Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on June 19th in the exercise class.

Exercise sheet 9

Exercise 9.1 (Ex. II.3.13. of [Har])

A morphism $f : X \longrightarrow Y$ of schemes is *locally of finite type*, if there exists a covering of Y by open affine subsets $V_i = \mathrm{Spec}(B_i)$ such that for each i, the open subschema $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \mathrm{Spec}(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra.
The morphism $f : X \longrightarrow Y$ is of *finite type*, if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the open affine subsets U_{ij}. We say that X *is of finite type over* Y.
Prove the following assertions:

(a) A closed immersion is of finite type.

(b) A quasi-compact open immersion is of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

(f) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, if $g \circ f$ is of finite type, and if f is quasi-compact, then f is of finite type.

(g) If $f : X \longrightarrow Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Exercise 9.2

Assume that all the schemes in the subsequent statements are noetherian. Under this hypothesis, prove the following assertions:

(a) A closed immersion is proper.

(b) A composition of two proper morphisms is proper.

(c) Proper morphisms are stable under base extension.

(d) If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are proper morphisms of S-schemes, then $f \times f' : X \times_S X' \longrightarrow Y \times_S Y'$ is also proper.

(e) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms, if $g \circ f$ is proper, and if g is separated, then f is proper.

(f) A morphism $f : X \longrightarrow Y$ is proper, if and only if Y can be covered by open subschemes V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for all i. We say that properness is *local on the base*.

Exercise 9.3

Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring and $M = \bigoplus_{d \in \mathbb{Z}} M_d$ a graded A-module. Show that the following three conditions for a submodule $N \subseteq M$ are equivalent:

(i) $N = \bigoplus_{d \in \mathbb{Z}} (N \cap M_d)$.

(ii) N is generated by homogeneous elements of M.

(iii) For all $n \in N$, all its homogeneous components belong to N.

We say that the submodule N of M is *homogeneous*.

Exercise 9.4

Let A be a graded ring. Prove the following assertions:

(a) Let $\mathfrak{p}, \mathfrak{p}' \subseteq A$ be relevant prime ideals. If $\mathfrak{p}_+ = \mathfrak{p}'_+$, then $\mathfrak{p} = \mathfrak{p}'$. A homogeneous ideal $\mathfrak{a} \subsetneq A_+$ is of the form $\mathfrak{p}_+$ for some relevant prime ideal $\mathfrak{p}$ of A if and only if for all homogeneous elements $a, b \in A_+ \setminus \mathfrak{a}$ one has $ab \notin \mathfrak{a}$.

(b) Let $S \subseteq A$ be a multiplicative set. Then, the set of homogeneous ideals $\mathfrak{a} \subsetneq A_+$ with $S \cap \mathfrak{a} = \emptyset$ has maximal elements and each such maximal element is of the form $\mathfrak{p}_+$ for a relevant prime ideal $\mathfrak{p}$.

(c) Let $\mathfrak{a} \subseteq A_+$ be a homogeneous ideal. Then, $\sqrt{\mathfrak{a}}_+ = \sqrt{\mathfrak{a}} \cap A_+$ is again a homogeneous ideal. Moreover, $\sqrt{\mathfrak{a}}_+$ is the intersection of A_+ with all relevant prime ideals containing $\mathfrak{a}$.