HU Berlin Summer Term 2012

Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on May 15th in the exercise class.

Exercise sheet 4

Exercise 4.1 (Ex. II.2.1. of [Har])

Let A be a commutative ring with 1 and let $X = \operatorname{Spec}(A)$. Show that for $f \in A$ the locally ringed space $(D(f), \mathcal{O}_X\big|_{D(f)})$ is isomorphic to $\operatorname{Spec}(A_f)$.

Exercise 4.2

Let X and Y be schemes, and let $\{U_i\}_{i\in I}$ be an open covering of X. Let $f_i : U_i \to Y$ $(i \in I)$ be a family of morphisms such that the restrictions of f_i and f_j to $U_i \cap U_j$ coincide for any $i, j \in I$. Show that there exists a unique morphism of schemes $f : X \to Y$ such that $f\big|_{U_i} = f_i$ for all $i \in I$.

Exercise 4.3 (Ex. II.2.12. of [Har])

Let $\{X_i\}_{i\in I}$ be a family of schemes. Suppose that for schemes X_i $(i \in I)$ there exist open subschemes $U_{ij} \subseteq X_i$ $(j \in I)$ and an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ $(i, j \in I)$ such that

(1) $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}$ $\quad (i \in I)$,

(2) $\varphi_{ji} = \varphi_{ij}^{-1}$ $\quad (i, j \in I)$,

(3) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ $\quad (i, j, k \in I)$,

(4) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$ $\quad (i, j, k \in I)$.

Show that there exists a unique scheme X, equipped with morphisms $\psi_i : X_i \to X$ $(i \in I)$, such that

(i) ψ_i yields an isomorphism from X_i onto an open subscheme of X $(i \in I)$,

(ii) $X = \bigcup_{i\in I} \psi_i(X_i)$,

(iii) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j) = \psi_j(U_{ji})$ $\quad (i, j \in I)$,

(iv) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij} $\quad (i, j \in I)$.

We say that X is obtained by *gluing the schemes* X_i along the isomorphisms φ_{ij}.

Exercise 4.4

Let k be an algebraically closed field. We consider two copies of the affine line $\mathbb{A}^1(k)$, which we distinguish by setting $X_1 = \operatorname{Spec}(k[s])$ and $X_2 = \operatorname{Spec}(k[t])$. Let $U_{12} := D(s) \subseteq X_1$ and $U_{21} := D(t) \subseteq X_2$. Let $\varphi_{12} : U_{12} \to U_{21}$ be induced by the isomorphism of rings

$$k[t, t^{-1}] \to k[s, s^{-1}]$$

sending t to s, and let $\tilde{\varphi}_{12}$ be induced by the isomorphism sending t to s^{-1}. Describe the scheme X obtained by gluing X_1 and X_2 along the isomorphisms φ_{12} and the scheme Y obtained by gluing along $\tilde{\varphi}_{12}$ instead. Show that X and Y are not isomorphic.