft$ such that $\beta c = \beta p_1 \circ \beta p_2$.*

(ii) *Morphisms $(f_1,\beta_1) \to (f_2,\beta_2)$ are 2–isomorphisms $\varphi\colon f_1 \to f_2$ such that $\beta_2 \circ \varphi s = \varphi t \circ \beta_1$.*

Then the functor

$$\mathsf{Hom}([U/R^{\mathrm{gpd}}],\mathscr{X}) \to \mathsf{eq}\big(\mathsf{Hom}(U,\mathscr{X}) \overset{s^*}{\underset{t^*}{\rightrightarrows}} \mathsf{Hom}(R,\mathscr{X})\big), \quad g \mapsto (f = g\pi,\ \beta = g\alpha),$$

(resp. the analogous functor for the sheaf U/R^{gpd}) is an equivalence of categories (resp. a bijection).

Before we pass to the proof, here are pictures for the 2–morphisms β and φ:

$$\begin{tikzcd}[column sep=8em]
R \times_{s,U,t} R \arrow[r, bend left=50, "fsp_2=fsc"] \arrow[r, "ftp_2=fsp_1" description] \arrow[r, bend right=50, "ftp_1=ftc"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \beta p_2"] \arrow[r, phantom, shift right=3ex, "\Downarrow \beta p_1"] \arrow[r, phantom, "\Downarrow \beta c"] & \mathscr{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
f_1 s \arrow[r, "\varphi s"] \arrow[d, "\beta_1"'] & f_2 s \arrow[d, "\beta_2"] \\
f_1 t \arrow[r, "\varphi t"] & f_2 t
\end{tikzcd}$$

Proof The result for sheaves is an easy consequence of the result for stacks and we content ourselves with a proof of the latter. Set $\mathsf{H} = \mathsf{Hom}([U/R^{\mathrm{gpd}}],\mathscr{X})$ and

$$\mathsf{E} = \mathsf{eq}(\mathsf{Hom}(U,\mathscr{X}) \rightrightarrows \mathsf{Hom}(R,\mathscr{X})), \quad \mathsf{E}_{\mathrm{gpd}} = \mathsf{eq}(\mathsf{Hom}(U,\mathscr{X}) \rightrightarrows \mathsf{Hom}(R^{\mathrm{gpd}},\mathscr{X})).$$

Let $F\colon \mathsf{H} \to \mathsf{E}$ be the functor in the statement.

Suppose first that $R \rightrightarrows U$ is a groupoid. For each pair (f,β), Lemma 77.23.2 in [28, Tag 044U] produces functorially a morphism $g\colon [U/R^{\mathrm{gpd}}] \to \mathscr{X}$ and a 2–isomorphism $\epsilon\colon g\pi \xrightarrow{\sim} f$. That is, we have a functor $G\colon \mathsf{E} \to \mathsf{H}$ and an isomorphism $\epsilon\colon FG \xrightarrow{\sim} \mathrm{id}$. Moreover the proof of loc. cit. shows that GF is equal to the identity; hence F and G are quasi-inverse equivalences.

We go back to the general case. The equivalence produced in the paragraph above identifies F with the functor

$$\mathsf{E}_{\mathrm{gpd}} \to \mathsf{E}$$

sending (f,β) to $(f,\beta r)$, where $r\colon R \to R^{\mathrm{gpd}}$ is the natural map.

Let κ be an isomorphism class of morphisms $U \to \mathscr{X}$; we denote by $\mathsf{E}_{\mathrm{gpd},\kappa}$ and E_κ the full subcategories of $\mathsf{E}_{\mathrm{gpd}}$ and E consisting of objects (f,β) with $f \in \kappa$. It suffices to show that the restriction map $F_\kappa\colon \mathsf{E}_{\mathrm{gpd},\kappa} \to \mathsf{E}_\kappa$ is an equivalence.

The class κ induces a groupoid of sheaves $G_\bullet\colon U \times_{\mathscr{X}} U \rightrightarrows U$, uniquely defined up to isomorphism of groupoids lying over the identity $U \to U$. The category E_κ is identified with the category $\mathsf{Hom}(X_\bullet, G_\bullet)$: an object (f,β) of E_κ corresponds to the map $X_\bullet \to G_\bullet$ which in degree 0 is the identity of U and in degree 1 is the map $R \to U \times_{f,\mathscr{X},f} U$ given by $(s,t,\beta\colon fs \to ft)$. Similarly, $\mathsf{E}_{\mathrm{gpd},\kappa}$ is identified with $\mathsf{Hom}(\mathrm{gpd}(X_\bullet), G_\bullet)$. The universal property of groupoidification stated in Corollary A.6.1 tells us exactly that the resulting functor $\mathsf{Hom}(\mathrm{gpd}(X_\bullet), G_\bullet) \to \mathsf{Hom}(X_\bullet, G_\bullet)$ is an equivalence. □

We conclude with an important observation: when one applies the groupoidification functor to a 2–étale space $X_\bullet$ with all the X_i quasicompact over S, the resulting groupoid in étale spaces may not have quasicompact terms. We give an example of this phenomenon:

A.7.2 Example Let $S = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field, so that $\mathbf{\acute{E}t}_S = \mathbf{Sets}$. We think of a 2–set as a particular kind of directed graph, as in Remark A.1.1. Let $n \geqslant 2$ and $X_\bullet$ be the graph with vertices $\{v_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}$, and one arrow $\alpha_{ij}\colon v_i \to v_j$ whenever $|i-j| \leqslant 1$. The composition law is defined to be the obvious one on pairs containing an identity loop, plus $\alpha_{ji} \circ \alpha_{ij} = \mathrm{id}_{v_i}$ for any i and j with $|i-j| \leqslant 1$.

In the groupoidification $\mathrm{gpd}(X_\bullet)$, the composition $v_1 \to v_2 \to v_3 \to \cdots \to v_n \to v_1$ is not equal to the identity of v_1, however. In particular, $\mathrm{gpd}(X)_1$ is infinite. More precisely, $\mathrm{gpd}(X)$ has the following description: its vertices are $v_1,\dots,v_n$; for any $i,j\in\mathbb{Z}/n\mathbb{Z}$ and $a\in\mathbb{Z}$ there is one arrow $\alpha_{i,j,a}\colon v_i\to v_j$, the identities being the arrows of the form $\alpha_{i,i,0}$. The composition is given by $\alpha_{j,k,b}\circ\alpha_{i,j,a}=\alpha_{i,k,a+b}$. It is immediately checked that this groupoid is equivalent to the groupoid $\mathbb{Z}\rightrightarrows\{\star\}$ with composition given by addition.

Appendix B Quasicompact presentations of quasicompact stacks

The definition of a τ–quasicompact topological space is given in Definition 5.3.1. The next two lemmas generalize classical results holding for a quasicompact quasiseparated (qcqs) base S to the case where S is only qs and τ–quasicompact.

B.1 Lemma *Let S be a τ–quasicompact and quasiseparated algebraic space, and $\mathscr{X}/S$ a quasicompact morphism of algebraic stacks. Let $f\colon\mathscr{Y}\to\mathscr{X}$ be a surjective, open S–morphism from an algebraic stack. Then there exists an open substack $\mathscr{U}\subset\mathscr{Y}$ such that $\mathscr{U}\to\mathscr{X}$ is surjective and $\mathscr{U}\to S$ is quasicompact.*

Proof We fix an open cover $\{S_i\}_{i\in I}$ of S as in Definition 5.3.1. Notice that each inclusion $S_i\to S$ is quasicompact, by quasiseparatedness of S.

Fix $i\in I$. Let $\mathscr{X}_i\subset\mathscr{X}$, $\mathscr{Y}_i\subset\mathscr{Y}$ and $f_i\colon\mathscr{Y}_i\to\mathscr{X}_i$ be the base changes via $S_i\subset S$. Let $g\colon\bigsqcup_{j\in J}W_j\to\mathscr{Y}_i$ be a smooth surjective morphism, where each W_j is an affine scheme. Then $\{f_i(g(W_j))\}_{j\in J}$ is an open cover of $\mathscr{X}_i$. The stack $\mathscr{X}_i=\mathscr{X}\times_S S_i$ is quasicompact because $\mathscr{X}\to S$ is. Also each $f_i(g(W_j))$ is quasicompact, being the image of a quasicompact scheme. Therefore there exists a finite subset $F_i\subset J$ such that $\{f_i(g(W_j))\}_{j\in F_i}$ covers $\mathscr{X}_i$. We write $\mathcal{W}_i=\bigcup_{j\in F_i}f_i(W_j)$, a quasicompact open substack of $\mathscr{Y}_i$.

Now we let i vary; write $\mathscr{U}=\bigcup_{i\in I}\mathcal{W}_i$, an open substack of $\mathscr{Y}$. It is immediate to see that the map $\mathscr{U}\to\mathscr{X}$ is surjective. It remains to show that $\mathscr{U}\to S$ is quasicompact. For this, it suffices to show that $\mathscr{U}\times_S S_{i_0}$ is quasicompact for every $i_0\in I$. The latter is $\bigcup_{i\in I}(\mathcal{W}_i\times_S S_{i_0})$. Let F be the finite subset of I of those i with $S_i\cap S_{i_0}\neq\varnothing$. Then the union above is equal to the finite union $\bigcup_{i\in F}(\mathcal{W}_i\times_S S_{i_0})$. It therefore suffices to show that $\mathcal{W}_i\times_S S_{i_0}$ is quasicompact for every $i\in F$. This holds by quasiseparatedness of S and quasicompactness of S_{i_0} and $\mathcal{W}_i$. □

B.2 Lemma *Let S be a τ–quasicompact, quasiseparated algebraic space. Let $\mathscr{X} \to S$ be a quasicompact algebraic stack locally of finite presentation. Then there exists a smooth surjective morphism $U \to \mathscr{X}$ from a finitely presented S–scheme. If $\mathscr{X} \to S$ is a Deligne–Mumford stack, $U \to \mathscr{X}$ may be taken étale.*

Proof We first take an open cover $\{S_i\}_{i \in I}$ as in Definition 5.3.1. The stack $\mathscr{X}_i := \mathscr{X} \times_S S_i$ is quasicompact because $\mathscr{X} \to S$ is so. Pick a surjective smooth morphism $U_i \to \mathscr{X}_i$ from an affine scheme. Let $U := \bigsqcup_{i \in I} U_i$, and $U \to \mathscr{X}$ the induced morphism. It is surjective and smooth. It remains to check that $U \to S$ is finitely presented. Each map in the composition $U_i \to \mathscr{X}_i \to S_i \to S$ is locally of finite presentation, and hence so is $U \to S$. To check that $U \to S$ is qcqs, it suffices to show that, for given $i_0 \in I$, $U \times_S S_{i_0} \to S_{i_0}$ is qcqs. As in the previous proofs, the latter is a union of $U_i \times_S S_{i_0}$ over finitely many i's. By quasiseparatedness of S and quasicompactness of U_i and S_{i_0}, we see that $U_i \times_S S_{i_0}$ is quasicompact. It is also quasiseparated, because base change of an affine via the quasiseparated open immersion $S_{i_0} \to S$.

The second part of the statement is clear, replacing the word "smooth" by "étale" where needed. □

B.3 Corollary *Let S be a τ–quasicompact, quasiseparated algebraic space. Let $\mathscr{X} \to S$ be a quasicompact algebraic stack locally of finite presentation, and $\mathscr{Y} \to \mathscr{X}$ a surjective, open S–morphism, locally of finite presentation, from an algebraic stack. Then there exists a smooth morphism $U \to \mathscr{Y}$ from a finitely presented S–scheme U such that $U \to \mathscr{Y} \to \mathscr{X}$ is surjective.*

Proof Apply first Lemma B.1 to obtain an open $\mathscr{U} \subset \mathscr{Y}$, and then Lemma B.2 with $\mathscr{X} = \mathscr{U}$. □

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