HU Berlin Summer Term 2012

# Exercises BMS Basic Course
# **Algebraic Geometry**

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Solution to be presented on June 12th in the exercise class.

## Exercise sheet 8

### Exercise 8.1 (Ex. II.3.9. of [Har])

(a) Let $k$ be a field. Show that

$$\mathbb{A}_k^n \times_{\operatorname{Spec}(k)} \mathbb{A}_k^m \cong \mathbb{A}_k^{n+m}$$

and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if $k$ is algebraically closed).

(b) Let $k$ be a field and $s$, $t$ indeterminates over $k$. Then, $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec}(k)$ are all one-point spaces. Describe the product scheme

$$\operatorname{Spec} k(s) \times_{\operatorname{Spec}(k)} \operatorname{Spec} k(t).$$

### Exercise 8.2 (Ex. II.3.10. of [Har])

(a) Let $f : X \longrightarrow S$ be a morphism of schemes and $s \in S$ a point. Show that $\operatorname{sp}(X_s)$ is homeomorphic to $f^{-1}(s)$ with the induced topology.

(b) Let $k$ be an algebraically closed field, $X := \operatorname{Spec} k[Y, Z]/(Y - Z^2)$, $S := \operatorname{Spec} k[Y]$, and $f : X \longrightarrow S$ the morphism induced by sending $Y \mapsto Y$. Prove the following assertions:

   (1) If $s \in S$ is the point $a \in k$ with $a \neq 0$, then the fiber $X_s$ consists of two points, with residue field $k$.

   (2) If $s \in S$ corresponds to $0 \in k$, then the fiber $X_s$ is a non-reduced one-point scheme.

   (3) If $\eta$ is the generic point of $S$, then $X_\eta$ is a one-point scheme, whose residue field is an extension of degree two of the residue field of $\eta$.

### Exercise 8.3

Let $S$ be a scheme, $X$ a scheme over $S$, and $p, q : X \times_S X \longrightarrow X$ the two projections. As usual, denote by $\Delta : X \longrightarrow X \times_S X$ the diagonal morphism giving rise to the subset $\Delta(X) \subseteq X \times_S X$. Further, consider the subset

$$Z := \{z \in X \times_S X \mid p(z) = q(z)\}$$

of $X \times_S X$. Show that the obvious inclusion $\Delta(X) \subseteq Z$ need not be an equality.

**Exercise 8.4 (Ex. II.3.11. of [Har])**

(a) Show that closed immersions are stable under base extension, i.e., if $f : Y \longrightarrow X$ is a closed immersion and if $X' \longrightarrow X$ is any morphism of schemes, then $f' : Y \times_X X' \longrightarrow X'$ is also a closed immersion.

(b) Let $Y$ be a closed subset of a scheme $X$, and give $Y$ the reduced induced subscheme structure. If $Y'$ is any other closed subscheme of $X$ with the same underlying topological space, show that the closed immersion $Y \longrightarrow X$ factors through $Y'$.
We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(c) Let $f : Z \longrightarrow X$ be a morphism of schemes. Show that there is a unique closed subscheme $Y$ of $X$ with the following property: the morphism $f$ factors through $Y$, and if $Y'$ is any other closed subscheme of $X$ through which $f$ factors, then $Y \longrightarrow X$ factors through $Y'$ also.
We call $Y$ the *scheme-theoretic image* of $f$. If $Z$ is a reduced scheme, then $Y$ is just the reduced induced structure on the closure of the image $f(Z)$.