HU Berlin Summer Term 2012

Exercises BMS Basic Course
Algebraic Geometry

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Solution to be presented on July 3rd in the exercise class.

Exercise sheet 11

Exercise 11.1 (Ex. II.1.14. of [Har])

Let $\mathcal{F}$ be a sheaf on a topological space X, and let $s \in \mathcal{F}(U)$ be a section over an open set $U \subseteq X$. The support $\mathrm{Supp}(s)$ of s is defined to be

$$\mathrm{Supp}(s) := \{P \in U \mid s_P \neq 0\},$$

where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. Show that $\mathrm{Supp}(s)$ is a closed subset of U. We define *the support* $\mathrm{Supp}(\mathcal{F})$ *of* $\mathcal{F}$ as

$$\mathrm{Supp}(\mathcal{F}) := \{P \in X \mid \mathcal{F}_P \neq 0\}.$$

It need not be a closed subset.

Exercise 11.2 (Ex. II.1.16. of [Har])

A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set $U \subseteq X$, the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

of abelian groups is also exact.

(c) If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

(d) If $f : X \longrightarrow Y$ is a continuous map of topological spaces, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_\star\mathcal{F}$ is a flasque sheaf on Y.

(e) Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the *sheaf of discontinuous sections of* $\mathcal{F}$ as follows. For each open set $U \subseteq X$, $\mathcal{G}(U)$ is the set of maps

$$s : U \longrightarrow \bigcup_{P \in U} \mathcal{F}_P$$

such that for each $P \in U$, $s(P) \in \mathcal{F}_P$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Exercise 11.3 (Ex. II.1.20. of [Har])

Let Z be a closed subset of a topological space X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support is contained in Z.

(a) Show that the presheaf given by the assignment

$$V \mapsto \Gamma_{Z \cap V}(V, \mathcal{F}\big|_V) \qquad (V \subseteq X, \text{ open})$$

is a sheaf. It is called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

(b) Let $U = X \setminus Z$, and let $j : U \longrightarrow X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_\star(\mathcal{F}\big|_U).$$

Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \longrightarrow j_\star(\mathcal{F}\big|_U)$ is surjective.