Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on April 24th in the exercise class.

Exercise sheet 1

Exercise 1.1 (Ex. II.1.2. of [Har])

(a) For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on a topological space X, show that $\ker(\varphi)_P = \ker(\varphi_P)$ and $\operatorname{im}(\varphi)_P = \operatorname{im}(\varphi_P)$, for each point $P \in X$.

(b) Show that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if the induced map on the stalks $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective (resp. surjective) for all $P \in X$.

(c) Show that a sequence

$$\ldots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \xrightarrow{\varphi^{i+1}} \ldots$$

of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Exercise 1.2 (Ex. II.1.3. of [Har])

(a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Show that φ is surjective if and only if the following condition holds: For every open set $U \subseteq X$ and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\varphi(t_i) = s\big|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ and an open set $U \subseteq X$ such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Exercise 1.3 (Ex. II.1.6. of [Har])

(a) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$ on a topological space X. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$. Thus, there is an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

(b) Conversely, if $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Exercise 1.4 (Ex. II.1.8. of [Har])

For any open subset U of a topological space X, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$ is an exact sequence of sheaves, then $0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'')$ is an exact sequence of abelian groups. We note that the functor $\Gamma(U, \cdot)$ need not be exact.