Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on June 5th in the exercise class.

Exercise sheet 7

Exercise 7.1 (Ex. II.2.18. of [Har])

(a) Let A be a commutative ring with 1, $X := \mathrm{Spec}(A)$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, and let $f : Y := \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A) =: X$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \longrightarrow X$ is a homeomorphism onto a closed subset and $f^\flat : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective, then φ is surjective.

Hint: Consider $X' := \mathrm{Spec}(A/\ker(\varphi))$, and use (b) and (c).

Exercise 7.2

Let A be a commutative ring with 1 and $\mathfrak{a} \subseteq A$ an ideal. Let $X := \mathrm{Spec}(A)$ and $Y := \mathrm{Spec}(A/\mathfrak{a})$.

(a) Show that the ring homomorphism $A \longrightarrow A/\mathfrak{a}$ induces a morphism of schemes $f : Y \longrightarrow X$, which is a closed immersion.

(b) Show that for any ideal $\mathfrak{a} \subseteq A$, we obtain a structure of a closed subscheme on the closed set $V(\mathfrak{a}) \subseteq X$.
In particular, every closed subset Y of X can have various subscheme structures corresponding to all the ideals $\mathfrak{a}$ for which $V(\mathfrak{a}) = Y$.

Exercise 7.3 (Ex. II.3.17. of [Har])

A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point. For example, let R be a discrete valuation ring and $T := \mathrm{sp}(\mathrm{Spec}(R))$ the underlying topological space of $\mathrm{Spec}(R)$. Then, T consists of two points, namely, t_0, the maximal ideal of R, and t_1, the zero ideal of R. The open subsets are $\emptyset$, $\{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.

(a) Show that if X is a noetherian scheme, then $\mathrm{sp}(X)$ is a Zariski space.

(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points*.

(c) Show that a Zariski space X satisfies the axiom T_0, i.e., given any two distinct points of X, there is an open set containing one but not the other.

(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.

(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \overline{\{x_1\}}$, then we say that x_1 *specializes to* x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization of* x_1 or that x_1 is a *generization of* x_0.
Now let X be a Zariski space. Show that the minimal points for the partial ordering determined by $x_1 > x_0$, if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points.
We say closed subsets are *stable under specialization*. Similarly, open subsets are *stable under generization*.

(f) Using the notation of the lecture, show that, if X is a noetherian topological space, then $t(X)$ is a Zariski space. Furthermore, X itself is a Zariski space if and only if the map $\alpha : X \longrightarrow t(X)$ is a homeomorphism.