Exercises BMS Basic Course
Algebraic Geometry

Prof. Dr. J. Kramer

Solution to be presented on May 29nd in the exercise class.

Exercise sheet 6

Exercise 6.1

Let R be a commutative ring with 1 and let $n \geq 0$ be an integer. Show that the following assertions are equivalent:

(i) $\operatorname{Spec}(R)$ is reduced (resp. irreducible, resp. integral).

(ii) $\mathbb{A}^n_R$ is reduced (resp. irreducible, resp. integral).

(iii) $\mathbb{P}^n_R$ is reduced (resp. irreducible, resp. integral).

Exercise 6.2 (Ex. II.2.3. of [Har])

For a commutative ring A with 1, we denote by A_{red} the quotient of A by its nilradical.

(a) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf given by the assignment

$$U \mapsto \mathcal{O}_X(U)_{\text{red}} \qquad (U \subset X, \text{open}).$$

Show that $X_{\text{red}} := (X, (\mathcal{O}_X)_{\text{red}})$ is a scheme, called *the reduced scheme associated to* X. Further, show that there is a morphism of schemes $X_{\text{red}} \longrightarrow X$, which is a homeomorphism on the underlying topological spaces.

(b) Let $f : X \longrightarrow Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \longrightarrow Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \longrightarrow Y$.

Exercise 6.3

Let k be a field, $A := k[X, Y, Z]$, and $X := \mathbb{A}^3_k = \operatorname{Spec}(A)$. Further, let $\mathfrak{p}_1 := (X, Y)$, $\mathfrak{p}_2 := (X, Z)$, and $\mathfrak{a} := \mathfrak{p}_1\mathfrak{p}_2$.

(a) Let $Z_1 := V(\mathfrak{p}_1)$, $Z_2 := V(\mathfrak{p}_2)$, and $Y := V(\mathfrak{a})$. Show that Z_1 and Z_2 are integral subschemes of X and show that $Y = Z_1 \cup Z_2$ (set-theoretically).

(b) Show that $Y = V(\mathfrak{a})$ is not reduced and describe Y_{red}.

Exercise 6.4

Recall that a *primitive integer solution* of the generalized Fermat equation

$$X^p + Y^q = Z^r \qquad (p, q, r \in \mathbb{Z}_{>0})$$

is a triple $(x, y, z) \in \mathbb{Z}^3$ satisfying $x^p + y^q = z^r$ with $\gcd(x, y, z) = 1$.

(a) Show that the affine scheme $\operatorname{Spec} \mathbb{Z}[X, Y, Z]/(X, Y, Z)$ can be identified with a closed subscheme T of the affine scheme $S := \operatorname{Spec} \mathbb{Z}[X, Y, Z]/(X^p + Y^q - Z^r)$.

(b) Consider the open subscheme $U := S \setminus T$. Prove that

$$U(\mathbb{Z}) := \operatorname{Hom}\big(\operatorname{Spec}(\mathbb{Z}), U\big)$$

is in bijection with the set of primitive integer solutions of $X^p + Y^q = Z^r$.