Solutions for Exercises, BMS Basic Course

Algebraic Geometry

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This is a collection of solutions to the exercises for the BMS basic course "Algebraic Geometry", given by Prof. Dr. Jürg Kramer in the summer semester 2012 at Humboldt University, Berlin.

The solutions are written up by the students who attended this course. We encourage any feedback from the readers.

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1 Solutions for Exercise Sheet-1

Exercise 1.1. *Prove the following:*

(a) *For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on a topological space X, show that $\ker(\varphi)_P = \ker(\varphi_P)$ and $\operatorname{im}(\varphi)_P = \operatorname{im}(\varphi_P)$, for each point $P \in X$.*

(b) *Show that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if the induced map on the stalks $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective (resp. surjective) for all $P \in X$.*

(c) *Show that a sequence*

$$\ldots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \xrightarrow{\varphi^{i+1}} \ldots$$

of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. **Solution by Karl Christ**

(a) Let $\langle U, s\rangle$ denote an equivalence class in $\mathcal{F}_P$ as characterized in the lecture. Then $\varphi_P(\langle U, s\rangle)$ is given by $\langle U, \varphi_U(s)\rangle$. Furthermore, it is obvious that

$$\{\langle U, s\rangle | \langle U, \varphi_U(s)\rangle = 0_P\} \supseteq \{\langle U, s\rangle | \varphi_U(s) = 0\}.$$

For $\langle U, s\rangle$ with $\langle U, \varphi_U(s)\rangle = 0_P$, $\exists V \subseteq U$ s.t. $\varphi_U(s)_{|V} = \varphi_V(s) = 0$. So

$$\langle U, s\rangle = \langle V, s\rangle \in \{\langle U, s\rangle | \varphi_U(s) = 0\},$$

which implies

$$\{\langle U, s\rangle | \langle U, \varphi_U(s)\rangle = 0_P\} \subseteq \{\langle U, s\rangle | \varphi_U(s) = 0\}.$$

Combining these observations, we find

$$\ker(\varphi_P) = \{\langle U, s\rangle | \varphi_P(\langle U, s\rangle) = 0_P\} = \{\langle U, s\rangle | \langle U, \varphi_U(s)\rangle = 0_P\} = \\ \{\langle U, s\rangle | \varphi_U(s) = 0\} = \{\langle U, s\rangle | s \in \ker(\varphi_U)\} \cong \varinjlim_{P \in U} \ker(\varphi_U) = \ker(\varphi)_P$$

By observations analogous to the ones above, for $\langle U, t\rangle \in \mathcal{F}_P$ and $\langle U, s\rangle \in \mathcal{G}_P$, we find

$$\operatorname{im}(\varphi_P) = \{\langle U, s\rangle | \exists \langle U, t\rangle : \varphi_P(\langle U, t\rangle) = \langle U, s\rangle\} = \\ \{\langle U, s\rangle | \exists \langle U, t\rangle : \langle U, \varphi_U(t)\rangle = \langle U, s\rangle\} = \{\langle U, s\rangle | \exists \langle U, t\rangle : \varphi_U(t) = s\} = \\ \{\langle U, s\rangle | s \in \operatorname{im}(\varphi_U)\} \cong \varinjlim_{P \in U} \operatorname{im}(\varphi_U) = \operatorname{im}(\varphi)_P,$$

where the last step is on presheaves, but since the associated sheaf to a presheaf and the presheaf itself coincide on stalks (as stated in the exercise class), this gives the desired equality.

(b) φ injective:

$$\ker(\varphi) = 0 \Leftrightarrow \ker(\varphi)_P = 0,\ \forall P \in X \quad \Leftrightarrow \\ \ker(\varphi_P) = 0,\ \forall P \in X \quad (\text{by(a)}) \Leftrightarrow \varphi_P \text{ injective},\ \forall P \in X.$$

φ surjective:

$$\operatorname{im}(\varphi) = \mathcal{G} \Leftrightarrow \operatorname{im}(\varphi)_P = \mathcal{G}_P,\ \forall P \in X \Leftrightarrow \\ \operatorname{im}(\varphi_P) = \mathcal{G}_P,\ \forall P \in X \quad (\text{by(a)}) \Leftrightarrow \varphi_P \text{ surjective},\ \forall P \in X.$$

(c) Let us assume that a sequence of sheaves

$$\ldots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \xrightarrow{\varphi^{i+1}} \ldots$$

is exact. Let φ denote the inclusion map

$$\varphi : \operatorname{im}(\varphi^{i-1}) \to \ker(\varphi^{i}).$$

Then we get

$$\begin{aligned}
&\operatorname{im}(\varphi^{i-1}) = \ker(\varphi^{i}) \Leftrightarrow \varphi \text{ isomorphism} \Leftrightarrow \\
&\varphi_P : \operatorname{im}(\varphi^{i-1})_P \to \ker(\varphi^{i})_P \text{ isomorphism } \forall P \in X \text{ (by(b))} \Leftrightarrow \\
&\operatorname{im}(\varphi^{i-1})_P = \ker(\varphi^{i})_P, \ \forall P \in X \Leftrightarrow \operatorname{im}(\varphi^{i-1}_P) = \ker(\varphi^{i}_P), \ \forall P \in X \text{ (by(a))},
\end{aligned}$$

which implies that the corresponding sequences of stalks are exact.

□

Exercise 1.2. *Prove the following:*

(a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Show that φ is surjective if and only if the following condition holds: For every open set $U \subseteq X$ and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\varphi(t_i) = s\big|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ and an open set $U \subseteq X$ such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Proof. **Solution by Karl Christ**

(a) We use the following description of an associated sheaf to a presheaf (defined in the exercise classes):

$$\mathcal{F}^{+}(U) = \{(s_x) \in \Pi_{x \in U} \mathcal{F}_x | \forall x \in U, \exists x \in W \subseteq U \text{ and } t \in \mathcal{F}(W) : s_x = t_x, \ \forall x \in W\}. \tag{1}$$

Suppose φ is surjective, and let $U \subseteq X$ be an open subset and $s \in \mathcal{G}(U)$. Then by surjectivity of φ, $s \in \operatorname{im}(\varphi)$. By the above description of an associated sheaf, we find that $\forall P \in U$, $\exists P \in V_P \subseteq U$, and $t^P \in \operatorname{im}(\varphi_{V_P})$ such that $\forall Q \in V_P$, we have $s_Q = t^P_Q$. This gives for every Q an open neighborhood V_Q, such that $s_{|V_Q} = t^P_{V_Q}$. Thus the V_Q form an open cover of V_P. Since $\mathcal{G}$ is a sheaf, $s_{V_P} = t^P$. Moreover as $t^P \in \operatorname{im}(\varphi_{V_P})$, we can choose $t_i \in \mathcal{F}(V_P)$, such that $\varphi_{V_P}(t_i) = t^P = s_{V_P}$. Varying P over U gives the desired covering.

Now let us assume the converse. Then for any $s \in \mathcal{G}(U)$, there exists an open covering $\{U_i\}$ of U, such that for every $x \in U$, there exists a $t_i \in \mathcal{F}(U_i)$ and $\varphi_{U_i}(t_i) = s_{U_i}$. This implies that $\varphi_{U_i}(t_i)_x = s_x \ \forall x \in U_i$. Taking the $\varphi_{U_i}(t_i)$ to be the t and U_i to be the W in the above description (1) of an associated sheaf, we derive that $s \in \operatorname{im}(\varphi)$.

(b) Let $\mathcal{F}$ be the sheaf of abelian groups under addition such that

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} | f \text{ holomorphic on } U\}$$

and $\mathcal{G}$ be the sheaf of abelian groups under multiplication such that

$$\mathcal{G}(U) = \{f : U \to \mathbb{C} | f \text{ holomorphic on } U \text{ and nowhere zero}\}$$

for any open subset U in the underlying topological space $\mathbb{C} \setminus \{0\}$. Observe that the map

$$\varphi : f \to e^f,$$

is a sheaf-homomorphism.

We first show that φ is surjective, by checking it on stalks: Let $\langle U, s \rangle \in \mathcal{G}_P$. Let $V' \subseteq s(U)$ and $s(P) \in V'$ be small enough such that a branch log is defined on V'. Put $V = s^{-1}(V')$. Then we find that

$$\varphi_P(\langle V, \log(s_{|V}) \rangle) = \langle V, s \rangle = \langle U, s \rangle,$$

so φ_P is surjective, and hence φ is surjective.

On the other hand, for the section $s(z) = z \in \mathcal{G}(\mathbb{C} \setminus \{0\})$, there is no preimage under φ. This of course is at the same time an example, where the image of a sheaf homomorphism is not itself a sheaf.

□

Exercise 1.3. *Prove the following:*

(a) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$ on a topological space X. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$. Thus, there is an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

(b) Conversely, if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Proof. **Solution by Albert Haase**

(a) First we will the prove the following useful fact.

Claim: For any $P \in X$ we have the following isomorphism of groups

$$(\mathcal{F}/\mathcal{F}')_P \cong \mathcal{F}_P/\mathcal{F}'_P. \tag{2}$$

Proof. When defining the sheaf $\mathcal{G}^+$ associated to a presheaf $\mathcal{G}$ on X we argued that $\mathcal{G}_P = \mathcal{G}^+_P$ for all $P \in X$. Let $U \subseteq X$ be open and define the presheaf

$$\mathcal{G}(U) := \mathcal{F}(U)/\mathcal{F}'(U).$$

Then for $P \in X$, we have $\mathcal{G}_P = \mathcal{G}^+_P \overset{\text{by def'n}}{=} (\mathcal{F}/\mathcal{F}')_P$. If we let brackets $\langle \cdot, \cdot \rangle$ denote the equivalence classes w.r.t. the relation which defines the stalk of $\mathcal{G}'$ at P then

$$\mathcal{G}_P = \{\, \langle U, \overline{s} \,\rangle | P \in U \subseteq X \text{ open}, \overline{s} \in \mathcal{F}(U)/\mathcal{F}'(U) \,\} =$$
$$\{\langle U, s + \mathcal{F}'(U) \,\rangle | P \in U \subseteq X \text{ open}, s \in \mathcal{F}(U)\} \cong \mathcal{F}_P/\mathcal{F}'_P$$
$$\text{via the map } \langle U, s + \mathcal{F}'(U) \,\rangle \mapsto \langle U, s \,\rangle + \mathcal{F}'_P.$$

□

The statement in (a) is now a corollary of Exercise 1.1 and the above claim. For all $P \in X$ we have the following exact sequence of abelian groups

$$0 \longrightarrow \mathcal{F}'_P \longrightarrow \mathcal{F}_P \longrightarrow \mathcal{F}_P/\mathcal{F}'_P \longrightarrow 0.$$

Hence by Exercise 1.1 (c) we can conclude that the sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0$$

is exact.

(b) We would like to show that for an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}/\mathcal{F}' \to 0$$

on a topological space X, $\mathcal{F}'$ is isomorphic to a subsheaf $\mathcal{G}$ of $\mathcal{F}$ and that $\mathcal{F}'' \cong \mathcal{F}/\mathcal{G}$. Exercise II.1.5 of [Har] goes to show that a morphism of sheaves is an isomorphism if and only if it is bijective. (Recall that [Har] defines an isomorphism of presheaves as a morphism with right and left inverses.) By Exercise 1.1 (c) we have the following exact sequence of groups

$$0 \longrightarrow \mathcal{F}'_P \xrightarrow{\phi} \mathcal{F}_P \xrightarrow{\psi} (\mathcal{F}/\mathcal{F}')_P \to 0.$$

Then by the isomorphism theorem for groups and equation (2), we find that the maps

$$\mathcal{F}'_P \longrightarrow \mathrm{im}(\phi_P) \quad \text{and} \quad \mathcal{F}_P/\mathrm{im}(\phi_P) \cong (\mathcal{F}/\mathrm{im}(\phi))_P \longrightarrow \mathcal{F}''_P$$

are bijective. Hence, applying Exercise 1.1 (a) and 1.1 (b) we get bijective maps

$$\mathcal{F}' \longrightarrow \mathrm{im}(\phi) \quad \text{and} \quad (\mathcal{F}/\mathrm{im}(\phi)) \longrightarrow \mathcal{F}''$$

which are isomorphisms. This concludes the proof of the exercise.

□

Exercise 1.4. *For any open subset U of a topological space X, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

is an exact sequence of sheaves, then

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'')$$

is an exact sequence of abelian groups. We note that the functor $\Gamma(U, \cdot)$ need not be exact.

Proof. **Solution by Albert Haase**

The exercise is to show that for any open set $U \subseteq X$, where X is a topological space, $\Gamma(U, \cdot)$ is a left exact covariant functor from the category of sheaves on X to the category of Abelian groups.

We know from class that $\Gamma(U, \cdot)$ sends sheaves $\mathcal{F}$ on X to groups $\mathcal{F}(U)$ and morphisms $\phi\colon \mathcal{F} \longrightarrow \mathcal{F}'$ of sheaves on X to homomorphisms of groups $\phi(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$.

Now let

$$0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \tag{3}$$

be an exact sequence of sheaves on X. We would like to show that the following sequence

$$0 \to \Gamma(U, \mathcal{F}') \xrightarrow{\phi} \Gamma(U, \mathcal{F}) \xrightarrow{\psi} \Gamma(U, \mathcal{F}'') \tag{4}$$

is exact.

Since ϕ is injective and $\ker(\phi) = 0$ is defined as the presheaf kernel $U \mapsto \ker(\phi)$, it must be the zero-sheaf. This implies that $\ker(\phi(U)) = 0$ and subsequently exactness at $\Gamma(U, \mathcal{F}')$. Furthermore, because $\phi(U)$ is injective, the presheaf image of ϕ is actually a sheaf and hence it coincides with $\mathrm{im}(\phi)$. By exactness of the sequence (3), we get

$$(U \mapsto \mathrm{im}(\phi(U))) = (U \mapsto \ker(\psi(U))) \,,$$

which proves the exactness of the sequence (4). $\square$

2 Solutions for Exercise Sheet-2

Exercise 2.1. *Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set and let $R(X) := k[X_1, \ldots, X_n]/I(X)$ denote its coordinate ring. Let $\mathcal{O}_X$ denote the sheaf of regular functions on X. For $f \in R(X)$, prove the equality*

$$\mathcal{O}_X\big(D(f)\big) = R(X)_f \subseteq \mathrm{Quot}\big(R(X)\big).$$

In particular, deduce that $\mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X) = R(X)$.

Proof. **Solution by Claudius Heyer**

Since X is irreducible, $I(X)$ is prime and hence $R(X)$ is an integral domain. Therefore $\mathrm{Quot}\big(R(X)\big)$ exists. Recall the definitions of $D(f)$, $R(X)_f$, and $\mathcal{O}_X\big(D(f)\big)$:

$$\begin{aligned} D(f) &= \big\{x \in X \,\big|\, f(x) \neq 0\big\}\,, \\ R(X)_f &= \left\{ \frac{g}{f^n} \,\middle|\, g \in R(X), n \in \mathbb{N}_0 \right\}, \\ \mathcal{O}_X\big(D(f)\big) &= \left\{ \varphi\colon D(f) \longrightarrow k \,\middle|\, \begin{array}{l} \forall x \in D(f)\colon \exists U \ni x \text{ open}\colon \exists g, h \in R(X)\colon \\ h(y) \neq 0 \ \forall y \in U \text{ and } \varphi(y) = \frac{g(y)}{h(y)} \quad \forall y \in U \end{array} \right\} \end{aligned}$$

It is obvious that $R(X)_f \subseteq \mathcal{O}_X\big(D(f)\big)$ (choose $U = D(f)$). It remains to show that $\mathcal{O}_X\big(D(f)\big) \subseteq R(X)_f$.

Let $\varphi \in \mathcal{O}_X\big(D(f)\big)$. For all $x_i \in D(f)$, there exist $x_i \in U_i$ open and $g_i, h_i \in R(X)$, such that $h_i(y) \neq 0$ and $\varphi(y) = \frac{g_i(y)}{h_i(y)}$ $\,,\forall y \in U_i$. Since the sets of the form $D(g)$, $g \in R(X)$ form a basis for the Zariski topology, we may assume that $U_i = D(p_i)$, for all $i \in I$. For all $i \in I$, we have $D(p_i) \subseteq D(h_i)$, i.e. $V(h_i) \subseteq V(p_i)$. From Hilbert's Nullstellensatz it follows that $\sqrt{p_i} \subseteq \sqrt{h_i}$. Therefore $p_i^n \in (h_i)$ for some $n \in \mathbb{N}$, i.e. $p_i^n = c \cdot h_i$ for some $c \in R(X)$, and hence $U_i = D(h_i)$.

Notice that we have $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ on $D(h_i) \cap D(h_j) = D(h_i h_j)$, for all $i, j \in I$, or equivalently $g_i h_j = g_j h_i$ in $R(X)$ (to see this, notice that $\frac{g_i}{h_i}$ and $\frac{g_j}{h_j}$ are equal in $R(X)_{h_i h_j}$). Hence, by definition of localization and since $R(X)$ is an integral domain, we get $g_i h_j = g_j h_i$ in $R(X)$]. By construction, we have $D(f) \subseteq \bigcup_{i \in I} D(h_i)$. From the lectures, it follows that

$$f^n = \sum_{i \in J} a_i h_i$$

for some finite $J \subseteq I$, $n \in \mathbb{N}$ and $a_i \in R(X)$. Putting $g := \sum_{i \in J} a_i g_i$, we get

$$g_j f^n = \sum_{i \in J} a_i h_i g_j = \sum_{i \in J} a_i h_j g_i = h_j g,$$

or equivalently, $\frac{g}{f^n} = \frac{g_j}{h_j}$ on $D(h_j)$ for all $j \in J$; and since the $D(h_j)$ cover $D(f)$, we get $\varphi = \frac{g}{f^n}$, i.e. $\varphi \in R(X)_f$. $\square$

Exercise 2.2. *Let X, Y be topological spaces and let $f\colon X \longrightarrow Y$ be a continuous map. Let all occurring (pre)sheaves be (pre)sheaves of abelian groups.*

(a) For a sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ by

$$f_*\mathcal{F}(V) := \mathcal{F}\big(f^{-1}(V)\big)$$

for any open subset $V \subseteq Y$. Show that $f_*\mathcal{F}$ is a sheaf on Y.

(b) *For a sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf*

$$f^{+}\mathcal{G}\colon U \longmapsto \varinjlim_{\substack{V\subseteq Y \text{open} \\ V\supseteq f(U)}} \mathcal{G}(V),$$

where $U \subseteq X$ is an open subset. Show that f^{-1} is a functor from the category of sheaves on Y to the category of sheaves on X.

Proof. **Solution by Claudius Heyer**

(a) Since f is continuous, $f^{-1}(V)$ is an open subset in X, for V an open subset of Y. Therefore, $f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$ is an abelian group. Furthermore, $f_*\mathcal{F}(\emptyset) = \mathcal{F}\big(f^{-1}(\emptyset)\big) = \mathcal{F}(\emptyset) = 0$.

If the restriction maps of $\mathcal{F}$ are denoted by ρ', then those of $f_*\mathcal{F}$ are given by $\rho_{UV} := \rho'_{f^{-1}(U)f^{-1}(V)}$ for all open $U, V \subseteq Y$.

For $W \subseteq V \subseteq U$ open subsets of Y, we have the following inclusion of open subsets of X, $f^{-1}(W) \subseteq f^{-1}(V) \subseteq f^{-1}(U)$. From this it is clear that $\rho_{UU} = \mathrm{id}_{f_*\mathcal{F}(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. Thus $f_*\mathcal{F}$ is a presheaf.

Now let $U \subseteq Y$ be an open subset, and $U = \bigcup_{i\in I} U_i$ be an open covering of U. Let $s \in f_*\mathcal{F}(U)$ such that $s\big|_{U_i} = 0$, for all $i \in I$. We may read this as $s \in \mathcal{F}\big(f^{-1}(U)\big)$ and $s\big|_{U_i} \in \mathcal{F}\big(f^{-1}(U_i)\big) = 0$, for all $i \in I$. Now $f^{-1}(U) = f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i)$ is an open covering. Since $\mathcal{F}$ is a sheaf $s = 0$, which proves sheaf property (iv) of $f_*\mathcal{F}$.

Now let $s_i \in f_*\mathcal{F}(U_i) = \mathcal{F}\big(f^{-1}(U_i)\big)$ such that $s_i\big|_{U_i\cap U_j} = s_j\big|_{U_i\cap U_j}$, for all $i, j \in I$. Again since $f^{-1}(U) = \bigcup_{i\in I} f^{-1}(U_i)$ is an open covering and $\mathcal{F}$ a sheaf, we find an $s \in \mathcal{F}\big(f^{-1}(U)\big) = f_*\mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$, for all $i \in I$. This shows sheaf property (v) of $f_*\mathcal{F}$.

(b) For functoriality of f^{-1}, we only need to show that for $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ ($\mathcal{F}, \mathcal{G}$ sheaves on Y), $f^{-1}\varphi$ lies in $\mathrm{Hom}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$ and that $f^{-1}(\psi\circ\varphi) = f^{-1}\psi\circ f^{-1}\varphi$ holds, whenever it makes sense.

So let $\varphi\colon \mathcal{G} \longrightarrow \mathcal{G}'$ and $\psi\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ be morphisms of sheaves on Y. Let $U \subseteq X$ be an open subset. For all $W \subseteq V \subseteq Y$ open subsets such that $f^{-1}(V) \supseteq f^{-1}(W) \supseteq U$, by the definition of the *direct limit*, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[rrr, "\varphi_V"] \arrow[dr, "\mu_V"] \arrow[dd, "\rho_{VW}"'] & & & \mathcal{G}'(V) \arrow[dl, "\mu'_V"'] \arrow[dd, "\rho'_{VW}"] \\
& f^{+}\mathcal{G}(U) \arrow[r, dashed, "\exists!\varphi^{+}_U"] & f^{+}\mathcal{G}'(U) & \\
\mathcal{G}(W) \arrow[rrr, "\varphi_W"] \arrow[ur, "\mu_W"'] & & & \mathcal{G}'(W) \arrow[ul, "\mu'_W"]
\end{tikzcd}$$

Furthermore, the process of sheafification yields the following commutative diagram

$$\begin{tikzcd} f^+\mathcal{G}(U) \arrow[r, "\varphi_U^+"] \arrow[d, "\iota_{(f^+\mathcal{G})_U}"'] \arrow[dr, phantom, "\circlearrowleft"] & f^+\mathcal{G}'(U) \arrow[d, "\iota_{(f^+\mathcal{G}')_U}"] \\ f^{-1}\mathcal{G}(U) \arrow[r, "\tilde{\varphi}_U"'] & f^{-1}\mathcal{G}'(U) \end{tikzcd}$$

Putting these things together shows that the following diagram is commutative

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\mu"'] & \mathcal{G}' \arrow[r, "\psi"] \arrow[d, "\mu'"'] & \mathcal{G}'' \arrow[d, "\mu''"] \\ f^+\mathcal{G} \arrow[r, "\varphi^+"] \arrow[d, "\iota_{(f^+\mathcal{G})}"'] & f^+\mathcal{G}' \arrow[r, "\psi^+"] \arrow[d, "\iota_{(f^+\mathcal{G}')}"'] & f^+\mathcal{G}'' \arrow[d, "\iota_{(f^+\mathcal{G}'')}"] \\ f^{-1}\mathcal{G} \arrow[r, "f^{-1}\varphi"] & f^{-1}\mathcal{G}' \arrow[r, "f^{-1}\psi"] & f^{-1}\mathcal{G}'' \end{tikzcd}$$

and that $f^{-1}(\psi \circ \varphi) = f^{-1}\psi \circ f^{-1}\varphi$. Hence, f^{-1} is a covariant functor from the category of sheaves on Y to the category of sheaves on X.

$\square$

Exercise 2.3. *Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on a topological space X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf*

$$U \longmapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $U \subseteq X$ is an open subset, is a sheaf. It is called the sheaf Hom and is denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof. **Solution by Claudius Heyer**

Note that for the inclusion $i\colon U \hookrightarrow X$, for all $V \subseteq U$ open subsets, we have

$$\mathcal{F}|_U(V) = i^{-1}\mathcal{F}(V) = \varinjlim_{\substack{W \subseteq U \text{ open} \\ W \supseteq i(V) = V}} \mathcal{F}(W) = \mathcal{F}(V).$$

For $V \subseteq U \subseteq X$ open subsets, let $\varphi, \psi \in \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, then define

$$(\varphi + \psi)_V \colon \mathcal{F}|_U(V) \longrightarrow \mathcal{G}|_U(V), \quad s \longmapsto \varphi_V(s) + \psi_V(s).$$

Since φ and ψ are morphisms of sheaves, for $W \subseteq V \subseteq U$ open subsets, and $s \in \mathcal{F}|_U(V)$, we have

$$\begin{aligned} (\varphi + \psi)_W \left(s|_W\right) &= \varphi_W \left(s|_W\right) + \psi_W \left(s|_W\right) \\ &= \left(\varphi_V(s)\right)|_W + \left(\psi_V(s)\right)|_W \\ &= \left(\varphi_V(s) + \psi_V(s)\right)|_W \\ &= \left((\varphi + \psi)_V(s)\right)|_W. \end{aligned}$$

Therefore the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}\big|_U(V) \arrow[r, "(\varphi+\psi)_V"] \arrow[d, "\rho_{VW}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}\big|_U(V) \arrow[d, "\rho'_{VW}"] \\
\mathcal{F}\big|_U(W) \arrow[r, "(\varphi+\psi)_W"'] & \mathcal{G}\big|_U(W)
\end{tikzcd}$$

and $\varphi+\psi\colon \mathcal{F}\big|_U \longrightarrow \mathcal{G}\big|_U$ is a morphism of sheaves. Since every $\psi \in \operatorname{Hom}\big(\mathcal{F}\big|_U, \mathcal{G}\big|_U\big)$ induces a morphism of sheaves $-\psi\colon \mathcal{F}\big|_U \longrightarrow \mathcal{G}\big|_U$ (by setting $(-\psi)_V := -\psi_V$ for $V \subseteq U$ open), we see that $0 \in \operatorname{Hom}\big(\mathcal{F}\big|_U, \mathcal{G}\big|_U\big)$. Hence $\operatorname{Hom}\big(\mathcal{F}\big|_U, \mathcal{G}\big|_U\big)$ is a group. It is abelian because $\mathcal{G}\big|_U(V)$ is abelian, for all $V \subseteq U$ open.
Equipped with the usual restriction maps, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ becomes a presheaf.

Now let $U \subseteq X$ be an open subset, and $U = \bigcup_{i\in I} U_i$ be an open covering of U.
Let $\varphi \in \operatorname{Hom}\big(\mathcal{F}\big|_U, \mathcal{G}\big|_U\big)$ such that $\varphi\big|_{U_i} = 0$, for all $i \in I$. We want to show that $\varphi = 0$, i.e. $\varphi_V = 0$ for all $V \subseteq U$ open. Notice that $\varphi\big|_{U_i} = 0$ implies

$$\varphi_{V\cap U_i} = \big(\varphi\big|_{U_i}\big)_{V\cap U_i} = 0, \quad \forall V \subseteq U \text{ open.}$$

For $V \subseteq U$ open subset, and $s \in \mathcal{F}\big|_U(V) = \mathcal{F}(V)$, we have

$$\big(\varphi_V(s)\big)\big|_{V\cap U_i} = \varphi_{V\cap U_i}\big(s\big|_{V\cap U_i}\big) = 0, \quad \forall i \in I.$$

Because $V = \bigcup_{i\in I} U_i \cap V$ and $\mathcal{G}\big|_U$ is a sheaf, it follows that $\varphi_V(s) = 0$. This shows the sheaf property (iv) of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

For $i, j \in I$, let $\varphi_i \in \operatorname{Hom}\big(\mathcal{F}\big|_{U_i}, \mathcal{G}\big|_{U_i}\big)$ be given such that $\varphi_i\big|_{U_i\cap U_j} = \varphi_j\big|_{U_i\cap U_j}$. We want to find $\varphi \in \operatorname{Hom}\big(\mathcal{F}\big|_U, \mathcal{G}\big|_U\big)$ such that $\varphi\big|_{U_i} = \varphi_i$, for all $i \in I$.

Let $V \subseteq U$ be an open subset and $s \in \mathcal{F}\big|_U(V)$. For $i \in I$, put $t_i := \varphi_{i,V\cap U_i}\big(s\big|_{V\cap U_i}\big) \in \mathcal{G}\big|_U(V \cap U_i)$. With this definition, for all $i, j \in I$, we have

$$\begin{aligned}
t_i\big|_{U_i\cap U_j} &= \varphi_{i,V\cap U_i}\big(s\big|_{V\cap U_i}\big)\big|_{U_i\cap U_j} \\
&= \varphi_{i,V\cap U_i\cap U_j}\big(s\big|_{V\cap U_i\cap U_j}\big) \\
&= \varphi_{j,V\cap U_i\cap U_j}\big(s\big|_{V\cap U_i\cap U_j}\big) \\
&= \varphi_{j,V\cap U_j}\big(s\big|_{V\cap U_j}\big)\big|_{U_i\cap U_j} \\
&= t_j\big|_{U_i\cap U_j}
\end{aligned}$$

Since $\mathcal{G}\big|_U$ is a sheaf and $V = \bigcup_{i\in I} U_i \cap V$, there exists $t \in \mathcal{G}\big|_U(V)$ such that

$$t\big|_{V\cap U_i} = t_i = \varphi_{i,V\cap U_i}\big(s\big|_{V\cap U_i}\big), \quad \forall i \in I.$$

Now put $\varphi_V(s) := t$. We still have to show that φ commutes with the restriction maps. For $i \in I$, $W \subseteq V \subseteq U$ open subsets and $s \in \mathcal{F}\big|_U(V)$, by writing $\varphi_V(s) = t$, $t_i = t\big|_{V\cap U_i}$ and $\varphi_W\big(s\big|_W\big) = \tilde{t}$, $\tilde{t}_i = \tilde{t}\big|_{W\cap U_i}$, we find

$$\begin{aligned}
\big(t\big|_W\big)\big|_{W\cap U_i} &= t\big|_{W\cap U_i} = t_i\big|_{W\cap U_i} \\
&= \varphi_{i,V\cap U_i}\big(s\big|_{V\cap U_i}\big)\big|_{W\cap U_i} \\
&= \varphi_{i,W\cap U_i}\big(s\big|_{W\cap U_i}\big) \\
&= \tilde{t}_i.
\end{aligned}$$

Again by using that $\mathcal{G}\big|_U$ is a sheaf and $W = \bigcup_{i\in I} U_i \cap W$, we get $t\big|_W = \tilde{t}$, i. e.

$$\varphi_V(s)\big|_W = \varphi_W\big(s\big|_W\big).$$

Therefore φ is a morphism of sheaves with the desired properties. This shows the sheaf property (v) of $\mathcal{H}om(\mathcal{F},\mathcal{G})$. $\square$

Exercise 2.4. *Let X be a topological space and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X. Furthermore, suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i and for each pair i,j an isomorphism*

$$\varphi_{ij}\colon \mathcal{F}_i\big|_{U_i\cap U_j} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_i\cap U_j}$$

of sheaves such that

(1) for each i: $\varphi_{ii} = \mathrm{id}$,

(2) for each i,j,k: $\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$ on $U_i\cap U_j\cap U_k$.

Show that there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms of sheaves $\psi_i\colon \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that, for each i,j, the equality $\psi_j = \varphi_{ij}\circ\psi_i$ holds on $U_i\cap U_j$. We say that $\mathcal{F}$ is obtained by glueing the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.

Proof. **Solution by Claudius Heyer**

Let $U \subseteq X$ be an open subset. We define $\mathcal{F}$ via

$$\mathcal{F}(U) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U\cap U_i) \;\middle|\; \forall i,j\in I\colon s_i\big|_{U\cap U_i\cap U_j} = \varphi_{ji,U\cap U_i\cap U_j}\big(s_j\big|_{U\cap U_i\cap U_j}\big) \right\}. \tag{5}$$

We show that $\mathcal{F}$ is a sheaf. $\mathcal{F}(\emptyset) = 0$ is clear. Observe that the restriction maps are given by

$$\rho_{UV}\colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V), \quad (s_i)_{i\in I} \longrightarrow \big(s_i\big|_V\big)_{i\in I} \quad \text{for } V\subseteq U\subseteq X \text{ open.}$$

From this, it is immediate that $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and $\rho_{UW} = \rho_{VW}\circ\rho_{UV}$. Hence $\mathcal{F}$ is a presheaf.

Let $U \subseteq X$ be an open subset and $U = \bigcup_{j\in J} V_j$ be an open covering of U. Let $s\in\mathcal{F}(U)$ such that $s\big|_{V_j} = 0$, for all $j\in J$. We have to show that $s = 0$. Notice that

$$\big(s_i\big|_{V_j}\big)_{i\in I} = s\big|_{V_j} = 0 = (0)_{i\in I}, \text{ i. e. } s_i\big|_{V_j} = 0,$$

for all $i\in I$, $j\in J$. Because $U\cap U_i = \bigcup_{j\in J} V_j\cap U_i$ and $\mathcal{F}_i$ is a sheaf, it follows that $s_i = 0$, for all $i\in I$, thus $s = 0$.

Let $s_j = (s_{ij})_{i\in I} \in \mathcal{F}(V_j)$, where $s_{ij}\in\mathcal{F}_i(V_j\cap U_i)$. Since $\mathcal{F}_i$ is a sheaf, there exists $s_i\in\mathcal{F}_i(U\cap U_i)$ such that $s_i\big|_{V_j\cap U_i} = s_{ij}$. Set $s = (s_i)_{i\in I}$. It is obvious that $s\big|_{V_j\cap U_i} = s_j$, for all $j\in J$. What is left to show is that s lies in $\mathcal{F}(U)$. So it suffices to prove that

$$s_i\big|_{U_i\cap U_k} = \varphi_{ki,U_i\cap U_k}\big(s_k\big|_{U_i\cap U_k}\big), \forall i,k\in I. \tag{6}$$

Recall that since φ_{ik} are morphisms of sheaves, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}_i\big|_{U_i\cap U_k}(U_i\cap U_k) \arrow[r, "\varphi_{ik,U_i\cap U_k}"] \arrow[d, "\rho_{U_i\cap U_k,V_j\cap U_i\cap U_k}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}_k\big|_{U_i\cap U_k}(U_i\cap U_k) \arrow[d, "\rho'_{U_i\cap U_k,V_j\cap U_i\cap U_k}"] \\
\mathcal{F}_i\big|_{U_i\cap U_k}(V_j\cap U_i\cap U_k) \arrow[r, "\varphi_{ik,V_j\cap U_i\cap U_k}"'] & \mathcal{F}_k\big|_{U_i\cap U_k}(V_j\cap U_i\cap U_k)
\end{tikzcd}$$

Thus, for $i,k\in I$, $j\in J$, we compute

$$\begin{aligned}
s_i\big|_{U_i\cap U_k}\Big|_{V_j\cap U_i\cap U_k} &= s_i\big|_{V_j\cap U_i}\Big|_{V_j\cap U_i\cap U_k}\\
&= s_{ij}\big|_{V_j\cap U_i\cap U_k}\\
&= \varphi_{ki,V_j\cap U_i\cap U_k}\left(s_{kj}\big|_{V_j\cap U_i\cap U_k}\right)\\
&= \varphi_{ki,V_j\cap U_i\cap U_k}\left(s_k\big|_{V_j\cap U_i\cap U_k}\right)\\
&= \varphi_{ki,U_i\cap U_k}\left(s_k\big|_{U_i\cap U_k}\right)\Big|_{V_j\cap U_i\cap U_k}.
\end{aligned}$$

Since $\mathcal{F}_i$ is a sheaf and $U_i\cap U_k=\bigcup_{j\in J}V_j\cap U_i\cap U_k$ is an open covering, (6) holds true. Therefore, we have $s\in\mathcal{F}(U)$ and thus, $\mathcal{F}$ is indeed a sheaf.

Fix $i\in I$, let $V\subseteq U_i$ be an open subset, then define the isomorphism $\psi_i\colon\mathcal{F}\big|_{U_i}\xrightarrow{\sim}\mathcal{F}_i$ by setting

$$\psi_{i,V}\colon\mathcal{F}\big|_{U_i}(V)\longrightarrow\mathcal{F}_i(V),\quad s=(s_i)_{i\in I}\longmapsto s_i.$$

The inverse is given by

$$\psi_{i,V}^{-1}\colon\mathcal{F}_i(V)\longrightarrow\mathcal{F}\big|_{U_i}(V),\quad s_i\longmapsto\Big(\varphi_{ij,V\cap U_j}\left(s_i\big|_{V\cap U_j}\right)\Big)_{j\in I}.$$

Firstly, check that ψ_i^{-1} is well-defined. Using property (2) of the φ_{ij}'s, for $j,k\in I$ and $V\subseteq U_i$ an open subset, we find that

$$\begin{aligned}
\varphi_{ij,V\cap U_j}\left(s_i\big|_{V\cap U_j}\right)\Big|_{V\cap U_j\cap U_k} &= \varphi_{ij,V\cap U_j\cap U_k}\left(s_i\big|_{V\cap U_j\cap U_k}\right)\\
&= \varphi_{kj,V\cap U_j\cap U_k}\Big(\varphi_{ik,V\cap U_j\cap U_k}\left(s_i\big|_{V\cap U_j\cap U_k}\right)\Big)\\
&= \varphi_{kj,V\cap U_j\cap U_k}\Big(\varphi_{ik,V\cap U_k}\left(s_i\big|_{V\cap U_k}\right)\Big|_{V\cap U_j\cap U_k}\Big).
\end{aligned}$$

But this is exactly the condition for $\psi_{i,V}^{-1}(s_i)$ to lie in $\mathcal{F}\big|_{U_i}(V)$. By property (1) it follows that $\psi_{i,V}\circ\psi_{i,V}^{-1}=\mathrm{id}_{\mathcal{F}_i(V)}$. Recall that for $s=(s_i)_{i\in I}\in\mathcal{F}\big|_{U_i}(V)$, by definition we get

$$s_j=s_j\big|_{V\cap U_j}=\varphi_{ij,V\cap U_j}\left(s_i\big|_{V\cap U_j}\right),\quad j\in I,$$

which proves $\psi_{i,V}^{-1}\circ\psi_{i,V}=\mathrm{id}_{\mathcal{F}|_{U_i}(V)}$. By the very definition, ψ_i (resp. ψ_i^{-1}) commutes with the restriction maps (notice that the φ_{ij}'s commute as well). Therefore, ψ_i is indeed an isomorphism of sheaves.

The fact that for all $s=(s_i)_{i\in I}\in\mathcal{F}(U)$, we have

$$s_i\big|_{U_i\cap U_j}=\varphi_{ji,U_i\cap U_j}\left(s_j\big|_{U_i\cap U_j}\right),$$

implies that $\psi_i = \varphi_{ji} \circ \psi_j$ on $U_i \cap U_j$.

One last thing still to prove is the uniqueness of $\mathcal{F}$. Let $\mathcal{G}$ be another sheaf on X, together with isomorphisms of sheaves $\tilde{\psi}_i \colon \mathcal{G}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ satisfying $\tilde{\psi}_j = \varphi_{ij} \circ \tilde{\psi}_i$ on $U_i \cap U_j$, for all $i, j \in I$. First of all, we get isomorphisms of sheaves

$$\sigma_i = \tilde{\psi}_i^{-1} \circ \psi_i \colon \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{G}\big|_{U_i}, \quad i \in I.$$

Hence, we get an isomorphism of sheaves

$$\sigma \colon \mathcal{F} \xrightarrow{\sim} \mathcal{G}$$

given by $\sigma_U \colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U), \quad (s_i)_{i\in I} \longmapsto \big(\sigma_{i,U}(s_i)\big)_{i \in I}$ for $U \subseteq X$ an open subset. Notice that σ commutes with the restriction maps, because ψ_i and $\tilde{\psi}_i$ commute with the restriction maps, for all $i \in I$. $\square$

Remark. Note that another equivalent way of formulating the sheaf $\mathcal{F}$ described in (5) is

$$\mathcal{F}(U) = \varprojlim_i \mathcal{F}_i(U \cap U_i) \quad (U \subseteq X, \text{ open}).$$

Hence, the following maps (which exist from the definition of inverse limit)

$$\mathcal{F}(V) \longrightarrow \mathcal{F}_i(V), \text{ where } V \subseteq U_i \subseteq X,$$

define morphism of sheaves

$$\psi_i : \mathcal{F}\big|_{U_i} \longrightarrow \mathcal{F}_i$$

satisfying $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

3 Solutions for Exercise Sheet-3

Exercise 3.1. *Let $X = \mathbb{C}$ be equipped with the Euclidean topology and consider the following (pre)sheaves on X: the locally constant sheaf $\underline{\mathbb{Z}}$ with group $\mathbb{Z}$, the sheaf $\mathcal{O}_X$ of holomorphic functions, and the presheaf $\mathcal{F}$ of functions admitting a holomorphic logarithm. Show that*

$$0 \longrightarrow 2\pi i\,\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{F} \longrightarrow 0,$$

where $2\pi i\,\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ is the natural inclusion, is an exact sequence of presheaves. Show that $\mathcal{F}$ is not a sheaf.

Proof. **Solution by Max Laum**

Let $X = \mathbb{C}$ be equipped with Euclidean topology. It is to show that the following sequence is an exact sequence of presheaves, and that $\mathcal{F}$ is not a sheaf.

$$2\pi i\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}.$$

From Exercise 1.1, we know that it suffices to prove the exactness at the level of stalks.

For $x \in \mathbb{C}$, consider the following sequence of stalks

$$2\pi i\underline{\mathbb{Z}}_x \xrightarrow{i_x} \mathcal{O}_{X,x} \xrightarrow{\exp_x} \mathcal{F}_x, \tag{7}$$

where i_x denotes the map $2\pi i\underline{\mathbb{Z}}_x = 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_{X,x}$, taking the group of integers $2\pi i\mathbb{Z}$ into the group of holomorphic functions at the point x. It is clear that i_x is injective.

Claim: $\operatorname{im}(i_x) = \ker(\exp_x)$.
For " $\subseteq$ " we see that $2\pi ik \longmapsto 2\pi ik \longmapsto \exp(2\pi ik) = 1$. Conversely, $\exp_x^{-1}(1) = \log 1 + 2\pi ik$. Hence $\operatorname{im}(i_x) = \ker(\exp_x)$.

The map $\exp_x$ is surjective because, by definition for every f that is holomorphic at x and admits a holomorphic logarithm, there exists a function g such that g is holomorphic at x and $\exp_x(f) = g$. This proves the exactness of the sequence (7).

To see that $\mathcal{F}$ is not a sheaf, look at the function $z \longmapsto z$ which has a logarithm on $U_1 = \mathbb{C} - \mathbb{R}_{\leq 0}$ and similarly on $U_2 = \mathbb{C} - \mathbb{R}_{\geq 0}$, but not on the entire complex plane $\mathbb{C}$ (see Exercise 1 .2(b)). ☐

Exercise 3.2. *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map.*

(a) Let $\mathcal{G}$ be a sheaf on Y. Construct explicitly an example such that the presheaf $f^+\mathcal{G}$ given by the assignment

$$U \mapsto \varinjlim_{\substack{V \subseteq Y,\ open \\ f(U) \subseteq V}} \mathcal{G}(V) \quad (U \subseteq X,\ open)$$

is not a sheaf.

(b) Let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a presheaf on Y. Show that there is a bijection

$$\operatorname{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathrm{PreSh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

of sets.

Proof. **Solution by Max Laum**

Let X, Y be topological spaces and $f : X \longrightarrow Y$ be continuous.

(a) Let $\mathcal{G}$ be a sheaf on Y. Define a presheaf $f^+\mathcal{G}$ on X by:

$$U \longmapsto \varinjlim_{\substack{V \subseteq Y \text{ open} \\ V \supseteq f(U)}} \mathcal{G}(V).$$

We want to construct an explicit example to show that this is in general not a sheaf.

Set $Y := \{g, s, t\}$ and a subset $U \subseteq Y$ is defined to be open if $U = \emptyset$ or $g \in U$. Let X be the closed subspace $X = \{s, t\}$ and $\mathcal{G}$ the sheaf associated to the presheaf $U \mapsto \mathbb{Z}$.

Let $f \colon X \hookrightarrow Y$ be the inclusion map. We observe that Y is connected, but X is not. Then

$$f^+\mathcal{G}(X) = \varinjlim_{V \supseteq f(X) \text{ open}} \mathcal{G}(V) = \mathbb{Z}, \tag{8}$$

as Y is the only open set containing X (in the topology of Y).

On the other hand, observe that

$$(f^+\mathcal{G})(\{s\}) = \mathbb{Z} \text{ and } (f^+\mathcal{G})(\{t\}) = \mathbb{Z}.$$

So by the glueing property of sheaves we would have that

$$(f^+\mathcal{G})(X) = \mathbb{Z}^2 = (f^{-1}\mathcal{G})(X),$$

which contradicts (8).

(b) Let $\mathcal{F}$ be a sheaf on X, and $\mathcal{G}$ a presheaf on Y. We show that the following map is a bijection:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &\longrightarrow \mathrm{Hom}_{\mathrm{PreSh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \phi &\longmapsto \phi^\flat \\ \psi^\# &\longleftarrow \psi, \end{aligned}$$

where the maps are defined as follows.

Let $\phi \colon f^{-1}\mathcal{G} \to \mathcal{F}$ be a morphism and $t \in \mathcal{G}(V)$ for some open $V \subseteq Y$. Since $f(f^{-1}(V)) \subseteq V$, we have a restriction map $\mathcal{G}(V) \to (f^+\mathcal{G})(f^{-1}(V))$. Then, we have a composition of maps:

$$\begin{aligned} \mathcal{G}(V) \to f^+\mathcal{G}(f^{-1}(V)) \to f^{-1}\mathcal{G}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) &= f_*\mathcal{F}(V) \\ s \mapsto \phi_{f^{-1}(V)}(s)&, \end{aligned}$$

and $\phi^\flat_V(t)$ is defined to be the image of t under the above composition.

Conversely, let $\psi \colon \mathcal{G} \longrightarrow f_*\mathcal{F}$ be a morphism of presheaves on Y. For $U \subseteq X$ open, an element in the direct limit $f^+\mathcal{G}(U)$ is represented by a pair $\langle V, s\rangle$ with $s \in \mathcal{G}(V)$ and $V \supseteq f(U)$. Then $\psi_V(s) \in (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ and we define $\psi^\#_U(\langle V, s\rangle) \in \mathcal{F}(U)$ to be the restriction $\psi_V(s)\big|_U$.

To show the bijection, it remains to check that the above defined maps are inverse to each other, which we leave as an exercise for the reader. This completes the proof of the exercise. $\square$

Exercise 3.3. *Let k be a field. Consider the projective space $\mathbb{P}^n(k) := (k^{n+1}\setminus\{0\})/\sim$, where the equivalence relation $\sim$ is given by*

$$(x_0,\dots,x_n)\sim(x_0',\dots,x_n') \iff \exists\lambda\in k\setminus\{0\}:\ x_i=\lambda x_i'\quad \forall i=0,\dots,n.$$

The equivalence class of a point $(x_0,\dots,x_n)$ is denoted by $[x_0:\dots:x_n]$. For $i=0,\dots,n$, we set

$$U_i := \big\{[x_0:\dots:x_n]\in\mathbb{P}^n(k)\,\big|\,x_i\neq 0\big\}\subset\mathbb{P}^n(k).$$

(a) We define the topology on $\mathbb{P}^n(k)$ by calling a subset $U\subseteq\mathbb{P}^n(k)$ open if $U\cap U_i$ is open in U_i for all $i=0,\dots,n$. Show that $\{U_i\}_{i=0,\dots,n}$ is an open covering of $\mathbb{P}^n(k)$.

(b) Prove that the map $U_i\to\mathbb{A}^n(k)$, given by

$$[x_0:\dots:x_n]\mapsto\Big(\frac{x_0}{x_i},\dots,\frac{\widehat{x_i}}{x_i},\dots,\frac{x_n}{x_i}\Big),$$

is a bijection; here, the hat means that the i-th entry has to be deleted. By means of this bijection we endow U_i with the structure of a locally ringed space isomorphic to $(\mathbb{A}^n(k),\mathcal{O}_{\mathbb{A}^n(k)})$ denoted by $(U_i,\mathcal{O}_{U_i})$.

(c) For an open set $U\subseteq\mathbb{P}^n(k)$, we set

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) := \big\{f:U\to k\,\big|\,f\big|_{U\cap U_i}\in\mathcal{O}_{U_i}(U\cap U_i)\,\forall i=0,\dots,n\big\}.$$

Show that

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^n(k)}(U)=\big\{f:U\to k\,\big|\,&\forall x\in U,\ \exists x\in V\subseteq U \text{ open},\\ &\exists g,h\in k[X_0,\dots,X_n]\text{ homogeneous: }\deg(g)=\deg(h),\\ &h(v)\neq 0, f(v)=g(v)/h(v)\ \forall v\in V\big\}.\end{aligned}$$

Conclude that $(\mathbb{P}^n(k),\mathcal{O}_{\mathbb{P}^n(k)})$ is a locally ringed space.

Proof. **Solution by Max Laum**

(a) Clearly, we have $\bigcup_{i=1}^n U_i\subseteq\mathbb{P}^n(k)$. Now let $x=[x_0:\dots:x_n]\in\mathbb{P}^n(k)$. Then, at least for one i we have $x_i\neq 0$, which implies that $x\in U_i$, and hence, $\mathbb{P}^n(k)\subseteq\bigcup_{i=1}^n U_i$. This proves that $\mathbb{P}^n(k)=\bigcup_{i=1}^n U_i$.

(b) It is to show that the map

$$\begin{aligned} g\colon U_i &\longrightarrow \mathbb{A}^n(k)\\ [x_0\colon \dots : x_n] &\longmapsto \left(\frac{x_0}{x_i},\dots,\frac{\hat{x}_i}{x_i},\dots,\frac{x_n}{x_i}\right)\end{aligned}$$

is a bijection.

To show this, we will construct an inverse of g. Consider the following map:

$$\begin{aligned} g^{-1}\colon \mathbb{A}^n(k) &\longrightarrow U_i\\ (a_0,\dots,\widehat{a_i},\dots,a_n) &\longmapsto [a_0\colon \dots : 1\colon \dots : a_n]\end{aligned}$$

Then, $g^{-1}\circ g=\mathrm{id}_{U_i}$ and $g\circ g^{-1}=\mathrm{id}_{\mathbb{A}^n(k)}$, since $[\frac{x_0}{x_i}:\dots:1:\dots:\frac{x_n}{x_i}]=[x_0:\dots:x_n]$ in $\mathbb{P}^n(k)$.

(c) We want to show that:

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f\colon U \to k | \forall x \in U, \exists x \in V \subseteq U \text{ open},\\ \exists g, h \in k\,[X_0, \ldots, X_n] \text{ homogeneous}: \deg(g) = \deg(h),\\ h(v) \neq 0, f(v) = g(v)/h(v)\, \forall v \in V\}.\end{aligned}$$

First, we will look at the concept of homogenization. A polynomial f is called homogeneous of degree d (written $f \in k[X_0, \ldots, X_n]^{(d)}$), if f is a sum of monomials of degree d.

For any $i \in \{0, \ldots, n\}$ the following map is a bijection (called dehomogenization):

$$\begin{aligned}\Phi_i^{(d)}\colon k[X_0, \ldots, X_n]^{(d)} &\longrightarrow \left\{ g \in k[T_0, \ldots, \hat{T}_i, \ldots, T_n] \middle| \deg(g) \leq d \right\}\\ f &\longmapsto f\left(T_0, \ldots, 1, \ldots, T_n\right).\end{aligned}$$

To prove this we will construct an inverse (called homogenization). Let g be a polynomial of the RHS and $g = \sum_{j=1}^{d} g_j$ its decomposition into homogeneous parts with respect to $T_0, \ldots, \hat{T}_i, \ldots, T_n$. So the $g_j \in k[T_0, \ldots T_n]^{(j)}$ and the map Ψ_i given by

$$\Psi_i^{(d)} := \sum_{j=0}^{d} X_i^{d-j} g_j(X_0, \ldots, \hat{X}_i, \ldots, X_n)$$

is indeed an inverse of Φ_i.

The definition of Φ can be extended to the field of fractions. Let Z be the subfield of $k(X_0, \ldots, X_n)$ that consists of elements that are of the form f/g, where $f, g \in k[X_0, \ldots, X_n]$ are homogeneous and of the same degree. We then have a well-defined isomorphism of k-extensions:

$$\begin{aligned}\Phi_i\colon Z &\longrightarrow k(T_0, \ldots, \hat{T}_i, \ldots, T_n)\\ \frac{f}{g} &\longmapsto \frac{\Phi_i(f)}{\Phi_i(g)}.\end{aligned}$$

Then, the proof of the claim just becomes an application of this bijection.

Let $f \in \mathcal{O}_{\mathbb{P}^n(k)}(U)$ and $x \in \mathbb{P}^n(k)$. Then, there exists an i such that $x \in U_i$ and $f\big|_{U\cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$. Therefore, we have that f is regular in a neighbourhood V of x, i.e. there exists a $V \subseteq U \cap U_i$ open with $x \in V$, such that there exists $\tilde{g}, \tilde{h} \in k[X_0, \ldots, \hat{X}_i, \ldots, X_n]$ with $\tilde{h} \neq 0$ and $f = \tilde{g}/\tilde{h}$ on V. Applying the inverse of Φ_i, gives us the element $\Phi_i^{-1}(\tilde{g}/\tilde{h}) = g/h$ which is of the desired form.

Conversely, if $f \in$ RHS, it is locally given by g/h on $U \cap U_i$ with $g, h \in k[X_0, \ldots, X_n]^{(d)}$, for some d. Applying the map Φ_i we get that f is of the form $\tilde{g}/\tilde{h}$ where $\tilde{g}, \tilde{h} \in k[X_0, \ldots, \hat{X}_i, \ldots, X_n]$. Hence, $f\big|_{U\cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$. $\square$

Exercise 3.4. *A locally ringed space $(X, \mathcal{O}_X)$ is called an affine scheme, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphic to $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$. A morphism of affine schemes is a morphism of locally ringed spaces. The category of affine schemes will be denoted by* (Aff)*, the category of commutative rings with* 1 *by* (Ring).

(a) Show that the assignment $A \mapsto (\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ induces a contravariant functor Spec : (Ring) $\to$ (Aff).

(b) Show that the assignment $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ *induces a contravariant functor* $\Gamma : (\mathrm{Aff}) \to (\mathrm{Ring})$.

(c) Prove that the functors Spec *and* Γ *define an anti-equivalence between the category* (Ring) *and the category* (Aff).

Proof. **Solution by Max Laum**

(a) It is to show that the assignment

$$\begin{aligned} \mathrm{Spec}\colon (\mathrm{Ring}) &\longrightarrow (\mathrm{Aff}), \\ A &\longmapsto (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \end{aligned}$$

is a contravariant functor.

Let $\varphi\colon A \to B$ be a morphism of rings. Then, we have already seen that the induced map

$$\begin{aligned} f\colon \mathrm{Spec}(B) &\to \mathrm{Spec}(A) \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

is a continuous map on the underlying topological spaces.

Now, we want to construct a morphism of sheaves $f^\flat\colon \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$. Observe that $\{D(s)\}_{s\in A}$ form a basis for the topology on $\mathrm{Spec}(A)$. Hence, it suffices to define $f^\flat$ on $D(s)$ ($s \in A$), such that the definition is compatible with restrictions to $D(t) \subseteq D(s)$. Now, for $s \in A$, we have

$$\begin{aligned} &\mathcal{O}_{\mathrm{Spec}(A)}(D(s)) = A_s \ \text{ and} \\ &f_*\mathcal{O}_{\mathrm{Spec}(B)}(D(s)) = \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(D(s))) = \mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(s))) = B_{\varphi(s)}, \end{aligned}$$

where the equality $f^{-1}(D(s)) = D(\varphi(s))$ is known by a proposition of the lecture. Using the above equalities, we define the following ring homomorphism:

$$\begin{aligned} f^\flat_{D(s)}\colon \mathcal{O}_{\mathrm{Spec}(A)}(D(s)) &\longrightarrow f_*\mathcal{O}_{\mathrm{Spec}(B)}(D(s)) \\ \frac{a}{s^r} &\longmapsto \frac{\varphi(a)}{\varphi(s)^r}. \end{aligned}$$

It can be shown that this homomorphism is compatible with the restriction maps, and thus we have a morphism of sheaves $f^\flat\colon \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$.

For every prime $\mathfrak{q} \in \mathrm{Spec}(B)$ the induced homomorphism

$$f^\sharp_{\mathfrak{q}} : \left(f^{-1}\mathcal{O}_{\mathrm{Spec}\,(A)}\right)_{\mathfrak{q}} = \mathcal{O}_{\mathrm{Spec}(A),f(\mathfrak{q})} = A_{\varphi^{-1}(\mathfrak{q})} \to \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}} = B_{\mathfrak{q}}$$

is a local homomorphism (i.e. $f^\sharp_{\mathfrak{q}}(\mathfrak{m}_{\varphi^{-1}(\mathfrak{q})}) \subseteq \mathfrak{m}_{\mathfrak{q}}$), where $\mathfrak{m}_{\mathfrak{q}}$ is the maximal ideal of $B_{\mathfrak{q}}$. Therefore

$$\mathrm{Spec}(\varphi) := (f, f^\flat)$$

is indeed a morphism of affine schemes. Now, it is straightforward to prove that $\mathrm{Spec}(\mathrm{id}_A) = \mathrm{id}_{(\mathrm{Spec}(A),\mathcal{O}_{\mathrm{Spec}(A)})}$ and that $\mathrm{Spec}(\varphi_2 \circ \varphi_1) = \mathrm{Spec}(\varphi_1) \circ \mathrm{Spec}(\varphi_2)$.

(b) Conversely, we show that the assignment

$$\begin{aligned} \Gamma\colon (\mathrm{Aff}) &\longmapsto (\mathrm{Ring}) \\ (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) &\longmapsto \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \end{aligned}$$

is a contravariant functor as well.

Let $f\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of affine schemes. Hence, we have a homomorphism of rings $f_Y^\flat\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$, which we will denote by $\Gamma(f)$.
This map is obviously functorial in the sense that for any morphism of schemes $g\colon (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Aff}}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{O}_Y, \mathcal{O}_X) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Aff}}(Y,Z) \arrow[r] & \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{O}_Z, \mathcal{O}_Y),
\end{tikzcd}$$

where the maps on the vertical arrow on the left is given by composition with g and on the right by composition with $g_Z^\flat\colon \mathcal{O}_Z \longrightarrow g_*\mathcal{O}_Y$.
This shows that Γ is a contravariant functor.

(c) We now show that the functors Spec and Γ define an anti-equivalence between the category of commutative rings with 1 and the category of affine schemes. For that we need to show that the functor Spec is essentially surjective and fully faithful.
The contravariant functor Spec is essentially surjective, by the definition of an affine scheme. It remains to show that it is fully faithful, i.e. the assignments

$$\mathrm{Hom}_{\mathrm{Ring}}(A,B) \rightleftarrows^{\mathrm{Spec}}_{\Gamma} \mathrm{Hom}_{\mathrm{Aff}}(\mathrm{Spec}(B), \mathrm{Spec}(A)),$$

are inverse to each other.

Clearly, we have $\Gamma \circ \mathrm{Spec} = \mathrm{id}$. Just set $s = 1$ in (a) to obtain

$$f_Y^\flat\colon \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) = B \to \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = A.$$

Conversely, let $(f, f^\flat)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism of affine schemes and $\Gamma(f) = \varphi\colon A \to B$ be the induced map. We want to show that $\mathrm{Spec}(\varphi) = (f_\varphi, f_\varphi^\flat)$, defined as above, equals our initial f.

For any prime $\mathfrak{q} \in B$ we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\
A_{f(\mathfrak{q})} \arrow[r, "f_{\mathfrak{q}}^\sharp"'] & B_{\mathfrak{q}}.
\end{tikzcd}$$

This shows that $\varphi^{-1}(\mathfrak{q}) \subseteq f(\mathfrak{q})$. Since the map $f_{\mathfrak{q}}^\sharp$ is also a local ring homomorphism, we find that

$$f_{\mathfrak{q}}^\sharp(f(\mathfrak{q})) \subseteq \mathfrak{q} \Rightarrow \varphi(f(\mathfrak{q})) \subseteq \mathfrak{q} \Rightarrow f(\mathfrak{q}) \subseteq \varphi^{-1}(\mathfrak{q}) \Rightarrow \varphi^{-1}(\mathfrak{q}) = f(\mathfrak{q}).$$

Therefore, f and f_φ coincide set-theoretically (as continuous maps).

Now $f_{\varphi,\mathfrak{q}}^\sharp$ by definition makes this diagram commute as well. Hence, $f_{\mathfrak{q}}^\sharp = f_{\varphi,\mathfrak{q}}^\sharp$, for all $\mathfrak{q} \in \mathrm{Spec}(B)$. It follows that $f^\sharp = f_\varphi^\sharp$, and hence, $f^\flat = f_\varphi^\flat$ as morphisms of sheaves, which concludes the proof. $\square$

4 Solutions for Exercise Sheet-4

Exercise 4.1. *Let A be a commutative ring with 1 and let $X = \mathrm{Spec}(A)$. Show that for $f \in A$ the locally ringed space $(D(f), \mathcal{O}_X\big|_{D(f)})$ is isomorphic to $\mathrm{Spec}(A_f)$.*

Proof. We need to construct a morphism $\phi : D(f) \to \mathrm{Spec}(A_f)$, and a morphism of sheaves $\phi^\flat : \mathcal{O}_{\mathrm{Spec}(A_f)} \to \phi_* \mathcal{O}_X\big|_{D(f)}$, and show that ϕ is a homemorphism and that $\phi^\flat$ is an isomorphism.

For $f \in A$, observe that the set of prime ideals in A_f are the prime ideals in A which do not intersect f. So the elements which are mapped to prime ideals in A_f under the map $A \to A_f$ are the prime ideals in A which do not contain f, which by definition is the set $D(f)$. This shows that the map $\phi : D(f) \to \mathrm{Spec}(A_f)$ induced by the map $A \to A_f$ is a bijection.

Furthermore for $\mathfrak{p}, \mathfrak{q} \in D(f)$ prime ideals, we have $\mathfrak{p} \subset \mathfrak{q}$ if and only if $\phi(\mathfrak{p}) \subset \phi(\mathfrak{q})$. This shows that our map ϕ is a homeomorphism.

Now for any $\mathfrak{p} \in D(f)$, as $f \notin \mathfrak{p}$, we have the isomorphism $A_\mathfrak{p} \cong (A_f)_{\phi(\mathfrak{p})}$. Using this isomorphism we can deduce that the morphism $\phi^\flat : \mathcal{O}_{\mathrm{Spec}(A_f)} \to \phi_* \mathcal{O}_X\big|_{D(f)}$ induced by the map ϕ is an isomorphism.

Hence, we can conclude that the locally ringed space $(D(f), \mathcal{O}_X\big|_{D(f)})$ is isomorphic to $\mathrm{Spec}(A_f)$. □

Exercise 4.2. *Let X and Y be schemes, and let $\{U_i\}_{i \in I}$ be an open covering of X. Let $f_i : U_i \to Y$ $(i \in I)$ be a family of morphisms such that the restrictions of f_i and f_j to $U_i \cap U_j$ coincide for any $i, j \in I$. Show that there exists a unique morphism of schemes $f : X \to Y$ such that $f\big|_{U_i} = f_i$ for all $i \in I$.*

Proof. There are many solutions available in the literature, for e.g. the solution given by Marco Lo Giudice in his notes. It has been proven as a proposition in Section 2.3.2 titled "**Gluing Morphisms**" on page 53.

Giudice's notes can be found at the following web-address `http://magma.maths.usyd.edu.au/users/kasprzyk/calf/pdf/My_Way.pdf`. □

Exercise 4.3. *Let $\{X_i\}_{i \in I}$ be a family of schemes. Suppose that for schemes X_i $(i \in I)$ there exist open subschemes $U_{ij} \subseteq X_i$ $(j \in I)$ and an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ $(i, j \in I)$ such that*

(1) $U_{ii} = X_i$ *and* $\varphi_{ii} = \mathrm{id} \quad (i \in I)$,

(2) $\varphi_{ji} = \varphi_{ij}^{-1} \quad (i, j \in I)$,

(3) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk} \quad (i, j, k \in I)$,

(4) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ *on* $U_{ij} \cap U_{ik} \quad (i, j, k \in I)$.

Show that there exists a unique scheme X, equipped with morphisms $\psi_i : X_i \to X$ $(i \in I)$, such that

(i) ψ_i *yields an isomorphism from* X_i *onto an open subscheme of* X $(i \in I)$,

(ii) $X = \bigcup_{i \in I} \psi_i(X_i)$,

(iii) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j) = \psi_j(U_{ji}) \quad (i, j \in I)$,

(iv) $\psi_i = \psi_j \circ \varphi_{ij}$ *on* U_{ij} $\quad (i, j \in I)$.

We say that X *is obtained by gluing the schemes* X_i *along the isomorphisms* φ_{ij}*.*

Proof. For a very precise and elaborate solution, we again refer the reader to the lemma on page 55 in Section 2.3.3 titled "**The Gluing Lemma**" of Giudice's notes. □

Exercise 4.4. *Let* k *be an algebraically closed field. We consider two copies of the affine line* $\mathbb{A}^1(k)$*, which we distinguish by setting* $X_1 = \operatorname{Spec}(k[s])$ *and* $X_2 = \operatorname{Spec}(k[t])$*. Let* $U_{12} := D(s) \subseteq X_1$ *and* $U_{21} := D(t) \subseteq X_2$*. Let* $\varphi_{12} : U_{12} \to U_{21}$ *be induced by the isomorphism of rings*

$$k[t, t^{-1}] \to k[s, s^{-1}]$$

sending t *to* s*, and let* $\tilde{\varphi}_{12}$ *be induced by the isomorphism sending* t *to* s^{-1}*. Describe the scheme* X *obtained by gluing* X_1 *and* X_2 *along the isomorphisms* φ_{12} *and the scheme* Y *obtained by gluing along* $\tilde{\varphi}_{12}$ *instead. Show that* X *and* Y *are not isomorphic.*

Proof. We refer the reader to Section 2.3.5 titled " **Gluing Affine Lines** " on page 58 of Giudice's notes for the solution. □

5 Solutions for Exercise Sheet-5

Exercise 5.1. *Consider the following affine schemes:*

(a) $X_1 = \operatorname{Spec} \mathbb{C}[X]/(X^2)$,

(b) $X_2 = \operatorname{Spec} \mathbb{C}[X]/(X^2 - X)$,

(c) $X_3 = \operatorname{Spec} \mathbb{C}[X]/(X^3 - X^2)$,

(d) $X_4 = \operatorname{Spec} \mathbb{R}[X]/(X^2 + 1)$.

For $i = 1, \dots, 4$, *describe the topological space* X_i *and its open subsets, and compute* $\mathcal{O}_{X_i}(U)$ *for all open subsets* $U \subseteq X_i$.

Proof. **Solution by Mattias Hemmig**

Consider $\mathbb{C}[X] \subset \mathbb{C}(X)$ and recall $\operatorname{Spec}(\mathbb{C}[X]) = \{(X - \alpha) \mid \alpha \in \mathbb{C}\} \cup \{(0)\}$. Hence to determine $\operatorname{Spec}(\mathbb{C}[X]/I)$ for any ideal $I \subset \mathbb{C}[X]$, it suffices to find the prime ideals in $\mathbb{C}[X]$ that contain I.

By the very definition of a sheaf of rings we have $\mathcal{O}_{X_i}(\emptyset) = 0$ for $i = 1, 2, 3, 4$.

(a) $X_1 = \{(\overline{X})\}$ and $\mathcal{O}_{X_1}(X_1) \cong \mathbb{C}[X]/(X^2)$.

(b) $X_2 = \{(\overline{X}), (\overline{X-1})\}$. These ideals are clearly maximal and, hence, closed in X_2. Therefore we have $\mathfrak{Top}(X_2) = \{\emptyset, \{(\overline{X})\}, \{(\overline{X-1})\}, X_2\}$.

- $\mathcal{O}_{X_2}((\overline{X})) = \mathcal{O}_{X_2}(D(\overline{X-1})) = (\mathbb{C}[X]/(X^2 - X))_{(\overline{X-1})}$
  $\cong \mathbb{C}[X]_{(X-1)}/(X^2 - X)_{(X-1)} = \mathbb{C}[X, \frac{1}{X-1}]/(X) \cong \mathbb{C}$;
- $\mathcal{O}_{X_2}((\overline{X-1})) = \mathcal{O}_{X_2}(D(\overline{X})) = (\mathbb{C}[X]/(X^2 - X))_{(\overline{X})}$
  $\cong \mathbb{C}[X]_{(X)}/(X^2 - X)_{(X)} = \mathbb{C}[X, \frac{1}{X}]/(X - 1) \cong \mathbb{C}$;
- $\mathcal{O}_{X_2}(X_2) \cong \mathbb{C}[X]/(X^2 - X)$.

(c) $X_3 = \{(\overline{X}), (\overline{X-1})\}$. By the same reasoning as in (b), we get $\mathfrak{Top}(X_3) = \{\emptyset, \{(\overline{X})\}, \{(\overline{X-1})\}, X_3\}$.

- $\mathcal{O}_{X_3}((\overline{X})) = \mathcal{O}_{X_3}(D(\overline{X-1})) = (\mathbb{C}[X]/(X^3 - X^2))_{(\overline{X-1})}$
  $\cong \mathbb{C}[X]_{(X-1)}/(X^3 - X^2)_{(X-1)} = \mathbb{C}[X, \frac{1}{X-1}]/(X^2) = \mathbb{C}[X]/(X^2)$
  For the last equality observe that $(\overline{X-1})^{-1} = \overline{-X-1}$ in $\mathbb{C}[X]/(X^2)$;
- $\mathcal{O}_{X_3}((\overline{X-1})) = \mathcal{O}_{X_3}(D(\overline{X})) = (\mathbb{C}[X]/(X^3 - X^2))_{(\overline{X})}$
  $\cong \mathbb{C}[X]_{(X)}/(X^3 - X^2)_{(X)} = \mathbb{C}[X, \frac{1}{X}]/(X - 1) \cong \mathbb{C}$;
- $\mathcal{O}_{X_3}(X_3) \cong \mathbb{C}[X]/(X^3 - X^2)$.

(d) Notice that $\mathbb{R}[X]/(X^2+1)$ has the structure of a field, indeed $\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$ and so $X_4 = \{(0)\}$ and $\mathcal{O}_{X_4}(X_4) \cong \mathbb{C}$. □

Remark. Comparing examples (a) and (d), or (b) and (c), observe that the introduction of nilpotent elements will give rise to more refined structure sheaves.

Notice further that in all of the above examples the underlying topological space carries the discrete topology.

Exercise 5.2. *Let k be an algebraically closed field and let $X = \operatorname{Spec} k[X_1, X_2]$ be an affine scheme. Show that $U = X \setminus V(X_1, X_2)$ is an open subscheme of X, which is not affine.*

Proof. **Solution by Mattias Hemmig**

First observe that the ideal $(X_1, X_2) \subset k[X_1, X_2]$ is maximal and hence the set $V(X_1, X_2) = \{(X_1, X_2)\}$ is closed in $X = \operatorname{Spec}(k[X_1, X_2])$. Thus $(U, \mathcal{O}_X|_U)$ is indeed an open subscheme of $(X, \mathcal{O}_X)$ — the affine plane over k with the origin removed.

We proceed by showing that U cannot be affine in two steps:

(i) We show that the restriction $k[X_1, X_2] = \mathcal{O}_X(X) \xrightarrow{\rho_{XU}} \mathcal{O}_X(U)$ is indeed an isomorphism of rings.

Injectivity: $k[X_1, X_2]$ is an integral domain and hence, $\operatorname{Spec}(k[X_1, X_2])$ is an integral scheme. It is a general fact that the restriction maps on an integral scheme are injective. For a proof consider the solution of Exercise 5.4.

Surjectivity: Let $s \in \mathcal{O}_X(U)$. Since $U = D(X_1) \cup D(X_2)$ we can find representations $s = \frac{\alpha}{X_1^m}$ on $D(X_1)$ and $s = \frac{\beta}{X_2^n}$ on $D(X_2)$ for some $\alpha, \beta \in k[X_1, X_2]$ and $m, n \in \mathbb{Z}_{\geq 0}$. On the intersection $D(X_1) \cap D(X_2) = D(X_1X_2)$ we have $\frac{\alpha}{X_1^m} = \frac{\beta}{X_2^n}$ and hence the equality $\alpha X_2^n = \beta X_1^m$. But $k[X_1, X_2]$ is a unique factorization domain and we find that $X_1^m \mid \alpha$ and $X_2^n \mid \beta$. Therefore there exists some $\gamma \in k[X_1, X_2]$ with $\gamma = \frac{\alpha}{X_1^m} = \frac{\beta}{X_2^n}$ and $\rho_{XU}(\gamma) = s$.

(ii) Assume now that $U = \operatorname{Spec}(A)$ is affine. We then consider the open immersion $U \hookrightarrow X$ and apply the functor Γ yielding the morphism of rings

$$\mathcal{O}_X(X) = k[X_1, X_2] \xrightarrow{\rho_{XU}} \mathcal{O}_X(U) = A,$$

which we know by step (i) to be an isomorphism. Applying now the functor Spec induces an isomorphism of affine schemes

$$\operatorname{Spec}(\mathcal{O}_X(U)) \xrightarrow{\sim} \operatorname{Spec}(\mathcal{O}_X(X)).$$

But this is clearly impossible as the inclusion $U \hookrightarrow X$ on the underlying topological spaces is not surjective. $\square$

Exercise 5.3. *Let R be a commutative ring with 1 and $R[X_1, \ldots, X_n]$ the polynomial ring in n variables over R. We define the affine space $\mathbb{A}^n_R$ of relative dimension n over R by*

$$\mathbb{A}^n_R := \operatorname{Spec} R[X_1, \ldots, X_n].$$

For $i = 0, \ldots, n$, let U_i be the affine spaces $\mathbb{A}^n_R$ of relative dimension n over R given by

$$U_i := \operatorname{Spec} R\Big[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\Big].$$

Further, let

$$U_{ij} := D_{U_i}\Big(\frac{X_j}{X_i}\Big) \subseteq U_i$$

for $i \neq j$ and $U_{ii} := U_i$ $(i, j = 0, \ldots, n)$. Finally, let $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i \neq j$, let $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ be the isomorphism of affine schemes induced by the equality

$$R\Big[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\Big]_{\frac{X_j}{X_i}} \longrightarrow R\Big[\frac{X_0}{X_j}, \ldots, \widehat{\frac{X_j}{X_j}}, \ldots, \frac{X_n}{X_j}\Big]_{\frac{X_i}{X_j}}.$$

(a) *Verify, that the given data constitute a gluing datum, i.e., they satisfy the assumptions (1)–(4) of Exercise 4.3.*

The scheme obtained by gluing the $n+1$ copies of $\mathbb{A}^n_R$ along the isomorphisms φ_{ij} is called the projective space $\mathbb{P}^n_R$ of relative dimension n over R.

(b) *Show that for $n > 0$ the scheme $\mathbb{P}^n_R$ is not affine.*

Proof. **Solution by Mattias Hemmig**

(a) We verify the gluing assumptions as given in Exercise 4.3:
Observe that $U_{ij} = U_{ji}$, and the morphisms of affine schemes $\varphi_{ij} : U_{ij} \to U_{ji}$, for $i, j = 0, \dots, n$ are just identities.

(1) $U_{ii} = U_i$ and $\varphi_{ii} = \mathrm{id}$, for $i = 0, \dots, n$ is true by definition.

(2) $\varphi_{ij} = \varphi_{ji}^{-1}$, for $i, j = 0, \dots, n$ is trivially true as these morphisms of affine schemes are just identities.

(3) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, for $i, j, k = 0, \dots, n$ as we have
$$U_{ij} \cap U_{ik} = U_{ji} \cap U_{jk} = \mathrm{Spec}\left(R\Big[X_0, \dots, X_n, \frac{1}{X_i X_j X_k}\Big]\right),$$
and as the φ_{ij} are identities.

(4) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$ for $i, j, k = 0, \dots, n$ is again trivial as all the morphisms considered are identities.

(b) We need to show that $\mathbb{P}^n_R$ is not affine. Define $V_i := U_0 \cup \ldots \cup U_i$ for $i = 1, \dots, n$. Let $s \in \mathcal{O}_{\mathbb{P}^n_R}(V_1)$. Then
$$R\Big[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\Big] \ni \rho_{V_1 U_0}(s) = \rho_{V_1 U_1}(s) \in R\Big[\frac{X_0}{X_1}, \frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}\Big]$$
on $U_{01} = U_0 \cap U_1$. But then $\rho_{V_1 U_0}(s) = \rho_{V_1 U_1}(s) = r \in R$, and so $s \in R$. Hence, $\mathcal{O}_{\mathbb{P}^n_R}(V_1) \subset R$ and similarly one gets that $\mathcal{O}_{\mathbb{P}^n_R}(V_i) \subset R$, for $i = 1, \dots, n$.

On the other hand consider $s_i \in \mathcal{O}_{\mathbb{P}^n_R}(U_i)$ with $s_i = r \in R$. Then clearly $\rho_{U_i U_{ij}}(s) = \rho_{U_j U_{ij}}(s) = r$, for $i \neq j$. So by gluing one obtains $R \subset \mathcal{O}_{\mathbb{P}^n_R}(V_i)$, for $i = 1, \dots, n$.

Now if $\mathbb{P}^n_R = V_n$ were indeed affine, then by the above considerations, one obtains $\mathbb{P}^n_R = \mathrm{Spec}(R)$. Since $\mathrm{Spec}(R) \subsetneqq \mathrm{Spec}(R[x_1, \dots, x_n]) = \mathbb{A}^n_R$, we arrive at a contradiction. $\square$

Exercise 5.4. *Let X be an integral scheme with generic point η and let $U = \mathrm{Spec}(A)$ be an affine open subset of X. Recall that the local ring $\mathcal{O}_{X,\eta}$ is a field, called the function field $K(X)$ of X.*

(a) *Show that* $\mathrm{Quot}(A) \cong \mathcal{O}_{X,\eta} = K(X)$.

(b) *By identifying $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,x}$ with subrings of $K(X)$, show that we have*
$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq K(X).$$

An element of $K(X)$ is called a rational function on X. We say that $f \in K(X)$ is regular at $x \in X$ if $f \in \mathcal{O}_{X,x}$.

(c) *Let k be an algebraically closed field. Describe the regular and the rational functions on $\mathbb{A}_k^n = \operatorname{Spec} k[X_1, \ldots, X_n]$.*

Proof. **Solution by Mattias Hemmig**

Notice first that integral schemes are irreducible and hence a generic point $\eta \in X$ exists.

(a) Since η is contained in any nonempty open subset of X, it is contained in the open affine subscheme U, in which it also lies dense. By the integrality condition, the ring $A = \mathcal{O}_X(U)$ is an integral domain, and $\eta \in U$ corresponds to $(0) \in \operatorname{Spec}(A)$. Thus we have

$$K(X) = \mathcal{O}_{X,\eta} = \mathcal{O}_{U,\eta} \cong A_{(0)} = \operatorname{Quot}(A).$$

(b) We prove (b) for an arbitrary open set $U \subset X$. We break down the proof into three steps:

(i) Prove the statement first for an affine open set $U = \operatorname{Spec}(A)$:

($\subset$): This is clear since $\mathcal{O}_X(U) = A \subset A_{\mathfrak{p}} = \mathcal{O}_{X,x}$, for all $\mathfrak{p} \in \operatorname{Spec}(A)$ with corresponding $x \in U$.

($\supset$): Let $\gamma \in \bigcap_{x \in U} \mathcal{O}_{X,x}$. Define the ideal

$$I := \{a \in A \mid a\gamma \in A\} \subset A.$$

Now take $\mathfrak{p} \in \operatorname{Spec}(A)$ corresponding to some $x \in U$. By the equality $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ we can find a representation $\gamma = \frac{\alpha}{\beta}$ with $\alpha \in A$ and $\beta \in A \backslash \mathfrak{p}$. It follows that $\beta \in I \backslash \mathfrak{p}$. Hence I is not contained in any prime (and in particular maximal) ideal of A and so $I = A$. But this means that $\gamma \in A$.

(ii) We show now that the restriction maps $\mathcal{O}_X(V) \xrightarrow{\rho_{VU}} \mathcal{O}_X(U)$ are injective for any open sets $\emptyset \neq U \subset V \subset X$. Indeed it suffices to show that the maps $\mathcal{O}_X(U) \xrightarrow{f \mapsto f_\eta} K(X)$ are injective for all open sets $\emptyset \neq U$. For $U = \operatorname{Spec}(A)$ affine, the map is simply the inclusion $A \hookrightarrow \operatorname{Quot}(A) \cong K(X)$.

(iii) Now take a general open set $U \subset X$. U can be covered by a family of affine open subsets $\{U_i\}_{i \in I}$. Using the injectivity of the restriction maps $\{\rho_{UU_i}\}_{i \in I}$ and the sheaf properties of $\mathcal{O}_X$, one immediately sees that $\mathcal{O}_X(U) = \bigcap_{i \in I} \mathcal{O}_X(U_i) \subset K(X)$. Together with step (i) this finishes the proof.

(c) Recall that $k[X_1, \ldots, X_n]$ is an integral domain and hence $\operatorname{Spec}(k[X_1, \ldots, X_n])$ is an integral scheme. So by part (b) the regular functions on $\mathbb{A}_k^n$ are given by

$$\bigcap_{x \in \mathbb{A}_k^n} \mathcal{O}_{\mathbb{A}_k^n, x} = \mathcal{O}_{\mathbb{A}_k^n}(\mathbb{A}_k^n) = k[X_1, \ldots, X_n];$$

the rational functions are simply

$$K(X) = \operatorname{Quot}(k[X_1, \ldots, X_n]) = k(X_1, \ldots, X_n).$$

□

6 Solutions for Exercise Sheet-6

Remark. The soltuions to this exercise have not been double checked as of yet, due to lack of time. However the solutions seem accurate, and we have put them up online so as to assist the students in preparing for the final exam.

Exercise 6.1. *Let R be a commutative ring with 1 and let $n \geq 0$ be an integer. Show that the following assertions are equivalent:*

(i) $\operatorname{Spec}(R)$ *is reduced (resp. irreducible, resp. integral).*

(ii) $\mathbb{A}^n_R$ *is reduced (resp. irreducible, resp. integral).*

(iii) $\mathbb{P}^n_R$ *is reduced (resp. irreducible, resp. integral).*

Proof. **Solution by Aaron Schöpflin**

From Proposition 3.1 in Chapter 2 of Hartshorne, it follows that a scheme is integral if and only if it is both reduced and irreducible. So it suffices prove the above equivalences for the property of being reduced and for the property of being irreducible.

Equivalence of being reduced
Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced. Equivalently X is reduced if for every open subset U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

Now we show $(a) \Leftrightarrow (b)$:

$$\begin{aligned}&\operatorname{Spec}(R) \text{ is reduced } \Leftrightarrow\ R \text{ is reduced } \Leftrightarrow\ R[X_1, ...X_n] \text{ is reduced } \Leftrightarrow\\ &\operatorname{Spec}\left(R[X_1, ..., X_n]\right) = \mathbb{A}^n_R \text{ is reduced.}\end{aligned}$$

Left to show $(b) \Leftrightarrow (c)$:
For $i = 0, ..., n$ let U_i be the affine spaces $\mathbb{A}^n_R$ of relative dimension n over R given by

$$U_i := \operatorname{Spec}\ R\left[\frac{X_0}{X_i}, ..., \frac{\hat{X}_i}{X_i}, ..., \frac{X_n}{X_i}\right].$$

Then we have

$$\begin{aligned}&\mathbb{A}^n_R = \operatorname{Spec}\left(R[X_1, ..., X_n]\right) \text{ is reduced } \Leftrightarrow\ R[X_1, ...X_n] \text{ is reduced } \Leftrightarrow\\ &R\left[\frac{X_0}{X_i}, ..., \frac{\hat{X}_i}{X_i}, ..., \frac{X_n}{X_i}\right] \text{ is reduced } \Leftrightarrow\ \operatorname{Spec}\left(R\left[\frac{X_0}{X_i}, ..., \frac{\hat{X}_i}{X_i}, ..., \frac{X_n}{X_i}\right]\right) \text{ is reduced } \Leftrightarrow\\ &U_i \text{ is reduced for all } i.\end{aligned}$$

Therefore the equivalence $(b) \Leftrightarrow (c)$ follows from the fact that the space $\mathbb{P}^n(\mathbb{R})$ is obtained from glueing the open sets U_i together.

Equivalence of being irreducible
We now show $(a) \Leftrightarrow (b)$:
We have to show that $\operatorname{Spec}(R)$ *is irreducible* $\Leftrightarrow$ $\operatorname{Spec}(R[X_1, ..., X_n])$ *is irreducible.* Since

$$R \text{ irreducible } \Leftrightarrow\ \operatorname{Spec}(R) \text{ is irreducible,}$$

it suffices to show that

$$R \text{ is irreducible } \Leftrightarrow\ R[X_1, ..., X_n] \text{ is irreducible.}$$

A ring is irreducible if its zero ideal is irreducible. Now the equivalence follows since the zero ideal of R equals the zero ideal of $R[X_1, ..., X_n]$.

Now we show $(b) \Leftrightarrow (c)$:
The equivalence follows from the fact that $\mathbb{P}^n(\mathbb{R})$ is obtained from glueing the finitely many affine open schemes $\mathbb{A}^n(\mathbb{R}) \cong U_i$, for all $i = 0, \ldots, n$. Hence, we have shown that

$$\mathrm{Spec}(R) \text{ is integral } \Leftrightarrow \mathbb{A}^n_R \text{ is integral } \Leftrightarrow \mathbb{P}^n_R \text{is integral.}$$

□

Exercise 6.2. *For a commutative ring A with 1, we denote by A_{red} the quotient of A by its nilradical.*

(a) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\mathrm{red}}$ be the sheaf associated to the presheaf given by the assignment

$$U \mapsto \mathcal{O}_X(U)_{\mathrm{red}} \qquad (U \subset X, \textit{open}).$$

Show that $X_{\mathrm{red}} := (X, (\mathcal{O}_X)_{\mathrm{red}})$ is a scheme, called the reduced scheme associated to X. Further, show that there is a morphism of schemes $X_{\mathrm{red}} \longrightarrow X$, which is a homeomorphism on the underlying topological spaces.

(b) Let $f : X \longrightarrow Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \longrightarrow Y_{\mathrm{red}}$ such that f is obtained by composing g with the natural map $Y_{\mathrm{red}} \longrightarrow Y$.

Proof. **Solution by Aaron Schöpflin**

(a) Let us denote the presheaf given by $U \mapsto (\mathcal{O}_X(U))_{\mathrm{red}}$ by $(\mathcal{O}_X)^p_{\mathrm{red}}$, so that $(\mathcal{O}_X)_{\mathrm{red}}$ is the sheafification of $(\mathcal{O}_X)^p_{\mathrm{red}}$.

Claim: The stalks of $(\mathcal{O}_X)^p_{\mathrm{red}}$ (and thus of $(\mathcal{O}_X)_{\mathrm{red}}$) at any $x \in X$ are canonically identified with $(\mathcal{O}_{X,x})_{\mathrm{red}}$; more precisely, the surjective presheaf morphism $\mathcal{O}_X \to (\mathcal{O}_X)^p_{\mathrm{red}}$ trivially induces a surjection on stalks $\mathcal{O}_{X,x} \to (\mathcal{O}_{X,x})^p_{\mathrm{red}}$, and the kernel is precisely the nilpotent elements of $\mathcal{O}_{X,x}$.

Proof. For an element of $\mathcal{O}_{X,x}$ represented by $\langle U, s\rangle$, let $\bar{s}$ denote the image of s in $(\mathcal{O}_X)^p_{\mathrm{red}}(U)$. The claim then follows from the following equivalences:
$\langle U, s\rangle$ is nilpotent in $\mathcal{O}_{X,x}$

$\Leftrightarrow$ there exists some neighborhood V of x contained in U such that $s_{|V}$ is nilpotent in $\mathcal{O}_X(V)$.

$\Leftrightarrow$ there exists some neighborhood V of x contained in U such that $s_{|V}$ maps to zero in $(\mathcal{O}_X)^p_{red}(V)$.

$\Leftrightarrow$ there exists some neighborhood V of x contained in U such that $\bar{s}_{|V} = 0$

$\Leftrightarrow$ $\langle U, \bar{s}\rangle = 0$ in $((\mathcal{O}_X)^p_{\mathrm{red}})_x$. □

Now since we have sheafified, it is automatic that $(X, (\mathcal{O}_X)_{\mathrm{red}})$ is a ringed space, and it suffices to show that it has an open cover by affine schemes. Given $x \in X$, let $U = \mathrm{Spec}(A)$ be an affine neighborhood of $x \in X$. We want to show that $(U, (\mathcal{O}_X)_{\mathrm{red}|U})$ is still an affine scheme, namely isomorphic to $\mathrm{Spec}(A_{\mathrm{red}})$. As a first step we observe that as topological spaces $\mathrm{Spec}(A_{\mathrm{red}})$ and $\mathrm{Spec}(A)$ are canonically homeomorphic, since the ideal of nilpotent elements is contained in all prime ideals, and any two ideals which agree modulo the nilpotent elements have the same set of primes containing them.

Claim: Taking the sheaf associated to a presheaf commutes with restriction to an open subset, so it is enough to see that the structure sheaf of $\mathrm{Spec}(A_{\mathrm{red}})$ is equal to $(\mathcal{O}_{\mathrm{Spec}(A)})_{\mathrm{red}}$.

Proof. We observe that if $\mathfrak{p} \subseteq A$ is a prime ideal, and $\bar{\mathfrak{p}}$ its image in A_{red}, then the surjection $A \to A_{\mathrm{red}}$ induces an isomorphism $(A_p)_{\mathrm{red}} \xrightarrow{\sim} (A_{\mathrm{red}})_{\bar{p}}$. Indeed, because $\mathfrak{p}$ is the preimage of $\bar{\mathfrak{p}}$ we have an induced map $A_p \to (A_{\mathrm{red}})_{\bar{p}}$ which is surjective. So it suffices to see that its kernel is precisely the nilpotens of $A_{\mathfrak{p}}$. Given $\frac{a}{f}$, with $a \in A$ and $f \notin \mathfrak{p}$, suppose $\frac{\bar{a}}{\bar{f}}$ the image of $\frac{a}{f}$ in $(A_{\mathrm{red}})_{\bar{p}}$ is 0. Then by definition of the local ring, there exists $\bar{g} \notin \bar{\mathfrak{p}}$ such that $\bar{g}\bar{a} = 0$ in A_{red}. Choose any $g \in A$ mapping to $\bar{g}$; we then conclude that ga is nilpotent in A. Moreover $\bar{g} \notin \bar{\mathfrak{p}}$ implies that $g \notin \mathfrak{p}$, so we see that a is nilpotent in A_p, and hence so is $\frac{a}{f}$ as desired. □

Now we can compare the structure sheaf of $\mathrm{Spec}(A_{\mathrm{red}})$ with $(\mathcal{O}_{\mathrm{Spec}\,A})_{\mathrm{red}}$.

We have morphisms

$$\begin{aligned} \mathcal{O}_{\mathrm{Spec}\,A} &\to (\mathcal{O}_{\mathrm{Spec}\,A})^p_{\mathrm{red}} \\ \text{and } \mathcal{O}_{\mathrm{Spec}\,A} &\to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}. \end{aligned}$$

We also know that $\mathrm{Spec}(A_{\mathrm{red}})$ is reduced, so nilpotent elements of $\mathcal{O}_{\mathrm{Spec}(A)}$ on any open subset U must be mapped to 0. So we conclude that the second map factors through the first map, giving us a presheaf morphism $(\mathcal{O}_{\mathrm{Spec}(A)})^p_{\mathrm{red}} \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$, which then by the universal property of sheafification induces a sheaf morphism $(\mathcal{O}_{\mathrm{Spec}(A)})_{\mathrm{red}} \to \mathcal{O}_{\mathrm{Spec}(A_{\mathrm{red}})}$. Finally, by our above calculation on stalks we see that the last two maps induce isomorphisms on stalks, so we conclude, that the last map is an isomorphism as desired.

We still want to show that there is a morphism of schemes $X_{\mathrm{red}} \to X$, which is a homeomorphism on the underlying topological spaces. To get the desired morphism, take the identity map on the underlying topological spaces, and it suffices to produce a map $\mathcal{O}_X \to (\mathcal{O}_X)_{\mathrm{red}}$ of sheaves which induces local homomorphisms on stalks. Since the underlying topological spaces are equal by definition, we may omit the pushforward. Now, the desired map is obtained by simply composing the canonical presheaf surjection $\mathcal{O}_X(U) \to (\mathcal{O}_X(U))_{\mathrm{red}}$ with the sheafification map. The sheafification is an isomorphism on stalks, the presheaf map simply gives $\mathcal{O}_{X,x} \to (\mathcal{O}_{X,x})_{\mathrm{red}}$ which is indeed a homomorphism of local rings.

(b) Obviously, f factors uniquely through Y_{red} at the level of topological spaces, since by definition the underlying space of Y_{red} is the same as that of Y. It thus remains to show that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ factors uniquely through the sheaf map $\mathcal{O}_Y \to (\mathcal{O}_Y)_{\mathrm{red}}$. Note that since the latter map is surjective, in fact the uniqueness is immediate (this one can see more easily at the level of stalks).

Now since X is reduced, for any open subset $U \subseteq Y$ we have that $f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$ has no nonzero nilpotents and we conclude that any nilpotents in $\mathcal{O}_Y(U)$ must map to 0 under $f^\#$. It follows that $f^\#$ factors through the presheaf morphism $\mathcal{O}_Y \to (\mathcal{O}_Y)^p_{\mathrm{red}}$, which is to say we have a presheaf morphism $(\mathcal{O}_Y)^p_{\mathrm{red}} \to f_*\mathcal{O}_X$ recovering $f^\#$ after composition. Then by the universal property of sheafification, this presheaf morphism factors through $(\mathcal{O}_Y)^p_{\mathrm{red}} \to (\mathcal{O}_Y)_{\mathrm{red}}$, giving the desired morphism $(\mathcal{O}_Y)_{\mathrm{red}} \to f_*\mathcal{O}_X$. □

Exercise 6.3. *Let k be a field, $A := k[X, Y, Z]$, and $X := \mathbb{A}^3_k = \mathrm{Spec}(A)$. Further, let $\mathfrak{p}_1 := (X, Y)$, $\mathfrak{p}_2 := (X, Z)$, and $\mathfrak{a} := \mathfrak{p}_1\mathfrak{p}_2$.*

(a) *Let $Z_1 := V(\mathfrak{p}_1)$, $Z_2 := V(\mathfrak{p}_2)$, and $Y := V(\mathfrak{a})$. Show that Z_1 and Z_2 are integral subschemes of X and show that $Y = Z_1 \cup Z_2$ (set-theoretically).*

(b) *Show that $Y = V(\mathfrak{a})$ is not reduced and describe Y_{red}.*

Proof. **Solution by Codrut Grosu**

(a) We first show that $Y = Z_1 \cup Z_2$ (set-theoretically).

Note that $\mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ implies $V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2) \subseteq V(\mathfrak{p}_1\mathfrak{p}_2)$. For the reverse inclusion, let $\mathfrak{q}$ be an arbitrary prime ideal in $V(\mathfrak{p}_1\mathfrak{p}_2)$. Without loss of generality, we may assume that $\mathfrak{p}_2 \not\subseteq \mathfrak{q}$. Then there exists $b \in \mathfrak{p}_2$ not in $\mathfrak{q}$. As $ab \in \mathfrak{q}$ for any $a \in \mathfrak{p}_1$, we must have $\mathfrak{p}_1 \subseteq \mathfrak{q}$, thus showing that $\mathfrak{q} \in V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2)$. Hence $Y = Z_1 \cup Z_2$ as claimed.

We will now show that Z_1 is an integral closed subscheme of X (the proof for Z_2 is similar and omitted). We will use the following assertion (for a proof, see [Hartshorne, p. 85, Example 3.2.3] or Exercise 7.2).

Assertion: Let A be a commutative ring with 1 and $\mathfrak{a}$ an ideal of A. Then $V(\mathfrak{a})$ is a closed subscheme of $\mathrm{Spec}(A)$ isomorphic to the affine scheme $\mathrm{Spec}(A/\mathfrak{a})$.

By above Assertion, Z_1 is a closed subscheme of X and $Z_1 \simeq \mathrm{Spec}(A/\mathfrak{p}_1)$. As $A/\mathfrak{p}_1$ is an integral domain, Z_1 is integral.

(b) By above Assertion, we may identify Y with $\mathrm{Spec}(A/\mathfrak{a})$.

Note that $\mathfrak{a} = (X^2, XY, XZ, YZ)$ and $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{p}_1} \cap \sqrt{\mathfrak{p}_2} = \mathfrak{p}_1 \cap \mathfrak{p}_2 = (X, YZ)$. In particular, $\mathrm{nil}(A/\mathfrak{a}) = \sqrt{\mathfrak{a}}/\mathfrak{a}$ is non-zero. Consequently Y is not reduced.

By Exercise 6.2, we know that $Y_{\mathrm{red}} \simeq \mathrm{Spec}(A/\sqrt{\mathfrak{a}})$. □

Exercise 6.4. *Recall that a primitive integer solution of the generalized Fermat equation*

$$X^p + Y^q = Z^r \qquad (p, q, r \in \mathbb{Z}_{>0})$$

is a triple $(x, y, z) \in \mathbb{Z}^3$ satisfying $x^p + y^q = z^r$ with $\gcd(x, y, z) = 1$.

(a) *Show that the affine scheme $\mathrm{Spec}\,\mathbb{Z}[X,Y,Z]/(X,Y,Z)$ can be identified with a closed subscheme T of the affine scheme $S := \mathrm{Spec}\,\mathbb{Z}[X,Y,Z]/(X^p + Y^q - Z^r)$.*

(b) *Consider the open subscheme $U := S \setminus T$. Prove that*

$$U(\mathbb{Z}) := \mathrm{Hom}\big(\mathrm{Spec}(\mathbb{Z}), U\big)$$

is in bijection with the set of primitive integer solutions of $X^p + Y^q = Z^r$.

Proof. **Solution by Codrut Grosu**

(a) Let $\mathfrak{a}$ be the ideal $(X,Y,Z)/(X^p + Y^q - Z^r)$ in the ring $A := \mathbb{Z}[X,Y,Z]/(X^p + Y^q - Z^r)$. By the Assertion in the solution to Exercise 6.3, $V(\mathfrak{a})$ is a closed subscheme T in S isomorphic to $\mathrm{Spec}(A/\mathfrak{a}) \simeq \mathrm{Spec}(\mathbb{Z}[X,Y,Z]/(X,Y,Z))$.

(b) We define

$$\mathcal{F} := \{(a,b,c) \in \mathbb{Z}^3_{>0} : (a,b,c) \text{ is a primitive solution to } X^p + Y^q - Z^r = 0\}.$$

We construct a map $H : \mathcal{F} \to \mathrm{Hom}(A, \mathbb{Z})$ by sending the triple (a,b,c) to the unique homomorphism $f : A \to \mathbb{Z}$ satisfying $f(X) = a$, $f(Y) = b$, and $f(Z) = c$. Then H is trivially injective. Also the gcd condition on the components of a primitive solution gives us the following

$$\begin{aligned}\operatorname{Im} H &= \{f : A \to \mathbb{Z} : \forall\, \mathfrak{p} \subseteq \mathbb{Z} \text{ prime ideal, not all of } f(X), f(Y), f(Z) \text{ are in } \mathfrak{p}\} \\ &= \{f : A \to \mathbb{Z} : \forall\, \mathfrak{p} \subseteq \mathbb{Z} \text{ prime ideal}, (X, Y, Z) \not\subseteq f^{-1}(\mathfrak{p})\}.\end{aligned}$$

Now recall the functors Spec and Γ defined in Exercise 3.4. We refer the reader to the solution of this exercise for the construction of the contravariant functors Spec and Γ. We shall use these two functors to construct a bijection between $\operatorname{Im} H$ and $U(\mathbb{Z})$.

Note that for any homomorphism $g \in \operatorname{Im} H$ the functor Spec gives us a morphism of schemes

$$(f(g), f(g)^{\#}) : \operatorname{Spec}(\mathbb{Z}) \to \operatorname{Spec}(A).$$

By construction, $f(g)(\mathfrak{p}) = g^{-1}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \subseteq \mathbb{Z}$. Hence, by our choice of g we have $\operatorname{Im} f(g) \subseteq U$. So we now define $F(g)$ to be the restriction of $(f(g), f(g)^{\#})$ to U. This gives us a map

$$F : \operatorname{Im} H \to U(\mathbb{Z}).$$

Now suppose we are given a morphism $(f, f^{\#}) \in U(\mathbb{Z})$. We compose it with the inclusion morphism $i : U \to S$ to get a morphism

$$(g, g^{\#}) : \operatorname{Spec}(\mathbb{Z}) \to \operatorname{Spec}(A).$$

Applying Γ, we obtain a homomorphism $h : A \to \mathbb{Z}$.

We claim $h \in \operatorname{Im} H$. Indeed, by construction we have for any prime ideal $\mathfrak{p} \subseteq \mathbb{Z}$, $h^{-1}(\mathfrak{p}) = g(\mathfrak{p}) = f(\mathfrak{p})$, and consequently $(X, Y, Z) \not\subseteq h^{-1}(\mathfrak{p})$. Then $h \in \operatorname{Im} H$, as claimed.

We set $G((f, f^{\#})) = h$, thus defining a map

$$G : U(\mathbb{Z}) \to \operatorname{Im} H.$$

As Spec and Γ are inverse to one another, we observe that

$$F \circ G = \mathrm{id}_{U(\mathbb{Z})} \text{ and } G \circ F = \mathrm{id}_{\operatorname{Im} H}.$$

This shows that $U(\mathbb{Z})$ is in bijection with $\operatorname{Im} H$, which in turn is in bijection with $\mathcal{F}$, the set of primitive solutions, proving the desired claim. □

For completeness, we also prove the following assertions.

Assertion Let R be a commutative ring with 1 and $X = \operatorname{Spec}(R)$ be an affine scheme. Then

1. X is reduced iff $\operatorname{nil}(R) = 0$.
2. X is irreducible iff $\operatorname{nil}(R)$ is a prime ideal.
3. X is integral iff R is an integral domain.

Proof. **Solution by Codrut Grosu**

(a) If X is reduced then by the equivalent definition of reduced schemes, $\mathcal{O}_X(X) \simeq R$ must be a reduced ring, and so $\operatorname{nil}(R) = 0$.

Conversely, assume $\mathrm{nil}(R) = 0$ and let $p \in X$ with $\mathcal{O}_{X,p} \simeq R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ in R. Let $\frac{f}{g} \in R_{\mathfrak{p}}$ with $\left(\frac{f}{g}\right)^n = 0$. Then there exists a $t \in R \setminus \mathfrak{p}$ such that $tf^n = 0$. So $(tf)^n = 0$. Then $tf \in \mathrm{nil}(R)$. Hence $tf = 0$ and then $\frac{f}{g} = 0$ in $R_{\mathfrak{p}}$. Hence X is reduced.

(b) Exercise I.20 [Atiyah, MacDonald] tells us that the irreducible components of X are the closed sets $V(\mathfrak{p})$, with $\mathfrak{p}$ a minimal prime ideal of R. So X is irreducible iff there is just one minimal prime ideal, which is equivalent to $\mathrm{nil}(R)$ being prime.

(c) This follows from the previous two statements and the fact that X is integral iff it is reduced and irreducible. $\square$

7 Solutions for Exercise Sheet-7

Remark. The soltuions to this exercise have not been double checked as of yet, due to lack of time. However the solutions seem accurate, and we have put them up online so as to assist the students in preparing for the final exam.

Exercise 7.1. *Prove the following:*

(a) Let A be a commutative ring with 1, $X := \operatorname{Spec}(A)$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, and let $f : Y := \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A) =: X$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \longrightarrow X$ is a homeomorphism onto a closed subset and $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective, then φ is surjective.

Hint: Consider $X' = \operatorname{Spec}(A/\ker(\varphi))$, and use (b) and (c).

Proof. (a) From the lectures we know that

$$\mathcal{O}_X\left(D(f)\right) = A_f.$$

The assertion follows from the fact that $A_f = \emptyset$, iff f is nilpotent.

(b) Let the homomorphism $\varphi : A \longrightarrow B$ be injective. For any $g \in A$, the sets $U = D(g)$ form a basis for the topological space X. So for any $g \in A$, it suffices to show that the following map of rings is injective

$$f^\flat\mid_U : \mathcal{O}_X(D(g)) = A_g \longrightarrow f_*\mathcal{O}_Y(D(g)) = \mathcal{O}_Y(f^{-1}D(g)) = B_{\varphi(g)}.$$

Now the injectivity of the map $f^\flat\mid_U : A_g \longrightarrow B_{\varphi(g)}$, follows from the injectivity of the map φ.

Conversely let us assume that the map

$$f^\flat\mid_U : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$$

is injective. Then we find that the map

$$\mathcal{O}_X(X) = A \longrightarrow f_*\mathcal{O}_Y(X) = B$$

is injective.

We now prove that the map f is dominant iff the map φ is injective. First let us assume that φ is injective. Let $U_\circ$ be an open set in $X\backslash f(Y)$, and x be any point in $U_\circ$. Then the map of structure sheaves $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is injective, from which we derive that the following map of local rings is injective

$$f^\flat : \mathcal{O}_{X,x} \longrightarrow \left(f_*\mathcal{O}_Y\right)_x.$$

Now the point $x \in X$ corresponds to a prime $\mathfrak{p} \in A$, so $\mathcal{O}_{X,x} = A_\mathfrak{p}$. As the set $U_\circ \subset X\backslash f(Y)$, we get

$$\left(f_*\mathcal{O}_Y\right)_x = \varinjlim_{x \in U} \mathcal{O}_Y(f^{-1}U) = 0,$$

which leads us to a contradiction. Hence, we find that $f(Y)$ is dense in X.

Conversely let us assume that the map $f(Y)$ is dense in X. Observe that $\overline{f(Y)}$ is the same as $\mathrm{Spec}(A/\ker(\varphi))$. Since $\overline{f(Y)} = X$, it follows that $\ker(\varphi) = 0$, when the rings A and B are reduced.

(c) Given the map $\varphi : A \longrightarrow B$ is surjective. Then we find that $A/\ker(\varphi) \cong B$. Using which we derive that the following map of topological spaces is bijective

$$\mathrm{Spec}(B) = Y \longrightarrow \mathrm{Spec}(A/\ker(\varphi)).$$

Hence, we find that Y is homoemorphic onto the topological space $\mathrm{Spec}(A/\ker(\varphi))$, and the latter space corresponds to the subscheme $V(\ker(\varphi))$. It is left to prove that the morphism of sheaves $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.

So for any $g \in A$, it suffices to show that the following map of rings is surjective

$$f^\flat |_U : \mathcal{O}_X(D(g)) = A_g \longrightarrow f_*\mathcal{O}_Y(D(g)) = \mathcal{O}_Y(f^{-1}D(g)) = B_{\varphi(g)}.$$

The surjectivity of the above map $f^\flat |_U : A_g \longrightarrow B_{\varphi(g)}$ follows from the surjectivity of the map φ.

(d) Assume that the map $f : Y \longrightarrow X$ is a homeomorphism onto a closed subset and the morphism $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.
The map $\tilde{\varphi} : A/\ker(\varphi) \longrightarrow B$ is injecitve. We need to show that $\tilde{\varphi}$ is surjective. Put $X' = \mathrm{Spec}(A/\ker(\varphi))$. Now the map f factors in the following way

$$Y \xrightarrow{\tilde{f}} X' \xrightarrow{\tilde{j}} X,$$

where the map $\tilde{j}$ is induced by the surjective morphism $A \longrightarrow A/\ker(\varphi)$. So from (c) it follows that X' is homeomorphic onto a closed subset of X. Since the map $\tilde{\varphi}$ is injective, from (b) it follows that the image $\tilde{f}(Y)$ is dense in X'.
Since the map $f(Y)$ is homemorphic onto a subset of X, it follows that $\tilde{f}(Y)$ is a closed subset of X', and hence $\tilde{f}(Y) = X'$.

Since the maps f and $\tilde{j}$ are homemorphisms, even the map $\tilde{f}$ is a homemorphism. Now the map $\tilde{\varphi} : A/\ker(\varphi) \longrightarrow B$ is injective, so from (b) we get an injective morphism of sheaves

$$\tilde{f}^\flat : \mathcal{O}_{X'} \longrightarrow f_*\mathcal{O}_Y.$$

Since the morphism $A \longrightarrow A/\ker(\varphi)$ is surjective, so from (c) it follows that the following morphism of sheaves is surjective

$$\tilde{j}^\flat : \mathcal{O}_X \longrightarrow \tilde{j}_*\mathcal{O}_{X'}.$$

Observe that the surjective morphism $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ factors in the following manner

$$\mathcal{O}_X \xrightarrow{\tilde{j}^\flat} \tilde{j}_*\mathcal{O}_{X'} \xrightarrow{\tilde{j}_*\tilde{f}^\flat} \tilde{f}_*\tilde{j}_*\mathcal{O}_Y.$$

Since $f^\flat$ and $\tilde{j}^\flat$ are surjective, so is the map $\tilde{f}^\flat$. Hence, we can conclude that the morphism $\tilde{f}^\flat$ is an isomorphism. From which we derive that the map $\tilde{\varphi}$ is an isomorphism, which implies that it is surjective. This completes the proof of the assertion. $\square$

Exercise 7.2. *Let A be a commutative ring with* 1 *and $\mathfrak{a} \subseteq A$ an ideal. Let* $X = \mathrm{Spec}(A)$ *and* $Y = \mathrm{Spec}(A/\mathfrak{a})$.

(a) *Show that the ring homomorphism $A \longrightarrow A/\mathfrak{a}$ induces a morphism of schemes $f : Y \longrightarrow X$, which is a closed immersion.*

(b) *Show that for any ideal $\mathfrak{a} \subseteq A$, we obtain a structure of a closed subscheme on the closed set $V(\mathfrak{a}) \subseteq X$.*
In particular, every closed subset Y of X can have various subscheme structures corresponding to all the ideals $\mathfrak{a}$ for which $V(\mathfrak{a}) = Y$.

Proof. (a) From Proposition 2.3 in Hartshorne, it follows that the surjective morphism of rings $A \longrightarrow A/\mathfrak{a}$ induces a morphism of schemes $f : Y \longrightarrow X$. We need to show that the morphism of schemes $f : Y \longrightarrow X$ is a closed immersion.

To show that the morphism is a closed immersion, we need to show that $f(Y)$ is a closed subset of X, Y is homemorphic to $f(Y)$, and that the following morphism is surjective

$$f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y.$$

Since the map $A \longrightarrow A/\mathfrak{a}$ is surjective, from Exercise 7.1 (c), it follows that Y is homemorphic to the closed subset $f(Y)$ and the morphism $f^\flat$ is surjective.

(b) We need to show that the closed subset $V(\mathfrak{a})$ is a closed subscheme of X. From (a), we know that the inclusion

$$i : Y = \mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow X = \mathrm{Spec}(A)$$

is a closed immersion. We need to show that $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}$, for $\mathcal{I}$ a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. As the morphism i is a closed immersion, it follows that the map $i^\flat : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ is surjective. So the kernel of the map $\ker(i^\flat)$ is a sub-sheaf, from which it follows that $\ker(i^\flat)$ is a sheaf of ideals and $i_*\mathcal{O}_Y \cong \mathcal{O}_X \backslash \ker(i^\flat)$, which completes the proof of the exercise. $\square$

Exercise 7.3. *A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point. For example, let R be a discrete valuation ring and $T = \mathrm{sp}(\mathrm{Spec}(R))$ the underlying topological space of $\mathrm{Spec}(R)$. Then, T consists of two points $t_0 =$ the maximal ideal of R, $t_1 =$ the zero ideal of R. The open subsets are $\emptyset$, $\{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.*

(a) *Show that if X is a noetherian scheme, then $\mathrm{sp}(X)$ is a Zariski space.*

(b) *Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these closed points.*

(c) *Show that a Zariski space X satisfies the axiom T_0, i.e., given any two distinct points of X, there is an open set containing one but not the other.*

(d) *If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.*

(e) *If x_0, x_1 are points of a topological space X, and if $x_0 \in \overline{\{x_1\}}$, then we say that x_1 specializes to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a specialization of x_1 or that x_1 is a generalization of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are stable under specialization.) Similarly open subsets are stable under generization.*

(f) *Using the notation of the lecture, show that, if X is a noetherian topological space, then $t(X)$ is a Zariski space. Furthermore, X itself is a Zariski space if and only if the map $\alpha : X \longrightarrow t(X)$ is a homeomorphism.*

Proof. **Solution by Jie Lin Chen**

(a) Note that $sp(X)$ is the underlying topological space of X. Since X is a noetherian scheme we know from the lectures that its underlying topological space is also noetherian.

So it is left to show that each irreducible closed subset Z has a unique generic point. First we make an observation. For $Z \subseteq X$ an irreducible and closed set, $U \subseteq X$ open, and η a generic point of Z, we find that either $\eta \in U$ or $U \cap Z = \emptyset$ (Assume $\eta \notin U \Rightarrow U^c$ is closed with $\eta \in U^c \Rightarrow \underbrace{\overline{\{\eta\}}}_{=Z} \subseteq U^c \Rightarrow U \cap Z = \emptyset$).

Now we can reduce to the affine case. Let $X = \mathrm{Spec}(A)$ be an affine scheme, we then find following bijection of sets

$$\{Z \subseteq X | Z \text{ irreducible closed }\} \overset{1:1}{\longleftrightarrow} \{\mathfrak{p} \subseteq A | \mathfrak{p} \text{ prime ideal}\}.$$

So for each Z irreducible closed subset there exists a corresponding unique $\mathfrak{p} \in \mathrm{Spec}(A)$ with $V(\mathfrak{p}) = Z$ ($V(\mathfrak{p}) := \{\mathfrak{q} \in \mathrm{Spec}(A) | \mathfrak{q} \supseteq \mathfrak{p}\}$ as shown in the lecture). The point corresponding to the prime ideal $\mathfrak{p}$ is the unique generic point of Z.

(b) Let $Z \neq \emptyset$ be a minimal closed subset of the Zariski space. Then we find that

$$Z \text{ minimal} \Longrightarrow Z\text{irreducible} \underset{\text{property}}{\overset{\text{Zariski space}}{\Longrightarrow}} \exists ! \eta \in Z \text{ such that} \overline{\{\eta\}} = Z.$$

Furthermore, we find that

$$\left.\begin{array}{c} \text{Let } x \in Z \\ Z \text{ minimal} \end{array}\right\} \Longrightarrow \overline{\{x\}} = Z \Longrightarrow x = \eta \text{ for all } x \in Z.$$

So Z only contains one point and hence, is a closed point.

(c) Let $x, y \in X$ be two distinct points. Then define $U := \overline{\{x\}}^C$ which is an open set not containing x. So if $y \in U$, then we are finished, so assume it is not, then $y \in \overline{\{x\}}$. Furthermore if $x \in \overline{\{y\}}^C$ we are also done, so assume also that $x \in \overline{\{y\}}$. Then $\overline{\{x\}} = \overline{\{y\}}$ and x and y are generic points for the same irreducible closed set. Since X is a Zariski space, it follows that $x = y$ which contradicts the assumption, that they are distinct. So if $y \in \overline{\{x\}}$ then $x \in \overline{\{y\}}^C$ and the claim holds true.

(d) Let us assume not. $\eta \notin U \Longrightarrow \eta \in U^C \Longrightarrow \overline{\{\eta\}} \subseteq U^C$ but $\overline{\{\eta\}} = X \Longrightarrow U^C = X \Longrightarrow U = \emptyset$.

(e) First we show that closed subsets are stable under specialization. Let $Z \subseteq X$ be closed and $x \in Z$. Then we find that $\overline{\{x\}} \subseteq Z$, so Z contains every specialization of its elements.

We now show that the maximal points are the generic points of the irreducible components of X. Let $X = \bigcup\limits_i Z_i$ with Z_i irreducible components of $X \Longrightarrow$ for all i the Z_i's are irreducible, maximal and closed.

Let η be the generic point of Z_i, and for any $x \in X$ let $\eta \in \overline{\{x\}}$. Then we find that $\eta \in \overline{\{x\}} \Longrightarrow Z_i \subseteq \overline{\{x\}} \underset{\text{maximal}}{\overset{Z_i \text{ is}}{\Longrightarrow}} Z_i = \overline{\{x\}} \overset{\text{Zariski}}{\Longrightarrow} \eta = x \Longrightarrow \eta$ is maximal.

Conversely let η be maximal. Then there exists an i so that $\eta \in Z_i$. Now let η' be the unique generic point of Z_i. Then $\eta \in \overline{\{\eta'\}}$ and since η is maximal it follows that $\eta = \eta'$.

Finally we show that the minimal points are the closed points. Let $x' \in X$ be minimal. Let $x \in \overline{\{x'\}}$, which means $x' \rightsquigarrow x$. Then by the minimality of x', we get $\overline{\{x'\}} = \{x'\}$ $\Leftrightarrow$, which implies that x' is a closed point.

(f) Note that $S \subseteq X$ is closed $\Longleftrightarrow t(S) \subseteq t(X)$ is closed. So we can see immediately that $t(X)$ is also noetherian. So for $t(X)$ to be a Zariski space we need to show that every irreducible closed subset has a unique generic point.

Consider $Z \subseteq X$ a closed irreducible set. The closure $\overline{\{Z\}}$ in $t(X)$ is just $\{Z\}$, since Z is closed and is the smallest closed subset of X containing Z. So every closed irreducible set in $t(X)$ is of the form $\{Z\}$ with $Z \subseteq X$ closed and irreducible. Hence, its unique generic point is Z itself, so $t(X)$ is a Zariski space.

Additionally if X is a Zariski space we get a bijection

$$\{x \in X\} \overset{1:1}{\longleftrightarrow} \{\text{closed irreducible sets in } X\} \overset{1:1}{\longleftrightarrow} \{\text{closed irreducible sets in } t(X)\}$$

The continuity of the direction (for $x \in X$) $x \mapsto \overline{\{x\}}$ holds, since for $t(S) \subseteq t(X)$ closed $\alpha^{-1}(t(S)) = S$ is closed and the inverse is continuos since for every closed $S \subseteq X$ the image $\alpha(S) = \overline{\{S\}}$ is closed. So we get that $\alpha : X \longrightarrow t(X)$ is a homeomorphism.

Conversely if α is a homeomorphism it is clear that X is Zariski, since in this case every irreducible closed subset corresponds to a unique generic point in X. $\square$

8 Solutions for Exercise Sheet-8

Remark. The soltuions to this exercise have not been double checked as of yet, due to lack of time. However the solutions seem accurate, and we have put them up online so as to assist the students in preparing for the final exam.

Exercise 8.1. *Prove the following:*

(a) Let k be a field. Show that

$$\mathbb{A}^n_k \times_{\mathrm{Spec}(k)} \mathbb{A}^m_k \cong \mathbb{A}^{n+m}_k$$

and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

(b) Let k be a field and s, t indeterminates over k. Then, $\mathrm{Spec}\, k(s)$, $\mathrm{Spec}\, k(t)$, *and* $\mathrm{Spec}(k)$ *are all one-point spaces. Describe the product scheme*

$$\mathrm{Spec}\, k(s) \times_{\mathrm{Spec}(k)} \mathrm{Spec}\, k(t).$$

Proof. **Solution by Fernando Santos Castelar and Imke Stühring**

Following the lectures (see proof of existence and uniqueness of fiber product), we note that

$$\mathbb{A}^n_k \times_{\mathrm{Spec}(k)} \mathbb{A}^m_k \cong \mathrm{Spec}\ (k[X_1, \ldots, X_n] \otimes_k k[Y_1, \ldots, Y_m]).$$

Observing that

$$k[X_1, \ldots, X_n] \otimes_k k[Y_1, \ldots, Y_m] \cong k[X_1, \ldots, Y_m],$$

we arrive at

$$\mathrm{Spec}\ (k[X_1, \ldots, X_n] \otimes_k k[Y_1, \ldots, Y_m]) \cong \mathrm{Spec}\ (k[X_1, \ldots, Y_m]) = \mathbb{A}^{n+m}_k.$$

Now we show that there exists no natural bijection between $\mathbb{A}^{n+m}_k$ and $\mathbb{A}^n_k \times_{\mathrm{Spec}(k)} \mathbb{A}^m_k$. Consider the natural injections:

$$\begin{tikzcd} & k[X_1, \ldots, Y_m] & \\ k[X_1, \ldots, X_n] \arrow[ur, "i_1"] & & k[Y_1, \ldots, Y_m] \arrow[ul, "i_2"'] \end{tikzcd}$$

and the induced homomorphism

$$\begin{aligned} f : \mathrm{Spec}\ k[X_1, \ldots, Y_m] &\to \mathrm{Spec}\ (k[X_1, \ldots, X_n] \otimes_k k[Y_1, \ldots, Y_m]) \\ p &\mapsto (i_1^{-1}(p), i_2^{-1}(p)). \end{aligned}$$

Notice that (0) as well as $\left(\prod_{i=1}^n \prod_{j=1}^m X_i \cdot Y_j + 1\right)$ are both mapped to $((0), (0))$. Hence, we can conclude that the induced morphism f is not injective.

(b) For $S = k[s] \setminus \{0\}$ and $T = k[t] \setminus \{0\}$, we have $k(s) = S^{-1}k[s]$ and $k(t) = T^{-1}k[t]$. Using tensor product properties, it follows that

$$\mathrm{Spec}\, k(s) \times_{\mathrm{Spec}(k)} \mathrm{Spec}\ k(t) \cong \mathrm{Spec}\ (k(s) \otimes_k k(t)) = \mathrm{Spec}\ (S^{-1}k[s] \otimes_k T^{-1}k[t]) =$$
$$\mathrm{Spec}\ (S^{-1}T^{-1}k[s] \otimes_k k[t]) = \mathrm{Spec}\ (S^{-1}T^{-1}k[s, t]).$$

Hence, we arrive at

$$\operatorname{Spec} k(s) \times_{\operatorname{Spec}(k)} \operatorname{Spec}\ k(t) \cong \operatorname{Spec}\ (k(s) \otimes_k k(t))\ =$$
$$\text{Prime ideals of the form } \left\{ \frac{f}{gh} \middle| f \in k[s,t],\ 0 \neq g \in k[s],\ 0 \neq h \in k[t] \right\}.$$

These are all irreducible polynomials $p \in k[s,t]$ with $p \notin k[s] \cup k[t]$. ☐

Exercise 8.2. *Prove the following:*

(a) Let $f : X \longrightarrow S$ be a morphism of schemes and $s \in S$ a point. Show that $\operatorname{sp}(X_s)$ is homeomorphic to $f^{-1}(s)$ with the induced topology.

(b) Let k be an algebraically closed field, $X := \operatorname{Spec} k[Y,Z]/(Y - Z^2)$, $S := \operatorname{Spec} k[Y]$, and $f : X \longrightarrow S$ the morphism induced by sending $Y \mapsto Y$. Prove the following assertions:

(1) If $s \in S$ is the point $a \in k$ with $a \neq 0$, then the fiber X_s consists of two points, with residue field k.

(2) If $s \in S$ corresponds to $0 \in k$, then the fiber X_s is a non-reduced one-point scheme.

(3) If η is the generic point of S, then X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η.

Proof. **Solution by Fernando Santos Castelar and Imke Stühring**

(a) Let us first assume that X and Y are affine, i.e. we have $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ for some rings A and B. This means that f is induced by a ring homomorphism $\phi : B \to A$ and the fiber product X_y can be written as $\operatorname{Spec}(A \otimes_B k(y))$. Since X and $\operatorname{Spec}(k(y))$ are Y-schemes we obtain the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & \operatorname{Spec}(A \otimes_B k(y)) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
f^{-1}(y) \subset \operatorname{Spec}(A) \arrow[dr, "f"'] & & \operatorname{Spec}(k(y)) = \{(0)\} \arrow[dl, "g"] \\
 & y \in \operatorname{Spec}(B) &
\end{tikzcd}$$

where p and q are the homomorphisms of the fiber product and $g((0)) = y$.

We now define $m^e := \phi(y)$ to be the image of y in A. This gives rise to an isomorphism $A \otimes_B k(y) \cong A/m^e \otimes_B k(y)$ by mapping $a \otimes b \mapsto [a] \otimes b$ (the inverse is well-definded because for $a \in m^e$, $b \in k(y)$, we have $a \otimes b = \phi(b') \otimes b = b' \cdot (1 \otimes b) = 1 \otimes b' \cdot b = 0$, where $a = \phi(b')$ and $b' \in y$). Furthermore, the image of p is contained in $f^{-1}(y)$ because of the commutativity of the above diagram. This implies the existence of a morphism $\widetilde{p} : \operatorname{Spec}(A/m^e \otimes_B k(y)) \to f^{-1}(y)$ which is induced by p.

Claim: $f^{-1}(y) = \{\mathfrak{p} \in \operatorname{Spec}(A) : m^e \subset \mathfrak{p}\}$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ with $f(\mathfrak{p}) = y$, and $m_\mathfrak{p}$, m_y denote the maximal ideals in the local rings $A_\mathfrak{p}$, B_y, respectively. The morphism f induces a morphism of local rings $f_y^b : \mathcal{O}_{Y,m_y} \longrightarrow \mathcal{O}_{X,m_\mathfrak{p}}$. So it follows that $f_y^b(m_y) \subset m_\mathfrak{p}$. Recalling the definition of m^e, we can conclude that m^e is contained in $\mathfrak{p}$.

Conversely let $\mathfrak{p} \in \operatorname{Spec}(A)$ with $m^e \subset \mathfrak{p}$. We define an ideal $\mathfrak{a}$ in $A/m^e \otimes_B k(y)$ by

$$\mathfrak{a} := \left\{ \sum_{i=0}^{n} [a_i] \otimes b_i | a_i \in \mathfrak{p}, b_i \in k(y) \right\}$$

This is a prime ideal since $\mathfrak{p}$ is prime and we have $\widetilde{p}(\mathfrak{a}) = \mathfrak{p}$. From the commutativity of the diagram, we have $f(\mathfrak{p}) = y$. □

As a result we get a bijection between $f^{-1}(y)$ and $\operatorname{Spec}(A/m^e \otimes_B k(y))$, therefore $\widetilde{p}$ is a homeomorphism.

By taking an affine open neighborhood of y without loss of generality, we can assume that Y is affine, i.e $Y = \operatorname{Spec}(B)$. We now cover $f^{-1}(\operatorname{Spec}(B))$ with open affine neighborhoods U_i, where $U_i = \operatorname{Spec}(A_i)$. Then by the above arguments, we obtain that $(U_i)_y$ is homeomorphic to $f^{-1}(y) \cap U_i$. Since X_y is covered by the open sets $(U_i)_y$, we can glue the homomorphisms to obtain a homeomorphism between X_y and $f^{-1}(y)$.

(b) (1) Let $y = (s - a) \in \operatorname{Spec}(k[s])$, with $0 \neq a \in k$. We define a surjective ring homomorphism by

$$\psi : k[s]_{(s-a)} \to k, \quad \frac{p}{q} \mapsto \frac{p(a)}{q(a)}.$$

Since $\ker(\psi) = m_y$ with m_y being the maximal ideal we obtain $k(y) = k[s]_{(s-a)}/m_y \cong k$.

X_y is affine with $X_y = \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k[s]_{(s-a)}/(s - a))$, and we can characterize the underlying ring as follows:

$$\begin{aligned} k[s,t]/(s - t^2) \otimes_{k[s]} k[s]_{(s-a)}/(s - a) &\cong k[t]/(t^2 - a) \\ &\cong k[t]/(t - \sqrt{a}) \oplus k[t]/(t + \sqrt{a}) \cong k \oplus k \end{aligned}$$

(where the first isomorphism is given by $t \mapsto t \otimes 1$ and the second is a result of the chinese reminder theorem). We now conclude that X_y consists of two elements which correspond to $((0, 1))$ and $((1, 0))$.

(2) Let $y = (s) \in \operatorname{Spec}(k[s])$. As above, we have

$$X_y = \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k[s]/(s)) = \operatorname{Spec}(k[t]/(t^2)).$$

Hence, it follows that $X_y = \operatorname{Spec}(k[t]/(t^2))$ consists of only one point (t), and is a non-reduced scheme.

(3) Let η be the generic point of the scheme S (which corresponds to the zero ideal). Then the residue field of S at η is given by $k[s]_{(0)} = k(s) = R^{-1}k[s]$, with $R = k[s]\setminus\{0\}$. Using the fact that $B \otimes_A R^{-1}A \cong R^{-1}B$, we get

$$\begin{aligned} &X_\eta = \operatorname{Spec}\left(k[s,t]/(s - t^2)\right) \times_{\operatorname{Spec} k[s]} \operatorname{Spec}\left(R^{-1}k[s]\right) = \\ &\operatorname{Spec}\left(k[s,t]/(s - t^2) \otimes_{k[s]} R^{-1}k[s]\right) = \operatorname{Spec}\left(R^{-1}k[s,t]/(s - t^2)\right) = \\ &\operatorname{Spec}\left((k[s] \setminus \{0\})^{-1}k[s][t]/(s - t^2)\right) = \operatorname{Spec}\left(k(s)[t]/(s - t^2)\right). \end{aligned}$$

As $(k(s)[t]/(s-t^2))$ is a field, X_η is a one-point scheme. Next observe that the residue field of η in X_η is $(k(s)[t]/(s-t^2))_{(0)} = k(s)[t]/(s-t^2)$ which is a degree two extension of the field $k(s)$. Hence, we can conclude that the residue field of η in X_η is a field extension of degree two of the residue field of η in S. □

Exercise 8.3. *Let S be a scheme, X a scheme over S, and $p, q : X \times_S X \longrightarrow X$ the two projections. As usual, denote by $\Delta : X \longrightarrow X \times_S X$ the diagonal morphism giving rise to the subset $\Delta(X) \subseteq X \times_S X$. Further, consider the subset*

$$Z := \{z \in X \times_S X \mid p(z) = q(z)\}$$

of $X \times_S X$. Show that the obvious inclusion $\Delta(X) \subseteq Z$ need not be an equality.

Proof. **Solution by Adeel Ahmad Khan**

Consider the affine schemes $X_1 = \operatorname{Spec} k[x]$ and $X_2 = \operatorname{Spec} k[y]$. Between the open subsets $U_1 = \{(x-a) : a \neq 0\} \subset X_1$ and $U_2 = \{(y-b) : b \neq 0\} \subset X_2$ there is a natural isomorphism $\phi : U_1 \to U_2$. Let X be the scheme obtained by gluing X_1 and X_2 along ϕ. Topologically, its points are equivalence classes of the disjoint union of X_1 and X_2, where the point $(x-a)$ is identified with its image $(y-a)$, for $a \neq 0$.

To compute the fiber product of X with itself over $\operatorname{Spec} k$, we follow the construction of Hartshorne of the fiber product of general schemes. We first glue $X_1 \times_{\operatorname{Spec} k} X_1$ and $X_2 \times_{\operatorname{Spec} k} X_1$ by certain open sets to obtain $X \times_{\operatorname{Spec} k} X_1$, and we glue $X_1 \times_{\operatorname{Spec} k} X_2$ and $X_2 \times_{\operatorname{Spec} k} X_2$ to obtain $X \times_{\operatorname{Spec} k} X_2$, and then we glue these together to obtain $X \times_{\operatorname{Spec} k} X$. Observe that $X_1 \times_{\operatorname{Spec} k} X_1 = \operatorname{Spec}(k[x] \otimes k[x]) \cong \operatorname{Spec} k[t_1, t_2]$ and $X_2 \times_k X_1 = \operatorname{Spec}(k[y] \otimes k[x]) \cong \operatorname{Spec} k[t_3, t_4]$.

Let p_1, p_2 be the first projection maps associated with $X_1 \times_{\operatorname{Spec} k} X_1$, $X_2 \times_{\operatorname{Spec} k} X_2$, respectively. Note that p_1 maps the point $\{(x-a), (y-b)\} \in X_1$ to $(x-a) \in X_1$. Now consider the open set $U_1' = p_1^{-1}(U_1) = \{(t_1 - a, t_2 - b) : a \neq 0, b \in k\}$. Similarly let $U_2' = p_2^{-1}(U_2) = \{(t_3 - a, t_4 - b) : a \neq 0, b \in k\}$. Hartshorne shows that the result of gluing $X_1 \times_{\operatorname{Spec} k} X_1$ and $X_2 \times_{\operatorname{Spec} k} X_1$ via the natural isomorphism $\phi' : U_1' \to U_2'$ is the fiber product $X \times_{\operatorname{Spec} k} X_1$. Topologically we find it is the disjoint union of $X_1 \times_{\operatorname{Spec} k} X_1$ and $X_2 \times_{\operatorname{Spec} k} X_1$ with $(t_1 - a, t_2 - b)$ identified with $(t_3 - a, t_4 - b)$ for all $a \neq 0, b \in k$. Analogously we find $X \times_{\operatorname{Spec} k} X_2$ to be the disjoint union of $X_1 \times_{\operatorname{Spec} k} X_2$ and $X_2 \times_{\operatorname{Spec} k} X_2$ with $(t_1 - a, t_2 - b)$ identified with $(t_3 - a, t_4 - b)$ for all $a \in k, b \neq 0$.

Now we glue $X \times_{\operatorname{Spec} k} X_1$ and $X \times_{\operatorname{Spec} k} X_2$. Let q_1 and q_2 be the second projection maps associated with $X \times_{\operatorname{Spec} k} X_1$ and $X \times_{\operatorname{Spec} k} X_2$, respectively. Note that q_1 maps equivalence classes $[(t_1 - a, t_2 - b)]$ and $[(t_3 - a, t_4 - b)]$ to $(x - b) \in X_1$ and similarly q_2 maps them to $(y - b) \in X_2$.

Let $U_1'' = q_1^{-1} U_1$. In this set the point $(t_1 - a, t_2 - b)$ is identified with $(t_3 - a, t_4 - b)$ for $a \neq 0$, $b \neq 0$. So U_1'' consists of equivalence classes $[(t_1 - a, t_2 - b)] = [(t_3 - a, t_2 - b)]$ for $a \neq 0$, $b \neq 0$, and $[(t_1, t_2 - b)]$, $[(t_3, t_4 - b)]$ for $b \neq 0$.

Similarly let $U_2'' = q_2^{-1} U_2$ consists of equivalence classes $[(t_1 - a, t_2 - b)] = [(t_3 - a, t_4 - b)]$ with $a \neq 0, b \neq 0$, along with $[(t_1 - a, t_2)]$ and $[(t_3 - a, t_4)]$ with $a \neq 0$. Thus, we have an isomorphism $\phi'' : U_1'' \to U_2''$ which maps $[(t_1 - a, t_2 - b)] \mapsto [(t_1 - b, t_2 - a)]$ and $[(t_3 - a, t_4 - b)] \mapsto [(t_3 - b, t_4 - a)]$. The result of gluing $X \times_k X_1$ and $X \times_k X_2$ via ϕ'' is the fiber product $X \times_k X$.

Note that we have the points $[(t_1, t_2)] \in X \times_k X_1$ and $[(t_1, t_2)] \in X \times_k X_2$ which are not identified in $X \times_k X$. This implies that we have two distict classes, say $[[(t_1, t_2)]_1], [[(t_1, t_2)]_2] \in X \times_k X$ and similarly $[[(t_3, t_4)]_1], [[(t_3, t_4)]_2] \in X \times_k X$. Now note that the projection maps associated to $X \times_k X$, say p and q, both map $[[(t_1, t_2)]_1]$ and $[[(t_1, t_2)]_2]$ to $[(x)] \in X$, and both map $[[(t_3, t_4)]_1]$ and $[[(t_3, t_4)]_2]$ to $[(y)] \in X$. We therefore find that all four of these points are contained in the set $Z = \{z \in X \times_k X : p(z) = q(z)\}$.

However, we can see that not all of them are in the image of the diagonal morphism $\Delta : X \to X \times_k X$. Observe that the morphism p is given by

$$\operatorname{Spec} k[t_1, t_2] \sqcup \operatorname{Spec} k[t_3, t_4] \sqcup \operatorname{Spec} k[t_1, t_2] \sqcup \operatorname{Spec} k[t_3, t_4] \to \operatorname{Spec} k[x] \sqcup \operatorname{Spec} k[y]$$
$$\text{which maps } (t_1 - a, t_2 - b) \mapsto (x - ab) \text{ and } (t_3 - a, t_4 - b) \mapsto (y - ab);$$

and the map Δ takes $(x - a) \mapsto (t_1 - \sqrt{a}, t_2 - \sqrt{a})$ and $(y - a) \mapsto (t_3 - \sqrt{a}, t_4 - \sqrt{a})$. So one can verify that $p \circ \Delta = \mathrm{Id}_X$ (to be precise we take the image of $(x - a)$ in the first copy of $\operatorname{Spec} k[t_1, t_2]$ and the image of $(y - a)$ in the second copy of $\operatorname{Spec} k[t_3, t_4]$). Therefore the image of Δ contains $[[(t_1, t_2)]_1] = \Delta([(x)])$ but not $[[(t_1, t_2)]_2]$; similarly it contains $[[(t_3, t_4)]_2] = \Delta([(y)])$ but not $[[(t_3, t_4)]_1]$. Hence, we see that in general $\Delta(X) \subsetneq Z$. $\square$

Exercise 8.4. *Prove the following:*

(a) Show that closed immersions are stable under base extension, i.e., if $f : Y \longrightarrow X$ is a closed immersion and if $X' \longrightarrow X$ is any morphism of schemes, then $f' : Y \times_X X' \longrightarrow X'$ is also a closed immersion.

(b) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \longrightarrow X$ factors through Y'.

We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(c) Let $f : Z \longrightarrow X$ be a morphism of schemes. Show that there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \longrightarrow X$ factors through Y' also.
We call Y the scheme-theoretic image of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Proof. **Solution by Adeel Ahmad Khan**

(a)

$$\begin{tikzcd} & Y \times_X X' \arrow[dl, "p"'] \arrow[dr, "f'"] & \\ Y \arrow[dr, "f"'] & & X' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Let $X = \operatorname{Spec} A$ be an affine scheme. Then we must have that Y is affine and the closed immersion $f : Y \to X$ is induced by a surjective homomorphism $\phi : A \to B$, where $Y = \operatorname{Spec} B$. Then $B \cong A/I$ where $I = \ker \phi$. If $X' = \operatorname{Spec} A'$ is also affine, then we have the commuting diagram

$$\begin{tikzcd}
 & A'/I'A' & \\
A' \arrow[ur] & & A/I \arrow[ul] \\
 & A \arrow[ul, "\psi"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where I' denotes the ideal $(\psi(I))$.

It follows that $Y \times_X X' = \operatorname{Spec} B \otimes_A A' = \operatorname{Spec} A'/I'A'$ and the map $f' : Y \times_X X' \to X'$ is induced by the natural surjection $A' \to A'/I'A'$. Thus f' is a closed immersion.

For the general case, we use the following lemma.

Lemma. (1) If $f : Y \to X$ is a closed immersion, then the restriction $f^{-1}(V) \to V$ for any open subset $V \subset X$ is also a closed immersion.

(2) If $f : Y \to X$ is a morphism and $\{V_i\}_i$ is an open cover of Y such that each restriction $f_i : f^{-1}(V_i) \to V_i$ is a closed immersion, then f is a closed immersion.

Proof. The assertions follows from the fact that a closed immersion is local on the target. ☐

Suppose $\{V_i\}_i$ is an affine open cover of X such that $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \to g^{-1}(V_i)$ are closed immersions.

$$\begin{tikzcd}
 & Y \times_X X' \arrow[dl, "p"'] \arrow[dr, "f'"] & \\
Y \arrow[dr, "f"'] & & X' \arrow[dl, "g"] \\
 & X &
\end{tikzcd}$$

$$\begin{tikzcd}
 & f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \arrow[dl] \arrow[dr] & \\
f^{-1}(V_i) \arrow[dr, "f"'] & & g^{-1}(V_i) \arrow[dl, "g"] \\
 & V_i &
\end{tikzcd}$$

Note that these maps are just the restrictions f' and g to open sets $f'^{-1}(g^{-1}(V_i))$ and $g^{-1}(V_i)$, respectively. Furthermore, the open sets $\{g^{-1}(V_i)\}_i$ cover X'. By the above lemma it follows that f' is a closed immersion.

Next, suppose X affine and let $\{V_i'\}_i$ be any affine open cover of X'.

$$\begin{tikzcd} & Y \times_X V_i' \arrow[dl] \arrow[dr] & \\ Y \arrow[dr, "f"'] & & V_i' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Since $Y \times_X V_i' \cong f'^{-1}(V_i')$, the second projection map $f'^{-1}(V_i') \to V_i'$ is a closed immersion by the affine case demonstrated at the beginning. By the lemma it then follows that f' is also a closed immersion.

Finally, suppose X is an arbitrary scheme and $\{V_i\}_i$ is an affine open cover for X. By above, $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \to g^{-1}(V_i)$ is a closed immersion for each i, and therefore it follows that f' is a closed immersion.

(b) Suppose X is affine with $X = \operatorname{Spec} A$. Then since $f : Y \to X$ is a closed immersion, Y is affine ($Y = \operatorname{Spec} B$) and there is a surjective homomorphism $\phi : A \to B$ that induces f. Therefore we can write $Y = \operatorname{Spec} A/I$, where $I = \ker \phi$. We must also have Y' affine, say $Y' = \operatorname{Spec} B' = \operatorname{Spec} A/J$ for some ideal $J \subseteq A$. Topologically, $Y = Y'$ which implies that $\operatorname{Spec} A/I = \operatorname{Spec} A/J$ from which we conclude that $\operatorname{Rad}(I) = \operatorname{Rad}(J)$. But since Y is reduced, $I = \operatorname{Rad}(I)$, so $I = \operatorname{Rad}(J)$. Then $A \to A/I$ factors as $A \to A/J \to A/\operatorname{Rad}(J) = A/I$ and correspondingly $Y \to X$ factors as $Y \to Y' \to X$.

For X an arbitrary scheme, let $\{V_i\}_i$ be an affine open cover of X. Give each $f^{-1}(V_i)$ the induced reduced subscheme structure associated to Y, and let $f_i : f^{-1}(V_i) \to V_i$, be the restriction of f to $f^{-1}(V_i)$. Observe that $\{f^{-1}(V_i)\}_i$ is an affine cover for Y. As Y' is homemorphic to Y, we can find an affine open cover $\{U_i\}_i$ of Y' such that U_i is homeomorphic to $f^{-1}(V_i)$, and f_i factors through U_i for each i.

Gluing these morphisms together, we derive that $f : Y \to X$ factors through Y'.

(c) First assume X is affine, $X = \operatorname{Spec} A$. Then $f : Z \to X$ is induced by a homomorphism $\phi : A \to \mathcal{O}_Z(Z)$. Consider the closed immersion $g : \operatorname{Spec} A/\ker\phi \to \operatorname{Spec} A$ induced by the natural projection $A \to A/\ker\phi$. It is clear that $f : Z \to \operatorname{Spec} A$ factors through $\operatorname{Spec} A/\ker\phi$, and if f factors through another closed immersion $\operatorname{Spec} A/I \to \operatorname{Spec} A$, then we must have $I \subset \ker\phi$. This implies that g also factors through $\operatorname{Spec} A/I \to \operatorname{Spec} A$. It follows that $\operatorname{Spec} A/\ker\phi$ is the scheme-theoretic image of f.

Finally let X be an arbitrary scheme. Let $\{V_i\}_i$ be an open cover of X. Let $g_i : Y_i \to V_i$ be the scheme-theoretic images of $f|_{f^{-1}(V_i)}$, given by the above. Then, also by the above, we get the scheme-theoretic image of $f^{-1}(V_i \cap V_j) \to V_i \cap V_j$ to be both $g_i^{-1}(V_i \cap V_j)$ and $g_j^{-1}(V_i \cap V_j)$. By uniqueness there must be an isomorphism $\phi_{ij} : g_i^{-1}(V_i \cap V_j) \to g_j^{-1}(V_i \cap V_j)$ for each i, j. Then we can glue the Y_i's together and the g_i's together along ϕ_{ij} to get a scheme Y and a unique morphism $g : Y \to X$, respectively. One can verify that this is the scheme-theoretic image of f. $\square$

9 Solutions for Exercise Sheet-9

Remark. The soltuions to this exercise have also not been double checked as of yet, due to lack of time. However the solutions seem accurate, and we have put them up online so as to assist the students in preparing for the final exam.

For the convenience of the reader, Balthasar Grabmayr and Marie Sophie Litz, the students who solved this exercise sheet would like to recall a few definitions and their equivalent reformulations.

Definition. A morphism $f : X \to Y$ is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

Definition. A morphism $f : X \to Y$ is *of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered **by a finite number** of open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra; equivalently if the statement holds for **every** open affine subset $V = \operatorname{Spec} B$ of Y; equivalently if f is locally of finite type and quasi-compact (by exercise II.3.3 in Hartshorne).

Definition. A scheme is locally noetherian if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a noetherian ring.

Definition. A scheme is noetherian if it is locally noetherian and $\operatorname{sp}(X)$ is quasi-compact.

Definition. A morphism $f : X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affine V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Equivalently, if for **every** open affine subset $V \subset Y$, $f^{-1}(V)$ is quasi-compact (by exercise II.3.2 Hartshorne).

Definition. A *closed immersion* is a morphism $f : X \to Y$ of schemes such that f induces a homeomorphism of $\operatorname{sp}(X)$ onto a closed subset of $\operatorname{sp}(Y)$, and furthermore the induced map $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y is surjective.

Definition. Let S be a fixed scheme (base scheme) and S' another base scheme, if $S' \to S$ is a morphism, then for any scheme X over S, we define $X' = X \times_S S'$ which is a scheme over S'. We say X' is obtained from X by making a *base extension* $S' \to S$. One says a property $\mathcal{P}$ is stable under base extension if the following holds: For every $f : X \to S$ with the property $\mathcal{P}$ and every base extension $S' \to S$, the induced scheme $f' : X' \to S'$ also has this property.

Exercise 9.1. *A morphism $f : X \longrightarrow Y$ of schemes is locally of finite type, if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec}(B_i)$ such that for each i, the open subschema $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec}(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra.*
The morphism $f : X \longrightarrow Y$ is of finite type, if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the open affine subsets U_{ij}. We say that X is of finite type over Y.
Prove the following assertions:

(a) A closed immersion is of finite type.

(b) A quasi-compact open immersion is of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

(f) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, if $g \circ f$ is of finite type, and if f is quasi-compact, then f is of finite type.

(g) If $f : X \longrightarrow Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Proof. **Solution by Balthasar Grabmayr and Marie Sophie Litz**

(a) Let $f\colon X \to Y$ be a closed immersion. Take an open affine cover $\{U_i\}_i$ of Y with $U_i = \operatorname{Spec} A_i$. The restriction $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is still a closed immersion, so it follows from Exercise 3.11 in Hartshorne that $f^{-1}(U_i)$ is affine, say $f^{-1}(U_i) = \operatorname{Spec} B_i$. Since the morphism $f^\flat$ of structure sheaves is surjective, the morphism at the level of stalks $(A_i)_\mathfrak{p} \to (B_i)_{\phi^{-1}(\mathfrak{p})}$ is surjective for every prime $\mathfrak{p} \in \operatorname{Spec} A_i$ ($\phi\colon B_i \to A_i$). From this we conclude that $A_i \to B_i$ is surjective. Hence, each B_i is a finitely generated A_i-algebra.

(b) Let $f\colon X \to Y$ be a quasi-compact open immersion and let $\{V_i\}_i$ be an open affine cover of Y with $V_i = \operatorname{Spec} B_i$. Then $f^{-1}(V_i)$ is quasi-compact for each i (see Exercise 3.2. of Hartshorne). Since f is an open immersion, it induces a homeomorphism $X \cong U$, where $U \subseteq Y$ an open subset. So $f^{-1}(V_i \cap U) = f^{-1}(V_i)$. Since $V_i \cap U$ is an open subset of V_i, we can write $V_i \cap U = \bigcup_\alpha D(f_\alpha)$. Let $W_\alpha = f^{-1}(D(f_\alpha))$. As f is a homeomorphism onto U, so we have $W_\alpha \cong \operatorname{Spec}(B_{i\,f_\alpha})$. Since $f^{-1}(V_i) = f^{-1}\left(\bigcup_\alpha D(f_\alpha)\right) = \bigcup_\alpha f^{-1}(D(f_\alpha)) = \bigcup_\alpha W_\alpha$, $\{W_\alpha\}$ is an open affine cover of $f^{-1}(V_i)$. Since $f^{-1}(V_i)$ is quasi-compact we can take a finite subcover $f^{-1}(V_i) = \bigcup_j W_j$. As each $(B_i)_{f_j}$ is a finitely generated B_i-algebra (e.g. generated by 1 and $\frac{1}{f_j}$), we can conclude that f is an open immersion.

(c) Let be $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of finite type. Let $\{V_i\}_i$ be an open affine cover of Z with $V_i = \operatorname{Spec} B_i$. By Ex. 3.3 of Hartshorne $g^{-1}(V_i)$ can be covered by finitely many open affine set $W_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. Since f is of finite type, $f^{-1}(W_{ij})$ can be covered by finitely many open affine sets $U_{ijk} = \operatorname{Spec} C_{ijk}$, where each C_{ijk} is a finitely generated A_{ij}-algebra. So $(g \circ f)^{-1}(V_i) = \bigcup_{j,k} U_{ijk}$, and C_{ijk} is a finitely generated A_{ij}-algebra which in turn is a finitely generated B_i-algebra. Hence, we can deduce that C_{ijk} is a finitely generated B_i-algebra, which implies that the morphism $g \circ f$ is of finite type.

(d) Suppose that $f\colon X \to S$ is a morphism of finite type and let $g : S' \to S$ be a base extension. We have to show that $q\colon X \times_S S' \to S'$ is of finite type.

First let us assume that X, S and S' are affine. Let $X = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(C)$. Since $X \to S$ is of finite type, A is a finitely generated C-algebra, so $A \otimes_C B$ is a finitely generated B-alebra, using which we deduce that q is of finite type.

If S, S' are affine then the claim holds true, since a finite open affine cover $U_i \subseteq X$ leads to a finite open affine cover $\{U_i \times_S S'\}_i$ of $X \times_S S'$. As we have just noted, if U_i is of finite type over S, then $U_i \times_S S'$ is of finite type over S'.

Now suppose that just S is affine and let $\{V_i\}_i$ be an open affine cover of S'. Then each $X \times_S V_i$ is of finite type over V_i. Since $\{V_i\}_i$ cover S' and $X \times_S V_i$ is the preimage of V_i in the map $X \times_S V_i \to V_i$, we see that $X \times_S S'$ is of finite type over S'.

So the only case left is when S is not affine. In this situation, take an open affine cover $\{U\}_i$ of S with $U_i = \operatorname{Spec} A_i$, and let $S'_i = g^{-1}(U_i)$ and $X_i = f^{-1}(U_i)$. From the above considerations we see that $X_i \times_{U_i} S'_i$ is of finite type over S'_i. But this is the same morphism as $X \times_S S'_i \to S'$ and so we have found an open cover on which q is of finite type.

(e) $X \times_S Y = \{(x,y) \in X \times Y | f(x) = g(y)\}$, so we can consider $X \times_S Y \to S$ as the composition $X \times_S Y \xrightarrow{\pi_Y} Y \xrightarrow{g} S$. The first map (the projection on the second coordinate) is a base extension of $f : X \to S$ and therefore of finite type by 9.1 (d). As g is of finite type, using 9.1 (c), we can conclude that the composition $q = \pi_Y \circ g$ is of finite type.

Alternative proof: Take an affine open cover $\{\operatorname{Spec} A_i\}$ of S. Since f and g are of finite type over S, using exercise II.3.3 in Hartshorne, we have that for every i the preimages $f^{-1}(\operatorname{Spec} A_i)$ and $g^{-1}(\operatorname{Spec} A_i)$ can be covered by a finite number of open affine subsets $\operatorname{Spec} F_{ij}$ and $\operatorname{Spec} G_{ik}$, respectively. Here each of the rings F_{ij}, G_{ik} are finitely generated A_i-algebras.

From the proof of the uniqueness of the fibre product, we know that $\operatorname{Spec} F_{ij} \times_S \operatorname{Spec} G_{ik} \cong \operatorname{Spec}(F_{ij} \otimes_{A_i} G_{ik})$. As F_{ij}, G_{ik} are finitely generated A_i-algebras, the fiber product $\operatorname{Spec} F_{ij} \times_S \operatorname{Spec} G_{ik}$ is of finite type over $\operatorname{Spec} A_i$. Since $\operatorname{Spec} F_{ij} \times_S \operatorname{Spec} G_{ik}$ form an open affine cover of $X \times_S Y$, therefore $X \times_S Y$ is of finite type over S.

(f) By exercise II.3.3 in Hartshorne, we need to show that f is locally of finite type (as f is quasi-compact). Take any open affine cover $\{\operatorname{Spec} C_i\}$ of Z. As $g \circ f$ is of finite type, we get a finite affine open cover $\{\operatorname{Spec} A_{ij}\}$ of the preimage $(g \circ f)^{-1}(\operatorname{Spec} C_i)$ for all i. Let $\{\operatorname{Spec} B_{ik}\}$ be an open affine cover of $g^{-1}(\operatorname{Spec} C_i)$.

Recall that an affine scheme $\operatorname{Spec} A$ can be covered by the principal open sets $\big\{D(a_l) = \operatorname{Spec} A_{a_l}\big\}$ with $a_l \in A$. So for each i, k we get an open covering of $f^{-1} \operatorname{Spec} B_{ik}$ of the form $\{\operatorname{Spec}((Aij)_{a_ijl})\}$ which gives us a sequence of ring homomorphisms $C_i \to B_{ik} \to (A_{ij})_{a_{ijl}}$. The local rings $(A_{ij})_{a_{ijl}}$ are finitely generated C_i-algebras, generated by the generators of the C_i-algebra A_{ij} (which are finitely many because $g \circ f$ is of finite type) and one more element, namely $\frac{1}{a_{ijl}}$.

In particular there are finitely generated B_{ik}-algebras (with the same set of generators). Thus we conclude that f is of finite type.

(g) Let $\{\text{Spec } B_i\}$ be a finite affine open cover of Y (which we get because Y is noetherian) with B_i noetherian for every i. Let $\{\text{Spec } A_{ij}\}_j$ be an finite affine open cover of $f^{-1}\text{Spec } B_i$ with A_{ij} a finitely generated B_i-algebra for all j. By a corollary of Hilbert's Basis Theorem, we see that the A_{ij} are noetherian. Since $\{\text{Spec } A_{ij}\}_{ij}$ is a finite open affine cover of X, it follows that X is noetherian. $\square$

Exercise 9.2. *Assume that all the schemes in the subsequent statements are noetherian. Under this hypothesis, prove the following assertions:*

(a) A closed immersion is proper.

(b) A composition of two proper morphisms is proper.

(c) Proper morphisms are stable under base extension.

(d) If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are proper morphisms of S-schemes, then $f \times f' : X \times_S X' \longrightarrow Y \times_S Y'$ is also proper.

(e) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, if $g \circ f$ is proper, and if g is separated, then f is proper.*

(f) *A morphism $f : X \longrightarrow Y$ is proper, if and only if Y can be covered by open subschemes V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for all i. We say that properness is local on the base.*

Proof. **Solution by Balthasar Grabmayr and Marie Sophie Litz**

(a) From Exercise 9.1 (a) a closed immersion is of finite type. We also know that a closed immersion is separated (by corollary in lecture class). It remains to show that a closed immersion is universally closed.

From Exercise 3.11 (a) in Hartshorne, we know that closed immersions are stable under base extension. Hence, we can conclude that a closed immersion is proper.

(b) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be proper morphisms. Since both are of finite type, also the composition is of finite type (by Ex. 9.1. (b)). Now we can apply the valuation criterion for being proper:

Let R be any valuation ring and K its quotient field. Set $U = \operatorname{Spec}(R)$ and $T = \operatorname{Spec}(K)$ and let $\alpha\colon T \to X$ and $\beta\colon U \to Y$ be morphisms such that the diagram

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\ U \arrow[r, "\beta"] & Y \end{tikzcd}$$

commutes. Since g is proper, there exists an unique morphism $\theta_g\colon U \to Y$ such that the diagram

$$\begin{tikzcd} T \arrow[r, "\alpha \circ f"] \arrow[d, hook] & Y \arrow[d, "g"] \\ U \arrow[r, "g \circ \beta"] \arrow[ur, dashed, "\exists! \theta_g"] & Z \end{tikzcd}$$

commutes. Now consider the following commutative diagram.

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\ U \arrow[r, "\theta_g"] & Y \end{tikzcd}$$

Since f is proper there exists an unique morphism $\theta_f\colon U \to X$ such that the whole diagram commutes. All in all we get the commuative diagram

$$
\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "g \circ f"] \\
U \arrow[r, "g \circ \beta"] \arrow[ur, dashed, "\exists! \theta_f"] & Z
\end{tikzcd}
$$

and by valuation criterion $g \circ f$ is proper.

(c) Let $f \colon X \to S$ be proper morphism and let $S' \to S$ be a morphism. We have to show that $f' \colon X \times_S S' \to S'$ is proper.

By Ex. 9.1. (d) f' is of finite type and we apply again the valuation criterion for being proper. Let R be any valuation ring and K its quotient field. Set $U = \mathrm{Spec}(R)$ and $T = \mathrm{Spec}(K)$ and let $\alpha \colon T \to X$ and $\beta \colon U \to Y$ be morphisms such that the diagram

$$
\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\
U \arrow[r, "\beta"] & Y
\end{tikzcd}
$$

commutes. Since f is proper, there exists an unique morphism $\theta_f \colon U \to X$ such that the diagram

$$
\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook] & X \times_S S' \arrow[r, "p"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
U \arrow[r, "\beta"] \arrow[urr, dashed, "\exists! \theta_f"] & S' \arrow[r] & S
\end{tikzcd}
\tag{9}
$$

commutes. Now consider the following commutative diagram.

$$
\begin{tikzcd}
U \arrow[r, "\beta"] \arrow[d, "\theta_f"] & S' \arrow[d] \\
X \arrow[r, "f"] & S
\end{tikzcd}
$$

By the universal property of the fiber product $X \times_S S'$ there exists an unique morphism $U \to X \times_S S'$ such that the diagram (9) commutes, which finishes the proof.

(d) The morphism $f \times f'$ is obviously of finite type. As $X \times_S X'$ can be considered as a subset of $X \times X'$, it is therefore noetherian. So we can use the valuation criterion for properness.

Given the following commutative diagram, we want to find the unique morphism $\theta^* : U \to X \times_S X'$ that makes the whole diagramm commutative. The maps α^* and β^* can be seen as tuples of maps $\alpha : T \to X$, $\alpha' : T \to X'$ and $\beta : U \to Y$, $\beta' : U \to Y'$, which induces two commutative diagrams with f and f'. Due to properness of f and f', there exist two unique morphisms $\theta : U \to X$ and $\theta : U \to X$, respectively, which make the diagrams commutative. Hence, from the universal property of the fiber product, there exists a a unique morphism $\theta^* : U \to X \times_S X'$ that makes the first diagram commutative.

$$\begin{tikzcd}
T \arrow[r, "\alpha^*"] \arrow[d, hook, "i"'] & X \times_S X' \arrow[d, "f \times f'"] \\
U \arrow[r, "\beta^*"'] \arrow[ur, dashed, "\exists!\theta^*"] & Y \times_S Y'
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
U \arrow[r, "\beta"'] \arrow[ur, dashed, "\exists!\theta"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & X' \arrow[d, "f'"] \\
U \arrow[r, "\beta'"'] \arrow[ur, dashed, "\exists!\theta'"] & Y'
\end{tikzcd}$$

(e) X is noetherian, so in particular, it is quasi-compact. So every morphism starting in X is quasi-compact. Using 9.1 (f) we see that f is of finite type. We use the valuation criterion. Given some commutative diagram for f, we want to find the unique morphism $\theta : U \to X$ that makes the whole diagram commutative.

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
U \arrow[r, "\beta"'] \arrow[ur, dashed, "\exists!\theta"] & Y
\end{tikzcd}$$

Look at the corresponding commutative diagram for $g \circ f$. Due to properness, there exists a unique morphism $\theta : U \to X$ which makes the diagram commutative. This induces a diagram for g. There exist two maps from U to X which make the diagram commutative, namely $f \circ \theta$ and β. By seperatedness of g, wo conclude $f \circ \theta = \beta$. Now we see that θ makes the first diagram commutative. In pictures:

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "g \circ f"] \\
U \arrow[r, "g \circ \beta"'] \arrow[ur, dashed, "\exists!\theta"] & Z
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & Y \arrow[d, "g"] \\
U \arrow[r, "g \circ \beta"'] \arrow[ur, bend left, "\beta"] \arrow[ur, bend right, "f \circ \theta"'] & Z
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
U \arrow[r, "f \circ \theta = \beta"'] \arrow[ur, "\theta"] & Y
\end{tikzcd}$$

(f) The statement holds for seperatedness (by Cor. 4.6), so we only need to check the properties "universally closed" and "of finite type".

$\Leftarrow$ The scheme X is noetherian, therefore quasi-compact, so f is quasi-compact. The right hand-side gives that f is locally of finite type, so it is of finite type by Exercise II.3.3. f is also universally closed which we see by the following argument:

Take a finite covering of Y by open subschemes $\{V_i\}$. Let $f' : X' \to Y'$ be obtained by base extension. Let $U' \subseteq X'$ be closed in $sp(X')$. Then

$$f'\Big|_{f'^{-1}(V_i')} : f'^{-1}(V_i') \to V_i'$$

is closed in Y. So in particular

$$f'\Big|_{f'^{-1}(V_i')}(U' \cap f'^{-1}(V_i')$$

is closed. The union of these sets (over finite i) is $f'(U')$ and it is again closed.

$\Rightarrow$ Trivial. $\square$

Exercise 9.3. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring and $M = \bigoplus_{d \in \mathbb{Z}} M_d$ a graded A-module. Show that the following three conditions for a submodule $N \subseteq M$ are equivalent:*

(i) $N = \bigoplus_{d \in \mathbb{Z}} (N \cap M_d)$.

(ii) N is generated by homogeneous elements of M.

(iii) For all $n \in N$, all its homogeneous components belong to N.

We say that the submodule N of M is homogeneous.

Proof. **Solution by Balthasar Grabmayr and Marie Sophie Litz**

(i) $\Rightarrow$ *(ii)* Each basis element generates a $N \cap M_d$, so all generators of N are homogeneous.

(ii) $\Rightarrow$ *(iii)* Let $n \in N \subseteq M = \bigoplus M_d$, so n can be uniquely written as a sum of homogeneous components $m_d \in M_d$. It can also be generated by homogeneous elements $\{n_i\}$ in N. So grouping together the elements of same degree, we get equations $m_d = \sum n_{i_d}$. So each homogeneous component is generated by those $n_{i_d} \in N$ and therefore lies in n itself.

(iii) $\Rightarrow$ *(i)* $N \cap M_d \subseteq N$ for all d, so $\bigoplus_{d \in \mathbb{Z}} (N \cap M_d) \subseteq N$. The other inclusion follows directly from *(iii)*. $\square$

Exercise 9.4. *Let A be a graded ring. Prove the following assertions:*

(a) Let $\mathfrak{p}, \mathfrak{p}' \subseteq A$ be relevant prime ideals. If $\mathfrak{p}_+ = \mathfrak{p}'_+$, then $\mathfrak{p} = \mathfrak{p}'$. A homogeneous ideal $\mathfrak{a} \subsetneq A_+$ is of the form $\mathfrak{p}_+$ for some relevant prime ideal $\mathfrak{p}$ of A if and only if for all homogeneous elements $a, b \in A_+ \setminus \mathfrak{a}$ one has $ab \notin \mathfrak{a}$.

(b) Let $S \subseteq A$ be a multiplicative set. Then, the set of homogeneous ideals $\mathfrak{a} \subsetneq A_+$ with $S \cap \mathfrak{a} = \emptyset$ has maximal elements and each such maximal element is of the form $\mathfrak{p}_+$ for a relevant prime ideal $\mathfrak{p}$.

(c) Let $\mathfrak{a} \subseteq A_+$ be a homogeneous ideal. Then, $\sqrt{\mathfrak{a}}_+ = \sqrt{\mathfrak{a}} \cap A_+$ is again a homogeneous ideal. Moreover, $\sqrt{\mathfrak{a}}_+$ is the intersection of A_+ with all relevant prime ideals containing $\mathfrak{a}$.

Proof. **Solution by Balthasar Grabmayr and Marie Sophie Litz**

We suggest that the reader also looks up at Görtz/Wedhorn.

(a) Recall that a homogeneous prime ideal $\mathfrak{p} \subseteq A$ is called relevant if it does not contain A_+, i.e. if $\mathfrak{p}_+ \subsetneq A_+$.

Let $\mathfrak{a} \subsetneq A_+$ be a homogeneous ideal and let $f \in A_+ \setminus \mathfrak{a}$. If $\mathfrak{a}$ is of the form $\mathfrak{p}_+$ we have

$$\mathfrak{p}_0 = \{a \in A_0 \mid a \cdot f^r \in \mathfrak{a}_{r \cdot \deg f} \text{ for all } r \geq 1\}.$$

This shows the uniqueness statement in (a).

It remains to show that $\tilde{\mathfrak{a}} = \mathfrak{p}_0 \oplus \mathfrak{a}$ is a prime ideal. It is clear that $\tilde{\mathfrak{a}}$ is an ideal. Let $g, g' \in A \setminus \tilde{\mathfrak{a}}$. Write g and g' as a sum of homogeneous elements:

$$g = g_0 + \cdots + g_h \quad \text{and} \quad g' = g'_0 + \cdots + g'_{h'}.$$

Since $\tilde{\mathfrak{a}}$ is homogeneous, it suffices to show that $g_h \cdot g'_h \notin \tilde{\mathfrak{a}}$. If $h \neq 0$, $h' \neq 0$, this follows from the hypothesis. If $h = 0$ (resp. $h' = 0$) we multiply g_h (resp. $g'_{h'}$) with a power of f and again use the hypothesis.

To show the converse let $a, b \in A_+ \setminus \mathfrak{a}$, so $a, b \notin \mathfrak{a}$. Assume $a \cdot b \in \mathfrak{a} = \mathfrak{p}_+ \subseteq \mathfrak{p}$. Then $a \in \mathfrak{p} \setminus \mathfrak{p}_+$ or $b \in \mathfrak{p} \setminus \mathfrak{p}_+$ (since $a, b \notin \mathfrak{p}_+$), which leads to a contradiction (note that $\deg a \neq 0$ and $\deg b \neq 0$).

(b) The existence of maximal elements follows from Zorn's lemma. We now use the equivalence of (a). Let be $a, b \in A_+ \setminus \mathfrak{a}$, so $a, b \notin \mathfrak{a}$. Hence, $\mathfrak{a} \subsetneq \mathfrak{a} + (a)$ and $\mathfrak{a} \subsetneq \mathfrak{a} + (b)$, so by maximality of $\mathfrak{a}$ we get

$$(\mathfrak{a} + (a)) \cap S \neq \emptyset \neq (\mathfrak{a} + (b)) \cap S.$$

Say $t = am + p$, $t' = bn + q$ with $t, t' \in S$ and $p, q \in \mathfrak{a}$. We have $tt' \in S$. If $ab \in \mathfrak{a}$, then $tt' = ambn + amp + pbn + pq \in \mathfrak{a}$, which is not possible because $S \cap \mathfrak{a} = \emptyset$. Thus $ab \notin \mathfrak{a}$.

(c) It suffices to show that $\sqrt{\mathfrak{a}}$ is the intersection $\mathfrak{p}$ of all relevant prime ideals containing $\mathfrak{a}$ (then $\sqrt{\mathfrak{a}}$ and hence $\sqrt{\mathfrak{a}}_+$ are homogeneous). We replace A by $A/\mathfrak{a}$ and can therefore assume $\mathfrak{a} = 0$. Clearly we have $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} \subseteq \mathfrak{p}$. Conversely, if $f \notin \sqrt{\mathfrak{a}}$, then an ideal maximal among those properly contained in A_+ and not meeting $\{1, f, f^2, \dots\}$ is a relevant prime ideal by (b). Thus f is not contained in the intersection of all relevant prime ideals. $\square$

10 Solutions for Exercise Sheet-10

Remark. The soltuions to this exercise have also not been double checked as of yet, due to lack of time. However the solutions seem accurate, and we have put them up online so as to assist the students in preparing for the final exam.

Notation In the following, let $A = \bigoplus_{d\geq 0} A_d$ be a graded ring and denote by $\operatorname{Proj} A$ the set of relevant ideals of A. We denote by A_+ the homogeneous ideal $\bigoplus_{d>0} A_d$ and for a homogeneous ideal $\mathfrak{a} \subset A$ we set $\mathfrak{a}_+ := \mathfrak{a} \cap A_+$. All rings considered shall be commutative and unitary.

Exercise 10.1. *Let $A = \bigoplus\limits_{d\geq 0} A_d$ be a graded ring and $\operatorname{Proj}(A)$ the set of relevant homogeneous prime ideals of A. For a homogeneous ideal $\mathfrak{a} \subseteq A$, let*

$$V_+(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj}(A) \,|\, \mathfrak{p} \supseteq \mathfrak{a}\}$$

denote the set of all relevant homogeneous prime ideals of A containing $\mathfrak{a}$.

(a) Show that the sets $V_+(\mathfrak{a})$, where $\mathfrak{a}$ ranges over the homogeneous ideals of A, satisfy the axioms for closed sets in $\operatorname{Proj}(A)$.

(b) Show that $V(\mathfrak{a}) \cap \operatorname{Proj}(A) = V_+(\mathfrak{a}^h)$, where $\mathfrak{a}^h$ is the homogeneous ideal generated by $\mathfrak{a}$. This shows that $\operatorname{Proj}(A)$ carries the topology induced from $\operatorname{Spec}(A)$.

Proof. **Solution by Isabel Müller and Robert Rauch**

(a) Clearly $V_+(0) = \operatorname{Proj} A$ and, since $\mathfrak{p} \not\supset A_+$ for all $\mathfrak{p} \in \operatorname{Proj} A$, we have $V_+(A_+) = \emptyset$. If $(I_i)_i$ is a family of homogeneous ideals in A, then $\sum_i I_i$ is homogeneous by Exercise 9.3 and for $\mathfrak{p} \in \operatorname{Proj} A$, we have

$$\mathfrak{p} \in \bigcap_i V_+(I_i) \Leftrightarrow \mathfrak{p} \supset \bigcup_i I_i \Leftrightarrow \mathfrak{p} \in V_+\left(\sum_i I_i\right), \quad \text{i.e.} \bigcap_i V_+(I_i) = V_+\left(\sum_i I_i\right). \tag{10}$$

Finally, if $I, J \subset A$ are homogeneous ideals then we claim that

$$V_+(I) \cup V_+(J) = V_+(I \cap J).$$

Indeed: if $\mathfrak{p} \in V_+(I) \cup V_+(J)$, then either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, in any case $I \cap J \subset \mathfrak{p}$, thus $\mathfrak{p} \in V_+(I \cap J)$. On the other hand, if $I \cap J \subset \mathfrak{p}$ and $I \not\subset \mathfrak{p}$, then we may fix some $a \in I \setminus \mathfrak{p}$ and for any $b \in J$ we have $ab \in I \cap J \subset \mathfrak{p}$, hence $b \in \mathfrak{p}$ because $\mathfrak{p}$ is prime and $a \notin \mathfrak{p}$.

Remark. As pointed out during the exercise class, what we have just proved follows trivially from the affine case, since $V_+(\mathfrak{a}) = \operatorname{Proj} A \cap V(\mathfrak{a})$ for all (homogeneous) ideals $\mathfrak{a} \subset A$.

(b) If $\mathfrak{p} \in V_+(\mathfrak{a}^h)$, then clearly $\mathfrak{p} \in \operatorname{Proj} A$ and $\mathfrak{p} \supset \mathfrak{a}^h \supset \mathfrak{a}$, thus $\mathfrak{p} \in V(\mathfrak{a}) \cap \operatorname{Proj} A$. Conversely, $\mathfrak{p} \in V(\mathfrak{a}) \cap \operatorname{Proj} A$ means that $\mathfrak{p}$ is relevant and $\mathfrak{p} \supset \mathfrak{a}$, hence $\mathfrak{p}^h \supset \mathfrak{a}^h$. But $\mathfrak{p}$ is homogeneous, thus also $\mathfrak{p} = \mathfrak{p}^h$, i.e. $\mathfrak{p} \in V_+(\mathfrak{a}^h)$. $\square$

Exercise 10.2. *Let $A = \bigoplus\limits_{d\geq 0} A_d$ be a graded ring, $A_+ := \bigoplus\limits_{d\geq 1} A_d$, and $\mathfrak{a}_+ := \mathfrak{a} \cap A_+$ for a homogeneous ideal $\mathfrak{a} \subseteq A$. Further, for a subset $Y \subseteq \operatorname{Proj}(A)$, define*

$$I_+(Y) := \Big(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\Big) \cap A_+.$$

Prove the following assertions:

(a) *If* $\mathfrak{a} \subseteq A_+$ *is a homogeneous ideal, then* $I_+(V_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}_+$. *If* $Y \subseteq \operatorname{Proj}(A)$ *is a subset, then* $V_+(I_+(Y)) = \overline{Y}$.

(b) *The maps*

$$Y \mapsto I_+(Y) \quad \text{and} \quad \mathfrak{a} \mapsto V_+(\mathfrak{a})$$

define mutually inverse, inclusion reversing bijections between the set of homogeneous ideals $\mathfrak{a} \subseteq A_+$ *such that* $\mathfrak{a} = \sqrt{\mathfrak{a}}_+$ *and the set of closed subsets of* $\operatorname{Proj}(A)$. *Via this bijection, the closed irreducible subsets correspond to ideals of the form* $\mathfrak{p}_+$, *where* $\mathfrak{p}$ *is a relevant prime ideal.*

(c) *If* $\mathfrak{a} \subseteq A_+$ *is a homogeneous ideal, then* $V_+(\mathfrak{a}) = \emptyset$ *if and only if* $\sqrt{\mathfrak{a}}_+ = A_+$. *In particular,* $\operatorname{Proj}(A) = \emptyset$ *if and only if every element in* A_+ *is nilpotent.*

(d) *The sets*

$$D_+(f) := \operatorname{Proj}(A) \setminus V_+(f)$$

for homogeneous elements $f \in A_+$ *form a basis of the topology of* $\operatorname{Proj}(A)$.

(e) *Let* $(f_i)_i$ *be a family of homogeneous elements* $f_i \in A_+$ *and let* $\mathfrak{a}$ *be the ideal generated by the* f_i. *Then, we have*

$$\bigcup_i D_+(f_i) = \operatorname{Proj}(A) \Longleftrightarrow \sqrt{\mathfrak{a}}_+ = A_+ \,.$$

Proof. **Solution by Isabel Müller and Robert Rauch**

(a) The first statement is a consequence of Exercise 9.3, since for any homogeneous ideal $\mathfrak{a} \subset A_+$, we have

$$\sqrt{\mathfrak{a}}_+ \overset{\text{Ex. 9.3}}{=} A_+ \cap \bigcap V_+(\mathfrak{a}) \overset{\text{def.}}{=} I_+(V_+(\mathfrak{a})).$$

To see the second statement, first note that for any homogeneous ideal $\mathfrak{a} \subset A$ and any subset $Y \subset \operatorname{Proj} A$, we have

$$\mathfrak{a} \subset \bigcap \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Proj} A, \mathfrak{a} \subset \mathfrak{p}\} = I_+(V_+(\mathfrak{a})) \tag{11}$$

$$Y \subset \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Proj} A, \bigcap \{\mathfrak{q} \mid \mathfrak{q} \in Y\} \subset \mathfrak{p}\} = V_+(I_+(Y)).$$

Therefore $\overline{Y} \subset V_+(I_+(Y))$, as $V_+(I_+(Y))$ is closed by definition. Conversely, let $V_+(\mathfrak{a}) \subset \operatorname{Proj} A$ be any closed set containing Y. Applying I_+ on this inclusion and using part (b) gives $I_+(V_+(\mathfrak{a})) \subset I_+(Y)$, hence $\mathfrak{a} \subset I_+(Y)$ by (11). Applying V_+ on this inclusion therefore yields

$$V_+(\mathfrak{a}) \supset V_+(I_+(Y)),$$

which means that $V_+(I_+(Y))$ is the smallest closed set containing Y, i.e. $V_+(I_+(Y)) = \overline{Y}$.

(b) Well-definedness: For any $\mathfrak{a} \in \operatorname{Rad}(A)$ the set $V_+(\mathfrak{a})$ is closed by definition. Conversely, assume $Y \subset \operatorname{Proj} A$ to be closed. We have to show $I_+(Y) \in \operatorname{Rad}(A)$, i.e. $I_+(Y) \subset A_+$ is homogeneous and $\sqrt{I_+(Y)} = I_+(Y)$. The first is immediate as arbitrary intersections of homogeneous ideals form a homogeneous ideal. As Y is closed, there is a homogeneous ideal $\mathfrak{a} \subset A_+$ such that $Y = V_+(\mathfrak{a})$. Thus

$$I_+(Y) = I_+(V_+(\mathfrak{a})) \overset{(a)}{=} \sqrt{\mathfrak{a}}_+.$$

As taking the radical of ideals is idempotent, we get

$$\sqrt{I_+(Y)}_+ = \sqrt{I_+(Y)} = \sqrt{\sqrt{\mathfrak{a}_+}} = \sqrt{\mathfrak{a}}_+ = I_+(Y).$$

V_+ and I_+ are mutually inverse: Assume $Y \in \mathrm{Cl}(A)$. Then

$$V_+(I_+(Y)) \overset{(a)}{=} \overline{Y} \overset{Y \text{ closed}}{=} Y.$$

On the other hand for $\mathfrak{a} \in \mathrm{Rad}(A)$ arbitrary, we get

$$I_+(V_+(\mathfrak{a})) \overset{(a)}{=} \sqrt{\mathfrak{a}} \overset{\mathfrak{a}\in\mathrm{Rad}(A)}{=} \mathfrak{a}.$$

The maps are inclusion reversing: Assume $\mathfrak{a}, \mathfrak{b} \in \mathrm{Rad}(A)$ with $\mathfrak{a} \subset \mathfrak{b}$. Then

$$V_+(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} A \mid \mathfrak{a} \subset \mathfrak{p}\} \overset{\mathfrak{a}\subseteq\mathfrak{b}}{\supset} \{\mathfrak{p} \in \operatorname{Proj} A \mid \mathfrak{b} \subset \mathfrak{p}\} = V_+(\mathfrak{b}).$$

Further for any $Y_1, Y_2 \in \mathrm{Cl}(A)$ with $Y_1 \subset Y_2$ it holds that

$$I_+(Y_1) = \bigcap_{\mathfrak{p}\in Y_1} \mathfrak{p} \cap A_+ \overset{Y_1\subseteq Y_2}{\supset} \bigcap_{\mathfrak{p}\in Y_2} \mathfrak{p} \cap A_+ = I_+(Y_2).$$

It still remains to show that the closed irreducible subsets correspond to ideals of the form $\mathfrak{p}_+$, where $\mathfrak{p}$ is a relevant prime ideal.

We first show that one can restrict the given maps to the corresponding sets, i.e. $\{\mathfrak{p}_+ \mid \mathfrak{p} \in \operatorname{Proj} A\} \subset \mathrm{Rad}(A)$ and $\{Y \subset \operatorname{Proj} A \mid Y \text{closed, irreducible}\} \subset \mathrm{Cl}(A)$. The latter is obvious. For the first statement take $\mathfrak{p}_+$ with $\mathfrak{p} \in \operatorname{Proj} A$. We can apply Exercise 9.4 (a) to conclude for all $a \in A$ and $n \in N$ that $a \in \mathfrak{p}_+$ if and only if $a^n \in \mathfrak{p}_+$, i.e. $\mathfrak{p}_+ = \sqrt{\mathfrak{p}_+}$. Thus we have $\mathfrak{p}_+ \in \mathrm{Rad}(A)$.

To show: $V_+(\mathfrak{p})$ is irreducible for all $\mathfrak{p} \in \operatorname{Proj} A$. Suppose that there is a decomposition of $V_+(\mathfrak{p}_+)$ into two closed sets, i.e. there are $\mathfrak{a}, \mathfrak{b} \subset A$ homogeneous such that

$$V_+(\mathfrak{p}_+) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b}).$$

We want to show that one of the right-hand sets is all $V_+(\mathfrak{p}_+)$. As $\mathfrak{p} \in \operatorname{Proj} A$ and $\mathfrak{p} \supset \mathfrak{p}_+$, we get that $\mathfrak{p} \in V_+(\mathfrak{p}_+) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$. Assume without loss of generality that $\mathfrak{p} \in V_+(\mathfrak{a})$, i.e. $\mathfrak{p} \supset \mathfrak{a}$. Note that $\sqrt{\mathfrak{p}_+}_+ = \sqrt{\mathfrak{p}}_+$. So from above we conclude

$$\sqrt{\mathfrak{a}}_+ \subset \sqrt{\mathfrak{p}}_+ = \sqrt{\mathfrak{p}_+}_+ \overset{\mathfrak{p}_+\in\mathrm{Rad}(A)}{=} \mathfrak{p}_+.$$

On the other hand as $V_+(\mathfrak{a}) \subset V_+(\mathfrak{p}_+)$ it follows that

$$\sqrt{\mathfrak{a}}_+ \overset{(a)}{=} I_+(V_+(\mathfrak{a})) \overset{(i)(3)}{\supset} I_+(V_+(\mathfrak{p}_+)) \overset{(i)(2)}{=} \mathfrak{p}_+.$$

Hence $\mathfrak{p}_+ = \sqrt{\mathfrak{a}}_+$ and

$$V_+(\mathfrak{p}_+) = V_+(\sqrt{\mathfrak{a}}_+) \overset{(a)}{=} V_+(I_+(V_+(\mathfrak{a}))) \overset{(i)(2)}{=} V_+(\mathfrak{a}).$$

It remains to show that for all $Y \subset \operatorname{Proj} A$ closed and irreducible there is a $\mathfrak{p} \in \operatorname{Proj} A$ such that $I_+(Y) = \mathfrak{p}_+$. Because of Exercise 9.4 a it suffices to show the following:

$$\forall a, b \in A \text{ homogeneous: } ab \in I_+(Y) \quad \text{iff atleast one of } \ a \ \text{ or } b \in I_+(Y).$$

As Y is closed, there is a homogeneous ideal $\mathfrak{a}$ such that $Y = V_+(\mathfrak{a})$. Thus

$$I_+(Y) = I_+(V_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}_+ \overset{Ex.9.4.c}{=} \bigcap\{\mathfrak{p} \in \operatorname{Proj} A \mid \mathfrak{a} \subset \mathfrak{p}\} \cap A_+.$$

Hence $ab \in I_+(Y)$ if and only if $ab \in \mathfrak{p}_+$ for all $\mathfrak{a} \subset \mathfrak{p} \in \operatorname{Proj} A$. In particular, as the $\mathfrak{p}$ are prime ideals, we get that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Proj} A$ containing $\mathfrak{a}$. Thus we have

$$Y = V_+(\mathfrak{a}) = V_+((\mathfrak{a} \cup \{a\})^h) \cup V_+((\mathfrak{a} \cup \{b\})^h).$$

As Y is irreducible, we conclude that one of the sets on the right hand-side has to be all Y. Assume without loss of generality that $Y = V_+((\mathfrak{a} \cup \{a\})^h)$. Then

$$a \in \sqrt{\mathfrak{a}} = I_+(V_+(\mathfrak{a})) = I_+(Y).$$

So $I_+(Y)$ is of the desired form.

(c) By Exercise 9.4c, we have

$$\sqrt{\mathfrak{a}}_+ = A_+ \cap \bigcap_{\mathfrak{q} \in V(\mathfrak{a})} \mathfrak{q}.$$

Therefore $\sqrt{\mathfrak{a}}_+ = A_+$ if and only if $\bigcap_{\mathfrak{q} \in V(\mathfrak{a})} \mathfrak{q} \supset A_+$, which is equivalent to $V_+(\mathfrak{a}) = \emptyset$, by definition of relevant prime ideals. In particular, we get

$$\operatorname{Proj} A = \emptyset \Leftrightarrow V_+(0) = \emptyset \iff \sqrt{0}_+ = A_+ \iff \operatorname{Nil} A_+ = A_+ \Leftrightarrow A_+ \subset \operatorname{Nil} A.$$

(d) Let U be open in $\operatorname{Proj} A$, i.e. there is a homogeneous ideal $I \subset A_+$ such that $U = \operatorname{Proj} A \setminus V_+(I)$. Choose a system (f_i) of homogeneous generators of I, then by (10) we get

$$\bigcap_i V_+(f_i) = V_+(I), \text{ i.e } \bigcup_i D_+(f_i) = \operatorname{Proj} A \setminus V_+(I) = U.$$

(e) Recall that $V_+(A_+) = \emptyset$. Thus, by taking complements and applying I_+ we get

$$\begin{aligned}\bigcup_i D_+(f_i) = \operatorname{Proj} A \Leftrightarrow \bigcap_i V_+(f_i) = \emptyset \Leftrightarrow V_+(\mathfrak{a}) = V_+(A_+)\\ \Leftrightarrow \sqrt{\mathfrak{a}}_+ = \sqrt{A_+}_+,\end{aligned}$$

but $\sqrt{A_+}_+ = A_+$ by part (c). □

Exercise 10.3. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring. With the previous notations, we define a presheaf of rings on* $\operatorname{Proj}(A)$ *by setting*

$$\mathcal{O}_{\operatorname{Proj}(A)}(D_+(f)) := A_{(f)} \tag{12}$$

for a homogeneous $f \in A_+$ and then defining

$$\mathcal{O}_{\operatorname{Proj}(A)}(U) := \varprojlim_{\substack{f \in A_+,\ homog. \\ D_+(f) \subseteq U}} \mathcal{O}_{\operatorname{Proj}(A)}(D_+(f))$$

for an open subset $U \subseteq$ $\operatorname{Proj}(A)$*.*

(a) Prove that $(\mathrm{Proj}(A), \mathcal{O}_{\mathrm{Proj}(A)})$ *is a ringed space.*

(b) Prove that $\mathrm{Proj}(A)$ *is a seperated scheme.*

Elaborate every step of your proof as detailed as possible.

Proof. **Solution by Isabel Müller and Robert Rauch**

As we have seen, the system $\mathcal{B} = \{D_+(f) \mid f \in A_+ \text{ homogeneous}\}$ forms a basis of $\mathrm{Proj}\, A$ as a topological space. Since the category of rings has inverse limits, (12) specifies a presheaf of rings on $\mathrm{Proj}\, A$, provided we specify *restriction mappings*

$$\rho_{fg} : A_{(f)} = \mathcal{O}_{\mathrm{Proj}\, A}(D_+(f)) \to \mathcal{O}_{\mathrm{Proj}\, A}(D_+(g)) = A_{(g)}$$

whenever $D_+(f) \supset D_+(g)$, i.e. the ρ_{fg} are morphisms of rings satisfying the cocycle conditions $\rho_{gh} \circ \rho_{fg} = \rho_{fh}$ and $\rho_{ff} = \mathrm{id}$. In fact, $D_+(f) \supset D_+(g)$ implies

$$V_+(f) \subset V_+(g) \Leftrightarrow \sqrt{f}_+ \supset \sqrt{g}_+ \Rightarrow \quad g^n = af \text{ for some } n \in \mathbb{N}, a \in A,$$

this means that $\frac{f}{1}$ is invertible in A_g with inverse $\frac{1}{f} = \frac{a}{g^n}$ (and a slight abuse of notation). Now, we can define $\rho_{fg} : A_{(f)} \to A_{(g)}$ by

$$\frac{a}{f^k} \mapsto \frac{a}{f^k} \cdot \frac{f^{d(g)}}{g^{d(f)}},$$

where $d(f)$, $d(g)$ denote the degree of the elements f, g, respectively.

Now check that the ρ_{fg} obtained this way are in fact morphisms of rings satisfying the cocycle conditions from above. In particular, this implies that $A_{(f)} \cong A_{(g)}$ via ρ_{fg} whenever $D_+(f) = D_+(g)$, so that (12) is indeed well-defined. To prove that $(\mathrm{Proj}\, A, \mathcal{O}_{\mathrm{Proj}\, A})$ is also a sheaf, you need to prove the sheaf axioms (*locality* and the *gluing-property*) for $U = \bigcup_i U_i$ with $(U_i) \subset \mathcal{B}$ (see Görtz/Wedhorn, proposition 2.20).

The crucial step is to establish an isomorphism of sheaves

$$\Phi_f : \left(D_+(f), \mathcal{O}_{\mathrm{Proj}\, A}|_{D_+(f)}\right) \to \left(\mathrm{Spec}\, A_{(f)}, \mathcal{O}_{\mathrm{Spec}\, A_{(f)}}\right),$$

proving that $\mathrm{Proj}\, A$ is a scheme. At the level of topological spaces, Φ_f is given by

$$D_+(f) \hookrightarrow D(f) = \mathrm{Spec}\, A_f \to \mathrm{Spec}\, A_{(f)}, \quad \text{i.e. } \mathfrak{p} \mapsto \mathfrak{p}_{(f)},$$

you need to verify that Φ_f is in fact continuous, open and that $\mathfrak{p}_{(f)} \in \mathrm{Spec}\, A_{(f)}$. The inverse of Φ_f (as a set-theoretic map) is then given by

$$\mathrm{Spec}\, A_{(f)} \ni \mathfrak{q} \mapsto \bigoplus_{n \geq 0} \mathfrak{p}_n \in D_+(f), \quad \text{where } \mathfrak{p}_n := \left\{ a \in A_n \,\middle|\, \frac{a}{f^n} \in \mathfrak{q} \right\}.$$

Here, it is not obvious that $\bigoplus_n \mathfrak{p}_n$ is actually in $D_+(f)$ (hint: use exercise 9.4a). What is missing now is the isomorphism $\Phi_f^\flat$ at the level of sheaves.

(b) By definition, $\mathrm{Proj}\, A$ is separated iff $\mathrm{Proj}\, A \to \mathrm{Spec}\, \mathbb{Z}$ is separated. The open affine cover $\mathrm{Proj}\, A = \bigoplus_f D_+(f)$ has the property that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine, so by a proposition from the lecture, $\mathrm{Proj}\, A$ is separated if and only if for any $f, g \in A_+$ homogeneous, the map

$$\begin{aligned} \Gamma(D_+(f), \mathcal{O}_{\mathrm{Proj}\, A}) \otimes_{\mathbb{Z}} \Gamma(D_+(g), \mathcal{O}_{\mathrm{Proj}\, A}) &\to \Gamma(D_+(fg), \mathcal{O}_{\mathrm{Proj}\, A}) \\ s \otimes t &\mapsto s|_{D_+(fg)} \cdot t|_{D_+(fg)} \end{aligned}$$

is surjective – yet another task for the reader. Notice that this map is given explicitly by

$$A_{(f)} \otimes_{\mathbb{Z}} A_{(g)} \ni \frac{a}{f^k} \otimes \frac{b}{g^l} \mapsto \frac{ab}{f^k g^l} \in A_{(fg)}.$$

$\square$

11 Solutions for Exercise Sheet-11

Exercise 11.1. *Let $\mathcal{F}$ be a sheaf on a topological space X, and let $s \in \mathcal{F}(U)$ be a section over an open subset $U \subseteq X$. The support* $\operatorname{Supp}(s)$ *of s is defined to be*

$$\operatorname{Supp}(s) := \{x \in U \mid s_x \neq 0\},$$

where s_x denotes the germ of s in the stalk $\mathcal{F}_x$. Show that $\operatorname{Supp}(s)$ *is a closed subset of U. We define the support* $\operatorname{Supp}(\mathcal{F})$ *of $\mathcal{F}$ as*

$$\operatorname{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Show that $\operatorname{Supp}(\mathcal{F})$ *need not be a closed subset of X.*

Proof. **Solution by Peter Patzt and Emre Sertöz**

Let $\mathcal{F}$ be a sheaf on X, $U \subset X$ open and $s \in \mathcal{F}(U)$. We now want to show that

$$X \setminus \operatorname{Supp}(s) = \tilde{U} := \bigcup_{\substack{V \text{ open in } U \\ s|_V = 0}} V,$$

which implies that $\operatorname{Supp}(s) \subset X$ is closed.

Let $x \in X \setminus \operatorname{Supp}(s)$, i.e. there is an open subset $V \subset U$ that contains x with $s|_V = 0$. In particular

$$x \in V \subset \bigcup_{\substack{V \text{ open in } U \\ s|_V = 0}} V.$$

Let $x \in \tilde{U}$ and $x \in V \subset \tilde{U}$ open with $s|_V = 0$. Now obviously $s_x = 0$, thus $x \notin \operatorname{Supp}(s)$.

We also want to show that there is a sheaf $\mathcal{F}$ such that

$$\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

is not closed. For this let $X = \{1, 2\}$ with the open sets $\emptyset, \{1\}, \{1, 2\}$. We now define the rings

$$\mathcal{F}(\{1\}) = \mathbb{Z} \text{ and } \mathcal{F}(\{1, 2\}) = 0.$$

With the unique homomorphisms this forms a sheaf. But $\mathcal{F}_1 = \mathbb{Z}$ and $\mathcal{F}_2 = 0$. Thus

$$\operatorname{Supp}(\mathcal{F}) = \{1\}$$

is not closed. □

Exercise 11.2. *A sheaf $\mathcal{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open subsets, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective.*

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of sheaves and if $\mathcal{F}'$ is flasque, then for any open subset $U \subseteq X$, the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

of abelian groups is also exact.

(c) If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

(d) If $f : X \longrightarrow Y$ is a continuous map of topological spaces and if $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

(e) Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the sheaf of discontinuous sections of $\mathcal{F}$ as follows. For each open subset $U \subseteq X$, the abelian group $\mathcal{G}(U)$ consists of the set of maps

$$s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x$$

such that for each $x \in U$, we have $s(x) \in \mathcal{F}_x$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Proof. **Solution by Peter Patzt and Emre Sertöz**

(a) Let X be irreducible and $\mathcal{F}$ the constant sheaf on X with abelian group A. That is

$$\mathcal{F}(U) = \{s \colon U \to A \text{ continuous}\}$$

with the discrete topology on A. With the next claim it is clear that $s \colon U \to A$ is constant, thus $\mathcal{F}(U) \cong A$ for every non-empty open $U \subset X$. This also means that the restriction maps are identity maps on A and thus surjective.

Claim If X is an irreducible topological space and $U \subset X$ is open, then U is connected.

Proof. Assume $U = V \cup W$ with non-empty open $V, W \subset U$ and $V \cap W = \emptyset$. Then V, W are also open in X and

$$X = X \setminus (V \cap W) = (X \setminus V) \cup (X \setminus W),$$

where both $X \setminus V$ and $X \setminus W$ are closed sets with empty intersection in X, contradicting the irreducibility of X. □

(b) Given that $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ are sheaves on X satisfying the short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$ a flasque sheaf. Then by Exercise 1.4, we already have exactness of

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U),$$

for all open $U \subset X$. It is enough to show the surjectiveness of the induced map ψ_U. Henceforth we can assume that $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, which follows from the injectiveness.

Let $s \in \mathcal{F}''(U)$. We now consider the pairs (V, t) with open subsets $V \subset U$ and sections $t \in \mathcal{F}(V)$ such that $\psi_V(t) = s|_V$. Let $\mathcal{S}$ be the set of all these pairs. On this we introduce the partial order

$$(V, t) \geq (V', t') :\iff V' \subset V \text{ and } t|_{V'} = t'.$$

Now on $\mathcal{S}$ we want to apply Zorn's Lemma. As the induced map $\mathcal{F}_P \to \mathcal{F}''_P$ is surjective for every $P \in U$, we find an open neighborhood $P \in V \subset U$ on which ψ is surjective and thus $\mathcal{S} \neq \emptyset$. Let (V_i, t_i) be a chain of elements in $\mathcal{S}$, then $\{V_i\}$ is certainly a cover of $V := \bigcup V_i$. On the other hand, we also have for $(V_i, t_i) \geq (V_j, t_j)$ the implication

$$V_i \cap V_j = V_j \implies t_i|_{V_i \cap V_j} - t_j|_{V_i \cap V_j} = t_i|_{V_j} - t_j = 0.$$

Thus by the second sheaf property of $\mathcal{F}$ we find a $t \in \mathcal{F}(V)$ with $t|_{V_i} = t_i$. Now since

$$\Big(\psi_V(t) - s|_V\Big)\Big|_{V_i} = \psi_{V_i}(t_i) - s|_{V_i} = 0,$$

we achieve

$$\psi_V(t) - s|_V = 0$$

by the first sheaf property.

This accumulates to the existence of a maximal element of $\mathcal{S}$. Now assume that this is (V,t) and $V \neq U$. Let $P \in U \setminus V$. Now as before there is an open subset $P \in W \subset U$ and a $u \in \mathcal{F}(W)$ such that $(W,u) \in \mathcal{S}$. Observing that

$$\psi_{V\cap W}\Big(t|_{V\cap W} - u|_{V\cap W}\Big) = 0,$$

we find that

$$t|_{V\cap W} - u|_{V\cap W} \in \mathcal{F}'(V \cap W),$$

and we therefore get a $v \in \mathcal{F}'(V)$ with

$$v|_{V\cap W} = t|_{V\cap W} - u|_{V\cap W}$$

because of the flasqueness of $\mathcal{F}'$. The pair $(W, u - v)$ is also an element of $\mathcal{S}$ as

$$\psi_W(u - v) = \psi_W(u) = s|_W$$

which follows from the fact that $v \in \mathcal{F}'(W) = \ker(\psi_W)$. Now observe that

$$t|_{V\cap W} = (u - v)|_{V\cap W},$$

which implies the existence of $\tilde{t} \in \mathcal{F}(V \cup W)$ satisfying the following condition

$$\tilde{t}|_V = t \quad \text{and} \quad \tilde{t}|_W = u - v.$$

Hence, we conclude that $(V \cup W, \tilde{t}) \in \mathcal{S}$. This follows from the first property of sheaves again. But since $(V \cup W, \tilde{t}) \gneqq (V,t)$, we have a contradiction to the maximality of the latter. This leaves us with surjectivity of ψ_U.

(c) Given that $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ are sheaves on X satisfying the short exact sequence

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$, and $\mathcal{F}$ are flasque sheaves. Then by (b) we have the following commutative diagram of two short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[r, "\psi_U"] \arrow[d, "\rho_{UV}"] & \mathcal{F}''(U) \arrow[r] \arrow[d, "\rho''_{UV}"] & 0 \\
0 \arrow[r] & \mathcal{F}'(V) \arrow[r] & \mathcal{F}(V) \arrow[r, "\psi_V"] & \mathcal{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

The restriction ρ_{UV} is surjective by falsqueness of $\mathcal{F}$. Therefore $\psi_V \circ \rho_{UV} = \rho''_{UV} \circ \psi_U$ is surjective, in particular so is ρ''_{UV}.

(d) Let $U \subset V$ be open subsets of Y, and ρ, ρ^* the restriction maps of $\mathcal{F}, f_*\mathcal{F}$, respectively. Then

$$\rho^*_{UV} = \rho_{f^{-1}(U)f^{-1}(V)}$$

is surjective.

(e) Let $\mathcal{G}$ be the sheaf of discontinuous section of the sheaf $\mathcal{F}$ on X. Then

$$\mathcal{G}(U) = \prod_{x \in U} \mathcal{F}(U)$$

for any open $U \subset X$. And the restriction is simply the restriction on the index set. Quite obviously this describes a presheaf structure. But we can also prove the two additional sheaf conditions.

Let $U \subset X$ open and $\{V_i\}_{i \in I}$ an open cover of U. Let $s \in \mathcal{G}(U)$ such that for every $i \in I$ the restriction $s|_{V_i} = 0$. It is to be shown that for every $x \in U$ the projection $s_x = 0$. This holds true as every $x \in U$ is covered by some V_i and

$$s_x = (s|_{V_i})_x = 0.$$

For every $i \in I$ let $s_i \in \mathcal{G}(V_i)$ such that

$$\forall i, j \in I \colon s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}.$$

We need to find an $s \in \mathcal{G}(U)$ that will comply with the restrictions

$$s|_{V_i} = s_i.$$

We take $s \in \mathcal{G}(U)$ with

$$s_x = (s_i)_x$$

for some $i \in I$ that satisfies $x \in V_i$. If $j \in I$ is another index such that $x \in V_j$, then $x \in V_i \cap V_j$. Thus

$$(s_i)_x = \left(s_i|_{V_i \cap V_j}\right)_x = \left(s_j|_{V_i \cap V_j}\right)_x = (s_j)_x$$

and $s|_{V_i} = s_i$ for every $i \in I$.

The restrictions for $V \subset U$ open in X are surjective because for some $t \in \mathcal{G}(V)$, we may take $s \in \mathcal{G}(U)$ with

$$s_x = \begin{cases} t_x, & \text{if } x \in V; \\ 0, & \text{otherwise} \end{cases}$$

as its preimage.

At last we want to investigate the natural homomorphism

$$\begin{aligned} \mathcal{F}(U) &\to \mathcal{G}(U) \\ s &\mapsto (\langle U, s \rangle_x)_{x \in U}. \end{aligned}$$

Assume that for every $x \in U$ the germ $\langle U, s \rangle_x = \langle U, t \rangle_x$. That means for every $x \in U$ we find an open set $V_x \subset U$ with $(s - t)|_{V_x} = 0$. But since $\{V_x\}_{x \in U}$ is an open cover of U, we get $s - t = 0$. □

Exercise 11.3. *Let Z be a closed subset of a topological space X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support is contained in Z.*

(a) Show that the presheaf given by the assignment

$$V \mapsto \Gamma_{Z \cap V}(V, \mathcal{F}\big|_V) \qquad (V \subseteq X,\ \textit{open})$$

is a sheaf. It is called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

(b) *Let $U = X \setminus Z$, and let $j : U \longrightarrow X$ be the inclusion. Show that there is an exact sequence of sheaves on X*

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*(\mathcal{F}\big|_U).$$

Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \longrightarrow j_*(\mathcal{F}\big|_U)$ is surjective.

Proof. **Solution by Peter Patzt and Emre Sertöz**

(a) Let $\mathcal{G}$ be the subpresheaf of $\mathcal{F}$ with the assignment

$$\mathcal{G}(U) = \Gamma_{Z\cap V}(U, \mathcal{F}|_V) = \{s \in \mathcal{F}(V) \mid \operatorname{Supp}(s) \subset Z\}.$$

As a subpresheaf we can take the first additional condition for sheaves for granted.

Let $U \subset X$ be open and $\{V_i\}_{i\in I}$ an open cover of U and $s_i \in \mathcal{G}(V_i)$ such that

$$\forall i, j \in I\colon s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}.$$

Now take $s \in \mathcal{F}(U)$ with $s|_{V_i} = s_i$, for every $i \in I$. It suffices to show that $\operatorname{Supp}(s) \subset Z$.

Let $x \in \operatorname{Supp}(s)$, i.e. $\langle U, s\rangle_x \neq 0$. Let $i \in I$ with $x \in V_i$. Since

$$\langle V_i, s_i\rangle_x = \langle U, s\rangle_x \neq 0,$$

we derive $x \in Z$.

(b) Let $\mathcal{H}^0_Z(\mathcal{F})$ be the sheaf $\mathcal{G}$ as described above. Furthermore, let $U = X \setminus Z$ and $j\colon U \to X$ be the inclusion map. We now want to prove the exactness of

$$0 \longrightarrow H^0_Z(\mathcal{F}) \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} j_*(\mathcal{F}|_U),$$

where φ and ψ are the natural morphisms.

The injectivity of φ is clear. Now the image of φ is $\mathcal{H}^0_Z(\mathcal{F})$, i.e.

$$(\operatorname{im}\varphi)(V) = \{s \in \mathcal{F}(V) \mid \operatorname{Supp}(s) \subset Z\},$$

as it is a subsheaf of $\mathcal{F}$. The kernel of ψ is determined by

$$(\ker\psi)(V) = \{s \in \mathcal{F}(V) \mid s|_{U\cap V} = s|_{V\setminus Z} = 0\}.$$

These two sets coincide because of the sequence of equivalences

$$s|_{V\setminus Z} = 0 \iff \forall x \in V \setminus Z\colon \langle V, s\rangle_x = 0 \iff \operatorname{Supp}(s) \subset Z.$$

Finally if $\mathcal{F}$ is flasque, we find that the map

$$\mathcal{F}(V) \to j_*(\mathcal{F}|_U)(V) = \mathcal{F}(U \cap V)$$

is surjective for every $V \subset X$ open. Now this implies that

$$\mathcal{F} \to j_*(\mathcal{F}|_U)$$

is surjective since already the presheaf image of ψ is $j_*(\mathcal{F}|_U)$. □

12 Solutions for Exercise Sheet-12

Exercise 12.1. *Let X be a topological space, $x \in X$ a point, and A an abelian group. Define a sheaf $i_x(A)$ on X by the assignment*

$$i_x(A)(U) = \begin{cases} A, & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases} \qquad (U \subseteq X, \text{ open})$$

Show that for the stalk $i_x(A)_y$ at a point $y \in X$, we have

$$i_x(A)_y = \begin{cases} A, & \text{if } y \in \overline{\{x\}}, \\ 0, & \text{otherwise;} \end{cases}$$

whence the name skyscraper sheaf originates. Show that the skyscraper sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{x\}}$ and $i : \overline{\{x\}} \longrightarrow X$ is the inclusion.

Proof. **Solution by Irfan Kadikoylu**

Let $y \in \overline{\{x\}}$ and U be an open set containing y. If $x \notin U$ then $X \backslash U$ is a closed set containing x, hence $\overline{\{x\}} \subseteq X \backslash U$ implying that $y \in X \backslash U$, which is a contradiction. Therefore $x \in U$. So we can deduce that

$$i_x(A)_y = \varinjlim_{\substack{y \in U \\ U \subseteq X}} i_x(A)(U) = \varinjlim_{\substack{y \in U \\ U \subseteq X}} i_x(A)(U) = \varinjlim_{\substack{y \in U \\ U \subseteq X}} A = A.$$

If on the other hand $y \notin \overline{\{x\}}$, the set $X \backslash \overline{\{x\}}$ is an open set containing y and $i_x(A)(X \backslash \overline{\{x\}}) = 0$. Moreover for any open set $V \subseteq X \backslash \overline{\{x\}}$ s.t $y \in V$, we have $i_x(A)(V) = 0$ Hence, $i_x(A)_y$ =0 as desired.

For the second claim observe that

$$i_*(A)(U) = \begin{cases} A, & \text{if } U \cap \overline{\{x\}} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} = i_x(A)(U),$$

because clearly $U \cap \overline{\{x\}} \neq \emptyset \Leftrightarrow x \in U$. □

Exercise 12.2. *Let X be a topological space, $Z \subseteq X$ a closed subset, and $i : Z \longrightarrow X$ the inclusion. Further, let $U = X \setminus Z$ be the complementary open subset and $j : U \longrightarrow X$ its inclusion.*

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that for the stalk $(i_*\mathcal{F})_z$ at a point $z \in Z$, we have

$$(i_*\mathcal{F})_z = \begin{cases} \mathcal{F}_z, & \text{if } z \in Z, \\ 0, & \text{otherwise;} \end{cases}$$

hence, we call the sheaf $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z.

(b) Let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf given by the assignment

$$j_!(\mathcal{F})(V) := \begin{cases} \mathcal{F}(V), & \text{if } V \subseteq U, \\ 0, & \text{otherwise;} \end{cases} \qquad (V \subseteq X, \text{ open}).$$

Show that for the stalk $j_!(\mathcal{F})_x$ at a point $x \in U$, we have

$$j_!(\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & \textit{if } x \in U, \\ 0, & \textit{otherwise}; \end{cases}$$

furthermore, show that $j_!(\mathcal{F})$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_!(\mathcal{F})$ the sheaf obtained by extending $\mathcal{F}$ by zero outside U.

(c) Let $\mathcal{F}$ be a sheaf on X. Show that there is the following exact sequence of sheaves on X

$$0 \longrightarrow j_!(\mathcal{F}\big|_U) \longrightarrow \mathcal{F} \longrightarrow i_*(\mathcal{F}\big|_Z) \longrightarrow 0.$$

Proof. **Solution by Irfan Kadikoylu**

(a) Let $z \notin Z$. Then $X \backslash Z$ is an open set containing z. Hence,

$$i_*\mathcal{F}(X \backslash Z) = \mathcal{F}(i^{-1}(X \backslash Z)) = \mathcal{F}(\emptyset) = 0$$

Moreover, $i_*\mathcal{F}(V) = 0$ for any other open set $V \subseteq X \backslash Z$. containing z. So $(i_*\mathcal{F})_z = 0$. Now let $z \in Z$. Then clearly $i_*\mathcal{F}(U) = \mathcal{F}(i^{-1}(U)) = \mathcal{F}(U \cap Z)$. Using this fact, we find

$$\varinjlim_{\substack{z \in U \\ U \text{open in } X}} i_*\mathcal{F}(U) = \varinjlim_{\substack{z \in U \\ U \text{open in } X}} \mathcal{F}(U \cap Z) = \varinjlim_{\substack{z \in U \\ U \text{open in } Z}} = \mathcal{F}(U) = \mathcal{F}_z.$$

(b) Let $\overline{\mathcal{F}}$ denote the presheaf on X, mentioned in the question. Let $x \in U$, then using the fact that a presheaf and its sheafification have the same stalk at every point, we have

$$j_!(\mathcal{F})_x = \overline{\mathcal{F}}_x = \varinjlim_{\substack{x \in V \\ V \text{open in X}}} \overline{\mathcal{F}}(V) = \varinjlim_{\substack{x \in V \\ V \text{open in } U}} \overline{\mathcal{F}}(V) = \varinjlim_{\substack{x \in V \\ V \text{open in } X}} \mathcal{F}(V) = \mathcal{F}_x.$$

Now let $x \notin U$. Then for any open set $V \subseteq X$ s.t $x \in V$, we have $V \nsubseteq U$ and hence, $\overline{\mathcal{F}}(V) = 0$. Using this, we get

$$j_!(\mathcal{F})_x = \overline{\mathcal{F}}_x = \varinjlim_{\substack{x \in V \\ V \text{open in } X}} \overline{\mathcal{F}}(V) = 0.$$

For the uniqueness statement, let $\mathcal{G}$ be another sheaf with the mentioned properties. Then since $\mathcal{G}|_U = \mathcal{F}$ we can define a map $f : \overline{\mathcal{F}} \to \mathcal{G}$ as $f_V = \mathrm{id}_V$ $\forall$ open sets $V \subseteq X$ and $f_V = 0$ for all other open sets V. Then by the universal property of the sheafification there exists a map φ making the following diagram commutative:

$$\begin{tikzcd} \overline{\mathcal{F}} \arrow[r, "i"] \arrow[dr, "f"'] & j_!(\mathcal{F}) \arrow[d, dotted, "\varphi"] \\ & \mathcal{G} \end{tikzcd}$$

Now clearly for any $P \in X$, the maps i_P and f_P are bijective and so is φ_P by the commutativity of the corresponding diagram on the stalks. So we conclude that φ is bijective.

(c) By the first exercise sheet, exactness of this sequence is equivalent to the exactness of the sequence of the stalks at P, $\forall P \in X$. So let $P \in U$. Then $i_*(\mathcal{F}|_Z)_P = 0$ by (a), and we need to show that the map $j_!(\mathcal{F}|_U)_P \to \mathcal{F}_P$ is bijective.

Let $\overline{\mathcal{F}|_U}$ denote the presheaf corresponding to $\mathcal{F}|_U$ as in part (b). Define $f : \overline{\mathcal{F}|_U} \to \mathcal{F}$ as $f_V = \mathrm{id}_V$ if $V \subseteq U$ and 0 otherwise. Again using the sheafification property and bijectivity of f_P (as in part (b)) we conclude that $j_!(\mathcal{F}|_U)_P \to \mathcal{F}_P$ is bijective.

Now let $P \notin U$. Then by (b) $j_!(\mathcal{F}|_U)_P = 0$, so we need to show that the map $\mathcal{F}_P \to i_*(\mathcal{F}|_Z)_P$ is bijective. Clearly for every open set $V \subseteq X$, we have a natural map $i_*(\mathcal{F}|_Z)(V) \to \mathcal{F}_P$, and it is easy to show that these maps satisfy the direct limit axioms (D1) and (D2). So by the direct limit property we get a map $i_*(\mathcal{F}|_Z)_P \to \mathcal{F}_P$ which is clearly an inverse to the map in the given sequence. $\square$

Exercise 12.3. *Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from exercise sheet 11.*

(a) Let A be a ring, M an A-module, $X = \mathrm{Spec}(A)$, and $\mathcal{F} = \widetilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\mathrm{Supp}(m) = V(\mathrm{Ann}(m))$.

(b) If A is a noetherian ring and M a finitely generated A-module, show that $\mathrm{Supp}(\mathcal{F}) = V(\mathrm{Ann}(M))$.

(c) Show that the support of a coherent sheaf on a noetherian scheme is closed.

(d) Again, let A be a ring and M an A-module. For an ideal $\mathfrak{a} \subseteq A$, we define the submodule $\Gamma_{\mathfrak{a}}(M)$ of M by

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \mid \exists n \in \mathbb{N} : \mathfrak{a}^n m = 0\}.$$

Show that if A is noetherian, $X = \mathrm{Spec}(A)$, and $\mathcal{F} = \widetilde{M}$, we have an isomorphism of $\mathcal{O}_X$-modules

$$\widetilde{\Gamma_{\mathfrak{a}}(M)} \cong \mathcal{H}^0_Z(\mathcal{F}),$$

where $Z = V(\mathfrak{a})$ and $\mathcal{H}^0_Z(\mathcal{F})$ is defined in Exercise 11.3.

(e) Let X be a noetherian scheme and $Z \subseteq X$ a closed subset. If $\mathcal{F}$ is a quasi-coherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathcal{H}^0_Z(\mathcal{F})$ is also quasi-coherent (respectively, coherent).

Proof. **Solution by Ana Maria Botero**

(a) Recall that $\mathrm{Supp}(m) := \{\mathfrak{p} \in \mathrm{Spec}(A) : m_{\mathfrak{p}} \neq 0\}$. Note that this condition is equivalent to asking $sm \neq 0$ for all $s \notin \mathfrak{p}$.

$\supset$: If $\mathfrak{p} \in V(\mathrm{Ann}(m))$ then $\mathfrak{p} \supset \mathrm{Ann}(m)$ which implies $sm \neq 0$ for all $s \notin \mathfrak{p}$. Hence, $\mathfrak{p} \in \mathrm{Supp}(m)$.

$\subset$: If $\mathfrak{p} \in \mathrm{Supp}(m)$ then $sm \neq 0$ for all $s \notin \mathfrak{p}$ which implies that $\mathrm{Ann}(m) \subset \mathfrak{p}$.

(b) By definition $\mathrm{Supp}(\mathcal{F}) := \{\mathfrak{p} \in \mathrm{Spec}(A) : M_{\mathfrak{p}} \neq 0\}$.

$\subset$: Suppose that $\mathfrak{p} \in \mathrm{Supp}(\mathcal{F})$. If $\mathfrak{p} \notin V(\mathrm{Ann}(M))$ then $\mathfrak{p} \not\supseteq \mathrm{Ann}(M)$ so there exists $s \in \mathrm{Ann}(M)$ such that $s \notin \mathfrak{p}$. This means that $sm = 0$ for all $m \in M$ and hence $M_{\mathfrak{p}} = 0$, a contradiciton. Hence, $\mathfrak{p} \in V(\mathrm{Ann}(M))$.

$\supset$: Suppose $\mathfrak{p} \notin \mathrm{Supp}(\mathcal{F})$. Then $M_{\mathfrak{p}} = 0$ which implies that for all $m \in M$, there exists $s_m \in A - \mathfrak{p}$ such that $ms_m = 0$. In particular, if $\{m_i\}$ is a finite set of generators for M, then $\Pi_i s_{m_i} \in A - \mathfrak{p}$, and $\Pi_i s_{m_i} \in \mathrm{Ann}(M)$. Hence, $\mathfrak{p} \not\supseteq \mathrm{Ann}(M)$.

(c) Note that $\mathrm{Supp}(\mathcal{F}) = \bigcup_i \mathrm{Supp}(\mathcal{F}|_{U_i})$, where $\{U_i\}$ is an open cover of X.

Now take an affine covering $\{U_i\}$, where $U_i = \mathrm{Spec}\, A_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$, for some finitely generated A_i-modules. Then, by part (b) of this exercise, $\mathrm{Supp}(\mathcal{F}|_{U_i}) = V(\mathrm{Ann}(M_i))$ is a closed subset. By noetherianity, this covering can be chosen to be finite. It follows that $\mathrm{Supp}(\mathcal{F})$ is a finite union of closed subsets and is thus a closed subset of X.

(d) Let $Z \subset X$ be a closed subset of X, and $\mathcal{F}$ a sheaf on X. Recall that the subgroup $\Gamma_Z(X, \mathcal{F}) \subset \Gamma(X, \mathcal{F})$ is defined as the set of global sections of $\mathcal{F}$ whose support is contained in Z. Also recall that $\mathcal{H}^0_Z$ is the sheaf $V \to \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ (see Exercise 11.3).

Now let $U = X - Z$ and let $j : U \to X$ be the inclusion. By Exercise 11.3, we have the following exact sequence of $\mathcal{O}_X-$modules

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$

Since A is noetherian, $X = \mathrm{Spec}(A)$ is a noetherian scheme. By definition $\mathcal{F}$ is quasi-coherent, and also by noetherianty it follows that j is quasi-compact. Hence, by a proposition in the lecture, we conclude that $j_*(\mathcal{F}|_U)$ is quasi-coherent. Hence, $\mathcal{H}^0_Z(\mathcal{F})$, being the kernel of a morphism of quasicoherent sheaves is quasi-coherent. Hence, it suffices to show that the modules of global sections are isomorphic. That is, we want an isomorphism between the following two modules:

$$\widetilde{\Gamma_\alpha(M)}(X) = \Gamma_\alpha(M) = \{m \in M : \alpha^n m = 0, \text{ for some } n > 0\}$$

and

$$\mathcal{H}^0_Z(\mathcal{F})(X) = \Gamma_Z(\mathcal{F}(X)) = \Gamma_Z(M) = \{m \in M : \mathrm{Supp}(m) \subset Z\}.$$

$\subset$: Suppose $m \in \Gamma_\alpha(M)$. Then $\alpha^n \in \mathrm{Ann}(m)$ for some $n > 0$. Hence,

$$Z = V(\alpha) = V(\alpha^n) \supset V(\mathrm{Ann}(m)) = \mathrm{Supp}(m),$$

where the last equality follows from part (a) of this excercise. Hence, $m \in \mathcal{H}^0_Z(\mathcal{F})(X)$.

$\supset$: Let $m \in \Gamma_Z(\mathcal{F})$. So $\mathrm{Supp}(m) \subset Z = V(\alpha)$. Hence, we have

$$V(\mathrm{Ann}(m)) = \mathrm{Supp}(m) \subset V(\alpha).$$

Since the radical of an ideal is the intersection of all prime ideals containing that ideal, we have

$$\sqrt{\mathrm{Ann}(m)} \supset \sqrt{\alpha} \supset \alpha.$$

Now since A is noetherian, the ideal α is finitely generated, say by n elements $\{a_i\}$. Since $\alpha \subset \sqrt{\mathrm{Ann}(m)}$ there exists a j_i for each i such that $a_i^{j_i} \in \mathrm{Ann}(m)$. Let $N := \max\{j_i\}$.

Observe that every element of α is of the form

$$\sum_{i=1}^{n} c_i a_i \text{ with } c_i \in A.$$

Hence, every element of α^{nN} is of the form

$$\sum_{1\le i_1<i_2\cdots<i_q\le n} c_{i_1 i_2\cdots i_q} a_{i_1}^{k_{i_1}} \cdots a_{i_q}^{k_{i_q}} \quad (c_{i_1 i_2\cdots i_q} \in A,\ k_{i_j} \in \mathbb{N}),$$

where $nN = \sum_{j=1}^{n} k_{i_j}$. Hence, one sees that every monomial contains a factor of the form a_i^k where $k \geq \max\{j_i\} \geq j_i$. It follows that $\alpha^{nN} \subset \mathrm{Ann}(m)$, which implies that $\alpha^{nN} m = 0$ which shows $m \in \Gamma_\alpha(M)$.

(e) Let $\{U_i\}$ be an affine open cover of X on which $\mathcal{F}$ is locally of the form $\widetilde{M_i}$, where M_i is an A_i-module. Since X is noetherian we apply part (d) of this excercise to find that $\mathcal{H}_Z^0(\mathcal{F}) = \widetilde{\Gamma_{\alpha_i}(M_i)}$, where α_i is the ideal corresponding to the closed affine subscheme $Z \cap U_i$ of U_i. This shows that $\mathcal{H}_Z^0(\mathcal{F})$ is quasi-coherent. The same proof works for the coherent case by taking the M_i's to be finitely generated. $\square$