
Problem 1.1. Let k be a field.

(a) Describe the Zariski topology on $\mathbb{A}^1(k)$. Starting from this, show that for $n > 1$ the Zariski topology on $\mathbb{A}^n(k)$ is the coarsest topology such that for all $f \in k[x_1, \dots, x_n]$ the map
$$f: \quad \mathbb{A}^n(k) \longrightarrow \mathbb{A}^1(k), \quad p \mapsto f(p) \quad \text{is continuous.}$$

(b) Show that if k is infinite, then the Zariski topology on the affine plane $\mathbb{A}^2(k)$ does *not* coincide with the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$.

Problem 1.2. Which of the following subsets are Zariski closed? In each case, determine also the annihilator ideal of the subset, and when the subset is not Zariski closed, find its Zariski closure:

$$\begin{aligned} Z_1 &:= \{(t, t^2, t^3) \mid t \in \mathbb{C}\} \subset \mathbb{A}^3(\mathbb{C}), \\ Z_2 &:= \{(t, \exp(t)) \mid t \in \mathbb{C}\} \subset \mathbb{A}^2(\mathbb{C}), \\ Z_3 &:= \{(\exp(2t), \exp(3t)) \mid t \in \mathbb{C}\} \subset \mathbb{A}^2(\mathbb{C}). \end{aligned}$$

Problem 1.3. Let k be an algebraically closed field and $\mathrm{Mat}(2\times 2, k)$ the space of 2×2 matrices over it, viewed as an affine space via

$$\mathbb{A}^4(k) \xrightarrow{\sim} \mathrm{Mat}(2 \times 2, k), \quad (a, b, c, d) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that the ideals

$$\begin{aligned} I_1 &:= (a^2 + bc, d^2 + bc, (a + d)b, (a + d)c) \trianglelefteq k[a, b, c, d] \\ I_2 &:= (ad - bc, a + d) \trianglelefteq k[a, b, c, d] \end{aligned}$$

satisfy $V(I_1) = V(I_2) = \{M \in \mathrm{Mat}(2 \times 2, k) \mid M \text{ is nilpotent}\}$ and that $\sqrt{I_1} = I_2$ but $I_1 \neq I_2$.

Problem 1.4. Let k be a field with algebraic closure $\overline{k}$.

(a) Let $I \ntrianglelefteq k[x_1, \dots, x_n]$ be a proper ideal. Show that $I \cdot \overline{k}[x_1, \dots, x_n] \ntrianglelefteq \overline{k}[x_1, \dots, x_n]$, and deduce
$$V(I) := \{\, p \in \mathbb{A}^n(\overline{k}) \mid \forall f \in I : f(p) = 0 \,\} \neq \varnothing.$$

(b) For $k = \mathbb{R}$, show that the ideals $I_1 = (x^2 + y^2), I_2 = (x, y) \trianglelefteq \mathbb{R}[x, y]$ are distinct and both radical ideals. What does this tell you about Hilbert's Nullstellensatz?

Problem 2.1. Let k be an algebraically closed field.

(a) Show that
$$f: \quad \mathbb{A}^1(k) \longrightarrow V(y^2 - x^3) \subset \mathbb{A}^2(k), \quad t \mapsto (t^2, t^3)$$
is a morphism of affine algebraic sets which is bijective but not an isomorphism.

(b) Consider the algebraic subsets
$$\begin{aligned} Z_1 &= V(w(v-1)-1) \quad \subset \quad \mathbb{A}^2(k) = \operatorname{Spec}(k[v,w]), \\ Z_2 &= V(y^2 - x^2(x+1)) \quad \subset \quad \mathbb{A}^2(k) = \operatorname{Spec}(k[x,y]). \end{aligned}$$
Show that
$$g: \quad Z_1 \longrightarrow Z_2, \quad (v,w) \mapsto (v^2 - 1, v(v^2-1))$$
is a well-defined morphism which for $\operatorname{char}(k) \neq 2$ is bijective but not an isomorphism.

Problem 2.2. Let A be a ring.

(a) Show that for any ideals $I, J \trianglelefteq A$ the following equivalences hold:
$$V(I) \subseteq V(J) \text{ in } \operatorname{Spec}(A) \iff J \subseteq \sqrt{I} \iff \sqrt{J} \subseteq \sqrt{I}$$

(b) Let $Z_1, \dots, Z_n \subseteq \operatorname{Spec}(A)$ be irreducible closed subsets. Let $I \trianglelefteq A$ be an ideal. Assume that for each i there is an $f_i \in I$ with $Z_i \not\subseteq V(f_i)$. Show that there is an $f \in I$ such that
$$Z_i \not\subseteq V(f) \quad \text{for all} \quad i \in \{1, \dots, n\}.$$

Problem 2.3. Let $\varphi : A \to B$ be a ring homomorphism with the property that every $b \in B$ can be written as a product of the form $b = e \cdot \varphi(a)$ with a unit $e \in B^\times$ and $a \in A$. Show that $\operatorname{Spec}(\varphi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a homeomorphism onto its image when the image is endowed with the subspace topology induced from $\operatorname{Spec}(A)$.

Problem 2.4. Let A be a ring.

(a) Let $\mathfrak{p} \trianglelefteq A$ be a minimal prime ideal. Show that the ideal $\mathfrak{p}A_\mathfrak{p} \trianglelefteq A_\mathfrak{p} = (A \setminus \mathfrak{p})^{-1}A$ is nilpotent, and deduce that every element of $\mathfrak{p}$ is a zero divisor.

(b) Conversely, show that if A is reduced, then every zero divisor of A lies in a minimal prime ideal of A. Does this also hold without the reducedness assumption?

Problem 3.1. Let $I \trianglelefteq R = k[x_1, \ldots, x_n]$ and $A = R/I$ for an algebraically closed field k.

(a) Show that the following topological spaces are homeomorphic:

- the affine algebraic subset $Y = V(I) \subseteq \mathbb{A}^n(k)$ with the Zariski topology,
- the subset $X_0 \subseteq X$ of closed points of $X = \operatorname{Spec}(A)$ with the subspace topology.

(b) Show that we have an equivalence of categories $\operatorname{Sh}(X) \xrightarrow{\sim} \operatorname{Sh}(X_0)$.

(c) Let $\mathcal{F}$ be a sheaf of abelian groups on X with $\mathcal{F}_x \simeq 0$ for all $x \in X_0$. Show that $\mathcal{F} \simeq 0$.

Problem 3.2. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X.

(a) Show that for any $s, t \in \mathcal{F}(X)$ the subset $\{x \in X \mid s_x = t_x\} \subseteq X$ is open.

(b) Give an example where the support $\operatorname{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \not\simeq 0\} \subseteq X$ is not closed.

Problem 3.3. Let X be a topological space, and B a basis of open subsets for its topology that is stable under finite intersections in the sense that $U \cap V \in B$ for all $U, V \in B$. Show that the category $\operatorname{Sh}(X)$ of sheaves of sets on X is equivalent to the category of functors

$$\mathcal{F} : \quad B^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

satisfying the sheaf axiom for all open covers of sets $U = \bigcup_{i \in I} U_i \in B$ by sets $U_i \in B$.

Problem 3.4. Let X be a topological space.

(a) Show that if X is irreducible, then for any presheaf $\mathcal{F}$ on X the following are equivalent:

- $\mathcal{F}$ is a constant sheaf on X.
- X can be covered by open subsets $U \subseteq X$ for which $\mathcal{F}|_U$ is a constant sheaf.
- $\mathcal{F}(\varnothing) \simeq \{0\}$, and $\operatorname{res}^X_U : \mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism for all open $\varnothing \neq U \subseteq X$.

(b) Conversely, show that if there exists a sheaf $\mathcal{F}$ of sets on X such that $|\mathcal{F}(X)| \geq 2$ and if the restriction maps

$$\operatorname{res}^X_U : \quad \mathcal{F}(X) \longrightarrow \mathcal{F}(U)$$

are bijective for every non-empty open subset $U \subseteq X$, then X is irreducible.

Problem 4.1. Let $\mathscr{F}$ be a presheaf on a topological space X. The disjoint union $F = \bigsqcup_{x \in X} \mathscr{F}_x$ becomes a topological space by taking as a basis of open subsets for the topology the subsets of the form $D(U, s) := \{s_x \in \mathscr{F}_x \mid x \in U\} \subseteq F$ with $U \subseteq X$ open and $s \in \mathscr{F}(U)$. Show that the projection $p : F \to X$ is a continuous map and the sheafification of the presheaf is given by

$$\mathscr{F}^{\mathrm{sh}}(U) \;=\; \{\, s : U \to F \mid p \circ s = \mathrm{id}_U \,\} \quad \text{for } U \subseteq X \text{ open.}$$

Problem 4.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space.

(a) We call a subset $U \subseteq X$ *clopen* if it is both open and closed. Show that for any such subset there exists a unique global section $e_U \in \Gamma(\mathscr{O}_X)$ with $e_U|_U = 1$ and $e_U|_{X \setminus U} = 0$, and deduce that we have a bijection

$$\big\{\text{clopen subsets of } X\big\} \;\xrightarrow{\sim}\; \big\{\text{idempotents of the ring } \Gamma(\mathscr{O}_X)\big\}, \quad U \;\mapsto\; e_U.$$

Show furthermore that for any two clopen subsets $U, V \subseteq X$ we have $e_U e_V = e_{U \cap V}$.

(b) Show that the following properties are equivalent:

- The topological space X is connected.
- The ring $\Gamma(\mathscr{O}_X)$ has no idempotents $\neq 0, 1$.
- The ring $\Gamma(\mathscr{O}_X)$ is not isomorphic to a product $R_1 \times R_2$ of two non-zero rings.

(c) Show that for any local ring A the spectrum $\mathrm{Spec}(A)$ is connected.

Problem 4.3. Let k be a field, and consider the localization $R = k[x]_{(x)} \subset K = k(x)$.

(a) Describe all open subsets $U \subseteq X = \mathrm{Spec}(R)$ and the corresponding rings $\mathscr{O}_X(U)$.

(b) Let $Z = \{p, q_1, q_2\}$ be a space with precisely three points, endowed with the coarsest topology where the subsets $U_i := \{p, q_i\} \subset Z$ are open for $i = 1, 2$. Define a sheaf $\mathscr{O}_Z$ of rings by

$$\mathscr{O}_Z(U_1) = \mathscr{O}_Z(U_2) = R \qquad \text{and} \qquad \mathscr{O}_Z(U_1 \cap U_2) = K,$$

with $\mathrm{res}^{U_i}_{U_1 \cap U_2}$ the natural inclusion. Show that $(Z, \mathscr{O}_Z)$ is a scheme but not affine.

Problem 4.4. Let X be a scheme. Show that for any prime p the following are equivalent:

(a) For every open $U \subseteq X$ the ring $\mathscr{O}_X(U)$ has characteristic p.

(b) The ring $\Gamma(\mathscr{O}_X)$ has characteristic p.

(c) The natural morphism $X \to \mathrm{Spec}\,\mathbb{Z}$ factors over $\mathrm{Spec}\,\mathbb{Z}/p\mathbb{Z}$.

Are they also equivalent to the residue field $\kappa(x)$ having characteristic p for all $x \in X$?

Problem 5.1. Let R be a discrete valuation ring, and let X be a scheme. Describe the set of morphisms of schemes

$$X(R) \;=\; \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), X).$$

Problem 5.2. Let k be a field.

(a) Let $A = k[u,v,x,y]/(ux,uy,vx,vy)$. Show that $X = \operatorname{Spec} A$ is the union of two affine planes meeting at a single point. Denote the complement of that point by $U \subseteq X$, and put

$$S \;:=\; \{f \in A \mid \forall x \in U : f_x \notin \mathfrak{m}_x \trianglelefteq \mathscr{O}_{X,x}\} \;\subset\; A.$$

Show that the natural homomorphism $S^{-1}A \longrightarrow \mathscr{O}_X(U)$ is *not* an isomorphism.

(b) Why doesn't this happen for the union of two affine *lines* meeting at a single point?

Problem 5.3. Let $B = k[x_1,x_2,y_1,y_2]/(x_1y_2 - x_2y_1)$ and $Z = \operatorname{Spec} B$. For $i = 1,2$, consider the sections

$$s_i \;:=\; \frac{x_i}{y_i} \;\in\; \mathscr{O}_Z(U_i) \quad \text{on the open subset} \quad U_i \;:=\; D(y_i) \;\subseteq\; Z.$$

Show that

(a) these two sections glue to a section $s \in \mathscr{O}_Z(U)$ on $U := U_1 \cup U_2$.

(b) the section s cannot be written as $\frac{a|_U}{b|_U}$ with $a, b \in B = \Gamma(\mathscr{O}_Z)$ and $b|_U \in (\mathscr{O}_Z(U))^\times$.

(c) the closed subset $W := V(y_1) \cap V(y_2) \subset |Z|$ is not of the form $V(f)$ for any $f \in B$.

Note: Nevertheless one can show for the Krull dimensions that $\dim W = \dim Z - 1$.

Problem 5.4. Let X be a scheme over a ring R, and let $f_0, \dots, f_n \in \Gamma(\mathscr{O}_X)$ be sections such that at every point $p \in |X|$ their germs generate the unit ideal of the local ring:

$$\mathscr{O}_{X,p} \;=\; \big(f_{0,p},\, \dots,\, f_{n,p}\big).$$

Show that

(a) $X = \bigcup_{i=0}^n D(f_i)$.

(b) there is a unique morphism of schemes $\varphi : X \to \mathbb{P}^n_R$ such that

- for all $i \in \{0,1,\dots,n\}$ we have $\varphi(D(f_i)) \subseteq U_i := \operatorname{Spec} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \subset \mathbb{P}^n_R$,
- the restriction $\varphi : D(f_i) \to U_i$ is given by $R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \to \mathscr{O}_X(D(f_i))$, $\frac{x_j}{x_i} \mapsto \frac{f_j}{f_i}$.

Problem 6.1. Let X be a scheme.

(a) Define the notion of finite unions and arbitrary intersections of closed subschemes of X, and show that they behave on the underlying topological spaces as expected.

(b) Describe the scheme-theoretic intersections

$$V(y-x^2)\cap V(y), \quad V(x-y^3)\cap V(x), \quad V(x^2-y^2)\cap V(y) \quad \subseteq \quad X \;=\; \operatorname{Spec}\mathbb{C}[x,y].$$

Explain via the first two examples how to see multiplicities and tangent directions, and via the third that in general we may have $(A\cup B)\cap C \neq (A\cap C)\cup(B\cap C)$.

Problem 6.2. Let $f: X\to Z$ be a quasi-compact morphism of schemes and $\mathscr{I} := \ker(f^\sharp)$.

(a) Show that $Y := (\operatorname{Supp}(\mathscr{O}_Z/\mathscr{I}), \operatorname{incl}^{-1}(\mathscr{O}_Z/\mathscr{I}))$ is a scheme.

(b) Show that Y is the smallest closed subscheme of Z over which f factors.

We call the closed subscheme $\operatorname{im}(f) := Y \subseteq Z$ the *scheme-theoretic image* of f.

Problem 6.3. Describe the sheaf of ideals $\mathscr{I} := \ker(f^\sharp : \mathscr{O}_Z \to f_*\mathscr{O}_X) \trianglelefteq \mathscr{O}_Z$ for the natural morphism

$$f: \quad X \;:=\; \bigsqcup_{n\in\mathbb{N}} \operatorname{Spec} k[t]/(t^n) \;\longrightarrow\; Z \;:=\; \operatorname{Spec} k[t]$$

of schemes. Is the ringed space $V(\mathscr{I}) := (\operatorname{Supp}(\mathscr{O}_Z/\mathscr{I}), \operatorname{incl}^{-1}(\mathscr{O}_Z/\mathscr{I}))$ a scheme? Compare

(a) the set-theoretic image $f(|X|)\subseteq |Z|$,

(b) the scheme-theoretic image $\operatorname{im}(f)\subseteq Z$.

Problem 6.4. Let $n,d\in\mathbb{N}$, and put $m := \binom{n+d}{n} - 1$.

(a) Show that there is a closed embedding $v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ which is given on points over a field k by

$$v_d: \quad \mathbb{P}^n(k) \;\hookrightarrow\; \mathbb{P}^m(k), \quad [a_0:\cdots:a_n] \;\mapsto\; [f_0(a):\cdots:f_m(a)],$$

where $f_0,\dots,f_m \in k[x_0,\dots,x_n]$ denote the homogenous monomials of degree d.

(b) Show that the zero locus of any homogenous polynomial of degree d in $\mathbb{P}^n$ is isomorphic to the intersection $v_d(\mathbb{P}^n)\cap H$ for some linear subspace $H\subset\mathbb{P}^m$.

We call v_d the *Veronese embedding of degree d.*

Problem 7.1. Show that:

(a) Every quasi-compact scheme has at least one closed point.

(b) A quasi-compact scheme X is reduced iff $\mathcal{O}_{X,p}$ is reduced for all closed points $p \in |X|$.

Problem 7.2. Let $K := \mathbb{Q}(x_1, x_2, \dots)$, the rational function field in countably many variables.

(a) Show that there is a unique valuation

$$v : \quad K^\times \longrightarrow \Gamma := (\oplus_{i\in\mathbb{N}} \mathbb{Z}, \succeq) \quad \text{(where } \succeq \text{ is the lexicographic order)}$$

such that $v(x_i) = e_i$ is the i-th standard basis vector for all $i \in \mathbb{N}$, and $v|_{\mathbb{Q}^\times} = 0$.

(b) Let $R := \{a \in K^\times \mid v(a) \succeq 0\} \cup \{0\}$. Show that for any $\mathfrak{p} \in \operatorname{Spec}(R)$,

- either $\mathfrak{p} \subseteq \mathfrak{p}_n := \{a \in K^\times \mid v(a) \succeq e_n\} \cup \{0\}$ for some $n \in \mathbb{N}$,
- or $\mathfrak{p} = \mathfrak{m}$ where $\mathfrak{m} := \bigcup_{n\in\mathbb{N}} \mathfrak{p}_n \trianglelefteq R$ is the maximal ideal of R.

(c) Show that $Y := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ is a scheme which has no closed point.

(d) Use this to give an example of a non-reduced scheme X without closed points.

Problem 7.3. Show that for a morphism $(f, f^\sharp) : X \to Y$ of integral schemes, the following are equivalent:

(a) The image of $f : |X| \to |Y|$ is dense in $|Y|$.

(b) The homomorphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is injective.

(c) $f^\sharp(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective for every open $V \subseteq Y$ and every open $U \subseteq f^{-1}(V)$.

(d) We have $f(\eta_X) = \eta_Y$ for the generic points $\eta_X \in |X|$ and $\eta_Y \in |Y|$.

(e) The image of $f : |X| \to |Y|$ contains the generic point $\eta_Y \in |Y|$.

If these properties are satisfied, we say that the morphism $(f, f^\sharp) : X \to Y$ is *dominant*.

Problem 7.4. Let $f : X \to Y$ be a dominant morphism between integral schemes.

(a) Show that f induces an embedding of function fields $f^* : K(Y) \hookrightarrow K(X)$.

(b) Show that if f is of finite type and f^* is an isomorphism, then there is an open $U \subseteq X$ such that

- the image $V := f(U) \subseteq Y$ is open, and
- the restriction $f|_U : U \to V$ is an isomorphism.

(c) What happens for the morphism $X = \operatorname{Spec}(\mathbb{Z}_{(2)}) \to Y = \operatorname{Spec}(\mathbb{Z})$?

Problem 8.1. Let k be a field and let $A = k[x, y_1, y_2, \dots]/(y_i^2, (x - a_i)y_{i+1} - y_i \mid i \in \mathbb{N})$, where $a_1, a_2, a_3, \dots \in k$ form a sequence of pairwise distinct elements of the base field. Show that

(a) the scheme $X := \mathrm{Spec}(A)$ is not Noetherian,

(b) the local rings $\mathcal{O}_{X,p}$ are Noetherian for all $p \in |X|$.

Problem 8.2. Let $f : Y \to X$ be a morphism of schemes which is locally of finite type, and let $\eta \in |X|$ be a generic point of an irreducible component of X. Show that the following properties are equivalent:

(a) The subset $f^{-1}(\eta) \subseteq |Y|$ is finite.

(b) There are an affine open subset $U \subseteq X$ with $\eta \in U$ and affine open subsets $V_i \subseteq f^{-1}(U)$ with $f^{-1}(\eta) \subseteq \bigcup_{i=1}^n V_i$ such that

$$f|_{V_i} : \quad V_i \longrightarrow U \quad \text{is a finite morphism for } i = 1, \dots, n.$$

We then say that the morphism f is *generically finite* over the point η.

Add-on. Show that if X and Y are Noetherian, then in (b) we can take $n = 1$ and $V_1 = f^{-1}(U)$.

Problem 8.3. Let G be a finite group of automorphisms of a ring A, and let $B = A^G \subseteq A$ be the subring of elements invariant under the group action.

(a) Let $p : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be the natural morphism. Show for all $x, y \in |\mathrm{Spec}(A)|$ the equivalence

$$p(x) = p(y) \iff \exists g \in G : \quad gx = y.$$

(b) Show that the morphism $p : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is integral and surjective.

(c) Show that $p(D(a)) = \bigcup_{i<d} D(b_i)$ for $a \in A$, where $b_1, \dots, b_{d-1} \in B$ are the coefficients of

$$\prod_{g \in G} (t - ga) = t^d + b_{d-1}t^{d-1} + \cdots + b_1 t + b_0 \in B[t].$$

Deduce that p is an open morphism and topologically $|\mathrm{Spec}(B)| \simeq |\mathrm{Spec}(A)|/G$.

(d) Show that for any open $U \subseteq \mathrm{Spec}(B)$ the group G acts on $V = p^{-1}(U) \subseteq \mathrm{Spec}(A)$, and that

$$\mathcal{O}_{\mathrm{Spec}(B)}(U) \simeq (\mathcal{O}_{\mathrm{Spec}(A)}(V))^G.$$

Problem 9.1. Let k be an algebraically closed field of characteristic $\mathrm{char}(k) \neq 2$.

(a) Let $f \in k[x_1, \ldots, x_n]$ be a homogenous polynomial of degree two. Show that there is an isomorphism

$$V(f) \simeq V(x_1^2 + \cdots + x_r^2) \subseteq \mathbb{A}_k^n \quad \text{for a unique} \quad r \in \{1, \ldots, n\}.$$

(b) For which r is the affine quadric $V(f) \subseteq \mathbb{A}_k^n$ integral? For which r is it normal?

Hint: In the integral case the function field of $V(x_1^2 + \cdots + x_r^2)$ is a degree two field extension of $k(x_2, \ldots, x_n)$. Look at minimal polynomials in that field extension...

Problem 9.2. Let k be a field and $C = \operatorname{Spec} k[x, y]/(y^2 - f(x))$ where $f \in k[x]$ is monic.

(a) Show that C is integral iff the polynomial $f(x)$ is not a square.

(b) Now assume that $\mathrm{char}(k) \neq 2$.

- Show that C is normal iff $f(x)$ has no square factor.
- In the cases where C is not normal, determine its normalization.

(c) Can you extend your results to perfect fields k of characteristic $\mathrm{char}(k) = 2$?

Problem 9.3. A scheme X is called *locally factorial* if $\mathscr{O}_{X,p}$ is factorial for all $p \in |X|$.

(a) Show that every locally factorial scheme is normal.

(b) Show that $X = \operatorname{Spec} \mathbb{C}[w, x, y, z]/(wz - xy)$ is normal but not locally factorial.

Problem 9.4. The property of being a factorial ring is not affine-local in the usual sense:

(a) Show that $X = \operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$ is locally factorial but $\Gamma(\mathscr{O}_X) = \mathbb{Z}[\sqrt{-5}]$ is not factorial.

(b) Bonus if you know some number theory: Show that for the cover $X = D(2) \cup D(3)$ by basic open subsets, both $\mathscr{O}_X(D(2))$ and $\mathscr{O}_X(D(3))$ are factorial.

Problem 10.1. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Show that the following are equivalent:

(a) The morphism f is surjective.

(b) For every algebraically closed field K the morphism $f : X(K) \to Y(K)$ is surjective.

(c) For every morphism $Z \to Y$ the base change $f_Z : X \times_Y Z \to Z$ is surjective.

Problem 10.2. Let X be a scheme of finite type over a field k with algebraic closure $\overline{k}$.

(a) Show that the following properties are equivalent:

- $X \times_k \overline{k}$ is irreducible.
- $X \times_k k_s$ is irreducible.
- $X \times_k K$ is irreducible for *every* field extension K/k.

Here k_s/k denotes the *separable closure*, the maximal separable subextension of $\overline{k}/k$.

(b) Show that the following properties are equivalent:

- $X \times_k \overline{k}$ is reduced.
- $X \times_k k_p$ is reduced.
- $X \times_k K$ is reduced for *every* field extension K/k.

Here k_p/k is the *perfect closure*, the maximal purely inseparable subextension of $\overline{k}/k$

(c) Find examples of integral schemes X where properties (a) respectively (b) fail.

Problem 10.3. Let $f : X \to Y$ a morphism of S-schemes, with Y separated over S. Show that

(a) the graph $\Gamma_f := (\mathrm{id}_X, f) : X \hookrightarrow X \times_S Y$ is a closed immersion.

(b) for any closed subscheme $Z \subseteq X$ proper[1] over S, the image $f(|Z|) \subseteq |Y|$ is closed.

Problem 10.4. Let $T \to S$ be a morphism of schemes.

(a) Find a left adjoint for the base change functor $(-)_T : \mathrm{Sch}_S \to \mathrm{Sch}_T$.

(b) In general the base change functor has no right adjoint. If for a scheme $X \in \mathrm{Sch}_T$ the functor

$$\mathrm{Res}_{T/S}(X) : \quad \mathrm{Sch}_S^{\mathrm{op}} \longrightarrow \mathrm{Sets}, \quad Y \mapsto \mathrm{Hom}_T(Y_T, X).$$

is representable by an S-scheme we call that S-scheme the *Weil restriction* of X. Show that for $T = \mathrm{Spec}(\mathbb{C}) \to S = \mathrm{Spec}(\mathbb{R})$ the Weil restriction of $X = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ exists and is given by

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(X) \simeq \mathrm{Spec}\ \mathbb{R}[x,y][\tfrac{1}{x^2+y^2}]$$

Hint: Show that

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(X)(A) = \left\{ \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right) \in \mathrm{GL}_2(A) \right\} \quad \text{for any } \mathbb{R}\text{-algebra } A.$$

[1] A morphism $Z \to S$ is *proper* if it is separated, of finite type and $X_T \to T$ is closed for all $T \to S$.

Problem 11.1. Let $X, Y \in \mathrm{Sch}_S$ be schemes over S.

(a) Show that if Y is reduced and X is separated over S, then for any $g : Z \to Y$ with dense image the map

$$\mathrm{Hom}_S(Y, X) \hookrightarrow \mathrm{Hom}_S(Z, X), \quad f \mapsto f \circ g \quad \text{is injective.}$$

(b) Show by examples that this may fail if Y is not reduced or if X is not separated over S.

Problem 11.2. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes.

(a) Show that the following "cancellation properties" hold:

- If $g \circ f$ is separated, then f is separated.
- If g is separated and $g \circ f$ is proper, then f is proper.

Illustrate by an example that in the second part the separatedness of g is needed.

(b) Show that if f is surjective and $g \circ f$ is universally closed, then g is universally closed.

Illustrate by an example that in general the surjectivity assumption on f is needed.

Problem 11.3. Show that for any quasi-compact morphism $f : X \to Y$ of schemes the following are equivalent:

(a) The image $f(|X|) \subseteq |Y|$ is closed.

(b) The image $f(|X|) \subseteq |Y|$ is stable under specialization.

Problem 11.4. Show that:

(a) A separated morphism $f : X \to S$ of finite type between Noetherian schemes is proper iff the base change

$$f_T : \quad X_T \;=\; X \times_S T \;\longrightarrow\; T$$

by all $T \to S$ *of finite type* is a closed morphism. Hint: You may use Chow's lemma.

(b) For morphisms of finite type between *Noetherian* schemes, the valuative criterion for separatedness and properness can be checked using only *discrete* valuation rings.

Problem 12.1. Let S be a graded ring.

(a) Show that $\mathrm{Proj}(S) = \varnothing$ iff all elements of the irrelevant ideal S_+ are nilpotent.

(b) More generally, show that for homogenous $f, f_i \in S$ $(i \in I)$ the following are equivalent:

- We have $D_+(f) \subseteq \bigcup_{i \in I} D_+(f_i)$.
- Some power of f lies in the ideal $(f_i \mid i \in I) \trianglelefteq S$.

Deduce that if S is Noetherian, then so is $\mathrm{Proj}(S)$. What about the converse?

(c) Show for the graded ring $S = \mathbb{Q}[\, x_n \,|\, n \in \mathbb{N}\,]$ that $\mathrm{Proj}(S)$ is *not* quasi-compact.

Problem 12.2. Let $S = \bigoplus_{d \in \mathbb{N}_0} S_d$ be a graded ring, and fix an integer $e > 0$.

(a) Show that $\mathrm{Proj}(S) = \mathrm{Proj}(S')$ for the graded subring $S' = \bigoplus_{d \in \mathbb{N}_0} S_{de} \subseteq S$.

(b) Suppose that

- S_e is finitely generated over a subring $A \subseteq S_0$,
- for each $d \in \mathbb{N}$ the group S_{de} is generated by products $b_1 \cdots b_d$ with $b_i \in S_e$.

Show that then there exists a closed immersion $\mathrm{Proj}(S) \hookrightarrow \mathbb{P}^n_A$ for some $n \in \mathbb{N}$.

NB. This generalizes the Veronese embedding that we have seen on problem sheet 6.

Problem 12.3. For $a_0, \ldots, a_n \in \mathbb{N}$ let $\mathbb{P}(a_0, \ldots, a_n) := \mathrm{Proj}\, \mathbb{Z}[t_0, \ldots, t_n]$ with $\deg(t_i) := a_i$ be the corresponding weighted projective space.

(a) Show that we have a finite surjective morphism

$$\pi : \mathbb{P}^n \;=\; \mathrm{Proj}\, \mathbb{Z}[x_0, \ldots, x_n] \;\longrightarrow\; \mathbb{P}(a_1, \ldots, a_n) \quad \text{defined by} \quad t_i \;\mapsto\; x_i^{a_i}.$$

(b) Show that $\mathbb{P}(a_0, a_1) \simeq \mathbb{P}^1$ (although π need not be an isomorphism).

(c) Show that $\mathbb{P}(1,1,2) \simeq V_+(x^2 - yz) \subset \mathbb{P}^2 = \mathrm{Proj}\, \mathbb{Z}[x, y, z]$, and that for any $a_0, a_1, a_2 \in \mathbb{N}$ we have

$$\mathbb{P}(a_0, a_1, a_2) \;\simeq\; \mathrm{Proj}(T)$$

where T is a graded ring that is generated by finitely many elements of degree one.

Problem 12.4. Compute the blowup $\mathrm{Bl}_Z(X)$ for $Z \subset X \subset \mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[x, y, z]$ given below:

(a) $X = V(x^2 - y^3) \supset Z = V(x, z)$,

(b) $X = V(x^2 - yz^2) \supset Z = V(x, y, z)$,

(c) $X = V(x^2 - yz^2) \supset Z = V(x, z)$.

Problem 13.1. Let $X = \mathrm{Spec}A$ for a ring A. Show that the functor $\tilde{\ }: \mathrm{Mod}(A) \to \mathrm{Mod}(\mathscr{O}_X)$ is left adjoint to the functor of global sections, i.e. we have a natural isomorphism

$$\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \simeq \mathrm{Hom}_A(M, \mathscr{F}(X)) \quad \text{for} \quad M \in \mathrm{Mod}(A), \quad \mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X).$$

Problem 13.2. Let X be a ringed space.

(a) Show that for any $\mathscr{O}_X$-module $\mathscr{F}$ of finite presentation, the support

$$\mathrm{Supp}(\mathscr{F}) := \{x \in X \mid \mathscr{F}_x \not\simeq 0\} \quad \text{is a closed subset of } X.$$

Give an example of a scheme and a quasicoherent sheaf whose support is not closed.

(b) Let $\mathscr{F} \in \mathrm{QCoh}(X)$ for an affine scheme $X = \mathrm{Spec}(A)$. Show that for any $s \in \mathscr{F}(X)$ we have

$$\mathrm{Supp}(s) := \{x \in U \mid s_x \neq 0 \text{ in } \mathscr{F}_x\} = V(\mathrm{Ann}(s)) \quad \text{where} \quad \mathrm{Ann}(s) := \{a \in A \mid a \cdot s = 0\}.$$

Problem 13.3. Let $f : X \to Y$ be a morphism of ringed spaces.

(a) Show that there exists a natural morphism of $\mathscr{O}_Y$-modules

$$(f_*\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{G})) \quad \text{for all} \quad \mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X), \quad \mathscr{G} \in \mathrm{Mod}(\mathscr{O}_Y).$$

(b) Show that if $\mathscr{G}$ is locally free of finite rank, then this is an isomorphism.

Problem 13.4. Let k be a field. Consider the sheaves on $X = \mathbb{P}^1_k$ defined by

$$\mathscr{O}_X(-1) := (U \mapsto \{f \in \mathscr{O}_X(U) \mid f(\infty) = 0\}) \quad \text{and} \quad \mathscr{O}_X(1) := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(-1), \mathscr{O}_X).$$

(a) Show that the sheaves $\mathscr{O}_X(\pm n) := (\mathscr{O}_X(\pm 1))^{\otimes n}$ are locally free for all $n \in \mathbb{N}$.

(b) Show that $\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \simeq \mathscr{O}_X(m+n)$ for all $m, n \in \mathbb{Z}$, and

$$\dim_k \Gamma(\mathscr{O}_X(n)) = \begin{cases} 0 & \text{for } n < 0, \\ n+1 & \text{for } n \geq 0. \end{cases}$$

(c) Deduce that in general $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})(X) \not\simeq \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{G}(X)$ for $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}(X)$.