Exercises BMS Basic Course
Algebraic Geometry

Prof. Dr. J. Kramer

Solution to be presented on May 22nd in the exercise class.

Exercise sheet 5

Exercise 5.1
Consider the following affine schemes:

(a) $X_1 = \operatorname{Spec} \mathbb{C}[X]/(X^2)$,

(b) $X_2 = \operatorname{Spec} \mathbb{C}[X]/(X^2 - X)$,

(c) $X_3 = \operatorname{Spec} \mathbb{C}[X]/(X^3 - X^2)$,

(d) $X_4 = \operatorname{Spec} \mathbb{R}[X]/(X^2 + 1)$.

For $i = 1, \ldots, 4$, describe the topological space X_i and its open subsets, and compute $\mathcal{O}_{X_i}(U)$ for all open subsets $U \subseteq X_i$.

Exercise 5.2
Let k be an algebraically closed field and let $X = \operatorname{Spec} k[X_1, X_2]$ be an affine scheme. Show that $U = X \setminus V(X_1, X_2)$ is an open subscheme of X, which is not affine.

Exercise 5.3
Let R be a commutative ring with 1 and $R[X_1, \ldots, X_n]$ the polynomial ring in n variables over R. We define the *affine space* $\mathbb{A}^n_R$ *of relative dimension* n *over* R *by*

$$\mathbb{A}^n_R := \operatorname{Spec} R[X_1, \ldots, X_n].$$

For $i = 0, \ldots, n$, let U_i be the affine spaces $\mathbb{A}^n_R$ of relative dimension n over R given by

$$U_i := \operatorname{Spec} R\Big[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\Big].$$

Further, let

$$U_{ij} := D_{U_i}\Big(\frac{X_j}{X_i}\Big) \subseteq U_i$$

for $i \neq j$ and $U_{ii} := U_i$ $(i, j = 0, \ldots, n)$. Finally, let $\varphi_{ii} = \mathrm{id}_{U_i}$ and for $i \neq j$, let $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ be the isomorphism of affine schemes induced by the equality

$$R\Big[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}\Big]_{\frac{X_i}{X_j}} \longrightarrow R\Big[\frac{X_0}{X_j}, \ldots, \widehat{\frac{X_j}{X_j}}, \ldots, \frac{X_n}{X_j}\Big]_{\frac{X_j}{X_i}}.$$

(a) Verify, that the given data constitute a gluing datum, i.e., they satisfy the assumptions (1)–(4) of Exercise 4.3.

The scheme obtained by gluing the $n+1$ copies of $\mathbb{A}^n_R$ along the isomorphisms φ_{ij} is called the *projective space* $\mathbb{P}^n_R$ *of relative dimension* n *over* R.

(b) Show that for $n > 0$ the scheme $\mathbb{P}^n_R$ is not affine.

Exercise 5.4

Let X be an integral scheme with generic point η and let $U = \operatorname{Spec}(A)$ be an affine open subset of X. Recall that the local ring $\mathcal{O}_{X,\eta}$ is a field, called the function field $K(X)$ of X.

(a) Show that $\operatorname{Quot}(A) \cong \mathcal{O}_{X,\eta} = K(X)$.

(b) By identifying $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,x}$ with subrings of $K(X)$, show that we have

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq K(X).$$

An element of $K(X)$ is called a *rational function* on X. We say that $f \in K(X)$ is *regular at* $x \in X$ if $f \in \mathcal{O}_{X,x}$.

(c) Let k be an algebraically closed field. Describe the regular and the rational functions on $\mathbb{A}^n_k = \operatorname{Spec} k[X_1, \ldots, X_n]$.