# Exercises BMS Basic Course
# **Algebraic Geometry**

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Solution to be presented on July 10th in the exercise class.

**Exercise sheet 12**

**Exercise 12.1 (Ex. II.1.17. of [Har])**

Let $X$ be a topological space, $x \in X$ a point, and $A$ an abelian group. Define a sheaf $i_x(A)$ on $X$ by the assignment

$$i_x(A)(U) = \begin{cases} A, & \text{if } x \in U, \\ 0, & \text{otherwise;} \end{cases} \qquad (U \subseteq X, \text{ open}).$$

Show that for the stalk $i_x(A)_y$ at a point $y \in X$, we have

$$i_x(A)_y = \begin{cases} A, & \text{if } y \in \overline{\{x\}}, \\ 0, & \text{otherwise;} \end{cases}$$

whence the name *skyscraper sheaf* originates. Show that the skyscraper sheaf could also be described as $i_\star(A)$, where $A$ denotes the constant sheaf $A$ on the closed subspace $\overline{\{x\}}$ and $i : \overline{\{x\}} \longrightarrow X$ is the inclusion.

**Exercise 12.2 (Ex. II.1.19. of [Har])**

Let $X$ be a topological space, $Z \subseteq X$ a closed subset, and $i : Z \longrightarrow X$ the inclusion. Further, let $U = X \setminus Z$ be the complementary open subset and $j : U \longrightarrow X$ its inclusion.

(a) Let $\mathcal{F}$ be a sheaf on $Z$. Show that for the stalk $(i_\star\mathcal{F})_z$ at a point $z \in Z$, we have

$$(i_\star\mathcal{F})_z = \begin{cases} \mathcal{F}_z, & \text{if } z \in Z, \\ 0, & \text{otherwise;} \end{cases}$$

hence, we call the sheaf $i_\star\mathcal{F}$ the *sheaf obtained by extending $\mathcal{F}$ by zero outside $Z$.*

(b) Let $\mathcal{F}$ be a sheaf on $U$. Let $j_!(\mathcal{F})$ be the sheaf on $X$ associated to the presheaf given by the assignment

$$j_!(\mathcal{F})(V) := \begin{cases} \mathcal{F}(V), & \text{if } V \subseteq U, \\ 0, & \text{otherwise;} \end{cases} \qquad (V \subseteq X, \text{ open}).$$

Show that for the stalk $j_!(\mathcal{F})_x$ at a point $x \in U$, we have

$$j_!(\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & \text{if } x \in U, \\ 0, & \text{otherwise;} \end{cases}$$

furthermore, show that $j_!(\mathcal{F})$ is the only sheaf on $X$ which has this property, and whose restriction to $U$ is $\mathcal{F}$. We call $j_!(\mathcal{F})$ the *sheaf obtained by extending $\mathcal{F}$ by zero outside $U$.*

(c) Let $\mathcal{F}$ be a sheaf on $X$. Show that there is the following exact sequence of sheaves on $X$
$$0 \longrightarrow j_!(\mathcal{F}\big|_U) \longrightarrow \mathcal{F} \longrightarrow i_\star(\mathcal{F}\big|_Z) \longrightarrow 0.$$

**Exercise 12.3 (Ex. II.5.6. of [Har])**

Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from exercise sheet 11.

(a) Let $A$ be a ring, $M$ an $A$-module, $X = \operatorname{Spec}(A)$, and $\mathcal{F} = \widetilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\operatorname{Supp}(m) = V(\operatorname{Ann}(m))$.

(b) If $A$ is a noetherian ring and $M$ a finitely generated $A$-module, show that $\operatorname{Supp}(\mathcal{F}) = V(\operatorname{Ann}(M))$.

(c) Show that the support of a coherent sheaf on a noetherian scheme is closed.

(d) Again, let $A$ be a ring and $M$ an $A$-module. For an ideal $\mathfrak{a} \subseteq A$, we define the submodule $\Gamma_{\mathfrak{a}}(M)$ of $M$ by
$$\Gamma_{\mathfrak{a}}(M) := \big\{ m \in M \,\big|\, \exists\, n \in \mathbb{N} : \mathfrak{a}^n m = 0 \big\}.$$
Show that if $A$ is noetherian, $X = \operatorname{Spec}(A)$, and $\mathcal{F} = \widetilde{M}$, we have an isomorphism of $\mathcal{O}_X$-modules
$$\widetilde{\Gamma_{\mathfrak{a}}(M)} \cong \mathcal{H}^0_Z(\mathcal{F}),$$
where $Z = V(\mathfrak{a})$ and $\mathcal{H}^0_Z(\mathcal{F})$ is defined in Exercise 11.3.

(e) Let $X$ be a noetherian scheme and $Z \subseteq X$ a closed subset. If $\mathcal{F}$ is a quasi-coherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathcal{H}^0_Z(\mathcal{F})$ is also quasi-coherent (respectively, coherent).