HU Berlin Summer Term 2012

Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on May 8th in the exercise class.

Exercise sheet 3

Exercise 3.1

Let $X = \mathbb{C}$ be equipped with the Euclidean topology and consider the following (pre)sheaves on X: the locally constant sheaf $\underline{\mathbb{Z}}$ with group $\mathbb{Z}$, the sheaf $\mathcal{O}_X$ of holomorphic functions, and the presheaf $\mathcal{F}$ of functions admitting a holomorphic logarithm. Show that

$$0 \longrightarrow 2\pi i \, \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{F} \longrightarrow 0,$$

where $2\pi i \, \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ is the natural inclusion, is an exact sequence of presheaves. Show that $\mathcal{F}$ is not a sheaf.

Exercise 3.2

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map.

(a) Let $\mathcal{G}$ be a sheaf on Y. Construct explicitly an example such that the presheaf $f^{+}\mathcal{G}$ given by the assignment

$$U \mapsto \varinjlim_{\substack{V \subseteq Y, \text{ open} \\ f(U) \subseteq V}} \mathcal{G}(V) \quad (U \subseteq X, \text{ open})$$

is not a sheaf.

(b) Let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a presheaf on Y. Show that there is a bijection

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathrm{PreSh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

of sets.

Exercise 3.3

Let k be a field. Consider the *projective space* $\mathbb{P}^n(k) := (k^{n+1} \setminus \{0\})/\sim$, where the equivalence relation $\sim$ is given by

$$(x_0, \ldots, x_n) \sim (x'_0, \ldots, x'_n) \iff \exists \lambda \in k \setminus \{0\} : x_i = \lambda x'_i \;\; \forall i = 0, \ldots, n.$$

The equivalence class of a point $(x_0, \ldots, x_n)$ is denoted by $[x_0 : \ldots : x_n]$. For $i = 0, \ldots, n$, we set

$$U_i := \big\{ [x_0 : \ldots : x_n] \in \mathbb{P}^n(k) \,\big|\, x_i \neq 0 \big\} \subset \mathbb{P}^n(k).$$

(a) We define the topology on $\mathbb{P}^n(k)$ by calling a subset $U \subseteq \mathbb{P}^n(k)$ open if $U \cap U_i$ is open in U_i for all $i = 0, \dots, n$. Show that $\{U_i\}_{i=0,\dots,n}$ is an open covering of $\mathbb{P}^n(k)$.

(b) Prove that the map $U_i \to \mathbb{A}^n(k)$, given by

$$[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}\right),$$

is a bijection; here, the hat means that the i-th entry has to be deleted. By means of this bijection we endow U_i with the structure of a locally ringed space isomorphic to $(\mathbb{A}^n(k), \mathcal{O}_{\mathbb{A}^n(k)})$ denoted by $(U_i, \mathcal{O}_{U_i})$.

(c) For an open set $U \subseteq \mathbb{P}^n(k)$, we set

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) := \left\{ f : U \to k \,\middle|\, f\big|_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i) \, \forall i = 0, \dots, n \right\}.$$

Show that

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^n(k)}(U) = \big\{ f : U \to k \,\big|\, & \forall x \in U, \exists x \in V \subseteq U \text{ open}, \\ & \exists g, h \in k[X_0, \dots, X_n] \text{ homogeneous: } \deg(g) = \deg(h), \\ & h(v) \neq 0, f(v) = g(v)/h(v) \ \forall v \in V \big\}. \end{aligned}$$

Conclude that $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a locally ringed space.

Exercise 3.4

A locally ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme*, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. A *morphism of affine schemes* is a morphism of locally ringed spaces. The category of affine schemes will be denoted by (Aff), the category of commutative rings with 1 by (Ring).

(a) Show that the assignment $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ induces a contravariant functor $\mathrm{Spec} : (\mathrm{Ring}) \to (\mathrm{Aff})$.

(b) Show that the assignment $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ induces a contravariant functor $\Gamma : (\mathrm{Aff}) \to (\mathrm{Ring})$.

(c) Prove that the functors Spec and Γ define an anti-equivalence between the category (Ring) and the category (Aff).