HU Berlin Summer Term 2012

Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on May 8th in the exercise class.

Exercise sheet 2

Exercise 2.1

Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set and let $R(X) := k[X_1, \ldots, X_n]/I(X)$ denote its coordinate ring. Let $\mathcal{O}_X$ denote the sheaf of regular functions on X. For $f \in R(X)$, prove the equality

$$\mathcal{O}_X(D(f)) = R(X)_f \subseteq \mathrm{Quot}(R(X)).$$

In particular, deduce that $\mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X) = R(X)$.

Exercise 2.2

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. Let all occurring (pre)sheaves be (pre)sheaves of abelian groups.

(a) For a sheaf $\mathcal{F}$ on X, we define the *direct image* sheaf $f_*\mathcal{F}$ by

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$$

for any open subset $V \subseteq Y$. Show that $f_*\mathcal{F}$ is a sheaf on Y.

(b) For a sheaf $\mathcal{G}$ on Y, we define the *inverse image* sheaf $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{\substack{V \subseteq Y \text{ open} \\ f(U) \subseteq V}} \mathcal{G}(V),$$

where $U \subseteq X$ is an open subset. Show that f^{-1} is a functor from the category of sheaves on Y to the category of sheaves on X.

Exercise 2.3 (Ex. II.1.15. of [Har])

Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on a topological space X. For any open set $U \subseteq X$, show that the set $\mathrm{Hom}(\mathcal{F}\big|_U, \mathcal{G}\big|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf

$$U \mapsto \mathrm{Hom}(\mathcal{F}\big|_U, \mathcal{G}\big|_U),$$

where $U \subseteq X$ is an open subset, is a sheaf. It is called the *sheaf Hom* and is denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Notation: If $U \subseteq X$ is regarded as a topological subspace with the induced topology and if $i : U \to X$ denotes the inclusion map, then we call the sheaf $i^{-1}\mathcal{F}$ the *restriction of* $\mathcal{F}$ *to* U and we denote it by $\mathcal{F}\big|_U$.

Exercise 2.4 (Ex. II.1.22. of [Har])
Let X be a topological space and let $\mathfrak{U} = \{U_i\}$ be an open cover of X. Furthermore, suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i and for each pair i, j an isomorphism

$$\varphi_{ij} : \mathcal{F}_i\big|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_i \cap U_j}$$

of sheaves such that

(1) for each i: $\varphi_{ii} = \mathrm{id}$,

(2) for each i, j, k: $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Show that there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms of sheaves $\psi_i : \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that, for each i, j, the equality $\psi_j = \varphi_{ij} \circ \psi_i$ holds on $U_i \cap U_j$. We say that $\mathcal{F}$ is obtained by *glueing* the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.