Algebraic Geometry
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Solutions

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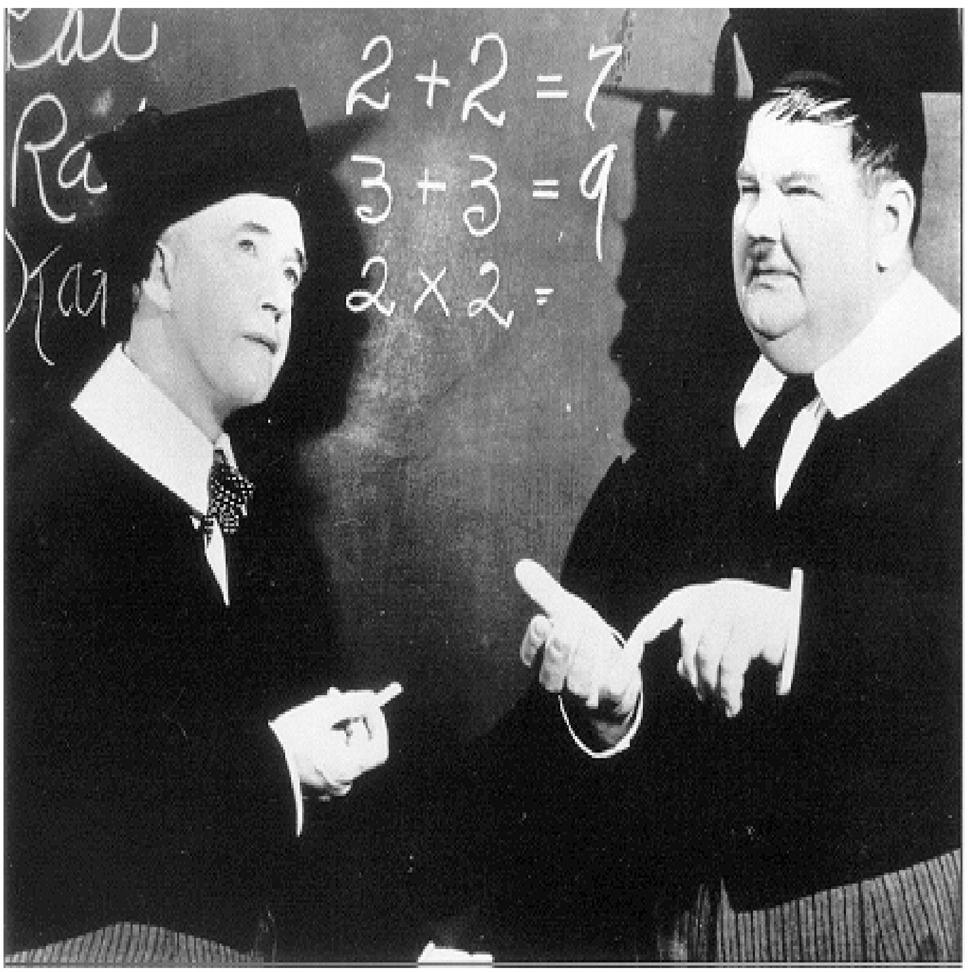


Foreword:

This is our attempt to put a collection of partially completed solutions scattered on the web all in one place. This started as our personal collection of solutions while reading Hartshorne. We were stuck (and are still) on several problems, which led to our web search where we found some extremely clever solutions by [SAM] and [BLOG] among others. Some solutions in this .pdf are all theirs and just repeated here for convenience. In other places the authors made corrections or clarifications. Due credit has tried to be properly given in each case. If you look on their websites (listed in the references) and compare solutions, it should be obvious when we used their ideas if not explicitly stated.

While most solutions are done, they are not typed at this time. I am trying to be on pace with one solution a day (...which rarely happens), so I will update this frequently. Check back from time to time for updates. As I am using this really as a learning tool for myself, please respond with comments or corrections. As with any math posted anywhere, read at your own risk!

1 Chapter 1: Varieties

1.1 Affine Varieties

1. (a) Let Y be the plane curve defined by $y = x^2$. Its coordinate ring $A(Y)$ is then $k[x, y]/(y - x^2) \cong k[x, x^2] \cong k[x]$.

   (b) $A(Z) = k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}]$, which is the localization of $k[x]$ at x. Any homomorphism of k-algebras $\varphi : k[x, \frac{1}{x}] \to k[x]$ must map x into k, since x is invertible. Then φ is clearly not surjective, so in particular, not an isomorphism.

   (c) Let $f(x, y) \in k[x, y]$ be an irreducible quadratic. The projective closure is defined by $z^2 f(\frac{x}{z}, \frac{y}{z}) := F(x, y, z)$. Intersecting this variety with the hyperplane at infinity $z = 0$ gives a homogeneous polynomial $F(x, y, 0)$ in two variables which splits into two linear factors. If F has a double root, the variety intersects the hyperplane at only one point. Since any nonsingular curve in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, $\mathcal{Z}(F)\backslash\infty = \mathbb{P}^1\backslash\infty \cong \mathbb{A}^1$. So $\mathcal{Z}(f) \cong \mathbb{A}^1$. If F has two distinct roots, say p, q, then the original curve is $\mathbb{P}^1$ minus 2 points, which is the same as $\mathbb{A}^1$ minus one point, call it p. Change coordinates to set $p = 0$ so that the coordinate ring is $k[x, \frac{1}{x}]$.

2. Y is isomorphic to $\mathbb{A}^1$ via the map $t \mapsto (t, t^2, t^3)$, with inverse map being the first projection. So Y is an affine variety of dimension 1. This also shows that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. I claim that the ideal of $Y, I(Y)$ is $(y - x^2, z - x^3)$. First note that for any $f \in f[x, y, z]$, I can write $f = h_1(y - x^2) + h^2(z - x^3) + r(x)$, for $r(x) \in k[x]$. To show this it is enough to show it for an arbitrary monomial $x^\alpha y^\beta z^\gamma = x^\alpha(x^2 + (y - x^2))^\beta(x^3 + (z - x^3))^\gamma = x^\alpha(x^{2\beta} +$ terms with $y - x^2)(x^{3\gamma} +$ terms with $z - x^3) = h_1(y - x^2) + h_2(z - x^3) + x^{\alpha+2\beta+3\gamma}$, for $h_1, h_2 \in k[y, y, z]$. Now, clearly $(y - x^2, z - x^3) \subseteq I(Y)$. So show the reverse inclusion, let $f \in I(Y)$ and write $f = h_1(y - x^2) + h^2(z - x^3) + r(x)$. Using the parametrization $(t, t^2, t^3), 0 = f(t, t^2, t^3) = 0 + 0 + r(t)$, so $r(t) = 0$.

3. Let $Y \subseteq \mathbb{A}^2$ be defined by $x^2 - yz = 0$ and $xz - x = 0$. If $x = 0$, then $y = 0$ and z is free, so we get a copy of the z-axis. If $z = 0$, then y is free, so we get the y-axis. If $x \neq 0, z = 1, y = x^2$. So $Y = \mathcal{Z}(x^2 - y, z - 1) \cup \mathcal{Z}(x, y) \cup \mathcal{Z}(x, z)$. Since each piece is isomorphic to $\mathbb{A}^1$, (see ex 1), the affine coordinate ring of each piece is isomorphic to a polynomial ring in one variable.

4. Let $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$. Consider the diagonal subvariety $X = \{(x, x) \mid x \in \mathbb{A}^1\}$. This is not a finite union of horizontal and vertical lines and points, so it is not closed in the product topology of $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$.

5. These conditions are all obviously necessary. If B is a finitely-generated k-algebra, generated by $t_1, \ldots, t_n$, then $B \cong k[x_1, \ldots, x_n]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of the polynomial ring defined by some $f_1, \ldots, f_n$. Let $X \subseteq \mathbb{A}^n$

be defined by $f_1 = \ldots = f_n = 0$. We prove that $\mathcal{I}_X = \mathcal{I}$ from which it will follow that $k[X] \cong k[x_1, \ldots, x_n]/\mathcal{I} \cong B$. If $F \in \mathcal{I}_X$, then $F^r \in \mathcal{I}$ for some $r > 0$ by the Nullstellensatz. Since B has no nilpotents, also $F \in \mathcal{I}$, thus $\mathcal{I}_X \subset \mathcal{I}$, and since obviously $\mathcal{I} \subset \mathcal{I}_X$, equality follows.

6. Let $U \subseteq X$ be a nonempty open subset with X irreducible. Assume U is not dense. Then there exists a nonempty open set $V \subseteq X$ such that $V \cap U = \emptyset$, namely $X \backslash \overline{U}$. Then $X = U^c \cup V^c$, contradicting the fact that X is irreducible. So U is dense. If U were not irreducible, write $U = Y_1 \cup Y_2$ where each Y_i is closed inside of U and proper. Then for two closed subsets $X_1, X_2 \subseteq X$, such that $Y_i = U \cap X_i, (X_1 \cup X_2) \cup U^c = X$, so X is reducible. Contradiction, so U is irreducible. Suppose Y is an irreducible subset of X and suppose $\overline{Y} = Y_1 \cup Y_2$. Then $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$, so by irreducibility of Y, we have WOLOG $Y = (Y_1 \cap Y)$. Since $\overline{Y}$ is the smallest closed subset of X containing Y, it follows that $\overline{Y} = Y_1$, so $\overline{Y}$ is irreducible.

7. (a) (i $\rightarrow$ ii) If X is a noetherian topological space, then X satisfies the D.C.C for closed sets. Let Σ be any nonempty family of closed subsets. Choose any $X_1 \in \Sigma$. If X_1 is a minimal element, then (ii) holds. If not, then there is some $X_2 \in \Sigma$ such that $X_2 \subset X_1$. If X_2 is minimal, (ii) holds. If not, chose a minimal X_3. Proceeding in this way one sees that if (ii) fails we can produce by the Axiom of Choice an infinite strictly decreasing chain of elements of Σ, contrary to (i).

   (ii $\rightarrow$ i) Let every nonempty family of closed subsets contain a minimal element. Then X satisfies the D.C.C. for closed subsets, so X is noetherian.

   (iii $\rightarrow$ iv) and (iv $\rightarrow$iii) Same argument as above.

   (i $\leftrightarrow$ iii) Let $C_1 \subset C_2 \subset \ldots$ be an ascending chain of open sets. Then taking complements we get $C_1^c \supset C_2^c \supset \ldots$, which is a descending chain of closed sets. So X is noetherian iff the closed chain stabilizes iff the open chain stabilizes.

   (b) Let $X = \bigcup U_\alpha$ be an open cover. Pick U_1 and U_2 such that $U_1 \subset (U_1 \cup U_2)$ (strict inclusion). Pick U_3 such that $U_2 \subset (U_1 \cup U_2 \cup U_3)$. Continue in this fashion to produce an ascending chain of open subsets. By part a), since X is noetherian, this chain must stabilize and we get a finite cover of X.

   (c) Let $Y \subseteq X$ be a subset of a noetherian topological space. Consider an open chain of subsets $V_0 \subseteq V_1 \subseteq \ldots$ in Y. By the induced topology, there exists open $U_i \subseteq X$ such that $U_i \cap Y = V_i$. Form the open sets $W_i = \bigcup_{i=1}^k U_i$. So $W_k \cap Y = \bigcup_{i=1}^k = \bigcup_{i=1}^k V_i = V_k$. The chain $W_0 \subseteq W_1 \subseteq \ldots$ in X stabilizes since X is Noetherian. So the chain $V_0 \subseteq V_1 \subseteq \ldots$ in Y, stabilizes, so by part a), Y is noetherian.

   which stabilizes since X is noetherian. Thus the original chain in Y stabilizes, so by part (a), Y is Noetherian.

(d) Let X be a noetherian space which is also Hausdorff. Let C be an irreducible closed subset. If C were not a point, then any $x, y \in C$ have disjoint open sets, which are dense by ex 1.6. So C is a finite union of irreducible closed sets, ie a finite set of points.

8. Let $Y \subseteq \mathbb{A}^n$ with dim $Y = r$. Let H be a hypersurface such that $Y \not\subseteq H$ and $Y \cap H \neq \emptyset$. Then $\mathcal{I}(H) \not\subseteq \mathcal{I}(Y)$. Let H be defined by $f = 0$. Irreducible components of $Y \cap H$ correspond to minimal prime ideals p in $k[Y]$ containing f. Since $Y \not\subseteq H$, f is not a zero-divisor, so by the Hauptidealsatz, every minimal prime ideal p containing f has height 1. Then Thm 1.8A, every irreducible component of $Y \cap H$ has dimension dim $Y - 1$.

9. Let $\mathfrak{a} \subseteq A = k[x_1, \dots, x_n]$ be an ideal which can be generated by r elements, say $a = (f_1, \dots, f_r)$. Then the vanishing of each f_i defines a hypersurface H_i. By applying the previous exercise r times, if the conditions are satisfied, then the dimension drops by 1 each time. If $Y \subseteq H_i$, then intersecting will not drop the dimension by 1. So we get the desired inequality.

10. (a) Let $Y \subseteq X$ and consider a strictly increasing chain of open sets $Y_0 \subset Y_1 \subset \dots \subset Y_n$ in Y, where $n =$ dim Y. Then each $Y_i = C_i \cap X$ for some closed $C_i \subset X$. Using the same replacement argument as in ex 7(c), we get a strictly increasing chain of open sets $(C_0 \cap Y) \subset (C_1 \cap Y) \subset \dots \subset (C_n \cap Y)$ in X. Then by definition, dim $Y \leq$ dim X.

(b) Let X be a topological space with open covering $\bigcup U_i$. By part a), we have dim $U_i \leq$ dim X, so sup dim $U_i \leq$ dim X. For any chain of irreducible closed subsets $C_0 \subset C_1 \subset \dots C_n$, choose an open set U_0 such that $C_0 \cap U_0 \neq \emptyset$. So $C_0 \cap U_0 \subset C_1 \cap U_0$. Continue in this way to construct a chain $(C_0 \cap U_0) \subset (C_1 \cap U_0) \subset \dots$ so that dim $U_0 \geq$ dim X. Then sup dim $U_i =$ dim X as desired.

(c) Let $X = \{0, 1\}$ with open sets $\emptyset, \{0\}, \{0, 1\}$. Then $\{0\}$ is open and its closure is all of X, so $\{0\}$ is dense. Clearly dim $\{0\} = 0$, but $\{1\} \subset \{0, 1\}$ is a maximal chain for $\{0, 1\}$, so dim $\{0, 1\} = 1$. So with $U = \{0\}$, dim $U <$ dim X.

(d) Let Y be a closed subset of an irreducible finite-dimensional topological space X such that dim $Y =$ dim X. Let $Y' \subset Y$ be irreducible with dim $Y' =$ dim Y. Let $C_0 \subset C_1 \subset \dots \subset C_n = Y'$ be a chain of irreducible closed sets. Then $C_0 \subset \dots \subset C_n \subset X$ is an irreducible closed chain which gives dim $X >$ dim Y'. Contradiction.

(e) For $n \in \mathbb{Z}_{\geq 0}$, let $U_n = \{n, n+1, n+2, \dots\}$. Then the set $\tau = \{\emptyset, U_0, U_1, \dots\}$ is a topology of open sets on $\mathbb{Z}_{\geq 0}$. In this space, if C and C' are closed sets, then it is easy to see that either $C \subseteq C^{'}$ or $C^{'} \subseteq C$, that every nonempty closed set is irreducible, and that every closed set other then $\mathbb{Z}_{\geq 0}$ is finite. So this is an example of a Noetherian infinite dimensional topological space.

11. Define $\varphi : k[x,y,z] \to k[t^3,t^4,t^5]$ by $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$. φ is surjective and $\ker \varphi = \mathcal{I}(Y)$. Since $k[t^3,t^4,t^5]$ is an integral domain, $\mathcal{I}(Y)$ is prime so Y is irreducible. Three elements of $\mathcal{I}(Y)$ of least degree are $xz-y^2, yz-x^3$, and z^2-x^2y. Since these 3 terms are linearly independent, no two elements can generate $\mathcal{I}(Y)$. See Kunz, "Introduction to Commutative Algebra and Algebraic Geometry", page 137, for a nice proof in full generality.

12. Let $f(x,y) = (x^2-1)^2 + y^2 = x^4 - 2x^2 + y^2$. Since $\mathbb{R}[x,y] \subset \mathbb{C}[x,y]$ and both are UFDs, and since $f(x,y)$ factors into irreducible degree 2 polynomials $(x^2-1+iy)(x^2-1-iy)$ in $C[x,y]$, $f(x,y)$ is irreducible over $\mathbb{R}[x,y]$. But $\mathcal{Z}(f) = \{(1,0),(-1,0)\} = \mathcal{Z}(x-1,y) \cup \mathcal{Z}(z+1,y)$, which is reducible.

1.2 Projective Varieties

1. Let $\mathfrak{a} \subset S$ be a homogeneous ideal, $f \in S$ a homogeneous polynomial with $\deg f > 0$ such that $f(P) = 0 \ \ \forall P \in \mathcal{Z}(\mathfrak{a})$. Then $(a_0 : a_1 : \ldots : a_n) \in \mathbb{P}^n$ is a zero of f iff $(a_0, \ldots, a_n) \in \mathbb{A}^{n+1}$ is a zero of f considered as a map $\mathbb{A}^{n+1} \to k$. By the affine Nullstellensatz, $f \in \sqrt{a}$.

2. Let $\mathfrak{a} \in k[x_0, \ldots, x_n]$ be a homogeneous ideal.

   (i $\leftrightarrow$ ii) By looking at the affine cone, $\mathcal{Z}(\mathfrak{a}) = \emptyset$ implies that $\mathfrak{a} = \emptyset$ or $\mathfrak{a} = 0$, in which case $\sqrt{\mathfrak{a}} = S$ or $\bigoplus_{d>0} S_d$ respectively.

   (ii $\to$ iii) If $\sqrt{\mathfrak{a}} = S$, then $1 \in \sqrt{\mathfrak{a}}$. So $1 \in \mathfrak{a}$ and thus $\mathfrak{a} = S$. But then $S_d \subseteq \mathfrak{a}$ for any d. Suppose $\sqrt{\mathfrak{a}} = \bigoplus_{d>0} S_d$. Then there's some integer m s.t. $x_i^m \in \mathfrak{a}$ for i=0,...,n. Every monomial of degree m(n+1) is divisible by x_i^m for some i so $S_d \subseteq \mathfrak{a}$ with d = m(n+1).

   (iii $\to$ i) Let $\mathfrak{a} \supseteq S_d, d > 0$. Then $x_i^d \in \mathfrak{a}$, i=0,...,n have no common zeroes in $\mathbb{P}^n$, so $\mathcal{Z}(\mathfrak{a}) = \emptyset$.

3. (a) Obvious.

   (b) Equally obvious.

   (c) See solutions to (a) and (b).

   (d) "$\subseteq$" is exercise 1. The reverse inclusion is obvious.

   (e) $Z(I(Y))$ is a closed set containing Y, so $Z(I(Y)) \supseteq \overline{Y}$. Conversely, let $P \notin \overline{Y}$. Then $\overline{Y} \subset \overline{Y} \cup \{P\}$ implies $I(\overline{Y}) \supset I(\overline{Y} \cup \{P\})$. So there's a homogeneous polynomial vanishing on $\overline{Y}$ (and hence Y), but not at P. Thus $P \notin Z(I(Y))$. Therefore $Z(I(Y)) \subseteq \overline{Y}$.

4. (a) This is the summary of ex 1, 2, 3(d), and 3(e).

   (b) Looking at the affine cone, this follows from Cor 1.4

   (c) $\mathfrak{U}_{\mathbb{P}^n} = (0)$, which is prime, so by part (b), $\mathbb{P}^n$ is irreducible.

5. (a) Let $C_0 \supseteq C_1 \supseteq \ldots$ be a descending chain of irreducible closed subsets of $\mathbb{P}^n$. Then by ex 2.3, they correspond to an ascending chain of prime ideals in $k[x_0, \ldots, x_n]$, which must stabilize since $k[x_0, \ldots, x_n]$ is a noetherian ring. So the chain $C_0 \supseteq C_1 \supseteq \ldots$ also stabilizes.

   (b) This is exactly the statement of (1.5).

6. Follow the hint. Choose i such that dim Y_i = dim Y. Exercise 1.10(b) says this is possible. For convenience we suppose $i = 0$. We can write any element $\frac{F}{x_0^n} \in S_{x_0}$ of degree 0 as the polynomial $F(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$, which is exactly the element $\alpha(F) \in A(Y_0)$, where α is defined in (2.2) and $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ are the coordinates on $\mathbb{A}^n$. Given a polynomial $f \in A(Y_0)$, we homogenize it to $F = \beta(f)$, where β is defined in (2.2). If deg F = d, we associate the degree zero element $\frac{F}{x_0^d} \in S_{x_0}$. The two processes are reversible, giving an isomorphism of $A(Y_0)$ with the subring of S_{x_0} of elements of degree 0. Clearly $S_{x_0} = A(Y_0)[x_0, \frac{1}{x_0}]$. The transcendence degree of $A(Y_0)[x_0, \frac{1}{x_0}]$ is one higher than that of $A(Y_0)$ so by (1.7) and (1.8A), dim S_{x_0} = dim $Y_0 + 1$. Since dim Y_i = dim Y, it follows that dim S_{x_0} = dim S. Thus dim S = dim $Y_0 + 1$.

7. (a) dim $S(\mathbb{P}^n)$ = n+1 so the result follows from exercise 6.

   (b) mimic the proof of (1.10) in the affine cone.

8. Let $Y \subseteq \mathbb{P}^n$ have dim $n-1$. Then dim $k[Y]$ = dim $Y + 1 = n$. In the affine cone, this corresponds to an $n-$dimensional variety in $\mathbb{A}^{n+1}$. By Prop 1.13, $\mathcal{I}(Y)$ is principal, generated by an irreducible polynomial f. So $Y = (Z)(f)$ in the affine cone and thus $Y = \mathcal{Z}(F)$ for the form homogenized form F corresponding to f. Conversely, let $f \in k[x_0, \ldots, x_n]$ be a non-constant irreducible homogeneous polynomial defining an irreducible variety $\mathcal{Z}(f)$. Its ideal (f) has height 1 by the Hauptidealsatz, so viewing this variety in the affine cone $\mathbb{A}^{n+1}$, by (1.8A), $\mathcal{Z}(f)$ has dimension n-1.

9. Let $Y \subseteq \mathbb{A}^n$ be an affine variety, $\overline{Y}$ its projective closure.

   (a) Let $F(x_0, \ldots, x_n) \in \mathcal{I}(\overline{Y})$. Then $f := \varphi(F) = F(1, x_1, \ldots, x_n)$ vanishes on $Y \subseteq \mathbb{A}^n$, the affine piece of $\mathbb{P}^n$ defined by $x_0 = 1$, so $f \in \mathcal{I}(Y)$ and clearly $\beta(f) = F$, so $F \in \beta(\mathcal{I}(Y))$. Similar for reverse inclusion.

   (b) We know from ex 1.1.2 that $(I)(Y) = (y - x^2, z - x^3)$. $\overline{Y} \subseteq \mathbb{P}^3 = \{(\frac{u}{v}, \frac{u^2}{v^2}, \frac{u^3}{v^3}, 1)\} = \{(u^3, uv^2, u^2v, v^3)\}$. Assume that $\mathcal{I}(\overline{Y}) = (wy - x^2, w^2z - x^3)$. Then $(0,1,1,0) \in \mathcal{Z}(\mathcal{I}(Y)) = \overline{Y}$, but $(0,1,1,0) \notin \{(u^3, uv^2, u^2v, v^3)\}$. So $\mathcal{I}(\overline{Y}) \notin (\beta(y - x^2), \beta(z - x^3))$.

10. (a) $C(Y) = \Theta^{-1}(Y) \cup \{(0, \ldots, 0)\}$. $\mathcal{I}(C(Y)) = \mathcal{I}(\Theta^{-1}(Y) \cup \{0, \ldots, 0)\} = \mathcal{I}(\Theta^{-1}(Y)) \cap \mathcal{I}(\{(0, \ldots, 0)\}) = \mathcal{I}(Y)$ for $Y \subseteq \mathbb{A}^{n+1}$ since $(0, \ldots, 0) \in Y$. So $C(Y)$ is an algebraic set, $C(Y) = \mathcal{Z}(\mathcal{I}(Y))$.

    (b) $C(Y)$ is irreducible iff $\mathcal{I}(C(Y))$ is prime iff $\mathcal{I}(Y)$ is prime by part $a)$ iff Y is irreducible.

(c) Let dim $Y = n$. Then there is a descending chain of irreducible proper varieties corresponding to an increasing chain of prime ideals in the polynomial ring. In $C(Y)$ the origin is added to the variety, which corresponds to the prime ideal $(x_0, \ldots, x_n)$ which is now added to the end of the chain of primes. So dim $Y + 1 =$ dim $C(Y)$

11. (a) $(i \to ii)$ Let $\mathcal{I}(Y) = (L_1, \ldots, L_m)$, where each L_i is a linear polynomial. Let $H_i = \mathcal{Z}(L_i)$. Then the H_i are hyperplanes and $Y = \bigcap H_i$. $(ii \to i)$ Let $Y = \bigcap H_i$. Do a linear transformation to get each H_i to be $\mathcal{Z}(x_i)$. Then $\mathcal{I}(Y) = \mathcal{I}(\bigcap H_i) = \mathcal{I}(\bigcap \mathcal{Z}(x_i)) = (x_1, \ldots, x_m)$.

    (b) By part a), Y is the intersection of hyperplanes. But by ex 1.1.9, the intersection of $\mathbb{P}^n$ with a hyperplane will at most drop the dimension of Y by 1. So if Y has dimension r, then Y is the intersection of at least $n - r$ hyperplanes, so $\mathcal{I}(Y)$ is minimally generated by $n - r$ linear polynomials.

    (c) This is the Projective Dimension Theorem, which is Prop 1.7.1 on page 48.

12. (a) $\mathfrak{a}$ is clearly homogeneous since the image of each y_i is sent to an element of the same degree. Since the quotient $k[y_0, \ldots, y_n]/\ker\theta$ is isomorphic to a subring of $k[x_0, \ldots, x_n]$, which is an integral domain, $\ker\theta$ is prime, and $\mathcal{Z}(\mathfrak{a})$ a projective variety.

    (b) If $f \in \ker\theta$, $f(M_0, \ldots, M_n) = 0$. Therefore f is identically zero on any point $(M_0(a), \ldots, M_n(a))$, so $Im(v_d) \subseteq \mathcal{Z}(\mathfrak{a})$. Conversely, $\mathcal{Z}(\mathfrak{a}) \subseteq Im(v_d)$ iff $\ker\theta \supseteq \mathcal{I}(Im(v_d))$. Let $f \in \mathcal{I}(Im(v_d))$. Then $f(x) = 0\ \forall x \in Im(v_d)$, ie $f(M_0, \ldots, M_n) = 0$, so $f \in \ker\theta$.

    (c) Since $\mathcal{Z}(\mathfrak{a}) = Im(v_d)$, and the d-uple embedding is an injective isomorphism, it is a homeomorphism.

    (d) The 3-uple embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$ maps (x_0, x_1) to $(x_0^3, x_0^2x_1, x_0x_1^2, x_1^3) = \{(\frac{u}{v}), (\frac{u}{v})^2, (\frac{u}{v})^3, 1)\} = \{(uv^2, vu^2, u^3, v^3)\}$, which is the projective closure of $\{(x_1, x_1^2, x_1^3)\}$

13. $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$ is given by $(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2)$. Let $C \subset \mathbb{P}^2$ be a curve defined by the homogeneous function $f(x_0, x_1, x_2) = 0$. Then $0 = f^2 \in k[x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2]$ defines a hypersurface $V \subset \mathbb{P}^5$. So $Z = v_2(C) = V \cap Y$.

14. To show that $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ is a closed set of P^N, write out its defining equations:(*) $w_{ij}w_{kl} = w_{kj}w_{il}$ for $0 \le i, k \le r, 0 \le j, l \le s$, where $\psi(x, y) = (w_{ij}), w_{ij} = a_ib_j$. Conversely, if w_{ij} satisfy (*), and say $w_{00} \neq 0$, then setting $k, l = 0$ gives $(w_{ij}) = \psi(x, y)$, where $x = (w_{00}, \ldots, w_{r0}), y = (w_{00}, \ldots, w_{0s})$. So $\psi(x, y)$ determines x and y uniquely, ie ψ is an embedding with image W a subvariety defined by (*).

15. Let $Q = \mathcal{Z}(xy - zw)$.

(a) For $r, s = 1, \psi(\mathbb{P}^1 \times \mathbb{P}^1)$ is defined by a single equation $w_{11}w_{00} = w_{01}w_{10}$, which is after an obvious change of coordinates $xy = zw$.

(b) For $\alpha = (\alpha_0, \alpha_1) \in \mathbb{P}^1$, the set $\psi(\alpha \times \mathbb{P}^1)$ is the line in $\mathbb{P}^3$ given by $\alpha_1 w_{00} = \alpha_0 w_{10}$. As α runs through $\mathbb{P}^1$, these lines give all the generators of one of the two families of lines of Q. Similarly, the set $\psi(\mathbb{P}^1 \times \beta)$ is a line of $\mathbb{P}^3$, and as β runs through all of $\mathbb{P}^1$, these lines give the generators of the other family.

(c) The curves $x = y$ of Q is not one of these families of lines. This closed curve is a closed subset of Q, but not closed in $\mathbb{P}^1 \times \mathbb{P}^1$.

16. (a) Let $Q_1 \subseteq \mathbb{P}^3$ be defined by $x^2 - yw = 0$, $Q_2 \subseteq \mathbb{P}^3$ defined by $xy - zw = 0$. Then in the affine piece $w = 1$, $x^2 - y = 0, xy = z$, therefore $y = x^2, z = x^3$. So $(x, y, z, w) = (x, x^2, x^3, 1)$, which is the twisted cubic. When $w = 0, x = 0$ and y, z are free, which is the line defined by $x = w = 0$.

(b) Let C be the conic in $\mathbb{P}^2$ defined by $x^2 - yz = 0$. Let L be defined by $y = 0$. Then $C \cap L$ is defined in the affine piece $z = 1$ by $y = 0$, which forces $x = 0$, which is the point $(0, 0, 1)$. $\mathcal{I}(P) = (x, y)$. $\mathcal{I}(C) + \mathcal{I}(L) = \{\alpha(x^2 - yz) + \beta(y)\} \not\ni x$.

17. (a) Let $Y = \mathcal{Z}(\mathfrak{a})$ be a variety in $\mathbb{P}^n$, and let $\mathfrak{a} = (f_1, \ldots, f_q)$. Show that $\dim Y \geq n - q$ by induction on q. If $q = 1$, then Y is a hypersurface, so $\dim Y \geq n - 1$ by ex 2.8. Now assume true for q: $\dim Y \geq n - q$. Let $\mathfrak{a} = (f_1, \ldots, f_q, f_{q+1})$, with $f_{q+1} \notin (f_1, \ldots, f_q)$. Then the hypersurface $\mathcal{Z}(f_{q+1})$ intersects Y, which reduces the dimension of Y by 1. So $\dim \mathcal{Z}(f_1, \ldots, f_q, f_{q+1}) = \dim Y - 1 \geq n - q - 1 = n - (q + 1)$.

(b) If $Y \in \mathbb{P}^n$ is a strict complete intersection, then $\mathcal{I}(Y) = (f_1, \ldots, f_{n-r})$. Each f_i defines a hypersurface $\mathcal{Z}(f_i)$ and $Y = \bigcap \mathcal{Z}(f_i)$, so Y is a set-theoretic complete intersection.

(c) Let Y be the twisted cubic $\{(x^3, x^2y, xy^2, y^3)\}$. No linear form vanishes on Y and the linearly independent quadratic forms $u_0u_3 - u_1u_2, u_1^2 - u_0u_2, u_2^2 - u_1u_3$ vanish on Y. Therefore any set of generators must have at least 3 elements.

Y is the intersection of $H_1 = \mathcal{Z}(x^2 - wy)$ and $H_2 = \mathcal{Z}(y^4 + wz^2 - 2xyz)$ as $(xy - wz)^3 = w(y^3 + wyz^2 - 2xyz) + y^2(x^2 - wy)$ and $(y^2 - xz)^2 = y(y^3 + wz^2 = 2xyz) + z^2(x^2 - wy)$ and $y^3 = wz^2 - 2xyz = y(y^2 - xz) + z(wz - xy)$. So $Y = H_1 \cap H_2$.

(d) Ingredients: 2 3/4 cups all-purpose flour, 1 teaspoon baking soda, 1/2 teaspoon baking powder, 1 cup butter, softened 1 1/2 cups white sugar, 1 egg, 1 teaspoon vanilla extract

Directions: Preheat oven to 375 degrees F (190 degrees C). In a small bowl, stir together flour, baking soda, and baking powder. Set aside. In a large bowl, cream together the butter and sugar until smooth. Beat in egg and vanilla. Gradually blend in the dry ingredients. Roll rounded teaspoonfuls of dough into balls, and place onto ungreased cookie sheets. Bake 8 to 10 minutes in the preheated oven, or until golden. Let stand on cookie sheet two minutes before removing to cool on wire racks.

1.3 Morphisms

1. (a) This follows from ex 1.1.1(c), Thm 2.3.2(a), and Cor 2.3.7

   (b) Any proper open set of $\mathbb{A}^1$ is $\mathbb{A}^1 \backslash S$, where S is a finite number of points. The coordinate ring of $\mathbb{A}^1 \backslash \{p_1, \dots, p_n\}$ is $k[x, \frac{1}{x-p_1}, \dots, \frac{1}{x-p_n}]$. This coordinate ring is not is not isomorphic to $k[x]$ since any isomorphism must take $x - p_i$ into k, since $x - p_i$ is a unit. Also, any automorphism must map p_i to k as well, so x would get mapped to k. So any automorphism wouldn't be surjective, contradiction. So $\mathbb{A}^1 \ncong \mathbb{A}^1 \backslash S$.

   (c) Let characteristic $k \neq 2$ and write the conic as $F(x, y, z) = az^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$. We have inserted the factor of 2 to write F in matrix form $F(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
   Since the conic is irreducible, this matrix has full rank. Since any symmetric matrix is diagonalizable, we can assume that $F(x, y, z) = x^2 + y^2 + z^2$. In particular, any two smooth projective plane conics are isomorphic, so to study conics, we can just pick one. Picking $F(x, y, z) = xz - y^2$, which is nonsingular we can say that any irreducible conic, up to isomorphism in $\mathbb{P}^2$, is the image of $\mathbb{P}^1$ under the 2-uple embedding $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$ and by ex 2.3.4, these are isomorphic.

   (d) In $\mathbb{P}^2$, any two lines intersect. So any homeomorphism from $\mathbb{A}^2$ to $\mathbb{P}^2$ would not have an inverse function defined at the point of intersection of the image two parallel lines in $\mathbb{A}^1$.

   (e) Let X be an irreducible affine variety, Y be a projective variety, and let $X \cong Y$. Then their rings of regular functions are isomorphic, and since Y is projective, by Thm 3.4(a), $\mathcal{O}(Y) = k$. So $\mathcal{O}(X) = k$ and then by ex 1.4.4, X must be a point.

2. (a) Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. φ is clearly bijective onto the curve $y^2 = x^3$. Also, since φ is defined by polynomials, it is continuous. The complement of a finite set gets mapped to the compliment of a finite set, so the map is open. Thus it is a bicontinuous morphism. However, the inverse function would have to be $(x, y) \mapsto y/x$, which is not defined at 0.

   (b) Let $\text{char}(k) = p$ and define φ to be the Frobenius morphism. φ is injective since if $x^p = y^p$, then $x^p - y^p = (x - y)^p = 0$, so $x = y$. Surjectivity follows from the fact that k is algebraically closed, thus perfect. So φ is bijective. φ is clearly continuous as well since it is defined by a polynomial t^p. The map is open by the same arguments as in part a) since we are dealing with curves in this case. φ is not an isomorphism however since the corresponding map on coordinate rings is not surjective.

3. (a) Let $\varphi : X \to Y$ be a morphism. Then there is an induced map on regular functions $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$ defined by $\varphi^*(f) = f \circ \varphi$, where f is regular on the image $\varphi(U)$ for some open $U \subseteq X$. Restricting this map to functions regular in neighborhoods of P gives the desired map.

   (b) Let φ be an isomorphism. Then viewed as a map on the topological spaces of X and Y, this map is a homeomorphism, and by part *a*), the induced map on local rings is an isomorphism. The converse is obvious.

   (c) Let the image $\varphi(X)$ be dense in Y. Define for some for some $f \in k(Y)$ and $P \in X$ $(\varphi^*(f))(P) = f(\varphi(P)) = 0$ for some $f \in k(Y)$. Assume $(\varphi^*(f)) = 0$. Then $f = 0$ on some neighborhood $\varphi(U) \subset Y$. If $f \neq 0$, then $\varphi(X) \subset \mathcal{Z}(f) \subsetneq Y$. Contradiction to $\varphi(X)$ being dense in Y,

4. This is easy to see for small n and d, but notationally annoying to type up in the general case. See Shafarevich I example 2 on page 52-53 for a proof.

5. Let $H \subset \mathbb{P}^n$ be a hypersurface of degree d. Then the d-uple embedding $v_d : \mathbb{P}^n \to \mathbb{P}^N$ is an isomorphism onto its image, and H is now a hyperplane section in $\mathbb{P}^N$. Since $\mathbb{P}^N$ minus a hyperplane is affine, $\mathbb{P}^n$ minus the hypersurface H is also affine.

6. Let $X = \mathbb{A}^2 - \{0, 0\}$. To show X is not affine, we will show that $\mathcal{O}_{\mathbb{A}^2}(X) = k[\mathbb{A}^2]$, ie that every regular function on X extends to a regular function on $\mathbb{A}^2$. (Over $\mathbb{C}$ this is Hartog's Theorem). Let f be a regular function on X. Cover X by the open sets $U_1 = \{x \neq 0\}$ and $U_2 = \{y \neq 0\}$, where x and y are coordinates in $\mathbb{A}^2$. Then the restriction of f to U_1 is of the form g_1/x^n, with g_1 a polynomial and $n \geq 0$. We can further assume that g_1 is not divisible by x^n. Similarly on U_2, $f = g_2/y^n$. Since the restrictions coincide on $U_1 \cap U_2$, we see that $x^n g_2 = y^m g_1$. Now, from the uniqueness of the decomposition into prime factors in the polynomial ring $k[x, y]$, $n = m = 0$ and $g_1 = g_2 = f$. So f extends over the origin and thus the ring of regular functions are isomorphic, implying that $\mathbb{A}^2$ is isomorphic to X, contradiction.

7. (a) This follows directly from the projective dimension theorem, Thm I.7.2

   (b) I'll just cut and paste this: This follows directly from the projective dimension theorem, Thm I.7.2. FYI- remember this result. It is used quite often to show that something is NOT projective.

8. $\mathbb{P}^n - (H_i \cap H_j) = \mathbb{A}_0^n \cup \mathbb{A}_1^n$. $\mathbb{A}_0^n \cap \mathbb{A}_1^n$ is a dense open set in $\mathbb{A}_0^n \cup \mathbb{A}_1^n$. In $\mathbb{A}_0^n \cap \mathbb{A}_1^n$, regular functions are of the form $\frac{h}{x_0^n x_1^m}$ of total degree 0. Since this function extends into both affine pieces, $n = m = 0$, forcing the degree on h to be 0, resulting in a constant function.

9. The homogeneous coordinate ring of $\mathbb{P}^1$ is $k[\mathbb{P}^1] = k[x, y]$. If Y is the image of $\mathbb{P}^1$ under the 2-uple embedding, then Y is the hypersurface defined by $xy = z^2$, so $k[Y] = k[x, y, z]/(xy - z^2)$. $k[Y] \ncong k[x, y]$ since the space of elements of degree 1 is 3 dimensional in $k[Y]$.

10. This question is stupid.

11. Let X be any variety and let $P \in X$. Irreducible varieties containing P correspond to prime ideals of $k[X]$ contained in the maximal ideal $\mathfrak{m}_{X,P}$, which in turn correspond to the prime ideas of the ring $k[X]_{\mathfrak{m}_{X,P}}$. By Thm 3.2(c), this is just $\mathcal{O}_{X,P}$, the local ring at P. This question is just the local statement of the last part of Corollary 1.4.

12. If P is a point on a variety X, then there is an affine neighborhood Y with dim Y = dim X. Since $\mathcal{O}_{X,P} = \mathcal{O}_{Y,P}$, dim X = dim Y = dim $\mathcal{O}_{Y,P}$ = dim $\mathcal{O}_{X,P}$ by Thm 3.2(c)

13. $\mathcal{O}_{Y,X}$ is clearly a local ring with maximal ideal $\mathfrak{m} = \{f \in \mathcal{O}_X(U) \mid f(P) = 0 \,\forall\, P \in U \cap Y\}$. The residue field is then $\mathcal{O}_{Y,X}/\mathfrak{m}_{Y,X}$, which consists of all invertible functions on Y, ie $k(Y)$. To prove the last statement, let X be affine and let $\mathfrak{a} = \{f \in k[X] \mid f|_Y = 0\}$. Then dim X = dim $k[X]$ = ht $\mathfrak{a}$+ dim $k[X]/\mathfrak{a}$. But the height of $\mathfrak{a}$ is equal to the height of $\mathfrak{m}_{Y,X}$ in $\mathcal{O}_{Y,X}$, and dim $k(X)/\mathfrak{a}$ = dim Y. Therefore dim $\mathcal{O}_{Y,X}$+ dim Y = dim X, ie dim $\mathcal{O}_{Y,X}$ = dim X - dim Y.

14. (a) By a change of coordinates let $\mathbb{P}^n$ be the hypersurface defined $x_0 = 0$ and let $P = (1, 0, \ldots, 0)$. If $x = (x_0, \ldots, x_n) \in \mathbb{P}^{n+1} - \{P\}, x_i \neq 0$ for some i. Therefore the line containing P and x meets $\mathbb{P}^n$ in $(0, x_1, \ldots, x_n)$, which is a morphism in a neighborhood $x_i \neq 0$, so φ is a morphism.

    (b) Let $Y \subseteq \mathbb{P}^3$ be the twisted cubic, which is the image of the 3-uple embedding of P^1. If the coordinates of $\mathbb{P}^1$ are (t, u), then Y is parameterized by $(x, y, z, w) = (t, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$ and let $\mathbb{P}^2$ be the hyperplane in $\mathbb{P}^3$ defined by $z = 0$. Then the projection of $Y = (t, t^2u, tu^2, u^3) \mapsto (t^3, t^2u, u^3) \in \mathbb{P}^2$, where the image is the variety $x_1^3 = x^2x_0^2$ For $x_2 \neq 0$, this is the same as $\frac{x_1^3}{x_2} = x_0^2$, ie $\frac{(x_1^3)^3}{x_2^3} = x_0^2, iey^3 = x^2$. This is the cuspidal cubic, with the cusp at $(0, 0)$ in affine coordinates or $(0, 0, 1)$ in projective coordinates.

15. Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Let $X \times Y \subseteq \mathbb{A}^{n+m}$. Assume that $X \times Y = Z_1 \cup Z_2$ for Z_i proper and closed in $X \times Y$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$. Then since Y is irreducible, $X = X_1 \cup X_2$, and X_i is closed since it is the image of the first projection. Since X is irreducible, $X = X_1$ or $X = X_2$, so $X \times Y = Z_1$ or Z_2, contradiction, so $X \times Y$ is irreducible.

(b) Define a homomorphism $\varphi : k[X] \otimes_k k[Y] \to k[X \times Y]$ by $(\sum f_i \otimes g_i)(x, y) = \sum f_i(x)g_i(y)$. The right hand side is regular on $X \times Y$, and it is clear that φ is onto since the coordinate functions are contained in the image of φ, and these generate $k[X \times Y]$ To prove that φ is one to one, it is enough to check that if f_i are linearly independent in $k[X]$ and g_j are linearly independent in $k[Y]$, then $f_i \otimes g_j$ are linearly independent in $k[X \times y]$. Now an equality $\sum_{i,j} c_{ij} f_i(x)g_j(y) = 0$ implies the relation $\sum_j c_{ij} g_j(y) = 0$ for any fixed y, and in turn that $c_{ij} = 0$.

(c) The projection maps are clearly morphisms and given a variety Z with morphisms $\varphi : Z \to X$ and $\phi : Z \to Y$, there is an induced map $\varphi \times \phi : Z \to X \times Y$ defined by $z \mapsto (\varphi(z), \phi(z))$.

(d) Let dim $X = n$, dim $Y = m$, t_i and u_i be coordinates of X and Y respectively. $k[X \times Y]$ is generated by $t_1, \dots, t_n, u_1, \dots u_m$, so we just need to show that all coordinate elements are algebraically independent. Suppose $f(t_1, \dots, t_n, u_1, \dots u_m) = 0$ on $X \times Y$. Then for $x \in X, f(x, u_1, \dots, u_m) = 0$, ie every coefficient $a_i(x) = 0$ on X. Therefore $a_i(t_i, \dots t_n) = 0$ on X, so $f(U, T) \equiv 0$, and all the $n + m$ coordinates are algebraically independent, so dim $X \times Y = n + m$.

16. (a) $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ and there are natural projections $p_1 : \mathbb{P}^n \times \mathbb{P}^m \to X, p_2 : \mathbb{P}^n \times \mathbb{P}^m \to Y$. The inverse of p_1 is $X \times \mathbb{P}^m$ and the inverse of p_2 is $\mathbb{P}^n \times Y$, which are both quasi-projective varieties since projections are regular maps. Therefore $X \times Y = (X \times \mathbb{P}^m) \cap (Y \times \mathbb{P}^n)$ so $X \times Y$ is quasiprojective.

(b) This follows from the same argument as in part a), replacing quasi-projective with projective.

(c) $X \times Y$ is a product in the category of varieties since restriction to open covers gives well defined projections and similarly, we can restrict to these open covers to get the universal property.

17. (a) Let X be a conic in $\mathbb{P}^2$. By ex 3.1(c), every plane conic is isomorphic to $\mathbb{P}^1$. The local rings over $\mathbb{P}^1$ are DVR's which are integrally closed (AM p 94), so X is normal.

(b) $Q_1 = \mathcal{Z}(xy - zw) \subseteq \mathbb{P}^3$ is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segue embedding. Since this is a nonsingular variety, Q_1 is normal since nonsingular implies normal for varieties (Shaf I, Thm II.5.1 p 126).

Let $Q_2 = \mathcal{Z}(xy - z^2)$. The matrix of this quadratic (as in ex 1) is $\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which has rank 3. Thus we can do a linear change of coordinates to let Q_2 be defined by the equation $x^2+y^2-z^2=0$, which is nonsingular everywhere except at $(0,0,0,1)$, so we can just check normality in the affine piece $w=1$. To do this, we need to show that $k[X]$ is integrally closed in $k(X)=\{u+vz \mid u,v\in k(x,y)\}$ and $k[X]=\{u+vz \mid u,v\in k[x,y]\}$. Hence $k[X]$ is a finite module over $k[x,y]$, and hence all elements of $k[X]$ are integral over $k[x,y]$. If $\alpha=u+vz\in k(X)$ is integral over $k[X]$ then it must also be integral over $k[x,y]$. Its minimal polynomial is $T^2-2uT+u^2-(x^2+y^2)v^2$, hence $2u\in k[x,y]$, so that $u\in k[x,y]$. Similarly, $u^2-(x^2+y^2)v^2\in k[x,y]$, and hence also $(x^2+y^2)v^2\in k[x,y]$. Now since $x^2+y^2=(x-iy)(x-iy)$ is the product of two coprime irreducibles, it follows that $v\in k[x,y]$ and thus $\alpha\in k[X]$.

(c) By Shaf I, Cor to Thm 3 on page 127, for curves, normal and nonsingular are equivalent, so since this cubic has a singular point at the origin, it is not a normal variety.

(d) This is Shaf I Ch 2 section 5, page 129-131

18. (a) If Y is projectively normal, then $k[Y]$ is integrally closed in its field of fractions. Since the localization of a integrally closed domain at a maximal ideal is again integrally closed (AM Prop 5.6 pg 61), $\mathcal{O}_P=k[Y]_{m_P}$ is integrally closed for $P\in Y$, and so Y is normal.

(b) The twisted quartic is just the image of $\mathbb{P}^1$ under the 4-uple embedding, which is an isomorphism. Since $\mathbb{P}^1$ is nonsingular, hence normal, so is the twisted quartic.

To show Y is not projectively normal, use (II 5.14(d)). The embedding of $\mathbb{P}^1\hookrightarrow\mathbb{P}^3$ is induced by a 4-dim linear subspace of $\mathrm{H}^0(\mathbb{P}^1,\mathcal{O}(4))$. The rational map $\Gamma(\mathbb{P}^3,\mathcal{O}(1))\to\Gamma(Y,\mathcal{O}_Y(1))\cong\Gamma(\mathbb{P}^1,\mathcal{O}(4))$ takes a 4 dimensional subspace to a 5 dimensional subspace, so therefore is not surjective. Therefore by (II.5.14(d)), Y is not projectively normal.

(c) The twisted quartic is just the image of $\mathbb{P}^1$ under the 4-uple embedding, which is an isomorphism. Since $\mathbb{P}^1$ is nonsingular, hence normal, so is the twisted quartic. Also, $k[\mathbb{P}^1]=k[x,y]$ is a UFD, hence integrally closed. Thus projective normality depends on the embedding.

19. (a) If $\varphi\in\mathrm{Aut}(\mathbb{A}^n)$, then each $f_i\notin k$, since then φ is not surjective. Therefore each f_i is a linear non-constant polynomial, so $J\in k^\times$

(b) 2 pounds ground beef, 1/2 pound fresh ground pork, 1 cup dry bread crumbs, 2 teaspoons salt, 1/2 teaspoon pepper, 1 large egg, 3 tablespoons butter, 1/2 cup hot water. Place a medium sized baking pan into a cool oven and heat oven to 350 degrees. Place the hot water into a large mixing bowl and add the butter. Stir until completely melted. Add all remaining ingredients and mix well. Shape mixture into a loaf and place in heated baking pan. Cook your meatloaf for approximately 40 minutes or until an internal temperature of 170 degrees has been reached.

20. Let Y be a variety of dimension > 2 and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

    (a) This is equivalent to saying that every morphism $f : (Y - P) \to \mathbb{A}^1$ extends to a morphism $\overline{f} : Y \to \mathbb{A}^1$. Regarding f as a rational map from Y to $\mathbb{P}^1$ and writing $\Gamma \subset Y \times \mathbb{P}^1$ for its graph, the set $\Gamma \cap (Y \times \{\infty\})$ is contained in $P \times \{\infty\}$. Hence its dimension is less then $\dim Y - 1$. On the other hand, $Y \times \{\infty\}$ is defined locally in $Y \times \mathbb{P}^1$ by one equation, so that $\dim(\Gamma \cap (Y \times \{\infty\})) \geq \dim \Gamma - 1 = \dim Y - 1$. This means that Γ does not meet $Y \times \{\infty\}$. Therefore the morphism $\Gamma \to Y$ is finite. As it is birational and X is normal, it is an isomorphism.

    (b) Over $\mathbb{C}$, $f(z) = \frac{1}{z}$ can not be extended over all of C by methods of elementary complex analysis.

21. (a) $\mathbb{G}_a$ is a group variety since $(\mathbb{A}^1, +)$ is a group and the inverse map defined by $y \mapsto -y$ is a morphism.

    (b) $\mathbb{G}_m$ is a group variety since $(\mathbb{A}^1 - \{0\}, x)$ is a group and the inverse map defined by $x \mapsto \frac{1}{x}$ is a morphism.

    (c) $\mathrm{Hom}(X, G)$ has a group structure given by defining for any $\varphi_1, \varphi_2 \in \mathrm{Hom}(X, G)$, $(\varphi_1 + \varphi_2)(x) = \mu(\varphi_1(x), \varphi_2(x)) \in G$.

    (d) $\varphi : \mathcal{O}(X) \cong \mathrm{Hom}(X, \mathbb{G}_a)$ defined by $f \mapsto f$ gives the required isomorphism.

    (e) $\varphi : \mathcal{O}(X)^{\times} \cong \mathrm{Hom}(X, \mathbb{G}_m)$ defined by $f \mapsto f$ gives the required isomorphism.

1.4 Rational Maps

1. Define $F = \begin{cases} f(p) & p \in U \\ g(p) & p \in V \end{cases}$. This defines a regular function on $U \cup V$.

2. If φ is a rational function, then U is the union of all open sets at which φ is regular. This is the same idea as in the previous question.

3. (a) Let $f : \mathbb{P}^2 \to k$ defined by $(x_0, x_1, x_2) \mapsto x_1/x_0$. This is a rational function defined where $x_0 \neq 0$, ie on the open affine set $\mathbb{A}^2_0$. The corresponding regular function $f|_{\mathbb{A}^2_0} \to \mathbb{A}^1$ is $(x_1, x_2) \mapsto x_1$.

    (b) Viewing φ now as a map from $\mathbb{P}^2 \to \mathbb{P}^1$, it is easy to see that φ is defined everywhere the image is nonzero. The projection map is $(x_0, x_1, x_2) \mapsto (x_0, x_1)$ and is defined everywhere except at the point $(0, 0, 1)$.

4. (a) By ex I.3.1(b), any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ and isomorphic implies birational.

   (b) The map $\varphi : \mathbb{A}^1 \to Y$ defined by $t \mapsto (t^2, t^3)$, with inverse $(x, y) \mapsto x/y$ gives a birational map between $\mathbb{A}^1$ and Y. Since $\mathbb{A}^1$ is birational to $\mathbb{P}^1$, so is the cubic Y.

   (c) Let Y be the nodal cubic defined by $y^2 z = x^2(x + z)$ in $\mathbb{P}^2$. Let φ be the projection from the point $(0, 0, 1)$ to the line $z = 0$. On the open set $\mathbb{A}^2$ where $z = 1$, we have the curve $y^2 = x^3 + x$ which is birational to a line by projecting from $(0, 0)$, given by setting $x = t^2 - 1$ and $y = t(t^2 - 1)$, which is found after setting $y = tx$. Therefore on Y, the projection map $(x, y, z) \mapsto (x, y)$, defined at all points $(x, y, z) \neq (0, 0, 1)$, gives a map to $\mathbb{P}^1$ and the inverse map is then $(x, y) \mapsto ((y^2 - x^2)x : (y^2 - x^2)y : x^3)$ for $(x, y) \neq (1, \pm 1)$.

5. By the projection map $\varphi : Q \to \mathbb{A}^2$ defined by $(w, x, y, z) \to (x/w, y/w)$ for $w \neq 0$, with inverse map $(x, y) \to (1 : x : y : xy)$, gives that Q is birational to $\mathbb{A}^2$, and thus $\mathbb{P}^2$. Q is not isomorphic to $\mathbb{P}^2$ since Q contains two families of skew lines, but any two lines in $\mathbb{P}^2$ intersect.

   Another way to see this is that Q is just $\mathbb{P}^2$ with two points blown-up and then blowing-down the line joining them. Since the blow-up is a birational map, Q and $\mathbb{P}^2$ are birational. A cool fancy way to see they are not isomorphic is to note that $K_Q^2 = 8$ and $K_{\mathbb{P}^2}^2 = 9$. (to be defined later)

6. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the Plane Cremona Transformation.

   (a) φ is $\mathbb{P}^2$ with 3 points blown up and then the lines connecting them blown down. See ex V.4.2.3. Since the blow-up and blow-down are birational, so is φ. $\varphi^2(x, y, z) = \varphi(yz, xz, xy) = (x^2yz, xy^2z, xyz^2) = (x, y, z)$ after dividing by xyz. Thus φ is its own inverse.

   (b) φ is isomorphic on the open set $\{(x, y, z) \,|\, xyz \neq 0\}$ by part a)

   (c) φ and φ^{-1} are defined on $\mathbb{P}^2$ everywhere except where the 2 coordinates are zero, ie $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$.

7. Let $f : X \to Y$. Let $f^* : \mathcal{O}_{P,X} \to \mathcal{O}_{Q,Y}$ be a k-algebra isomorphism. Then this induces an isomorphism on the fraction fields of the local rings $k(X) \cong k(Y)$. So X and Y are birational. It is easy to see that the corresponding morphism f maps Q to P (since f^* is an isomorphism) and thus f is an isomorphism on some open neighborhoods U and V of P and Q respectively.

8. (a) Since $\mathbb{A}^n = \bigoplus^n k, |\mathbb{A}^n| = |k|$ since cardinality holds over finite direct sums. Since $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, $|\mathbb{P}^n| \geq |\mathbb{A}^n|$. But $\mathbb{A}^{n+1} - \{0\} \twoheadrightarrow \mathbb{P}^n$, so $|\mathbb{P}^n| \leq |\mathbb{A}^{n+1} - \{0\}| = |\mathbb{A}^{n+1}| = |\mathbb{A}^n|$. Therefore $|\mathbb{P}^n| = |\mathbb{A}^n| = |k|$.
   Since any curve X is birational to a plane curve, $|X| \leq |\mathbb{P}^2| = |k|$. Pick a point not on the curve and project now to $\mathbb{P}^1$. This map is surjective, so $|X| \geq |\mathbb{P}^1| = |k|$. Thus $|X| = |k|$. The rest follows by induction, using Prop 4.9 for the inductive step.

(b) Any two curves have the same cardinality as k, and the finite complement topology. Thus they are homeomorphic.

9. Let M of dimension $n-r-1$ be a linear space disjoint from X defining a projection $p_M : X \to \mathbb{P}^r$. p_M is surjective, hence it induces an inclusion of the function fields $k(X) \hookrightarrow k(\mathbb{P}^r)$ and since both have transcendence degree r, $k(X)$ is a finite algebraic extension of $k(\mathbb{P}^r)$. If $x_0, \dots x_n$ are homogeneous coordinates on $\mathbb{P}^n$ such that $M = \mathcal{Z}(x_0, \dots, x_r)$, hence $x_0, \dots, x_r$ are coordinates on $\mathbb{P}^r$, then $k(X)$ is generated over $k(\mathbb{P}^r)$ by the images of the functions $\frac{x_{r+1}}{x_0}, \dots, \frac{x_n}{x_0}$. By the theorem of the primitive element, it is generated by a suitable linear combination $\sum_{i=r+1}^n \alpha_i(\frac{x_i}{x_0})$. Let $L = M \cap \mathcal{Z}(\sum_{i=r+1}^n \alpha_i x_i)$. Then $\mathbb{P}^n - M \stackrel{p_L}{\to} \mathbb{P}^{r+1} - \{x\} \stackrel{p_x}{\to} \mathbb{P}^r$ with p_M being the composition and x the image by p_L of the center M of p_M. $X \subset \mathbb{P}^n - M \stackrel{p_L}{\to} \mathcal{Z}(F)$, where $\mathcal{Z}(F)$ is the hypersurface which is the image of X. This gives the inclusions of function fields $k(\mathbb{P}^r) \hookrightarrow k(\mathcal{Z}(F)) \stackrel{\alpha}{\hookrightarrow} k(X)$. Here $k(X) = k(\mathbb{P}^r)(\frac{x_{r+1}}{x_0}, \dots, \frac{x_n}{x_0})$ and $k(\mathcal{Z}(F)) = k(\mathbb{P}^r)(\frac{\sum \alpha_i x_i}{x_0})$. By assumption, α is a surjection. Therefore, since p_L is a dominating regular map, with an open set $X \subset \mathbb{P}^n - M$ such that the cardinality of the fiber p_L^{-1} (defined to be the degree, which is equivalent to the degree of the corresponding function field inclusion) is 1, $p_L|_X \to \mathcal{Z}(F)$ is almost everywhere one-to-one, hence is birational.

10. Let Y be the cuspidal cubic $y^2 = x^3$. Let (t, u) be the coordinates on $\mathbb{P}^1$. Then X, the blowing up of Y at $(0,0)$ is defined by the equation $xu = ty$ inside of $\mathbb{A}^2 \times \mathbb{P}^1$. Denote the exceptional curve $\varphi^{-1}(0)$ by E. In the open set $t \neq 0$, set $t = 1$ to get $y^2 = x^3, y = xu \Rightarrow x^2u^2 = x^3 \Rightarrow x^2(u^2 - x) = 0$. We get two irreducible components, one defined by $x = 0, y = 0, u$ free, which is the exceptional curve E. The other component is defined by $u^2 = x, y = xu$. This is $\widetilde{Y}$, which meets E at $u = 0$. $\widetilde{Y}$ is defined by $y = u^3$, which is non-singular and isomorphic to $\mathbb{A}^1$ by projection on to the first coordinate.

1.5 Nonsingular Varieties

1. (a) Setting the partials equal to 0 gives the only singular point at $(0,0)$. This is the tacnode.

   (b) Setting the partials equal to 0 gives the only singular point at $(0,0)$. Since the degree 2 term xy is the product of two linear factors, this is the node.

   (c) Setting the partials equal to 0 gives the only singular point at $(0,0)$. Since the degree 2 term y^2 is a perfect square, this is a cusp.

   (d) Setting the partials equal to 0 gives the only singular point at $(0,0)$. Intersecting this curve with a line at the origin $y = mx$, a t^3 factors out of $f(t, mt)$, so we have a triple point.

2. (a) Setting the partials equal to 0 gives $y = z = 0$, x free. So the singular points lie on the x-axis and we have a pinched point

   (b) Setting the partials equal to 0 gives the singular point at $(0,0,0)$, which is the conical double point.

   (c) Here the singular locus is the line $x = y = 0$ with z free, which corresponds to the double line.

3. (a) $\mu_P(Y) = 1 \leftrightarrow f = f_1 + f_2 + \ldots f_d \leftrightarrow f(x,y)$ has a term of degree 1, namely $\alpha x + \beta y$ for $\alpha, \beta \neq 0 \leftrightarrow f_x^{'} = \alpha, f_y^{'} = \beta \not\equiv 0 \leftrightarrow P$ is nonsingular.

   (b) The multiplicity at $P = (0,0)$ is the smallest degree term that appears. The multiplicity of P for $5.1(a),(b),(c)$ is 2, and 3 for 5.1(d).

4. (a) $(Y \cdot Z)_P$ is finite if the length of the $\mathcal{O}_p$-module $\mathcal{O}_p/(f,g)$ is finite. Let $\mathfrak{a}_P \subseteq k[U]$ be the ideal of P in the affine coordinate ring of some open affine neighborhood U containing P and no other point of intersection of Y and Z. By the Nullstellensatz, $\mathfrak{a}_P^r \subset (f,g)$ for some $r > 0$. Then $\mathcal{O}_P = k[U]_{\mathfrak{a}_P}$. It follows that in O_P, $\mathfrak{m}^r \subset (f,g)$. To show that $l(\mathcal{O}/(f,g)) < \infty$, it is enough to show that $l(\mathcal{O}/\mathfrak{m}^r) < \infty$. To show this, it is sufficient to show that $\mathcal{O}/\mathfrak{m}^r$ is a finite dimensional k-vector space (AM Prop 6.10). Do this by filtrating (inside $\mathcal{O}/\mathfrak{m}^r$) $0 = \mathfrak{m}^r \subseteq \mathfrak{m}^{r-1} \subseteq \ldots \subseteq \mathfrak{m} \subseteq \mathcal{O}/\mathfrak{m}^r$. Since $\mathcal{O}$ is Noetherian, each quotient is a finite k-vector space, and thus $\mathcal{O}/\mathfrak{m}^r$ is finite dimensional.

   Now show that $(Y \cdot Z)_P \geq \mu_P(Y) \cdot \mu_P(Z)$. For the case that P is nonsingular on both Y and Z, see Shafarevich Bk 1, p 225 ex 3. For the case that P is singular on one of Y or Z, see Shaf Bk 1, p 226 ex 4. For the general case, let f be homogeneous of degree m and let g be homogeneous of degree n, with $m \leq n$. Start with linearly independent monomials in $k[x,y] : \{1, x, y, x^2, y^2, xy, \ldots\}$. Mod out by (f,g) and take the maximal set of linearly independent terms. Label these terms $M_0, \ldots, M_a$ in $k[x,y]/(f,g)$ and count the number of terms of fixed degree:

$\underline{Deg}$	$\underline{No. Terms}$
0	1
1	2
$\vdots$	$\vdots$
$m-1$	m
m	$m = m+1-\{f_m\}$
$m+1$	$m = m+2-\{xf_m, yf_m\}$
$\vdots$	$\vdots$
$n-1$	m
n	$m-1 = m-\{g_n\}$
$n+1$	$m-2 = m-\{xg_n, yg_n\}$
$\vdots$	$\vdots$
$n+m-2$	1

Therefore the total number of terms, adding up the right column, is just mn. So we have a chain $(0) \subset (M_a) \subset (M_a, M_{a-1}) \subset \ldots \subset (M_a, M_{a-1}, \ldots, M_1)$ of length a in $k[x,y]/(f,g)$, which extends to a chain of length a in $(k[x,y]/(f,g))_{(x,y)} \cong \mathcal{O}_p/(f,g)$. Therefore $l(\mathcal{O}_p/(f,g)) \geq a = mn = \mu_P(Y) \cdot \mu_P(Z)$

(b) Let $L_1, \ldots, L_m$ be the distinct linear factors appearing in the lowest term of the equation of Y. Then if L is not one of these and r is the multiplicity, then $m_x^r \subseteq (f, L)$ by counting dimensions in the table above. The table then gives a sum of r 1's, so the intersection multiplicity is r.

(c) The fact that $(Y \cdot L) = m$ follows exactly from Bezout's Theorem. However, doing it their way, if we set L to be the line defined by $y = 0$, then for $z \neq 0$, Y is defined by $f(x) + yg(x,y) = 0$, where f is a polynomial in x of deg n. If x is a root of multiplicity m, $(L \cdot Y)_{(x,0)} = m$, so the sums of their intersection multiplicities along the x-axis is equal to the number of roots of f, which is n. But at $(0,1,0)$, the intersection multiplicity id $d-n$ since the equation for f is locally $z^{d-n} + \ldots + xg(x,y) = 0$. So $\sum (L \cdot Y)_P = n + d - n = d$.

5. If char $k = 0$ or char $k = p$ does not divide d, then $x^d + y^d + z^d = 0$ defines a nonsingular hyperplane of degree d. If p divides d, then $xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$ works.

6. (a) i. Let Y be defined by $x^6 + y^6 - xy$ (node). Blow-up Y at $(0,0)$: Let t, u be the homogeneous coordinates on $\mathbb{P}^1$. Then $\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1$ is defined by $xu = ty$. Call the exceptional curve E. In the affine piece $t = 1$, we get $y = xu$ and $x^6 + y^6 - xy = x^2(x^4 + x^4u^6 - u) = 0$. We get two irreducible components. The exceptional curve is defined by $x = y = 0, u$ free. $\tilde{Y}$ is defined by $(x^4 + x^4u^6 - u) = 0, y = xu$, which meets E at (0,0,0). Replacing $y = xu$, we get $\widetilde{Y}$

is defined by $x^4+y^4u^2-u=0$. An easy check shows the partial derivatives never vanish, so $\widetilde{Y}$ is non-singular.

ii. Let Y be defined by $y^2+x^4+y^4-x^3$ (cusp). Blow-up Y at $(0,0)$: Let t,u be the homogeneous coordinates on $\mathbb{P}^1$. Then $\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1$ is defined by $xu = ty$. Call the exceptional curve E. In the affine piece $t=1$, we get $y = xu$ and $x^2u^2+x^4+x^4u^4-x^3 = x^2(u^2+x^2+x^2u^4-x)=0$. We get two irreducible components. The exceptional curve is defined by $x=y=0, u$ free. $\widetilde{Y}$ is defined by $u^2+x^2+x^2u^4-x=0, y=xu$, which meets E at (0,0,0). Replacing $y=xu$, we get $\widetilde{Y}$ is defined by $x^4+y^4u^2-u=0$ which meets E only at $(0,0,0)$. An easy check shows the matrix of the partial derivatives evaluated at $(0,0,0)$ is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has rank = codim $Y=2$, so $\widetilde{Y}$ is non-singular. An easy check shows that $\widetilde{Y} \cap E = \emptyset$ in the affine piece $u \neq 0$, so $\widetilde{Y}$ is nonsingular.

(b) Points on the exceptional curve correspond to tangent lines. Since a node has 2 distinct tangent lines, we expect the blowup of the curve to intersect the exceptional divisor twice. By a change of coordinates, Y is defined by $xy+f(x,y)=0$ where $f(x,y)$ has only terms of degree greater than 2. Let $P=(0,0)$. Blow-up $\mathbb{A}^2$ at P: $\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1$ is defined by $xu=yt$. In the affine piece $t=1$, $y=xu, xy+f(x,y)=0 \Rightarrow x^2u = f(x,ux) = x^2(u+g(x,xu))=0$. Therefore we get 2 irreducible components. One is the exceptional curve E defined by $x=0, y=0, u$ free. $\widetilde{Y}$ is defined by $u=g(x,xu)=0, xu=y$, which meets E at $(0,0,0) \in \mathbb{A}^3_{t\neq 0}$. Similar arguments in the affine piece $u=1$ show that $\widetilde{Y} \cap E = (0,0,0) \subset \mathbb{A}^3_{u\neq 0}$.. An easy check on the Jacobian shows that these points are nonsingular. Thus $\varphi^{-1}(P) \cap E = \{(0,0,1,0),(0,0,0,1)\}$.

(c) Let $P=(0,0,0)$ and Y defined by $x^2=x^4+y^4$ have a tacnode. The blowup $\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1$ is defined by $xu=yt$. In the affine piece $t=1$, we have $y=xu$ and $x^4+y^4-x^2=0$, which give $x^2(x^2+x^2u^4-1)=0$. We get 2 irreducible components: the exceptional curve E defined by $x=y=0$, u free, and $\widetilde{Y}|_{t\neq 0}$ defined by $x^2+x^2u^4-1=0$. $\widetilde{Y}|_{t\neq 0} \cap E = \emptyset$. In the affine piece $u=1$, we get $x=yt$ and $x^4+y^4-x^2=0$, which gives $y^2(y^2t^4+y^2-t^2)=0$. This defines two irreducible components: the exceptional curve E defined by $y=x=0, t$ free. $\widetilde{Y}|_{u\neq 0}$ defined by $y^2t^4+y^2-t^2=0$, which intersects E at $(0,0,0) \subseteq \mathbb{A}^3_{u\neq 0}$. At this point, the lowest degree terms are $y^2-t^2=(y-t)(y+t)$, so $(0,0,0)$ is a node and by (b), can be resolved in one blow-up. So the tacnode can be resolved by two successive blowups.

(d) Let Y be the plane curve $y^3=x^5$, which has a higher order cusp at 0. Since the lowest term is of degree 3, $(0,0)$ is clearly a triple point. The blowup $\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1$ is defined by $xu=yt$. In the affine piece

$t = 1$, $y = xu, y^3 = x^5$ gives $x^3(x^2 - u^3) = 0$. We get two irreducible components: the exceptional curve E is defined by $x = y = 0, u$ free. $\widetilde{Y}$ is defined by $x^2 = u^3$, which has a cusp. BY part a), blowing up a cusp gives a nonsingular strict transform. Therefore 2 blow-ups resolve the singularity.

7. (a) At $(0, 0, 0)$, at least one partial derivative is nonzero by assumption. At $(0, 0, 0)$, the partials all vanish since the degree > 1. Thus $(0, 0, 0)$ is clearly a singular point, and it is the only singular point since Y is a nonsingular curve in $\mathbb{P}^2$.

   (b) Blow up X at $(0, 0, 0)$ to get $\widetilde{X} \subset \mathbb{A}^3 \times \mathbb{P}^2$, where the coordinates on $\mathbb{P}^2$ are (t, u, v) is defined by $xu = yt, xv = zt, yv = zu$. Look at the affine piece $t = 1$. Then $\widetilde{X}$ is defined by $f(x, xu, xv) = 0$, which becomes $x^d f(1, u, v) = 0$. The exceptional curve E is defined by $x^d = 0$. $\widetilde{X} \backslash E$ is defined by $f(1, u, v) = 0$. So dim $\widetilde{X} = 2$ inside of $\mathbb{A}^3_{t \neq 0}$. Therefore the Jacobian of partials is just $\begin{pmatrix} 0 & f_u^{'} & f_v^{'} \end{pmatrix}$ which has rank one since both $f_u^{'}$ and $f_v^{'} \neq 0$ since X is nonsingular in $\mathbb{P}^2$. Applying the same argument for the other affine covers, w get that $\widetilde{X}$ is nonsingular.

   (c) In each affine piece, the strict transform is defined by the equations $f(1, u, v) = 0, f(t, 1, v) = 0, f(t, u, 1) = 0$. These define $Y = \varphi^{-1}(P) = \bigcup Y|_{\mathbb{A}^3_i} = Y \subset \mathbb{P}^2$.

8. Partials of a homogeneous polynomial are again homogeneous. In the affine piece $a_0 = 1$, the matrix of partials becomes $n \times t$ instead of $n+1 \times t$. However, by Euler's Theorem, the rank of the matrix does not change since the deleted row is a multiple of the others.

9. Assume that f is reducible, say $f = g \cdot h$. By ex 2.7, there exists P such that $g(P) = h(P) = 0$. Then $f'_x(P) = g(P)h'_x(P) + h(P)g_x^{'}(P)$. So if $f(P) = g(P)h(P) = 0, f_x^{'} = 0$. Similar for partials with respect to y and z. Therefore all derivatives would vanish, which contradicts the fact that Sing Y is proper. Thus f is irreducible and Y is non-singular.

10. (a) This is the argument in the second paragraph in Shaf I, II.1.4 (the bottom of page 92).

    (b) Let $\varphi : X \to Y$ be defined by $x \mapsto (f_1(x), \ldots, f_n(x))$. Define $\varphi^* : m_{\varphi(x)}/m^2_{\varphi(x)} \to m_x/m_x^2$ on the cotangent space defined by $f \mapsto f \circ \varphi$. This is well-defined since if $f \in m_{\varphi(x)}$, ie $f(\varphi(x)) = 0$, then $\varphi^*(f) = f(\varphi(x)) = 0$, so $\varphi^*(f) \in m_x$. It is easy to see that $\varphi^*(m^2_{\varphi(x)}) \subseteq m_x^2$, so taking the dual of this map gives a map $\Theta_{P,X} \to \Theta_{\varphi(P),Y}$

    (c) Let $\varphi : \mathcal{Z}(x - y^2) \to \{x\text{−axis}\}$. be defined by $(x, y) \to x$. As in a), define the dual map $\varphi^* : m_0/m_0^2 \to m_0/m_0^2$ by $f \mapsto f \circ \varphi$. $m_x = (x)$ and $\varphi(x) = x$. But $x = y^2$, so $x \in m_x^2$. Therefore $\varphi^* = 0$. Thus the map defined on the cotangent spaces is the zero map, so the dual map is again the zero map.

11. Let $Y = \mathcal{Z}(x^2 - xz - yw, yz - xw - zw) \subset \mathbb{P}^3$. Let $P = (0,0,0,1)$. Let φ denote the projection from P to the plane $w = 0$, ie $\varphi : Y \to \mathbb{P}^2 = \mathcal{Z}(w)$ is defined by $(w, y, z, w) \mapsto (x, y, z)$. To see that $\varphi(Y) \subseteq \mathcal{Z}(y^2z - x^3 + xz^2)$, we just have to note that $y^2z - x^3 + xz^2 = y(yz - xw - zw) + (x+z)(x^2 - xz - yw)$. Solving for w we get that in Y, $w = \frac{x^2 - xz}{y}$ and $w = \frac{yz}{x+z}$. In the image, these are both equal, so we have $\varphi^{-1} : \mathcal{Z}(y^2x - x^3 + xz^2) \to Y \backslash P$ defined by $(x, y, z) \mapsto (x, y, z, \frac{x^2 - xz}{y}) = (x, y, z, \frac{yz}{x+z})$, which is not defined at $(1, 0, -1)$.

12. (a) Generalizing the idea in Ex 1.3.1(c), we can write any conic as a symmetric matrix, which from Linear Algebra, we know we can diagonalize. A change of basis corresponds to a linear transformation, so we can always write a conic f as a sum of squares $x_0^2 + \ldots + x_r^2$ where r is the rank of the matrix.

    (b) f is obviously reducible for $r = 0, 1$. Now the hypersurface $\mathcal{Z}(f)$ is irreducible iff its defining equation is irreducible. If f factors, then clearly $x_{r+1}, \ldots, x_n$ don't appear in the factorization. So it's enough to check irreducibility in $k[x_0, \ldots, x_r]$. This is equivalent to f defining an irreducible hypersurface in $\mathbb{P}^r$. But f defines an nonsingular, hence irreducible, hypersurface so we're done.

    (c) Sing Q is the zero locus of the partial derivatives, each of which has degree 1 since f is a conic and the characteristic of $k \neq 2$. Thus the variety defined by them is linear. By ex 2.6, dim Z = dim $S(Z)$ - 1 = dim $k[x_0, \ldots, x_n]/(x_0, \ldots, x_r) - 1 = n - r - 1$.

    (d) For $r < n$, define $Q' \subseteq \mathbb{P}^r$ by $\mathcal{Z}(f)$ and embed $\mathbb{P}^r \hookrightarrow \mathbb{P}^n$ as the first r coordinates. Then the rest is clear since in P^2 for instance the line joining $(a, b, 0)$ and $(0, 0, c)$ is $\{(sa, sb, sc) \mid s, t \in k^\times\}$.

13. Since this question is local we can assume that X is affine. By the finiteness of integral closure, the integral closure of $k[X]$, $\overline{k[X]}$ is finitely generated, say with generators $f_1, \ldots, f_n$. Then for any $x \in X$, $\overline{\mathcal{O}_x}$ is generated by the images of $f_1, \ldots, f_n$. Denote the image of f_i in the stalk again by f_i. Then $\mathcal{O}_x$ is integrally closed iff $f_i \in \mathcal{O}_x$ for every i. Any rational function is defined on a nonempty open set, and a finite intersection of these is again open and is nonempty since X is irreducible. Thus the normal locus is a nonempty open set, forcing the non-normal locus to be proper and closed.

14. (a) Let $P \in Y, Q \in Z$ be analytic isomorphic plane curve singularities. Then $\widehat{\mathcal{O}}_{P,Y} \cong \widehat{\mathcal{O}}_{Q,Z}$, where $\widehat{\mathcal{O}}_{P,Y} \cong k[[x,y]]/(f_r + \ldots + f_d), \widehat{\mathcal{O}}_{Q,Z} \cong k[[x,y]]/(g_s + \ldots + g_d)$ where $Y = \mathcal{Z}(f_r + \ldots + f_d)$ and $Z = \mathcal{Z}(g_s + \ldots + g_d)$. The isomorphism between the completion of the local rings must map $x \mapsto \alpha x + \beta y + h.o.t$ and $y \mapsto \alpha' x + \beta' y + h.o.t$ for $\alpha, \alpha', \beta, \beta' \neq 0$. This is to guarantee that x, y are in the image and that they span a 2 dimensional subspace in the image as well. Therefore $f(x, y) \mapsto$

$f(\alpha x+\beta y+h.o.t, \alpha^{'}x+\beta^{'}y+h.o.t) := \widetilde{f}(\Phi_1, \Phi_2)$. Therefore $\widetilde{f} = ug$ since any automorphism of $\mathbb{A}^2$ is given by $F(\Phi_1, \Phi_2) = GU$ for some unit U (see Shaf Bk 1, p 113, ex 10). Therefore $\widetilde{f}$ and g have the same lowest term, so $r = s$ and $\mu_P(Y)\mu_Q(Z)$.

(b) Let $f = f_r + \ldots \in k[[x, y]], f_r = g_s h_t$, where g_s, h_t are forms of degree s, t, with no common linear factor. Construct $g = g_s + g_{s+1} + \ldots, h = h_t + h_{t+1} + \ldots \in k[[x, y]]$ step by step as in the example. Then $f_{r+1} = h_t g_{s+1} + g_s h_{t+1}$ since $s + t = r$. This is possible since g_s, h_t generate the maximal ideal of $k[[x, y]]$. Continue in this way to construct g and h such that $f = gh$.

(c) Let Y be defined by $f(x, y) = 0$ in $\mathbb{A}^2$. Let $P = (0, 0)$ be a point of multiplicity r on Y. Write $f = f_r + hot$. Let Q be another point of multiplicity r, for $r = 2, 3$. From Linear Algebra, if $f = (\alpha x + \beta y)(\alpha^{'}x+\beta^{'}y)+hot$ centered at P and $f = (\gamma x+\delta y)(\gamma^{'}x+\delta^{'}y)+hot$ centered at Q, then $\alpha x+\beta y, \alpha^{'}x+\beta^{'}y, \gamma x+\delta y, \gamma^{'}x+\delta^{'}y$ are all lines in $\mathbb{P}^1$, and in $\mathbb{P}^1$ (or $\mathbb{A}^2$), any 2 or 3 pairs of lines can be moved to each other by a linear transformation. However, for 4 or more lines, this can not be done in $\mathbb{P}^1$ or $\mathbb{A}^2$. Therefore the one parameter family is the fourth line that cannot be mapped via a linear transformation after equating the other three lines.

(d) Ingredients: 1 Chicken and giblets, cut up; 1 tb Salt; 4 Carrots, chopped; 6 Celery stalks w/leaves; chop 1 Onion, med., chopped; 1 Garlic clove, minced 1 cup Rice or noodles.
Directions: Put chicken pieces in large pot with water to cover. Add salt and bring to a boil. Reduce heat to simmer and skim off fat. Add vegetables and garlic, cover and cook until tender. Remove chicken and either serve separately or dice and return to soup. Season to taste. Add rice or noodles and cook until tender.

15. (a) $(x_0, \ldots, x_N) \in \mathbb{P}^N \mapsto x_0x^d + x_1y^d + x_2z^d + \ldots + x_Nyz^{d-1}$ with the reverse correspondence clear.

(b) The correspondence is one-to-one if f has no multiple factors, ie if f is irreducible. By elimination theory, the points in $\mathbb{P}^N$ such that f and $\nabla f = 0$ correspond to the set $\{g_1, \ldots, g_r\}$ of polynomials with integer coefficients which are homogeneous in each f_i. Therefore the points where $f, \nabla f \neq 0$ are in one-to-one correspondence with $g_1, \ldots, g_r \neq 0$ which defines an open set in $\mathbb{P}^N$. Since $\nabla f \neq 0$, the curve is non-singular.

1.6 Nonsingular Curves

1. (a) Let Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$. By Prop 6.7, Y is isomorphic to an abstract nonsingular curve. Therefore Y is a subset of the complete abstract nonsingular curve

Z of its function field. But Z is birational to $\mathbb{P}^1$, so in fact $Z \cong \mathbb{P}^1$, so $Y \subseteq \mathbb{P}^1$. Since Y is not complete, it must be inside some $\mathbb{A}^1$.

(b) Embed $Y \hookrightarrow \mathbb{P}^1$. Since Y is not isomorphic to $\mathbb{P}^1$, $Y \subseteq \mathbb{A}^1$. Since by part a), $Y = \mathbb{A}^1$ minus a finite number of points, Y is a principle open subset, which is affine.

(c) Since by part b), Y is a principal open set, say $\mathbb{A}^1 \backslash \{\alpha_1, \ldots, \alpha_n\}$, $A(Y) = \mathcal{O}(Y) = k[t, \frac{1}{t-\alpha_1}, \ldots, \frac{1}{t-\alpha_n}]$ This is the localization of a UFD, which is again a UFD.

2. Let Y be defined by $y^2 = x^3 - x$ in $\mathbb{A}^2$, with char $k \neq 2$.

(a) $f_x^{'} = 3x^2 - 1, f_y^{'} = -2y$. The zero locus of the partial derivatives is the points $(\pm 1/\sqrt{3}, 0)$, which is not on the curve. So Y is nonsingular. $A(Y) = k[x,y]/(y^2 - x^3 + x)$ is integrally closed since Y is nonsingular, and in dimension 1, nonsingular and normal are equivalent. (Shaf 1, Corollary p 127)

(b) Since k is algebraically closed, x is transcendental over k, and thus $k[x]$ is a polynomial ring. Since $y^2 \in k[x], y \in \overline{k[x]}$. So $A \subseteq \overline{k[x]}$. Since $k[x] \subseteq A$, by taking the integral closure of both sides gives $\overline{k[x]} \subseteq \overline{A} = A$ (since A is integrally closed by part a). So $A = \overline{k[x]}$.

(c) $\sigma : A \to A$ defined by $y \mapsto -y$ is an automorphism due to the y^2 term and clearly leaves x fixed. Let $a = f(x,y) = yf(x) + g(x) \in A$. Then $N(f(x,y)) = f(x,y)f(x,-y) = (yf(x) + g(x))(-yf(x) + g(x)) = -y^2 f^2(x) + g^2(x) = -(x^3 - x)f^2(x) + g^2(x) \in k[x]$. $N(1) = 1$ is clear, and $N(ab) = (ab)\sigma(ab) = a\sigma(a)b\sigma(b) = N(a)N(b)$.

(d) If a is a unit in A, then $aa^{-1} = 1$. Taking norms of both sides, we get $N(aa^{-1}) = N(a)N(a^{-1}) = N(a)N(a)^{-1} = N(1) = 1$. So if a is a unit, its norm must have an inverse in k, ie lie in $k^{\times}$. Assume x is reducible, ie $x = ab$ for both a, b irreducible. Then takin norms, $N(x) = x^2 = N(a)N(b)$. Since there does not exist any a, b whose norm is a degree 1 polynomial, x must be irreducible. Similar argument for y. A is not a UFD since $y^2 = x(x^2 - 1)$, so $x | y^2$. If A were a UFD, then $x = uy$ for some unit u. But by comparing norms as before, this can not happen. So A is not a UFD.

(e) A is neither trivial nor a UFD, so by ex 1, Y is not rational.

3. (a) Let dim $X \geq 2$. Let $X = \mathbb{A}^2$. Then the map $\varphi : \mathbb{A}^2 \backslash (0,0) \to \mathbb{P}^1$ defined by $(x,y) \mapsto (x : y)$. Then this map is not regular at the origin.

(b) Let $Y = \mathbb{A}^1$. Then $\varphi : \mathbb{P}^1 \backslash \mathbb{A}^1$ defined by $(x : y) \mapsto x/y$. If φ had an extension, then the identity map and φ would agree on some dense open set, and thus be equal, which is a contradiction.

4. Let Y be a nonsingular projective curve. Let f be a nonconstant rational function on Y. Let $\varphi : Y \to \mathbb{P}^1$ defined by $x \mapsto f(x)$ in the affine

piece. Since Y is projective, the image must be closed in $\mathbb{P}^1$. Since f is nonconstant and Y is irreducible, the image must be all of $\mathbb{P}^1$. Since φ is dominant, it incudes an inclusion $k(Y) \hookrightarrow k(\mathbb{P}^1)$. Since both fields are finitely generated extension fields of transcendence degree 1 of k, $k(\mathbb{P}^1)$ must be a finite algebraic extension of $k(Y)$. To show that φ is quasi-finite (ie $\varphi^{-1}(P)$ is a finite set), look at any open affine set V in $\mathbb{P}^1$. Its coordinate ring is $k[V]$, and by Finiteness of Integral Closure Thm (I.3.9), $\overline{k[V]}$ is a finite $k[V]$-module. The corresponding affine set to $\overline{k[V]}$ is isomorphic to an open subset U of Y. Clearly $U = \varphi^{-1}(V)$, and thus φ is a quasi-finite morphism.

5. Let X be a nonsingular projective curve. Then embed $X \hookrightarrow Y$ by a regular map. Since the image of a projective variety is closed under regular mappings, X is closed in Y.

6. (a) If $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is defined by $x \mapsto (ax+b)/(cx+d)$, then the inverse map is then given by $\frac{1}{ad-bc}(xd-b)/(a-xc)$.

   (b) Any $\varphi : \mathbb{P}^1 \cong \mathbb{P}^1 \in \mathrm{Aut}(\mathbb{P}^1)$, clearly induces an isomorphism $\varphi^* : k(x) \cong k(x)$ defined by $f \mapsto f \circ \varphi$. Conversely, given an automorphism φ of $k(x)$, this induces a birational map of $\mathbb{P}^1$ to itself. But any birational map of non-singular projective curves is an isomorphism.

   (c) If $\varphi \in$Aut $k(x)$, $\varphi(x) = f(x)/g(x)$ for $(f,g) = 1$. If deg $g, f > 1$, the map won't be injective, so both f and g are linear, say $f(x) = ax+b$ and $g(x) = cx+d$ and by $(f,g) = 1$, $ad - bc \neq 0$. Therefore PGL(1) $\cong$ Aut $k(x)$ $\cong$Aut $\mathbb{P}^1$.

7. If $\mathbb{A}^1 \backslash P \cong \mathbb{A}^1 \backslash Q$, then there is an induced birational map between $\mathbb{P}^1$ and $\mathbb{P}^1$. But any birational map between nonsingular projective curves is an isomorphism, so in particular, it is injective and surjective. Thus $|P| = |Q|$. The converse is not true for $r > 3$ since any set of at most 3 points in P^1 can be mapped to any other set of the same size under Aut $\mathbb{P}^1$. Any isomorphism between $\mathbb{P}^1$ and $\mathbb{P}^1$ fixes at most 2 points, so if $r > 3$, the map must be the identity isomorphism. If P and Q only have 3 elements in common, with other elements different, then $\mathbb{A}^1 \backslash P \ncong \mathbb{A}^1 \backslash Q$.

1.7 Intersections in Projective Space

1. (a) By ex 2.12, the homogeneous coordinate ring is isomorphic as a graded algebra with the subalgebra of $k[x_0, \ldots, x_n]$ generated by monomials of degree d. Thus $\varphi_Y(l) = \binom{n+dl}{n}$, so $P_Y(z) = \binom{n+dz}{n}$. So the degree is $n! \cdot \frac{d^n}{n!} = d^n$.

   (b) By ex 2.14, the homogeneous coordinate ring is isomorphic as a graded algebra to the subring of $k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ generated by $\{x_i, y_k\}$ with M_k being the set of polynomials of degree $2k$. Each monomial is made up of half x's and half y's, so $\varphi_Y(l) = \binom{r+l}{l}\binom{s+l}{l} = \binom{r+l}{r}\binom{s+l}{s}$. So the degree $= (r+s)! \cdot \frac{1}{r!s!} = \binom{r+s}{r}$.

2. (a) $p_a(\mathbb{P}^n) = (-1)^r(\binom{n}{n} - 1) = 0$

   (b) Let Y be a plane curve of deg d. Then $\dim k[x,y]/(f)_l = \begin{cases} \binom{l+2}{2} & l \leq d \\ \binom{l+2}{2} - \binom{l+d+2}{2} & l > d \end{cases}$
   Then $P_Y(z) = \binom{z+2}{2} - \binom{z-d+2}{2}$. So $P_Y(0) = \binom{2}{2} - \binom{-d+2}{2} = 1 - \frac{(d-1)(d-2)}{2}$. Therefore $p_a(Y) = (-1)(1 - \frac{(d-1)(d-2)}{2} - 1) = \frac{(d-1)(d-2)}{2}$. This result is sometimes called Plücker's formula.

   (c) $p_a(H) = (-1)^{n-1}[\binom{n}{n} - \binom{n-d}{n} - 1] = (-1)^n\binom{n-d}{n} = (-1)^n\frac{(n-d)(n-d-1)\ldots(1-d)}{n!} = \frac{(d-1)\ldots(d-n)}{n!} = \binom{d-1}{n}$.

   (d) Let $Y = X_1 \cap X_2$, with $X_i = \mathcal{Z}(f_i)$. Then $X_1 \cup X_2 = \mathcal{Z}(f_1f_2)$, so deg $X_1 \cup X_2 = a + b$. From the exact sequence

$$0 \to S/(f_1f_2) \to S/(f_1) \oplus S/(f_2) \to S/(f_1, f_2) \to 0$$

   we get $P_Y = P_{X_1} + P_{X_2} - P_{X_1 \cup X_2}$. So $p_a(Y) = -1[\binom{3-1-b}{3} - \binom{3-a}{3} - \binom{3-b}{3}] = \frac{1}{2}a^2b + \frac{1}{2}ab^2 - 2ab + 1 = \frac{1}{2}ab(a+b-4) + 1$.

   (e) The graded ring here is isomorphic to $\bigoplus_{i=0} M_i \otimes N_i \subseteq k[x_0, \ldots, x_n] \otimes k[y_0, \ldots y_m]$.. Tensor products multiply dimensions, so $\varphi_{Y \times Z}(l) = \varphi_Y(l)\varphi_Z(l)$ and so $\varphi_{Y \times Z} = \varphi_Y\varphi_Z$. Thus $p_a(Y \times Z) = (-1)^{r+s}(P_Y(0)P_Z(O) - 1) = (-1)^{r+s}[(P_Y(0) - 1)(P_Z(0) - 1) + (P_Y(0) - 1) + (P_Z(0) - 1)] = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z)$.

3. If $P = (a_0', a_1', a_2')$, then the tangent line $T_p(Y)$ is defined by $\frac{\partial F}{\partial a_0}|_P(a_0 - a_0') + \frac{\partial F}{\partial a_1}|_P(a_1 - a_1') + \frac{\partial F}{\partial a_2}|_P(a_2 - a_2') = 0$. This line is unique since P is a nonsingular point. The intersection multiplicity is the highest power of t, where $L = t\overrightarrow{\alpha}$ and $Y \cup L = F(t\alpha_1, t\alpha_2, t\alpha_3)$ after looking in $\mathbb{A}^2$ of the point $P = (0,0)$. P is singular iff $F = F_2 + \ldots + F_d$. Therefore multiplicity is ≥ 2. The mapping $P^2 \to (\mathbb{P}^2)^*$ is defined by $(x_0, x_1, x_2) \mapsto \nabla f|_{(x_0,x_1,x_2)} \neq 0$, ie P is non-singular.

4. By Bezout's Theorem, any line not tangent to Y and not passing through a singular point meets Y in exactly d distinct points. Since Sing Y is closed and proper, the lines intersecting with Sing Y are closed in $(\mathbb{P}^2)^*$. By ex 3, the tangent lines to Y are contained in proper closed subsets of $(\mathbb{P}^2)^*$, so there exists $U \neq \emptyset$ open in $(\mathbb{P}^2)^*$ intersecting Y in d points.

5. (a) Assume there exists a point P with multiplicity $\geq d$. Pick any line through the singular point P of multiplicity $\geq d$ and any other point Q. Then $(C.L) = (C.L)_P + (C.L)_Q > d$, which contradicts Bezout's Theorem.

   (b) Let Y be an irreducible curve of deg $d > 1$, with P having multiplicity $d - 1$. Assume that Y is defined by $f = f(x,y) + g(x,y)$, where deg $f(x,y) = d-1$, deg $g(x,y) = d$. Let $t = y/x, y = -f(t,1)/g(t,1)$ and $x = yt$. This is just the projection from a point and gives a birational map to $\mathbb{A}^1$.

6. Let dim $Y = 1$. By Prop 7.6(b), Y is irreducible. Pick any two points on Y and pass a hyperplane through them. Then by Thm 7.7, we must have $Y \subseteq H$. Since this is true for any hyperplane through these points, Y is the line through these two points. Now suppose the assumption is true for dimension r varieties and let dim $Y = r + 1$. Let $P, Q \in Y$ and H a hyperplane through P and Q not containing Y. Then by Thm 7.7 again, $Y \cap H$ is linear, so Y contains the line through P and Q. So Y is linear.

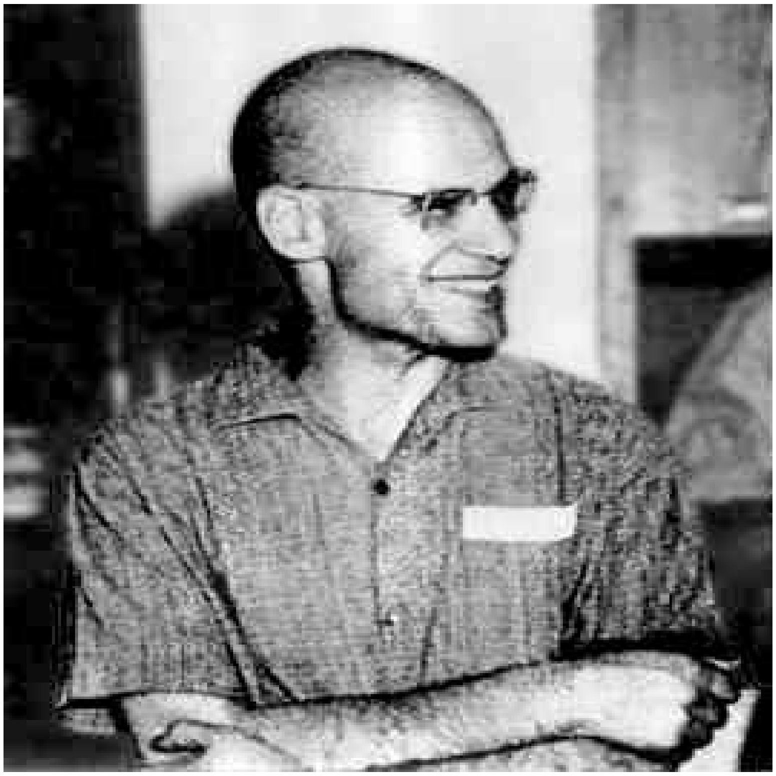
   (a) Fix P and consider the projection map to P^1. X is parameterized by the fibers of this map, of which are same dimension and irreducible, so X is a variety of dim $r + 1$.

   (b) For dim $Y = 0$, Y consists of d points, so X is $d - 1$ lines. So the deg $X = d - 1$. Now suppose dim $Y = r$. Choose a hyperplane H through P not containing Y so that the intersection multiplicity alone any component of $X \cap H$ is 1. Then by Them 7.7 and 7.6(b), deg $X \cap H =$ deg X, and deg $Y \cap H \leq$ deg $Y = d$. $X \cap H$ is the cone over $Y \cap H$ so by induction, def $X \cap H \leq$ deg $Y \cap H = d$. So deg $X < d$.

7. Let $Y^r \subseteq \mathbb{P}^n$ be a variety of deg 2. By ex 7, Y is contained in a degree 1 variety H of dimension $r + 1$ in $\mathbb{P}^n$. By ex 6, this is a linear variety and thus isomorphic to $\mathbb{P}^{r+1}$.

1.8 What is Algebraic Geometry?

Answer: Understanding this guy:



2 Chapter 2: Schemes

2.1 Sheaves

1. Given the constant presheaf $\mathscr{F} : U \mapsto A$, for A an abelian group with restriction maps the identity, construct $\mathscr{F}^+$ as in 1.2. Since the constant sheaf $\mathcal{A}$ is the sheaf of locally constant functions, $\mathcal{A}$ satisfies the conditions of $\mathscr{F}^+$ and by uniqueness, $\mathcal{A} \cong \mathscr{F}^+$.

2. (a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. We have the commutative diagram:
$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}_p \arrow[r, "\varphi_p"] & \mathscr{G}_p \end{tikzcd}$$
Let $s \in \ker\, \varphi(U)$. Let $\overline{s}$ be its image in $\mathscr{F}_p$. Now since $s \mapsto 0 \mapsto \overline{0}$ in $\mathscr{G}_p$ and since the diagram commutes, $\overline{s} \mapsto 0$. Thus $\overline{s} \in \ker\, \varphi_p$ and $(\ker\, \varphi)_p \subseteq \ker\, \varphi_p$. To show the reverse inclusion, let $\overline{s} \in \ker\, \varphi_p$. Pull back $\overline{s}$ to $s \in \mathscr{F}(U)$. Say $\varphi(U)(s) = t$, where $\overline{t} = 0 \in \mathscr{G}_p$. Therefore in some neighborhood, say $V \subseteq U$, $t|_V = 0$. Therefore we have the commutative diagram
$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[d] & \mathscr{G}(V) \arrow[d] \\ \mathscr{F}_p \arrow[r, "\varphi_p"] & \mathscr{G}_p \end{tikzcd}$$
where now $\varphi(V)(s) = 0$. Therefore $\overline{s} \in (\ker\, \varphi)_p$ when restricted to a small enough open set. So $\ker\, \varphi_p \subseteq (\ker\, \varphi)_p$ and equality follows.

   (b) If φ is injective, then $\ker\, \varphi = 0$. Therefore $(\ker\, \varphi)_p = 0$ and so by part a), $\ker\, \varphi_p = 0$ and φ_p is injective. Converse is obvious. Now if φ is surjective, $\operatorname{im}\, \varphi = \mathscr{G}$. ie $(\operatorname{im}\, \varphi)_p = \mathscr{G}_p$, so by part a), $\operatorname{im}\, \varphi_p = \mathscr{G}_p$ and φ_p is surjective. Converse is obvious.

   (c) The sequence $\ldots \to \mathscr{F}^{i-1} \stackrel{\varphi^{i-1}}{\to} \mathscr{F}^i \stackrel{\varphi^i}{\to} \mathscr{F}^{i+1} \to \ldots$ of sheaves is exact iff $\operatorname{im}\, \varphi^{i-1} = \ker\, \varphi^i$ iff $(\operatorname{im}\, \varphi^{i-1})_p = (\ker\, \varphi^i)_p$ iff $\operatorname{im}\, \varphi_p^{i-1} = \ker\, \varphi_p^i$ iff $\ldots \to \mathscr{F}_p^{i-1} \stackrel{\varphi_p^{i-1}}{\to} \mathscr{F}_p^i \stackrel{\varphi_p^i}{\to} \mathscr{F}_p^{i+1} \to \ldots$ is exact.

3. (a) Let $\varphi : \mathscr{F} \to G$ be a morphism of sheaves on X. Suppose for every open $U \subseteq X, s \in \mathscr{G}(U), \exists$ a covering $\{U_i\}$ of U with $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$. To show φ is surjective, we just have to show (by 1.2(b)) φ_p is surjective. Consider the commutative diagram:
$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}_p \arrow[r, "\varphi_p"] & \mathscr{G}_p \end{tikzcd}$$
. Pick $s_p \in \mathscr{G}_p$ and pull it back to some $s \in \mathscr{G}(U)$. By assumption, $t_i \in \mathscr{F}(U_i)$ exist such that $\varphi(t)_i = s|_{U_i}\, \forall\, i$. Mapping

these t_i to $\mathscr{F}_p$, we see that $\exists \bar{t}_i \in \mathscr{F}_p$ such that $\varphi_p(\bar{t}_i) = s_p$. ie φ_p is surjective. Conversely, if φ is surjective, then φ_p is surjective $\forall p \in X$. Let $s \in \mathscr{G}(U)$. Then there exists $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. ie there exists a neighborhood U_p of p such that $\varphi(t|_{U_p}) = s|_{U_p}$, so U_p is a covering of U and the condition holds.

(b) The standard example here is $\varphi : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $f \mapsto e^{2\pi i f}$. The stalks are surjective because since by choosing a small enough neighborhood, every nonzero-holomorphic function has a logarithm. So by 1.2b, φ is surjective. But for $U = \mathbb{C}^*$, $\varphi(\mathbb{C}^*)$ is not surjective since z is a non-zero holomorphic function on $\mathbb{C}^*$ but does not have a global logarithm.

4. (a) By construction, $\mathscr{F}_p = \mathscr{F}_p^+$ and $\mathscr{G}_p = \mathscr{G}_p^+$. If φ in injective, then $\varphi_p = \varphi_p^+$ is injective for all p. By 1.2(b), φ^+ is injective.

   (b) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then there is an injective morphism $\varphi(\mathscr{F}) \hookrightarrow \mathscr{G}$ where $\varphi(\mathscr{F})$ is the image presheaf. By part a), im $\varphi \to \mathscr{G}$ is injective, so im φ is a subsheaf of $\mathscr{G}$.

5. This follows immediately from Prop 1.1 and ex 1.2b

6. (a) Let $\mathscr{F}'$ be a subsheaf of $\mathscr{F}$. Since the map on stalks $\mathscr{F}_p \twoheadrightarrow (\mathscr{F}/\mathscr{F}')_p$ is clearly surjective, so is the natural map $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ with obvious kernel $\mathscr{F}'$. Thus the sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$ is exact.

   (b) If the sequence $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is exact, then the im $\varphi =$ ker ψ. By 1.4.(b), im φ is a subsheaf of $\mathscr{F}$ and $\mathscr{F}' \cong$ im φ. By 1.7a), im $\psi \cong \mathscr{F}/$ker ψ, and therefore $\mathscr{F}'' \cong \mathscr{F}/\mathscr{F}'$.

7. (a) Apply the first isomorphism theorem to the stalks and then use Prop 1.1.

   (b) The stalks are isomorphic, so done.

8. Let $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$ be exact. Then for any open $U \subseteq X$, since φ is injective, ker $\varphi = 0$, so in particular ker $\varphi(U) = 0$ and the sequence $0 \to \Gamma(U, \mathscr{F}') \xrightarrow{\varphi(U)} \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is exact. Thus the functor $\Gamma(U, \cdot)$ is left exact.

9. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X and let $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ be a presheaf. Let $\{U_i\}$ be an open cover for $U \subseteq X$. If $s = (t, u) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ restricted to U_i equals 0 for every U_i, then $(t|_{U_i}, u|_{U_i}) = 0 \,\forall i$. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves, $(t, u) = (0, 0) = s = 0$ on all of U. If $s_i = (t_i, u_i) \in \mathscr{F}(U_i) \oplus \mathscr{G}(U_i)$, $s_j = (t_j, u_j) \in \mathscr{F}(U_j) \oplus \mathscr{G}(U_j)$ agree on $U_i \cap U_j$, by a similar argument as before, since both $\mathscr{F}$ and $\mathscr{G}$ are sheaves, there exists $s = (t, u) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ whose restriction on U_i and U_j agree with s_i and s_j respectively.

10. Let $\mathscr{F}_i$ be a direct system of sheaves and morphisms on X. Define the direct limit of the system $\varinjlim \mathscr{F}_i$ to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. This has the universal property from the corresponding statement for abelian groups at the level of stalks. (See Dummit and Foote, 7.6.8(c))

11. Since each $\mathscr{F}_n$ is a sheaf, given any open $U \subseteq X$, we can choose a finite open cover $\{U_i\}^n$ of U and write $\mathscr{F}_n(U)$ as $\varinjlim \mathscr{F}_n(U_{ij})$. Here the limit is indexed by double intersections with inclusions as morphisms. Since X is noetherian, this limit is finite, so we have

$$\varprojlim_{ij}(\varinjlim_n \mathscr{F}_n)(U_{ij}) = \varprojlim_{ij}(\varinjlim_n \mathscr{F}_n(U_{ij})) = \varinjlim_n(\varprojlim_{ij} \mathscr{F}_n(U_{ij})) = \varinjlim_n \mathscr{F}_n(U) = (\varinjlim_n \mathscr{F}_n)(U)$$

12. This is the same argument as in the previous exercise, but since arbitrary limits commute, we don't need to assume the cover to be finite.

13. [BLOG] Let U be an open subset of X and consider $s \in \mathscr{F}^+(U)$. We must show that $s : U \to \text{Spé}(\mathscr{F})$ is continuous. Let $V \subseteq \text{Spé}(\mathscr{F})$ be an open subset and consider the preimage $s^{-1}V$. Suppose $P \in X$ is in the preimage of V. Since $s(Q) \in \mathscr{F}_Q$ for each point $Q \in X$, we see that $P \in U$. This means that there is an open neighborhood U' of P contained in U and a section $t \in \mathscr{F}(U')$ such that for all $Q \in U'$, the germ $t_{U'}$ of t at U' is equal to $s|_{U'}$, ie $s|_{U'} = t$. So we have $s|_{U'}^{-1} = t^{-1}(V)$, which is open since by definition of the topology on $\text{Spé}(\mathscr{F})$, t is continuous. So there is an open neighborhood $t^{-1}(V)$ of P that is contained in the preimage. P was arbitrary so every in the preimage $s^{-1}V$ has an open neighborhood contained within the preimage $s^{-1}V$. Hence it is the union of these open neighborhoods and therefore open itself. So s is continuous.

    Now suppose that $s : U \to \text{Spé}(\mathscr{F})$ is a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. First we show that for any open V and any $t \in \mathscr{F}(U)$, the set $t(V) \subset \text{Spé}(\mathscr{F})$ is open. To see this, recall that the topology on $\text{Spé}(\mathscr{F})$ is defined as the strongest such that every morphism of this kind is continuous. If we have the topology $\mathcal{U}$, where $\mathcal{U}$ is the collection of open sets on $\text{Spé}(\mathscr{F})$ such that each $t \in \mathscr{F}(U)$ is continuous and $W \in \text{Spé}(\mathscr{F})$ has the property that $t^{-1}W$ is open in X for any $t \in \mathscr{F}(V)$ and any open V, then the topology generated by $\mathcal{U} \cup \{W\}$ also has the property that each $t \in \mathscr{F}(U)$ is continuous. So since we are taking the strongest topology such that each $t \in \mathscr{F}(U)$ is continuous, if a subset $W \subset \text{Spé}(\mathscr{F})$ has the property that $t^{-1}W$ is open in U for each $t \in \mathscr{F}(U)$, then W is open in $\text{Spé}(\mathscr{F})$. Now fix one $s \in \mathscr{F}(U)$ and consider $t \in \mathscr{F}(V)$. For a point $x \in t^{-1}s(U), s(x) = t(x)$. That is, the germs of t and s are the same at x. This means that there is some open neighborhood W of x contained in both U and V such that $s|_W = t|_W$ and hence $s = t$ for every $y \in W$, so $S \subset t^{-1}s(U)$. Since every point in $t^{-1}s(U)$ has an open neighborhood in $t^{-1}s(U)$, we see that $t^{-1}s(U)$ is open and therefore by above we get that $s(U)$ is open in $\text{Spé}(\mathscr{F})$.

Now let $s : U \to \text{Spé}(\mathscr{F})$ be a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. For every point $x \in U$, the image of x under s is some germ (t, W) in $\mathscr{F}_x$. That is, an open neighborhood W of x which we can choose small enough to be contained in U and $t \in \mathscr{F}(W)$. Since s is continuous and we have seen that $t(W)$ is open, it follows that $s^{-1}(t(W))$ is open in X. This means that there is an open neighborhood W' of x on which $t|_{W'} = s|_{W''}$. Since s is locally representable by sections of $\mathscr{F}$, it is a well-defined section of $\mathscr{F}^+$.

14. Let $\mathscr{F}$ be a sheaf on X, $s \in \mathscr{F}(U)$. Then the compliment of Supp s is the set $\{P \in U \mid s_P = 0\}$. For $P \in U$, pick an open neighborhood V such that $s|_V = 0$. For any other $P' \in V, s_{P'} = 0$. Therefore $(\text{Supp } s)^c$ is open and Supp s is closed.

    Define Supp $\mathscr{F} = \{p \in X \mid \mathscr{F}_p \neq 0\}$. An example where it need not be closed can be given by example 19b.

15. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For open $U \subset X, \varphi, \psi \in Hom(\mathscr{F}|_U, \mathscr{G}|_U)$, let $(\varphi+\psi)(s) = \varphi(s)+\psi(s)$, which is abelian since $\mathscr{G}(U)$ is abelian. So $Hom(\mathscr{F}|_U, \mathscr{G}|_U)$ is an abelian group. To show the presheaf $U \mapsto Hom(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf, let $\{U_i\}$ be an open cover of U. Let $s \in Hom(\mathscr{F}|_U, \mathscr{G}|_U)$ such that $s|_{U_i} = 0$ for all i. That is, $s(f) = 0$ on all U_i, or equivalently, $s(f|_{U_i}) = 0$. Since $\mathscr{F}$ is a sheaf, $\exists f' \in \mathscr{F}(U)$ such that $s(f') = 0$ on U. Therefore $s|_U = 0$. Now suppose $\psi_i \in Hom(U_i)$ such that for all $i, j, \psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$. For an open $W \subseteq U$, the compatibility of ψ_i give rise to some $\psi \in Hom(U)$ which coincides on the restrictions to U_i for all i. Therefore Hom is a sheaf.

16. A sheaf $\mathscr{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open set, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

    (a) If X is irreducible, then the restriction maps $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ are just the identity maps $id : A \to A$, which are clearly surjective.

    (b) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of sheaves, with $\mathscr{F}'$ flasque. By ex. 1.8, $\Gamma(U, \cdot)$ is a left exact functor, so we just need to show that $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Consider open subsets $V, V' \subset U$ and a section $t \in \mathscr{F}''(U)$. Assume that t can be lifted to section $s \in \mathscr{F}(V)$ and $s' \in \mathscr{F}(V')$. Then, on $V \cap V'$, those lifting differ by an element $r \in \mathscr{F}'(V \cap V')$. Since $\mathscr{F}'$ is flasque, we can extend r to a section $\widetilde{r}$, and take $s' + \widetilde{r}$ in place of s', which is also a lifting of $t|_{V'}$. Then s and s' coincide on $V \cap V'$, thus defining a lifting of t over $V \cup V'$. Conclude the proof by transfinite induction over a cover of U.

(c) Let $V \subseteq U$. By (a), the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d, "\rho'"] & \mathscr{F}(U) \arrow[r] \arrow[d, "\rho"] & \mathscr{F}''(U) \arrow[r] \arrow[d, "\rho''"] & 0 \\ 0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0 \end{tikzcd}$$

has exact rows. Since ρ and ρ' are surjective, so is ρ''.

(d) For any open $V \subseteq U$ of Y, $(f_*\mathscr{F}(V) \to f_*\mathscr{F}(U)) = (\mathscr{F}(f^{-1}V) \to \mathscr{F}(g^{-1}(U))$, which is surjective since $\mathscr{F}$ is flasque.

(e) For any $s \in \mathscr{G}(V)$, define $s' \in \mathscr{G}(U)$ by $s' = \begin{cases} s\, on\, V \\ 0\, else \end{cases}$ Then clearly $\mathscr{G}(U) \twoheadrightarrow \mathscr{G}(V)$, so $\mathscr{G}$ is flasque. For any open $U \subseteq X$, define $\mathscr{F}(U) \to \mathscr{G}(U)$ by $x \mapsto (P \mapsto x_P)$. Suppose that the map $P \mapsto x_P$ is the zero map for $x \in \mathscr{F}(U)$. Then for all $P \in U, \exists$ open neighborhood U_P such that $x|_{U_P} = 0$. Since $\{U_P\}$ cover U and $\mathscr{F}$ is a sheaf, $x = 0$. Therefore $\mathscr{F}(U) \hookrightarrow \mathscr{G}(U)$ is injective for all U, so $\mathscr{F} \hookrightarrow \mathscr{G}$.

17. The stalk of $i_P(A)$ at $Q \in \{P\}^-$ is just $\varinjlim i_P(A)(U) = A$ for $Q \in \{P\}^-$ and 0 if $Q \notin \{P\}^-$. Now let $i : \{P\}^- \hookrightarrow X$ be the inclusion. Then the stalk of $i_*(A)$ is A on $\{P\}^-$. So for every stalk, $i_P(A)_P \cong i_*(A)_P$. So $i_P(A) \cong i_*(A)$.

18. $(f^{-1}f_*\mathscr{F})(U) = \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V))$. Define a map $h_1 : f^{-1}f_*\mathscr{F} \to \mathscr{F}$ by $h_{1_U}(\overline{\sigma}_{f^{-1}(V)}) = \rho_U^{f^{-1}(V)}(\sigma_{f^{-1}(V)})$. $(f_*f^{-1}\mathscr{G})(V) = \varinjlim_{U \supseteq f(f^{-1}(V))} \mathscr{G}(U)$. Since $V \supseteq f(f^{-1}(V))$, define $h_2 : \mathscr{G} \to f_*f^{-1}\mathscr{G}$ by $h_{2_V}(\sigma_V) = \overline{\sigma}_V$. Any $h : f^{-1}\mathscr{G} \to \mathscr{F}$ induces $f_*h : f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$. Pre-composing with h_2 we get $f_*h \circ h_2 : \mathscr{G} \to f_*\mathscr{F}$. Any $h : \mathscr{G} \to f_*\mathscr{F}$ induces $f^{-1}h : f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F}$ and composing with h_1, we get $h_1 \circ f^{-1}h : f^{-1}\mathscr{G} \to \mathscr{F}$. So the Hom groups are isomorphic.

19. (a) Obvious since $i_*\mathscr{F}(U) = \mathscr{F}(U \cap Z)$

(b) If $P \in U$ then for every open V containing P, there exists an open set $V' \subseteq U$ containing P and so every element (V, s) of the stalk is equivalent to an element $(V', s|_{V'})$ of the stalk $\mathscr{F}_P$.

(c) By the previous two exercises, the sequence of stalks is exact regardless if P is in U or Z.

20. (a) Let $\{V_i\}$ be an open cover of $V \subseteq X$. Let $s \in \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ such that $s|_{V_i} = 0 \,\forall\, i$. Therefore supp $s|_{V_i}$ in $V_i = \emptyset$. So supp s in V is empty since $s_p = (s|_{V_i})_p \,\forall\, i$ and thus $s_p = 0 \,\forall\, p \in V$. Therefore $s = 0$ since $\mathscr{F}$ is a sheaf. Let $s_i \in \Gamma_{Z\cap V_i}(V_i, \mathscr{F}|_{V_i})$ such that $\forall\, i, j, s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Since $\mathscr{F}$ is a sheaf, $\exists$ a unique $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$. For $p \in V - Z, p \in V_i$, therefore $s_p = (s|_{V_i})_p = (s_i)_p$. Since supp s_i in $V_i \subseteq V_i \cap Z$, $(s_i)_p = 0$. Therefore supp $s \subseteq Z \cap V$, so $s \in \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ and $\mathcal{H}_Z^0(\mathscr{F})$ is a sheaf.

(b) Let $U = X - Z$, $j{:}U \hookrightarrow X$ be the inclusion, and let $V \subseteq X$ be open. Since $\mathcal{H}^0_Z(\mathscr{F})(V) \subseteq \mathscr{F}(V)$, define $\mathcal{H}^0_Z(\mathscr{F}) \overset{\varphi}{\hookrightarrow} \mathscr{F}$. Since $\mathscr{F}(U \cap V) = (\mathscr{F}|_U)(U \cap V) = (\mathscr{F}|_U)(j^{-1}(V)) = (j_*(\mathscr{F}|_U))(V)$, we can define the map $\mathscr{F} \xrightarrow{\psi} j_*(\mathscr{F}|_U)$ to be given by the restriction maps of $\mathscr{F}$. Therefore $0 \to \mathcal{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$ is exact.

If $\mathscr{F}$ is flasque, ψ is surjective. Since φ is injective, $V \mapsto \text{im } \varphi(V)$ is a sheaf so by 1.4(b), it is enough to show that $\text{im } \varphi(V) = \ker(\psi(V))$ for all V. If $x \in \ker \psi(V)$ for some V, then $x|_{U\cap V} = 0$. Therefore $\text{supp } V \subseteq Z \cap V$ and thus $x \in \text{im } \varphi(V)$. If $x \in \text{im } \varphi(V)$, then for all $Q \in V \backslash Z, x_Q = 0$. So there exists a neighborhood $V_Q \subseteq U \cap V$ such that $x|_{V_Q} = 0$. Since $\{V_Q\}$ is a cover of $U \cap V$ and $j_*(\mathscr{F}|_U)$ is a sheaf, $\psi(V)(x) = x|_{U\cap V} = 0$. Therefore $x \in \ker(\psi(V))$.

21. (a) $\mathcal{I}_Y$ is just the kernel of the sheaf morphism $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$, which is a sheaf.

(b) Let $i : Y \hookrightarrow X$ be the inclusion map. Define $\varphi : \mathcal{O}_x \to i_*(\mathcal{O}_Y)$ by restricting $f \in \mathcal{O}_X$ to Y. This map is surjective with kernel consisting of functions that vanish on Y, ie $\mathcal{I}_Y$. Therefore by the first isomorphism theorem, $\mathcal{O}_X/\mathcal{I}_Y \cong i_*(\mathcal{O}_Y)$.

(c) The initial sequence is clearly exact, with the first map being the inclusion and the second map is just the restriction $f \mapsto (f, f)$, where if $f \notin \mathcal{O}_P(U)$, set $f = 0$. Same for Q. The induced map on global sections is in fact not surjective since $k \cong \Gamma(X, \mathcal{O}_X)$ which has dimension 1 and $\Gamma(Y, \mathscr{F}) \cong k \oplus k$ has dimension 2.

(d) A regular function on U is a function $f : U \to k$, such that is an open cover $\{U_i\}$ of U on which $f|_{U_i}$ is a rational function with no poles in U_i. Since the f_i are restrictions of f as functions, they agree on intersections U_{ij} and therefore define a section of $\mathcal{K}(U)$. The morphism $\mathcal{K} \to \sum_{P\in X} i_P(I_P)$ is clear. To show exactness it is enough to show exactness on the stalks, which takes the form

$$0 \to \mathcal{O}_P \to \mathcal{K}_P \to (\sum_{Q\in X} i_Q(I_Q))_P \to 0$$

Since $\mathcal{K}$ is a constant sheaf, it takes the value K at every stalk. On the right, we have a sum of skyscraper sheaves, all which vanish except at $Q = P$, which by definition is $K/\mathcal{O}_P$. Hence the sequence is

$$0 \to \mathcal{O}_P \to K \to K/\mathcal{O}_P \to 0$$

which is exact.

(e) We know $\Gamma(X, \cdot)$ is left exact so we just need to show the map $\Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O})$ is surjective. Using the description of $\mathcal{K}/\mathcal{O}$ from the previous part as $\sum i_P(I_P)$, we have to show that given a rational function $f \in K$ and a point P, there exists another rational function $f' \in K$ such that $f' \in \mathcal{O}_Q$ for every $Q \neq P$ and $f' - f \in \mathcal{O}_P$. Since $K \cong k(x)$, we can write $f = \frac{\alpha(x)}{\beta(x)} = \frac{\prod_{i=1}^{n}(x-a_i)}{(x-b_i)}$ and then the points in $\mathbb{A}^1 \subset \mathbb{P}^1$ for which $f \notin \mathcal{O}_Q$ are just b_i, and $f \notin \mathcal{O}_\infty$ if $m < n$. In fact, we can write f as $f = x^{-\nu}\frac{\alpha}{\beta'}$ with $x \nmid \alpha, \beta$. By a linear transformation, we can pick P to be $0 \in \mathbb{A}^1$. If $\nu \leq 0$, then choosing $f' = 1$ satisfies the required conditions. If $\nu > 0$, then choose $f' = \frac{\sum_{i=0}^{\nu} c_i}{x^\nu}$ with c_i defined iteratively via $c_0 = \frac{\alpha_0}{\beta_0}$ and $c_i = \beta_0^{-1}(a_i - \sum_{j=0}^{i-1} c_j\beta_{i-j})$, where α_i and β_i are the coefficients for $\alpha = \sum \alpha_i x^i$ and $\beta' = \sum \beta_i x^i$ respectively. Our chosen f' satisfies the requirement that $f' \in \mathcal{O}_Q$ for all $Q \neq P$ and so consider $f - f'$. We have $f - f' = \frac{\alpha}{x^\nu \beta'} - \frac{\sum_{i=1}^{\nu} c_i}{x^\nu} = \frac{\alpha - \beta' \sum_{i=0}^{\nu} c_i}{x^\nu \beta'}$. The ith coefficient of the numerator for $i \leq \nu$ is $\alpha_i - \sum_{j=0}^{i} c_j\beta_{i-j}$, which is zero due to our careful choice of the c_i. So the x^ν in the denominator vanishes and we see that $f - f' \in \mathcal{O}_p$ since $x \nmid \beta'$.

22. See Shaf II page 31-32 for everything you ever wanted to know about gluing sheaves together.

2.2 Schemes

1. $\mathcal{D}(f) \subset X = \{\mathfrak{p} \subseteq A \,|\, \mathfrak{p} \not\ni f\}$ and Spec $A_f = \{p \in A \,|\, (p) \cap (f) = \emptyset\}$, ie such that $f \notin (p)$. Therefore, as topological spaces, $\mathcal{D}(f) \overset{homeo}{\approx}$ Spec A_f. By Prop 2.2b, $\mathcal{O}_X(\mathcal{D}(f)) = A_f$, so $\mathcal{O}_X|_{\mathcal{D}(f)} \cong \mathcal{O}_{A_f}$. Thus as locally ringed spaces, $(\mathcal{D}(f), \mathcal{O}_X|_{\mathcal{D}(f)}) \cong$ Spec A_f.

2. Pick $x \in U$ and let $V =$ Spec A be an affine neighborhood of x. Pick $f \in A$ such that $\mathcal{D}(f) \subseteq V \cap U$, which you can do since the principal open sets form a basis for the topology. Since by the previous exercise $\mathcal{D}(f) \cong$ Spec A_f, $\mathcal{D}(f)$ is an affine neighborhood of x in U and $(U, \mathcal{O}_U)$ is a scheme.

3. (a) Let $(X, \mathcal{O}_X)$ be reduced. Then by definition, the nilradical $\eta(\mathcal{O}_X(U)) = 0$ for any open $U \subseteq X$. Let $P \in X$ and let $U' \subseteq X$ be an open affine neighborhood of P. Then $\eta(\mathcal{O}_{X,P}) = \eta(\mathcal{O}_X(U)_P) = \eta(\mathcal{O}_X(U))_P = 0_P = 0$, so $\mathcal{O}_{X,P}$ has no nilpotents. (Note: fact that localization commutes with radicals is from AM p 42)
   Conversely, let $\eta(\mathcal{O}_{X,p}) = 0$ for all $p \in X$. For any open $U \subseteq X$, pick a section $s \in \mathcal{O}_X(U)$ and assume that $s^n - 0$ for some n. Then looking at the stalk, we see that $s_p = 0$ for all $p \in U$. By the sheaf property, since s is zero on a cover of X,s is 0 everywhere and $(X, \mathcal{O}_X)$ is reduced.

   (b) Since $\eta(A_f) = (\eta(A))_f$, any open affine $U =$ Spec A becomes $U =$ Spec $A/\eta(A)$ in X_{red}. Thus it is a scheme. Define the natural morphism $(f, f^\#) : X_{red} \to X$ by letting f be the identity on sp(X_{red}) and $f^\#$ be the quotient map by the nilradical.

(c) Let X be reduced and let $f : X \to Y$ be a morphism. Then define $g : X \to Y_{red}$ by letting g be the same as $f$on the points of X and by defining the sheaf map $\mathcal{O}_{Y_{red}}(U) \to g_*\mathcal{O}_X$ as in part b. This is well-defined since X is reduced and the map $\mathcal{O}_U(U) \to \mathcal{O}_X(U)$ takes $\eta(\mathcal{O}_Y(U))$ to 0, so it factors through $\eta(\mathcal{O}_Y(U))$.

4. Picking any U in an affine cover U_i of X, we get a ring map $A \xrightarrow{\phi} \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho} \Gamma(U, \mathcal{O}_U)$. The associated map is then $\phi_U^* : \text{Spec}\,(\Gamma(U, \mathcal{O}_U)) \to \text{Spec}\, A$. Since U is affine, $U \cong \text{Spec}\, \Gamma(U, \mathcal{O}_U)$. Glue all the ϕ_U^* to get a map $\phi_X^* : X \to \text{Spec}\, A$. Then α is a bijection since ϕ_X^* is its inverse.

5. $\text{Spec}\, \mathbb{Z} = \{0\} \cup \{(p) \,|\, p \text{ is prime in } \mathbb{Z}\}$. $\{0\}$ is open and (p) are closed since $\mathbb{Z}$ is a PID and non-zero prime ideals are maximal. Now let X be a scheme. Any ring has a unique homomorphism from $\mathbb{Z}$ so by ex 4, there is a unique morphism $X \to \text{Spec}\, \mathbb{Z}$ for any scheme X.

6. $\text{Spec}\, 0 = \emptyset$ since there are no prime ideals. The unique map $\emptyset \to X$ is the trivial map on points and sheaves, so Spec 0 is an initial object in the category of schemes.

7. Let X be a scheme and let K be any field and let $(f, f^\#) : \text{Spec}\, K \to X$ be a morphism of schemes. Since Spec K consists of just one point O, f maps O to some $x \in X$. The map on stalks is $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{Spec\,K,x} \cong K$. The map on the corresponding residue fields is then $\widetilde{f_x^\#} : k(x) = \mathcal{O}_{X,x}/m_{X,x} \to \mathcal{O}_{Spec\,K,x}/m_{Spec\,K,x} \cong K/0 \cong K$. The isomorphism $\mathcal{O}_{Spec\,K,x}/m_{Spec\,K,x} \cong K/0$ follows since $f_x^\#$ is a local morphism. Now $f_x^\#$ is an inclusion since we have a non-zero homomorphism of fields.

   Conversely, let $x \in X$ and $k(x) \hookrightarrow K$ be given. Define the continuous map on topological spaces by $f : \text{Spec}\, K \to X$ by setting $f(O) = x$. To construct $f^\# : \mathcal{O}_x \to f_*\mathcal{O}_{Spec\,K}$, define it locally. If $x \in U \subseteq X$, define $f^\#(U) : \mathcal{O}_X(U) \to \mathcal{O}_{Spec\,K}(f^{-1}(U)) \cong K$ by $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_{X,x} = k(x) \hookrightarrow K$. If $x \notin U, f_*\mathcal{O}_{Spec\,K}(U) = \mathcal{O}_{Spec\,K}(f^{-1}(U)) = \mathcal{O}_{Spec\,K}(\emptyset) = 0$, therefore we only need to define the map for open $U \subseteq X$ containing x. $f_p^\#$ is a local homomorphism since for all $p \in X, \mathcal{O}_p = \mathcal{O}_x$ if some open neighborhood of p contains x and thus $(f, f^\#)$ is a morphism of schemes.

8. See Shaf II, example 2 on page 36

9. Let X be a scheme, $Z \subseteq X$ closed and irreducible. If $U \subseteq Z$ is open and $\zeta \in U$ such that $\overline{\zeta} = U$, then $\overline{\zeta} = Z$ in X since Z is irreducible. So we can assume that $X = \text{Spec}\, A$ is affine and $Z = \text{Spec}\, A/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq A$. Now we can further assume that $Z = X = \text{Spec}\, A$ is irreducible. It follows that there can only be one minimal prime ideal belonging to the nilradical $\eta(A)$, whose closure is then all of X. Uniqueness is clear from the uniqueness of the nilradical.

10. $\mathbb{R}[x]$ is a PID, so all irreducible elements correspond to prime ideals. Thus Spec $\mathbb{R}[x]$ has a point for every irreducible polynomial and the generic point corresponds to (0). Closed points correspond to maximal ideals, which are of the form $(x-\alpha)$, where $\alpha \in \mathbb{R}$ as well as $(x+\beta)(x+\overline{\beta})$ for $\beta \in \mathbb{C}$. The residue field at the real numbers is $\mathbb{R}$ and at the complex numbers is $\mathbb{C}$. The only non-trivial proper closed sets are finite sets.

11. Spec $k[x] = \{0\} \cup \{(f)\}$, where f is an irreducible monic polynomial and (0) is the generic point. The residue field of a point corresponding to a polynomial of degree d is $\mathbb{F}_{p^d}$. Given a residue field, the number of points can be determined by using the Möbius Inversion formula, which is done in Dummit and Foote page 588

12. Yes, you can glue. See Shaf II page 31-32 for everything you ever wanted to know about gluing sheaves together.

13. (a) Assume X is a noetherian topological space. By ex I.1.7c, any $U \subseteq X$ is noetherian, and by ex I.7b, U is quasi-compact. Conversely, let $U_1 \subset U_2 \subset \ldots$ be a chain of quasi-compact subsets. Define $U = \bigcup U_i$. By assumption, U is quasi-compact, so $U = \bigcup_n U_i$ so the chain must stabilize and X is noetherian.

    (b) We can refine any given cover into a cover of principal open sets $\mathcal{D}(f_\alpha)$. If Spec $A = \bigcup \mathcal{D}(f_\alpha)$, then $\emptyset = \bigcap \mathcal{V}(f_\alpha) = \mathcal{V}(f_\alpha)$, so $1 \in (f_\alpha)$. Write $1 = a_1 f_1 + \ldots + a_n f_n$. Then $1 \in (f_1, \ldots f_n)$, so Spec $A = \cup_{i=1}^n \mathcal{D}(f_i)$. Thus Spec A is quasi-compact.

    An example of a non-noetherian affine scheme is Spec $k[x_1, x_2, \ldots]$ which has a decreasing chain of closed subsets $\mathcal{V}(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, x_3) \supset \ldots$.

    (c) If $\mathcal{V}(a_1) \supseteq \mathcal{V}(a_2) \supseteq \ldots$ is a decreasing sequence of closed subsets, then it terminates since the corresponding increasing sequence of ideals $a_1 \subseteq a_2 \subseteq \ldots$ terminates since A is noetherian.

    (d) Let $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$. Then each $x_i \in \eta(A)$ and thus every $p \in$ Spec A contains x_i and since $(x_1, x_2, \ldots)$ is maximal, there is only one prime ideal. So Spec A is trivially noetherian, but A is not noetherian since there is an increasing chain $(x_1) \subset (x_1, x_2) \subset \ldots$ which does not stabilize.

14. (a) If S_+ is nilpotent, then every prime ideal contains S_+ so Proj $S = \emptyset$. Now suppose that Proj $S = \emptyset$ and let $f \in S_+$ be a homogeneous polynomial. Then $\mathcal{D}(f) = \emptyset$ so Spec $S_{(f)} \cong \mathcal{D}(f) = \emptyset$. Thus $S_{(f)} = 0$, which implies that $\frac{1}{f^n} = 0$ and hence $f^n(1) = 0$ for some n. Thus f is nilpotent. S_+ is generated by homogeneous elements so $S_+ \subseteq \eta(S)$.

    (b) Let $p \in U$ be some prime ideal. Then $\varphi(S_+) \nsubseteq p$ and so unless $S_+ = 0$, there is some $f \in S_+$ such that $\varphi(f) \notin p$. If for every homogeneous component f_i of f, $\varphi(f_i) \in p$. Then $\varphi(f) \in p$, so there must be some homogeneous component f_i such that $\varphi(f_i) \notin p$.

So there is a principal open set $\mathcal{D}_+(\varphi(f_i))$ containing p which is contained in U since every prime ideal in $\mathcal{D}_+(\varphi(f_i))$ does not contain $\varphi(f_i)$ and thus does not contain $\varphi(S_+)$. These principal open sets cover U and since U is a union of open sets, U is open in Proj T.

For $p \in U$ define $f(p) = \varphi^{-1}(p)$. Since $p \not\supseteq \varphi(S_+), \varphi^{-1}(p) \not\supseteq S_+$ so the morphism is well-defined. This morphism takes closed sets to closed sets so it is continuous and the induced morphism on sheaves is induced by $S_{(\varphi^{(-1)}(p))} \to T_{(p)}$.

(c) First lets show that the open set U is in fact Proj T. Let $\varphi_d : S_d \to T_d$ be an isomorphism for all $d \geq d_0$. Pet $\mathfrak{p}$ be any homogeneous prime ideal of T and suppose that $\mathfrak{p} \subseteq \varphi(S_+)$. Let $x \in T_+$ be a homogeneous element of deg $\alpha > 0$. For some n, $n\alpha \geq d_0$, so $x^n \in T_{n\alpha} = \varphi(S_{n\alpha}) \subseteq \mathfrak{p}$. So $x \in \mathfrak{p}$ and thus $T_+ \subseteq \mathfrak{p}$. So $U =$ Proj T. The continuous map on the topological spaces $f :$ Proj $T \to$ Proj S is given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

Show surjectivity: Let $\mathfrak{p} \in$ Proj S and define $\mathfrak{q}$ to be the radical of the homogeneous ideal generated by $\varphi(\mathfrak{p})$. (Note that the radical of homogeneous ideals are again homogeneous). First show that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. The inclusion $\mathfrak{p} \subseteq \varphi^{-1}(\mathfrak{q})$ is clear, so suppose we have $a \in \varphi^{-1}(\mathfrak{q})$. Then $\varphi(a^n) \in (\varphi(\mathfrak{p}))$ for some n. This means that $\varphi(a^n) = \sum b_i \varphi(s_i)$ for some $b_i \in T$ and $s_i \in \mathfrak{p}$. For $m >> 0$, every monomial in the b_i will be in $T_{\geq d_0}$, and since we have $T_d \cong S_d$ for $d \geq d_0$, this means that these monomials correspond to some $c_j \in S$. The element $(\sum b_i \varphi(s_i))^m$ is a polynomial in the $\varphi(s_i)$ whose coefficients are monomials of degree m in the b_i and this corresponds in S to a polynomial in the s_i with coefficients in the c_j, which is in $\mathfrak{p}$, as all the s_i are. Hence, $\varphi(a^{nm}) \in \varphi(\mathfrak{p})$ and so $a^{nm} \in \mathfrak{p}$ and therefore $a \in \mathfrak{p}$. Thus $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$ and combining this with the other inclusion leads to the equality $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. To show that $\mathfrak{q}$ is prime, suppose that $ab \in \mathfrak{q}$ for some $a, b \in T$. Then using the same reasoning as before, we see that $(ab)^{nm} \in \varphi(\mathfrak{p})$ for some n, m such that $(ab)^{nm} \in T_{\geq d_0}$. If necessary, take a higher power so that $a^{nmk}, b^{nmk} \in T_{\geq d_0}$ as well. Using the isomorphism $T_{\geq d_0} \cong S_{\geq d_0}$, this means that a^{nmk}, b^{nmk} correspond to elements of S and we see that their product is in $\mathfrak{p}$. Hence one of a^{nmk} or b^{nmk} are in $\mathfrak{p}$, say a^{nmk}. Then $a^{nmk} \in \varphi(\mathfrak{p})$ and so $a \in \mathfrak{q}$ and $\mathfrak{q}$ is prime.

Show injectivity: Suppose that $\mathfrak{p}, \mathfrak{q} \in$ Proj T have the same image under $f :$ Proj $T \to$ Proj S. Then $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$. Consider $t \in \mathfrak{p}$. Since $t \in \mathfrak{p}$, we have $t^{d_0} \in \mathfrak{p}$ and since φ_d is an isomorphism for $d \geq d_0$, it follows that there is a unique $s \in S$ with $\varphi(s) = t^{d_0}$. The element s is in $\varphi^{-1}(\mathfrak{p})$ and so since $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q}), s \in \varphi^{-1}(\mathfrak{q})$. So $\varphi(s) = t^{d_0} \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime, $t \in \mathfrak{q}$ and $\mathfrak{p} \subseteq \mathfrak{q}$. Similarly, $\mathfrak{q} \subseteq \mathfrak{p}$ and equality follows.

Show Isomorphism of structure sheaves: Since Proj S is covered by open affine of the form $D_+(s)$ for some homogeneous $s \in S$, it is enough to check the isomorphism on these principal open sets. Note that $D_+(s) = D_+(s^i)$ so we can assume that the degree of s is $\geq d_0$. With this assumption, $f^{-1}D_+(s) = D_+(t) \subseteq \text{Proj } T$, where t is the element of T corresponding to s under the isomorphism $S_{deg\, s} \to T_{deg\, s}$ since a homogeneous prime ideal $\mathfrak{q} \subset T$ gets sent to $D_+(s)$ iff s is not in the preimage iff $t \notin \mathfrak{q}$. So we have to show that the morphism $S_{(s)} \to T_{(t)}$ is an isomorphism. If $\frac{f}{s^n}$ gets sent to zero then $0 = t^m\varphi(f) = \varphi(s^m)\varphi(f)$ for some m. Choose $m > 0$ so we do not have to handle the case deg $f = 0$ separately, and so $s^m f \in \ker\varphi$. Taking a high enough power of $s^m f$ puts it in one of the S_d for which $S_d \to T_d$ is an isomorphism and so $s^m f = 0$ and therefore $\frac{f}{s^n} = 0$ and our morphism is injective. Now suppose that $\frac{f}{t^n} \in T_{(t)}$. This is equal in $T_{(t)}$ to $\frac{t^{d_0}f}{t^{n+d_0}}$ and now $t^{d_0}f$ has degree high enough to have a preimage in S. So our morphism is surjective.

(d) This follows from prop II.4.10

15. (a) Let V be a variety over an algebraically closed field k. Let $P \in t(V)$. Assume the residue field if k. Then $\{x\}$ is closed iff $\{x\} \cap U_i$ is closed in each U_i for some open cover $\{U_i\}$ of X. We can assume this cover to be an affine open cover, so each $U_i = \text{Spec } A_i$. Since the residue field of P is k, P corresponds to a maximal ideal $\mathfrak{m}_i$ in each Spec A_i and is therefore a closed point.

Conversely, if P is a closed point of X, then it is closed in some open affine neighborhood Spec A. Then P corresponds to a maximal ideal in Spec A, and so its residue field $k(P) = \mathcal{O}_{P,X}/\mathfrak{m}_P = k$.

(b) Let $f : X \to Y$ be a morphism of schemes over k and let $P \in X$ is a point with residue field k. Then $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a morphism of residue fields $k(f(P)) \to k(P)$. Since X and Y are schemes over k, these residue fields are both extensions of k and since $k(P) = k$, we have the field extensions $k \hookrightarrow k(f(P)) \hookrightarrow k$. So $k(f(P)) \cong k$.

(c) $\text{Hom}_{var}(V, W) \to \text{Hom}_{Sch/k}(t(V), t(W))$ is defined by $\varphi \mapsto \varphi^*$, where by part b, closed points map to closed points. Thus $\varphi^*(P) = \varphi(P)$. For an irreducible subvariety Y, $\varphi^*(Y) = \overline{\varphi(Y)}$. The maps on schemes over k are extensions of $\varphi : V \to W$, so injectivity is clear. To show surjectivity, given any $\varphi^* : t(v) \to t(W)$, we know that closed points map to closed points, so we can define φ to be $\varphi^*|_V$. Now, we need to show that φ is regular. Let $p \in V$, $\varphi(P) = Q$. Choose an open affine neighborhood $U = \text{Spec } A$ of P. Then $P \in U' \subseteq f^{-1}(U)$ for some affine neighborhood $U \subseteq t(V)$. So $f|_{U'}$ is a map f : Spec $A' \to \text{Spec } A$ which is induced by a the map $A \to A'$ on rings. This in turn induces a map of varieties φ and thus φ is regular.

[BLOG]

16. Let X be a scheme. $f \in \Gamma(X, \mathcal{O}_x)$ and define $X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_x\}$.

   (a) $x \in U \cap X_f$ iff $x \in U$ and $f_x \notin \mathfrak{m}_x$. Since U is affine, we can take x to be a prime $\mathfrak{p} \in \text{Spec } B$ and so the maximal ideal of the local ring is $\mathfrak{m} = \mathfrak{p}B_{\mathfrak{p}}$. $\overline{f} \in \mathfrak{m}$ iff $\overline{f} \in \mathfrak{p}$ and so $U \cap X_f = D(\overline{f})$. Since a subset of a topological space is open iff it is open in every element of an open cover, X_f is open in X.

   (b) Now assume that X is quasi-compact. Let $U_i = \text{Spec } A_i$ be a affine cover of X, which can be taken to be finite since X is quasi-compact. The restriction of a to $U_i \cap X_f = \text{Spec } (A_i)_f$ is zero for each i and so $f^{n_i}a = 0$ in A_i for some n_i. Choose an $n > n_i$ for all i. Then $f^n a = 0$ in each Spec A_i. Since $X = \bigcup \text{Spec } A_i$ and since $\mathcal{O}_X$ is a sheaf, $f^n a = 0$.

   (c) Let $U_i = \text{Spec } A_i$. Then $b|_{X_f \cap U_i} = \frac{b_i}{f^{n_i}}$ for each i. Since there are finitely many affines, we can choose the expression so that all the n_i's are the same, say n. In other words, $\exists b_i \in A_i$ such that $f^n b|_{U_i \cap X_f} = b_i$. Now consider $b_i - b_j$ on $U_i \cap U_j := U_{ij}$. Since U_{ij} is quasi-compact and the restriction of $b_i - b_j$ to $U_{ij} \cap X_f = (U_{ij})_f$ vanishes, we can apply the previous part to find m_{ij} such that $f^{m_{ij}}(b_i - b_j) = 0$ on U_{ij}. Again, we choose m bigger than all the m_{ij} so that they are all the same. So the now we have sections $f^m b_i$ on each U_i that agree on intersections. Hence they lift to some global section $c \in \Gamma(X, \mathcal{O}_X)$. Consider $c - f^{n+m}b$ on X_f. Its restriction to each $U_i \cap X_f$ is $f^m b_i - f^m b_i = 0$ and so $c = f^{n+m}b$ on X_f. Hence $f^{n+m}b$ is the restriction of the global section c.

   (d) Consider the morphism $A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$. If an element $\frac{a}{f^n}$ is in the kernel then $a|_{X_f} = 0$ and so by part b), we have $f^m a = 0$ as global sections for some m. Hence $\frac{a}{f^n}$ is zero and the morphism is injective. Now suppose we have a section b on X_f. By part c) there is an m such that $f^m b$ is the restriction of some global section, say c. Hence we have found $\frac{c}{f^m} \in A_f$ that gets sent to b so the morphism is surjective.

17. A criterion for Affineness [BLOG]

   (a) Let $f : X \to Y$ be a morphism of schemes and let U_i be an open cover of Y. Let $f^{-1}(U_i) \cong U_i$ for all i. Then f is a homeomorphism since for any open $V \subset X$, $V = \bigcup (V \cap f^{-1}(U_i))$ which is open. Since $f^{-1}(U_i) \cong U_i$, $f(V) = \bigcup f(V \cap f^{-1}(U_i))$ is open in Y. So f is a homeomorphism. Now for any $p \in X$, $p \in U_i$ for some i. Again, since $f^{-1}(U_i) \cong U_i$, the map on stalks $f_p^{-1} \to (U_i)_p$ is an isomorphism. Gluing gives an isomorphism on stalks $f_p : X_p \to Y_p$, so $f : X \to Y$ is an isomorphism.

   (b) If A is affine we can take $f_1 = 1$. Conversely, let $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that each open subset X_f is affine and $(f_1, \ldots, f_r)$

generate the unit ideal in A. Consider the morphism $f : X \to \text{Spec } A$. Since the f_i generate A, the principal open sets $\mathcal{D}(f_i) = \text{Spec } A_{f_i}$ cover Spec A. Their pre-images are X_{f_i}, which by assumption are affine, isomorphic to Spec A_i. So the morphism restricts to the morphism $\varphi_i : \text{Spec } A_i \to \text{Spec } A_{f_i}$. Now we just need to show that φ_i is an isomorphism so that the result follows from part a). Equivalently, we need to show that $\varphi_i : \Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_X)$ is an isomorphism for each i. Show injectivity: Let $\frac{a}{f_i^n} \in A_{f_i}$ and suppose that $\varphi_i(\frac{a}{f_i^n}) = 0$, for $\frac{a}{f_i^n} \in A_{f_i}$). This means that it also vanishes in each of the intersection $X_{f_i} \cap X_{f_j}$ =Spec $(A_j)_{f_i}$. So for each j there is some n_j such that $f_a^{n_j} = 0$ in A_j. Choosing m big enough, the restriction of $f_i^m a$ to each open set in a cover vanishes. So $f_i^m a = 0$ and in particular, $\frac{a}{f_i^n} = 0$ in A_{f_i}.

Show surjectivity: Let $a \in A_i$. For each $j \neq i$, we have $\mathcal{O}_X(X_{f_i f_j}) \cong (A_j)_{f_i}$ so $a|_{X_{f_i f_j}}$ can be written as $\frac{b_j}{f_i^{n_j}}$ for some $b_j \in A_j$. That is, we have elements $b_j \in A_j$ whose restrictions to $X_{f_i f_j}$ is $f_i^{n_j} a$. Since there are finitely many, we can choose them so that all the n_i are the same, say n. Now on the triple intersections $X_{f_i f_j f_k} = \text{Spec } (A_j)_{f_i f_k}$ $= \text{Spec } (A_k)_{f_i f_j}$ we have $b_j - b_k = f_i^n a - f_i^n a = 0$ and so we can find some integer m_{jk} such that $f_i^{m_{jk}}(b_j - b_k) = 0$ on $X_{f_j f_k}$. Replacing each m_{jk} by a large enough m, we have a section $f_i^m b_j$ for each X_{f_j} for $j \neq i$ together with a section $f_i^{n+m} a$ on X_{f_i} and these sections all agree on intersections. This gives us a global section d whose restriction to X_{f_i} is $f_i^{n+m} a$ and so $\frac{d}{f_i^{n+m}}$ gets mapped to a by φ_i.

18. (a) The nilradical $\eta(A)$ of A is the intersection of all prime ideals of A, so this result clearly follows.

    (b) If the map of sheaves is injective, then in particular, $A \cong \Gamma(X, \mathcal{O}_X) \to \Gamma(X, f_* \mathcal{O}_Y) \cong B$ is injective. Conversely, let $A \hookrightarrow B$ be injective. Let $\mathfrak{p} \in \text{Spec } A$ and consider $f_{\mathfrak{p}}^{\#} : A_{\mathfrak{p}} \to (f_* \mathcal{O}_{Spec\, B})_{\mathfrak{p}}$. Then $(f_* \mathcal{O}_{Spec\, B})_{\mathfrak{p}}$ is $S^{-1}B = B \otimes_A A_{\mathfrak{p}}$ where $S = A/\mathfrak{p}$. This follows since we can shrink every open subset U containing $\mathfrak{p}$ to one of the form $\mathcal{D}(a)$ for some $a \in A$. Then we can compute the stalk by taking the direct limit over these. Since the preimage of $\mathcal{D}(a)$ is $D(\varphi(a)) \subset \text{Spec } B$, $(f_* \mathcal{O}_{Spec\, B})_{\mathfrak{p}}$ is then the colimit of $\mathcal{O}_{Spec\, B}$ evaluated at open sets $\mathcal{D}(a)$ with $a \notin \mathfrak{p}$. That is, the colimit $B_{\varphi(a)}$ for $a \notin \mathfrak{p}$, which is exactly $S^{-1}B$. Equality with the tensor product follows from the universal product of the tensor product. So now the injectivity of the map on stalks $f_{\mathfrak{p}}^{\#} : A_{\mathfrak{p}} \to S^{-1}B$ follows from the injectivity of $A \to B$.

    (c) We immediately have a bijection between primes of A containing I and primes of $A/I \cong B$ where I is the kernel of φ. We already know that Spec $B \to$ Spec A is continuous so we just need to see that is is open to show that it is a homeomorphism. Note that for

$f+I \in A/I$, the preimage of $D(f) \subset$ Spec A is $D(f+I) \subset$ Spec (A/I). So principal open sets of Spec (A/I) are open in the image with the induced topology. Since arbitrary unions of open sets are open, and principal open sets for a base for the topology, the image of every open set is open. The stalk $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$ of the sheaf morphism at $\mathfrak{p} \in$ Spec A is clearly surjective.

(d) If $f^{\#}$ is surjective, then it is surjective on each stalk. So for an element $b \in B$, for each point $\mathfrak{p}_i \in$ Spec A, there is an open neighborhood which we can take to be a principal open set $D(f_i)$ of Spec A such that the germ of b is the image of some $\frac{a_i}{f_i^{n_i}} \in A_{f_i}$. That is, $f_i^{m_i}(a_i - f_i^{n_i}b) = 0$ in B. Since all affine schemes are quasi-compact, we can find a finite set of the $D(f_i)$ that cover Spec A, so we can assume all the n_i and m are the same, say n and m. Since $D(f_i)$ is a cover, the f_i generate A and therefore so do the f_i^{n+m}, so we can write $1 = \sum g_i f_i^{n+m}$ for some $g_i \in A$. We now have $b = \sum g_i f_i^{n+m} b = \sum g_i f_i^m a_i \in$ image φ. So φ is surjective.

19. ($1 \Rightarrow 3$) If Spec A is disconnected, then it is the disjoint union of 2 closed sets, say U and V. U and V both correspond to ideals, say I and J, so $U =$ Spec A/I and $V =$ Spec A/J. It follows that Spec $A = \mathrm{Spec}(A/I) \coprod$ Spec A/J and therefore $A = A/I \times A/J$. (In general, Spec $(A \times B) \cong$ Spec $A \coprod$ Spec B).

($3 \Rightarrow 2$) Choose $e_1 = (1,0)$ and $e_2 = (0,1)$.

($2 \Rightarrow 1$) Since $e_1 e_2 = 0$, for every prime, either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$. The closed sets $V((e_1))$ and $V((e_2))$ cover Spec A. If a prime $\mathfrak{p}$ is in both these closed sets, then $e_1, e_2 \in \mathfrak{p}$ and therefore $1 = e_1 + e_2 \in \mathfrak{p}$ and so $\mathfrak{p} = A$. So the closed sets $V((e_1))$ and $V((e_2))$ are disjoint. Since we have a cover of Spec A by disjoint closed sets, Spec A is disconnected.

2.3 First Properties of Schemes

1. ($\Rightarrow$) Let $F : X \to Y$ denote the morphism of schemes. Let $Y = \bigcup V_i = \bigcup$ Spec B_i such that $F^{-1}V_i$ is covered by open affines Spec A_{ij}, where each A_{ij} is a finitely generated B_i-algebra. Each $V_i \cap V$ is open in V_i and so is a union of principal open sets Spec $(B_i)_{f_{ik}}$ of V_i since they form a base of the topology of Spec B_i. Considering f_{ik} as an element of A_{ij} under the morphism $B_i \to A_{ij}$, the preimage of Spec $(B_i)_{f_{ik}}$ is Spec $(A_{ij})_{f_{ik}}$, and the induced ring morphisms make each $(A_{ij})_{f_{ik}}$ a finitely generated $(B_i)_{f_{ik}}$-algebra.

So we can cover Spec B with open affines Spec C_i whose preimages are covered with open affines Spec D_{ij} such that each D_{ij} is a finitely generated C_i-algebra. Now given a point $\mathfrak{p} \in$ Spec B, $\mathfrak{p}$ is contained in some Spec C_i. Since these are open, there is a principal open affine Spec $B_{g_{\mathfrak{p}}} \subseteq$ Spec C_i that contains $\mathfrak{p}$. Associating $g_{\mathfrak{p}}$ with its image under the induced ring homomorphisms $B \to C_i$ and then $C_i \to D_{ij}$, it can be seen that Spec $(C_i)_{g_\mathfrak{p}} \cong$ Spec $B_{g_\mathfrak{p}}$. The preimage of these sets is Spec $(D_{ij})_{g_\mathfrak{p}}$, and $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra. Spec $(D_{ij})_{g_\mathfrak{p}}$ cover the preimage of Spec B, and since $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra, $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated B-algebra (adding $g_\mathfrak{p}$ to the generating set). Hence the preimage of Spec B can be covered by open affine Spec A_i such that each A_i is a finitely generated B-algebra.

($\Leftarrow$) Follows from by definition.

2. ($\Rightarrow$) Let $f : X \to Y$ be a quasi-compact morphism. Let V_i be an open affine covering of Y such that $f^{-1}(V_i)$ is quasi-compact. Given any open affine $U \subseteq Y$, cover $U \cap V_i$ by open sets in both U and V_i. Since U is affine, and hence quasi-compact, we can pick a finite number of open sets. Therefore $f^{-1}(U)$ is a finite union of the preimages of these open sets. So it is enough to show each distinguished open set has a quasi-compact preimage. Thus we are reduced to the case $f : X \to Y$ where X is quasi-compact and $Y =$ Spec B is affine. Cover X with finitely many Spec A_i. Let f_i : Spec $A_i \to Y$ be the restriction of f. Choose $D(g) \subseteq Y$. Then $f_i^{-1}(D(g)) = D(f_i^{\#} g)$. Finally, $f^{-1}(D(g)) = \bigcup f_i^{-1}(D(g))$ and each $D(f_i^{\#} g)$ is quasi-compact since it is isomorphic to Spec $(A_i)_{f_i^{\#} g}$, so $f^{-1}(D(g))$ is a finite union of quasi-compact spaces and is thus quasi-compact.

   ($\Leftarrow$) Follows from by definition.

3. (a) We only need to show that if f is of finite type then it is quasi-compact. The others follow immediately from the definitions. Since f is of finite type, there is a cover of Y by open affines Spec B_i whose preimages are covered by finitely many open affines Spec A_{ij}. By ex 2.2.13(b) that each Spec A_{ij} is quasi-compact. In general, if a space can be covered by finitely many quasi-compact opens, then it itself is quasi-compact, so we have found an open affine cover of Y whose preimages are quasi-compact. Hence f is quasi-compact.

   (b) Follows directly from Ex 2.3.1, 2.3.2, and 2.3.3(a)

   (c) Cover $f^{-1}(V)$ by affines $U_i =$ Spec A_i such that each A_i is a finitely generated B-algebra. We can cover each of the intersections $U_i \cap U$ with distinguished open sets in both U and U_i. Let Spec $A_{f_i} =$ Spec $(A_i)_{g_i}$ be a cover of U by these principal open sets, which we can choose to be finite since this morphism is quasi-compact. Since each A_i is a finitely generated B-algebra, $(A_i)_{g_i} = A_{f_i}$ is a finitely generated B algebra, and therefore, since the Spec A_{f_i} form a finite cover of U, the ring A is a finitely generated B-algebra.

4. Let $V_i =$ Spec B_i be an affine cover of Y such that each preimage $f^{-1}V_i = U_i =$ Spec A_i is affine, with each A_i a finitely generated B_i-module. Cover each intersection $U \cap U_i$ with distinguished opens $D(f_{ij}) = (B_i)_{f_{ij}}$ of U_i. Note that the preimage of $D(f_{ij}) =$ Spec $(A_i)_{f_{ij}}$, where f_{ij} is associated

with its image in A_i. Since A_i is a finitely generated B_i-module, it follows that $(A_i)_{f_{ij}}$ is a finitely generated $(B_i)_{f_{ij}}$-module.

Now we have a cover of $V = \text{Spec } B$ by opens Spec B_{g_i} that are principal in V and each of the preimages is Spec C_i, with each C_i a finitely generated B_{g_i}-module. Use the affine criterion from ex 2.2.17. Since Spec B is affine, by ex 2.2.13(b), it is quasi-compact. So there is a finite subcover $\{$ Spec $B_{g_i}\}^n$. Since this is a cover, the $g_i, \ldots, g_n$ generate the unit ideal. This mean their image in $\Gamma(U, \mathcal{O}_U)$, where $U = f^{-1}\text{Spec } B$ also generate the unit ideal. Furthermore, the preimage of each Spec B_{g_i} is in fact U_{g_i}, where we associated g_i with its image in $\Gamma(U, \mathcal{O}_U)$. So by the affine criterion, U is affine.

Now let $U = \text{Spec } A$. We need to show that A is a finitely generated B-module. But this follows from the fact that if $f_1, \ldots f_n \in B$ are elements which generate the unit ideal, and A_{f_i} is a finitely generated B_{f_i}-module for every i, then A is a finitely generated B-module.

5. (a) Let $\mathfrak{p} \in Y$ be a point. Since the morphism is by assumption finite, there is an open affine Spec B containing $\mathfrak{p}$ such that the pre-image $f^{-1}\text{Spec } B$ is affine, say Spec A, where A is a finite B-module. So we can immediately reduce to the case where $X = \text{Spec } A$ and $Y =$ Spec B. To show that the preimage of $\mathfrak{p}$ is finite, it is enough to show that the fiber Spec $A \otimes k(\mathfrak{p})$ has finitely many primes. Since A is a finite B-module, $A \otimes_B k(\mathfrak{p})$ is a finite $k(p)$-module. That is, a vector space of finite dimension. Hence there are a finite number of prime ideals since $A \otimes_B k(\mathfrak{p})$ is Artinian and thus the morphism is quasi-finite.

   (b) We can assume that Y is affine and it suffices to show that $f(X)$ is closed in Y. To say a finite morphism is closed is equivalent to showing that if $y \notin f(X)$, then there is a function $g \in k[Y]$ such that $g(y) = 1$ and $f(X) \subseteq \mathcal{Z}(g)$. That is, $k[X]$ is annihilated by $f^*(g)$. Let $A = k[Y], B = k[X]$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point y. By the Nullstellensatz, $y \notin f(X)$ iff $f^*(\mathfrak{m})B = B$. Now, since B is a finite A-module, the required assertion follows from Nakayama's Lemma.

   (c) Let X be the bug-eyed line (two copies of $\mathbb{A}_k^1$ glued at the compliment of a point P) and let $Y = \mathbb{A}_k^1 = \text{Spec } k[x]$. Let $f : X \to Y$ be the morphism defined by gluing $\mathbb{A}_k^1 \mapsto \mathbb{A}^1$ outside of some fixed point P. Then f is surjective and quasi-finite since it is the identity outside of P and $f^{-1}(P)$ consists of 2 points. f is of finite type since Y is affine and $f^{-1}(Y)$ has a covering of open affines Spec $k[x]$, where $k[x]$ is a finite $k[x]$-algebra. Since $f^{-1}(Y)$ is not affine, f is not finite by ex 2.3.4.

6. Let $U = \text{Spec } A$ be an open affine subset of X. By definition, A is an integral domain so (0) is a prime ideal. A closed subset $V(I)$ contains (0)

iff (0) contains I, thus the closure of (0) is $V((0))$, ie Spec A. Hence, by uniqueness, (0) is the generic point η of X. $\mathcal{O}_X(U)_{(0)} = \mathcal{O}_\eta$ is the fraction field of $\mathcal{O}_X(U)$.

7. [BLOG] Let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes, with X and Y both irreducible.

   Step 1: Show k(X) is a finite field extension of k(Y): Choose an open affine Spec $B = V \subset Y$ and an open affine in its preimage Spec $A = U \subset f^{-1}V$ such that A is a finitely generated B-algebra (by the finite type hypothesis). Since X is irreducible, so is U, so A is integral.

   Now A is finitely generated over B and therefore so is $k(B) \otimes_B A \cong B^{-1}A$. By Noether Normalization, there is an integer n and a morphism $k(B)[t_1, \ldots, t_n] \to B^{-1}A$ for which $B^{-1}A$ is integral over $k(B)[t_1, \ldots, t_n]$. Since $B^{-1}A$ is integral over $k(B)[t_1, \ldots, t_n]$, the induced morphism of affine schemes is surjective. But Spec $B^{-1}A$ has the same underlying topological space as $f^{-1}(\eta_Y) \cap U$, which is finite by assumption. By the Going-Up Theorem, Spec $B^{-1}A \to$ Spec $k(B)[t_1, \ldots, t_n]$ is surjective ($B^{-1}A$ is integral and integral over $k(B)[t_1, \ldots, t_n]$) we see that $n = 0$ and moreover, $B^{-1}A$ is integral over $k(B)$. Since it is also of finite type, this implies that it is finite over $k(B)$. By clearing the denominators from elements of A we get that $k(B^{-1}A) = k(A)$ is finite over $k(B)$.

   Step 2: Show for X and Y both affine: Let $X =$ Spec A, and $Y =$ Spec B and consider a set of generators $\{a_i\}$ for A over B. Considered as an element of $k(A)$, each generator satisfies some polynomial in $k(B)$ since it is a finite field extension. Clearing denominators, we get a set of polynomials with coefficients in B. Let b be the product of the leading coefficients in these polynomials. Replacing B and A with B_b and A_b, all these leading coefficients become units, and so after multiplying by their inverses, we can assume that the polynomials are monic. That is, A_b is finitely generated over B_b and there is a set of generators that satisfy monic polynomials with coefficients in B_b. Hence, A_b is integral over B_b and therefore a finitely generated B_b-module.

   Step 3: The general case: If X and Y are not necessarily affine, then take an affine subset $V =$ Spec B of X and cover $f^{-1}V$ with finitely many affine subsets $U_i =$ Spec A_i. By Step 2, for each i there is a dense open subset of V for which the restriction of f is finite. Taking the intersection of all these gives a dense open subset V' of V such that $f^{-1}V' \cap U_i \to V'$ is finite for all i. Furthermore, by the previous step, we see that V' is in fact a distinguished open of set of V. Shrink V' if necessary so that $f^{-1}V'$ is affine and replace V with V' and similarly replace U_i with $U_i \cap f^{-1}V'$. Since V' is a distinguished open in V, we still have an open affine subset of Y and the $U_i \cap f^{-1}V'$, now written as U_i, form an affine cover of $f^{-1}V'$.

   Let $U' \subseteq \bigcap U_i$ be an open subset that is open in each of the U_i. Then there are elements $a_i \in A_i$ such that $U' =$ Spec $(A_i)_{a_i}$ for each i. Since each A_i is finite over B, there are monic polynomials g_i with coefficients in B that the a_i satisfy. Take g_i of smallest possible degree so that the constant terms b_i are nonzero and define $b = \prod b_i$. Now the preimage of Spec B_b is Spec $((A_i)_{a_i})_b$ (any i gives the same open) and $((A_i)_{a_i})_b$ is a finitely generated B_b module. So we are done.

8. We have to check the patching condition. Let U and V be two open affine subschemes of X. Let $\widetilde{U} =$ Spec $\widetilde{A}$ and $\widetilde{V} =$ Spec $\widetilde{B}$. We have to show a canonical isomorphism $\varphi : U' \to V'$ where U' is the inverse image of $U \cap V$ in $\widetilde{U}$ and V' is the inverse image of $U \cap V$ in $\widetilde{V}$.

   Since it suffices to construct a conical morphism on an open cover, we can assume that U and V are open affines of some common affine scheme $W =$ Spec C and that $A = C_f$ and $B = C_g$, where $f, b \in C$. It suffices to check that if $\widetilde{A}$ is the integral closure of A, then $\widetilde{A}_f$ is the integral closure of A_f. It is clear that any element of $\widetilde{A}_f$ is integral over A_f. Indeed, if $a/f^k \in \widetilde{A}_f$, where $a \in \widetilde{A}$ satisfies the monic polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then a/f^k satisfies the monic polynomial $x^n + b_{n-1}x^{n-1} + \ldots + b_0$, where $b_i = a_i/f^{n(k-i)}$. On the other hand, if u belongs to the integral closure of A_f, then u is a root of a monic polynomial $x^n + b_{n-1}x^{n-1} + \ldots + b_0$, where each $b_i \in A_f$. Clearing denominators, it follows that $a = f^l u \in \widetilde{A}$ for some power of f. Thus one can glue the schemes $\widetilde{U}$ together to get a scheme $\widetilde{X}$. The inclusion $A \hookrightarrow \widetilde{A}$ induces a morphism of schemes $\widetilde{U} \to U$, and thus a morphism of schemes $\widetilde{U} \to X$. Arguing as before, these morphisms agree on overlaps. It follows that there is an induced morphism $\widetilde{X} \to X$.

   Now suppose that there is a dominant morphism of schemes $Z \to X$, where Z is normal. This induces a dominant morphism $Z_U \to U$, where U is an open affine subscheme and Z_U is the inverse image of U. Thus it suffices to prove the universal property of X in the case when X is affine. Covering Z by open affines, it suffices to prove this result when Z is affine. Using the equivalence of categories, we are reduced to proving that if $A \hookrightarrow \widetilde{A}$ is the inclusion of A inside its integral closure, and $A \to B$ is a ring homomorphism, with B integrally closed, then there is a morphism $\widetilde{A} \to B$. Clearly there is such a morphism into the field of fractions L of B. On the other hand, any element of the image is obviously integral over the image of A, and so integral over B. But then the image of $\widetilde{A}$ lies in B, as B is integrally closed. Suppose that X is of finite type. Clearly we may assume that $X =$ Spec A is affine. We are reduced to showing that the integral closure $\widetilde{A}$ of a finitely generated k-algebra A is a finitely generated A-module. Since this is a well known result in algebra, we are done.

9. (a) $\mathbb{A}^2_k =$ Spec $k[x, y] =$ Spec $(k[x] \otimes k[y]) = \mathbb{A}^1_k \times \mathbb{A}^1_k$. The points of $\mathbb{A}^1_k$ consist of the maximal ideals $\mathfrak{m}_a$ and the generic point η. The points

of the product of sets are then ordered pairs

Points	Closure
$(\mathfrak{m}_a, \mathfrak{m}_b)$	$\{(\mathfrak{m}_a, \mathfrak{m}_b)\}$
$(\mathfrak{m}_a, \eta)$	$\{(m_a, \mathfrak{m}_b) \mid b \in k\} \cup \{(\mathfrak{m}_a, \eta)\}$
$(\eta, \mathfrak{m}_b)$	$\{(\mathfrak{m}_a, \mathfrak{m}_b) \mid a \in k\} \cup \{(\eta, \mathfrak{m}_b)\}$
(η, η)	The whole space

Look at the prime ideal $(xy-1)$. Its closure is the set $\{(\mathfrak{m}_a, \mathfrak{m}_b) \mid ab = 1\} \cup \{\eta\}$. Thus $(xy-1)$ is not a point of the product of the two sets.

(b) As a topological space, $X = \text{Spec}\ (k(s) \times k(t))$ contains many points. $k(s) \times k(t)$ is the localization of $k[s,t]$ by the multiplicative set S generated by irreducible polynomials in s and t. But this leaves many irreducible polynomials in both s and t which are not inverted, and each of these will generate a prime ideal.

10. Let $f : X \to Y$ be a morphism, $y \in Y$ a point, and $k(y)$ be the residue field of y. Let Spec $k(y) \to Y$ be the natural morphism.

(a) Then $X_y = X \times_Y \text{Spec}\ k(y) \cong f^{-1}(V) \times_{\text{Spec}\ A} \text{Spec}\ k(Y)$, where $y \in V = \text{Spec}\ A \subseteq Y$ some open affine. Then if $f^{-1}(V) = \bigcup \text{Spec}\ B_i$,

$$\begin{aligned} f^{-1}(V) \times_{\text{Spec}\ A} \text{Spec}\ k(y) &= (\textstyle\bigcup \text{Spec}\ B_i) \times_{\text{Spec}\ A} \text{Spec}\ k(y) \\ &= \textstyle\bigcup (\text{Spec}\ B_i \times_{\text{Spec}\ A} \text{Spec}\ k(y)) \\ &= \textstyle\bigcup \text{Spec}\ (B_i \otimes_A k(y)) \\ &= \textstyle\bigcup f^{-1}|_{\text{Spec}\ B_i}(y)\ \textit{(by claim below)} \\ &= f^{-1}(y) \end{aligned}$$

Claim: Spec $(B_i \otimes_A k(y)) = f^{-1}|_{\text{Spec}\ B_i}(y)$.
Proof: Let $B_i = B, \mathfrak{p} = y \in \text{Spec}\ A$. Then $\text{Spec}\ (B \otimes_A (A/\mathfrak{p})_{\mathfrak{p}}) = \text{Spec}\ (B_{\mathfrak{p}} \otimes_A A/\mathfrak{p}) = \text{Spec}\ (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Now, $B_{\mathfrak{p}} = \{\frac{b}{d} \mid d \notin f(p), d \in f(A)\}$, so $\text{Spec}\ B_{\mathfrak{p}} = \{\mathfrak{q} \in \text{Spec}\ B \mid \mathfrak{q} \cap f(A) \subseteq f(\mathfrak{p})\} = \{\mathfrak{q} \in \text{Spec}\ B \mid f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}\}$.
Therefore

$$\begin{aligned} \text{Spec}\ (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) &= \{\mathfrak{q} \in \text{Spec}\ B \mid f^{-1}(y) \subseteq \mathfrak{p}, \mathfrak{q} \supseteq f(\mathfrak{p})\} \\ &= \{\mathfrak{q} \in \text{Spec}\ B \mid f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}, f^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}\} \\ &= \{\mathfrak{q} \in \text{Spec}\ B \mid f^{-1}(y) = \mathfrak{p}\} \\ &= f^{-1}(\mathfrak{p}) \end{aligned}$$

Therefore Spec $(B_i \otimes_A k(y)) = f^{-1}|_{\text{Spec}\ B_i}(y)$

(b) Let $X = \text{Spec}\ k[s,t]/(s-t^2)$. Let $Y = \text{Spec}\ k[s]$. Let $f : X \to Y$ be defined by $s \mapsto s$. Let $y \in Y$ be the point $a \in k^\times$. Then

$$\begin{aligned}
X_y &= X_a \\
&= \text{Spec } k[s,t]/(s-t^2) \times_{\text{Spec } k[x]} \text{Spec } k(y) \\
&= \text{Spec } (k[s,t]/(s-t^2) \otimes_{k[s]} k(a)) \\
&= \text{Spec } (k[s,t]/(s-t^2) \otimes_{k[s]} k[s]_{(s-a)}/(s-a)k[s]_{(s-a)}) \\
&= \text{Spec } (k[s,t]/(s-t^2) \otimes_{k[s]} k[s]/(s-a)) \\
&= \text{Spec } (k[s,t]/(s-t^2, s-a)) \;\; (\textit{since}\, M \otimes A/I \cong M/IM) \\
&= \text{Spec } (k[t]/(a-t^2))
\end{aligned}$$

Now, if $a = 0$, Spec $(k[t]/t^2)$. The only prime ideal containing t^2 is (t), which is nilpotent, and thus we get a non-reduced one point scheme.

If $a \neq 0$, Spec $(k[t]/(a-t^2))$ = Spec $(k[t]/(\sqrt{a}-t)(\sqrt{a}+t))$ = Spec $(k[t]/(\sqrt{a}-t)) \times k[t]/(\sqrt{a}+t)$ = Spec $k \times$ Spec k. Thus X_y consists of two points, $(0,1)$ and $(1,0)$. The residue field $k(a) = k(s-a) = k[s]_{(s-a)}/(s-a)k[s]_{(s-a)} = (k[s]/(s-a))_{(s-a)} = k_{(s-a)} = k$.

Let η be the generic point of Y, corresponding to the (0) ideal in Spec Y. Then

$$\begin{aligned}
X_\eta &= \text{Spec } (k[s,t]/(s-t^2) \otimes_{k[s]} k(s)) \\
&= \text{Spec } (k[s,t]/(s-t^2) \otimes_{k[s]} k[s]_0) \\
&= \text{Spec } (k[s]\backslash 0)^{-1} k[s,t]/(s-t^2) \;\; (\textit{since}\, B \otimes_A S^{-1}A \cong S^{-1}B) \\
&= \text{Spec } (k(s)[t]/(s-t^2)) \\
&= \text{Spec of field}
\end{aligned}$$

and thus we have a point point scheme, with residue field itself, so the degree is 2 since $s - t^2$ has degree 2 in t.

11. (a) Let $Y' = Y \times_X X'$, $g : X' \to X$ any morphism. To show that the base change $f : Y' \to X'$ is a closed immersion, we can replace X' by an affine open neighborhood U' of a point of $f'(Y')$. Furthermore, we may assume that $U' \subseteq g^{-1}(U)$ for an affine open set U of Y. Set $U' = \text{Spec } A'$ and $U = \text{Spec } A$. Since f is a closed immersion, we can write $f^{-1}(U) = \text{Spec } B$, where $B \cong A/I$ for some ideal I in A. Then $f'^{-1}(U') = \text{Spec } (A' \otimes_A B) \cong \text{Spec } (A'/IA')$. Hence $f' : Y' \to X'$ is a closed immersion.

    (b) See Shaf Bk 2, page 33

    (c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. Let Y' be any other subscheme of X with the same underlying topological space. Let $f : Y' \to X$ be the closed immersion. Then clearly, as a map on topological spaces, $f : Y' \to Y \to X$ gives $\text{sp}(Y)' \overset{homeo}{\approx} \text{sp}(Y) \overset{homeo}{\approx} \text{sp}(V(\mathfrak{a})) \subset \text{sp}(X)$. For any open set U in $V(\mathfrak{a}) \subset X$, since $Y = V(\mathfrak{a})$, U open in Y, the surjective map $\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_{Y'}$ extends to a surjective map $\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_{Y'} \twoheadrightarrow f_*\mathcal{O}_Y$. For the case when X is not affine, glue.

(d) Let $f : Z \to X$ be a morphism. If Z is reduced, then the unique closed subscheme Y of X such that f factors is clearly the reduced induced structure on the closure of $f(Z)$ by part c). If Z is not reduced, factor f as $f' : Z \to Z_{\text{red}} \to X$ and then use the reduced induced structure of $f'(Z_{\text{red}})^-$.

12. (a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Then $\varphi(S_+) = T_+$. By definition, $U = \{p \in \text{Proj } T \mid p \not\supseteq \varphi(S_+)\}$, and thus $U = \text{Proj } T$. The map $f : \text{Proj } T \to \text{Proj } S$ is defined by $p \mapsto \varphi^{-1}(p)$.

Show f is injective: Let $\varphi^{-1}(p) = \varphi^{-1}(q)$ for $p, q \in \text{Proj } T$. If $p \neq q$, choose $x \in q \backslash p$. Since φ is surjective, $\varphi^{-1}(x) \neq \emptyset$. If $\varphi^{-1} \subseteq p, \varphi(\varphi^{-1}(p))$ is strictly bigger then p, which is a contradiction, so f is injective.

Claim: $f(\text{Proj }) = V(\mathfrak{a})$, where $\mathfrak{a} = \bigcap_{p \in \text{Proj } T} \varphi^{-1}(p)$. Let $q \supseteq \mathfrak{a}$ and let q' be the inverse image of $\varphi(q)$. Note that $\varphi(q)$ is a homogeneous prime ideal of B since φ is surjective. That is, if $ab \in \varphi(q)$, with both a and b homogeneous ideals, then a and b have homogeneous pre-images whose product is contained in q, so at least one of a or b is contained in $\varphi(q)$. By definition, $q' \supseteq q$. If the inclusion is proper, pick $x \in q' \backslash q$. Then there exists $y \in q$ such that $\varphi(s) = \varphi(y)$. But then $x - y \in q' \backslash q$ and $\varphi(x - y) = 0$. But $0 \subseteq p$ for all prime ideals p in B. Thus $x - y \subseteq \mathfrak{a}$ which is a contradiction and thus $q' = q$. Therefore the claim that $f(\text{Proj } T) = V(\mathfrak{a})$, where $\mathfrak{a} = \bigcap_{p \in \text{Proj T}} \varphi^{-1}(p)$ is proven and $f(\text{Proj } T)$ is closed.

Thus f is a bijection, φ preserves inclusions of ideals, and thus f is a homeomorphism. Finally the map on stalks is the same as the localization map $\varphi(p)$: $S_{(p)} \to T \otimes_S S_{(p)}$, which is surjective since φ is surjective. Thus f is a closed immersion.

(b) Let $I \subseteq S$ be a homogeneous ideal and let $T = S/I$. Let Y be the closed subscheme of $X = \text{Proj } S$ defined as the image of the closed immersion $\text{Proj } S/I \to X$. There is a commutative diagram of graded rings where the maps are projections:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S/I' \arrow[dl] \\ S/I & \end{tikzcd}$$

This corresponds to a commutative diagrams of schemes:

$$\begin{tikzcd} \text{Proj } S & \text{Proj } S/I' \arrow[l] \\ \text{Proj } S/I \arrow[u] \arrow[ur] & \end{tikzcd}$$

The map $S/I' \to S/I$ is an isomorphism for degree $d \geq d_0$, so by ex 2.14(c), the map Proj $S/I \to$ Proj S/I' is an isomorphism. The commutative diagram shows that I and I' determine the same closed subscheme.

13. Properties of Morphisms of finite type

    (a) Let $f : X \to Y$ be a closed immersion and identify X with a closed subset $V \subseteq Y$. Cover Y by open affines $U_i = \text{Spec } A_i$. Locally on each U_i we have a closed immersion $f^{-1}(V \cap U_i) \to U_i$ which looks like $A_i \to A_i/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i \subseteq A_i$. Then $A_i/\mathfrak{a}_i$ is a finitely generated A_i-algebra, so f is a morphism of finite type.

    (b) Let $f : X \to Y$ be a quasi-compact open immersion. Identify X with an open affine $U \subseteq Y$. For any open affine $V \subseteq Y$, $f^{-1}(V) = U \cap V$. Cover this intersection with open sets distinguished in both U and V. Since f is quasi-compact, we can choose a finite number of these distinguished opens. If $V = \text{Spec } A$, then each distinguished open in $U \cap V$ is Spec A_f for some $f \in A$ and A_f is a finitely generated A-algebra with generating set $\{\frac{1}{f}\}$, so f is of finite type.

    (c) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of finite type. Let $h = g \circ f$ and let $U = \text{Spec } C$ be an open affine of Z. By ex 3.3(b), $g^{-1}(U)$ can be covered by finitely many Spec B_i such that B_i is a finitely generated C-algebra. Then $f^{-1}(\text{Spec } B_i)$ can be covered by finitely many Spec A_{ij} such that A_{ij} is a finitely generated B_i-algebra. Then we have $C \to B_i \to A_{ij}$, so A_{ij} is a finitely generated $C-$algebra. To see this, it is enough to note for some n, m, there exists a surjective homomorphism $B_i[x_1, \ldots, x_n] \twoheadrightarrow A_{ij}$ and $C[y_i, \ldots, y_n] \twoheadrightarrow B$. This gives a surjective homomorphism $C[x_1, \ldots, x_n, y_i, \ldots, y_m] \twoheadrightarrow A_{ij}$. Since $h^{-1}(U) = \bigcup \text{Spec } A_{ij}$, h is a morphism of finite type.

    (d) Let $f : X \to S$ and $g : S' \to S$ be morphisms such that f is of finite type. Let $f' : X' \to S'$, where $X' = X \times_S S'$. Pick an open affine $U = \text{Spec } A \subseteq S$, with $g^{-1}(U) \neq \emptyset$, and $U' = \text{Spec } A' \subseteq g^{-1}(U)$ such that $f'^{-1}(U') \neq \emptyset$. Cover $f^{-1}(U)$ be finitely many open affines $V_i = \text{Spec } B_i$ such that B_i is a finitely generated A-algebra. Now, $f'^{-1}(U')$ is covered by $V_i \times_U U' = \text{Spec } (B_i \otimes_A A')$. If $\{b_1, \ldots, b_r\}$ is a finite generating set for B_i as an A-algebra, then $\{b_i \otimes_A 1\}$ is a finite generating set for $B_i \otimes_A A'$ as an A'-algebra. Cover S with open affines U_i and let $g^{-1}(U_i)$ be a cover for S'. Then we can cover each $g^{-1}(U_i)$ with open affines $V_{ij} = \text{Spec } A'_{ij}$ whose preimage under f' can be covered by finitely many $W_{ijk} = \text{Spec } B'_{ijk}$ such that each B'_{ijk} is a finitely generated A'_{ij}-algebra. So f' is a morphism of finite type.

    (e) The morphism $X \times_S Y \to S$ can be factored $X \times_s Y \overset{p_2}{\to} Y \to S$. The first map is of finite type since $X \to S$ is of finite type and by part d). The second map is of finite type by assumption, so part c) then gives that their composition $X \times_S Y \to S$ is a morphism of finite type.

(f) Let $f : X \to Y$ be a quasi-compact morphism. Let $g : Y \to Z$ be a morphism such that $h = g \circ f$ is of finite type. Pick Spec $C \subseteq Z$, Spec $B \subseteq g^{-1}(\text{Spec } C)$, Spec $A \subseteq f^{-1}(\text{Spec B})$, each non-empty. Then Spec $A \subseteq h^{-1}(\text{Spec } C)$, so by ex 3.3c), A is a finitely generated C-algebra and we get homomorphisms $C \to B \to A$. If $\{a_1, \dots, a_n\}$ are the generators for A as a C-algebra, there is a surjective morphism $C[x_1, \dots, x_n] \to A$ defined by mapping $x_i \mapsto a_i$. Then this factors through a map $B[x_1, \dots, x_n] \to A$, where $x_i \mapsto a_i$. Since the $C[x_1, \dots, x_n] \to B[x_1, \dots, x_n] \to A$ is a surjective map from $C[x_1, \dots, x_n] \to A$, $B[x_1, \dots, x_n] \to A$ is surjective, so A is a finitely generated B-algebra. Finally, if Spec C_i is a cover of Z, then there exists a cover Spec B_j of Y such that Spec $B_j \subseteq g^{-1}(\text{Spec } C_i)$ for some i. So by the above argument, $f^{-1}(\text{Spec } B_j)$ can be covered by finitely many Spec A_{jk} such that A_{jk} is a finitely generated B_j-algebra, so f is locally of finite type. By assumption, f is also quasi-compact, so f is of finite type.

(g) Since Y is noetherian, it is quasi-compact, so we can cover it with finitely many open affines Spec B_i. Then each $f^{-1}(\text{Spec } B_i)$ can be covered by finitely many open affines Spec A_{ij} each of which is quasi-compact and such that $f^{-1}(\text{Spec } B_i)$ cover X. So X is a finite union of quasi-compact sets, so X is quasi-compact. Also, each A_{ij} is a finitely generated B_i-algebra. Then $A_{ij} \cong B[x_1, \dots, x_n]/\mathfrak{a}$ for some n and some ideal $\mathfrak{a}$. Since Y is noetherian, B_i is a noetherian ring, and so by the Hilbert Basis Theorem, $B[x_1, \dots, x_n]$ is noetherian. Since homomorphic images of noetherian rings are again noetherian, we have covered X by noetherian rings and have shown it to be quasi-compact. Thus X is a noetherian scheme.

14. We need to show that every open subset in a basis of the topology contains a closed point and we can assume that X is affine. Clearly every affine open set contains a closed point in its own topology. Such a closed point is closed in the whole subscheme since closed points are precisely those whose residue fields are finite extensions of k.

    This is not true for an arbitrary scheme. Consider Spec $k[X]_{(x)} = \{0, (x)\}$. Then (x) is a closed point and 0 is not, so the set of closed points is not dense.

15. See "Algebraic Geometry and Arithmetic Curves" by Qing Liu section 3.2.2 pg 89.

16. Let X be a noetherian topological space. Let P be a property of closed subsets of X. Define $S = \{V \subseteq X \mid V \neq \emptyset, V \text{ is closed and does not have property } P\}$. If $S \neq \emptyset$, then S has a minimal element with respect

to inclusion since X is noetherian. If every proper closed subset of Z satisfies P, then so does Z be assumption. However, if there is a proper closed subset of Z that does not satisfy P, then Z is not minimal, which contradicts the choice of Z. So $S = \emptyset$ and X has property P.

17. (a) We have already seen in Caution 3.1.1 that $\mathrm{sp}(X)$ is a noetherian topological space, so we just need to show that each closed irreducible subset has a unique generic point. Note that for a closed irreducible subset Z of any topological space and an open subset U, either U contains the generic points of Z, or $U \cap Z = \emptyset$ (since if $\eta \notin U$, then U^c is a closed subset containing η and so $\overline{\{\eta\}} \subseteq U^c$ and therefore $U \cap Z = \emptyset$). So we can reduce to the affine case.

    Let X be affine. Then the irreducible closed subsets correspond to ideals I with the property that $\sqrt{I} = \sqrt{JK} \Rightarrow \sqrt{I} = \sqrt{J}$ or $\sqrt{K}$. We claim that the ideals with this property are prime. To see this suppose that $fg \in \sqrt{I}$. Then $\sqrt{I} = ((f)+\sqrt{I})((g)+\sqrt{I})$ and so either $\sqrt{I} = (f)+\sqrt{I}$ or $\sqrt{I} = (g)+\sqrt{I}$. Hence, either $f \in \sqrt{I}$ or $g \in \sqrt{I}$. It is straightforward that $\mathfrak{p}$ is a generic point for $V(\mathfrak{p})$ so we just need to show uniqueness. Suppose that $\mathfrak{p}, \mathfrak{q}$ are two generic points for a closed subset determined by an ideal I. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{I} = \sqrt{\mathfrak{q}}. = \mathfrak{q}$

    (b) Let Z be a minimal nonempty closed subset. Since Z is minimal it is irreducible and therefore, by the previous part has a unique generic point η. For any point $x \in Z$, again since Z is minimal, we have $Z = \overline{\{x\}}$ and so $x = \eta$ by uniqueness of the generic point.

    (c) Let x, y be the two distinct points and let $U = \overline{\{x\}}^c$. If $y \in U$, we are done, so assume not. Then $y \in \overline{\{x\}}$. If $x \in \overline{\{y\}}$, then x and y are both generic points for the same closed irreducible subset, which contradicts the assumption they were distinct. Hence $x \in \overline{\{y\}}^c$.

    (d) If $\eta \notin U$, then $\eta \in U^c$, a closed subset, and so $X = \overline{\{\eta\}} \subseteq U^c$. Therefore $U = \emptyset$.

    (e) Let $X = \bigcup Z_i$ be the expression of X as a union of its irreducible closed subsets. In particular, the Z_i are the maximal irreducible closed subsets. Let η be the generic point of Z_i and x a point such that $\eta \in \overline{\{x\}}$. This implies that $Z_i \subseteq \overline{\{x\}}$ and so since the Z_i are maximal, $Z_i = \overline{\{x\}}$. Since the generic points of irreducible closed subsets are unique, this implies that $\eta = x$. So η is maximal. Conversely, suppose that η is maximal. η is in Z_i for some i. If η' is the unique generic point of Z_i, then $\eta \in \overline{\{\eta'\}}$ and so since η is maximal, $\eta = \eta'$.

    Let Z be a closed subset and $z \in Z$. Since $\overline{\{z\}}$ is the smallest closed subset containing z, we have $\overline{\{z\}} \subseteq Z$.

    (f) Since the lattice of closed subsets of $t(X)$ is the same as the lattice of closed subsets of X, we immediately have the $t(X)$ is noetherian. Now consider η, a closed irreducible subset of X, and its closure $\overline{\{\eta\}}$

in $t(X)$. This is the smallest closed subset of X containing η. Since η is itself a closed subset of X, we see that this is η. So if η' is a generic point for $\overline{\{\eta\}} \subseteq t(X)$, then $\overline{\{\eta\}} = \overline{\{\eta'\}}$, and so $\eta = \eta'$. Hence each closed irreducible subset has a unique generic point. If X is itself a Zariski space, then there is a one-to-one correspondence between points and irreducible closed subsets. Hence α is a bijection on the underlying sets. It is straightforward to see that its inverse is also continuous.

18. [BLOG] Let X be a Zariski topological space. A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$

    (a) Consider $\coprod_{i=1}^n Z_i \cap U_i \subseteq X$, where Z_i are closed subsets of X and U_i are open subsets of X. Note that (1)+(3) implies that all closed subsets of X are in $\mathfrak{F}$ and (2)+(3) implies that finite unions of elements of $\mathfrak{F}$ are in $\mathfrak{F}$. Hence, as long as the $Z_i \cap U_i$ are disjoint, $\coprod_{i=1}^n Z_i \cap U_i = \bigcup_{i=1}^n Z_i \cap U_i \in \mathfrak{F}$.
    Let $\mathfrak{F}'$ be the collection of subsets of X that can be written as a finite disjoint union of locally closed subsets. We have just shown that $\mathfrak{F}' \subset \mathfrak{F}$, so by definition, if $\mathfrak{F}'$ satisfies (1), (2), and (3), then $\mathfrak{F}' = \mathfrak{F}$. We immediately have that (1) is satisfied since $U \cap X = U$ and X is closed. If $\coprod_{i=1}^n Z_i \cap U_i$ and $\coprod_{i=1}^n Z_i' \cap U_i'$ are two elements of $\mathfrak{F}'$, then their intersection is

    $$\left(\coprod_{i=1}^n Z_i \cap U_i\right) \cap \left(\coprod_{i=1}^n Z_i' \cap U_i'\right) = \coprod_{i,j=1}^n (Z_i \cap Z_j') \cap (U_i \cap U_j')$$

    which is in $\mathfrak{F}'$ so (2) is satisfied. Show (3) by induction on n. Let $\mathfrak{F}_n' \subset \mathfrak{F}$ be the collection of subsets of X that can be written as a finite disjoint union of n locally closed subsets. Note that $\bigcup_n \mathfrak{F}_n' = \mathfrak{F}'$ and that we have already shown that the intersection of an element of $\mathfrak{F}_n'$ and an element of $\mathfrak{F}_m'$ is in $\mathfrak{F}'$. Let $S \in \mathfrak{F}_1'$. So $S = U \cap Z$. Then its complement is

    $$S^c = (U \cap Z)^c = U^c \cup Z^c = U^c \coprod (Z^c \cap U)$$

    which is in $\mathfrak{F}'$. Now let $S \in \mathfrak{F}_n'$ and suppose that for all $i < n$, complements of members of $\mathfrak{F}_i'$ are in $\mathfrak{F}'$. We can write S as $S = S_{n-1} \coprod S_1$ for some $S_{n-1} \in \mathfrak{F}_{n-1}'$ and $S_1 \in \mathfrak{F}_1'$. The complement of S is then $S_{n-1}^c \cap S_1^c$. We know that S_{n-1}^c and S_1^c are in $\mathfrak{F}'$ by the inductive hypothesis and we know that their intersection is in $\mathfrak{F}'$ by (2) which we proved above. Hence S^c is in $\mathfrak{F}'$ and we are done.

    (b) Let $S \in \mathfrak{F}$. if the generic point η is in S, then $\overline{S} \supseteq \overline{\{\eta\}} = X$, so S is dense.

For the converse, use the fact that for an irreducible Zariski space, every non-empty open subset contains the generic point (Ex 3.17(d)). Suppose $S = \coprod_{i=1}^n Z_i \cap U_i$ is dense. The closure $\overline{S}$ is the smallest closed subset that contains S. So any closed subset, in particular $\bigcup Z_i \supseteq S$, contains its closure. Hence $\bigcup Z_i \supseteq \overline{S} = X$. But since X is irreducible, $Z_i = X$ for some i. So up to re-indexing, $S = U_n \coprod (\coprod_{i=1}^{n-1} Z_i \cap U_i)$. Since every non-empty set contains the generic point, S contains the generic point.

(c) It is immediate that the closed (resp. open) subsets are constructible and stable under specialization (resp. generalization). Suppose that $S = \coprod_{i=1}^n Z_i \cap U_i$ is a constructible set stable under specialization and let x be the generic point of an irreducible component of Z_i that intersects U_i non-trivially. Since S is closed under specialization, S contains every point in the closure of $\{x\}$. So S contains every point of every irreducible component of each Z_i. That is $S \supseteq \bigcup Z_i$. Now consider a point $x \in S$. It is contained in some Z_i, and so $S \subseteq \bigcup Z_i$. Hence $S = \bigcup Z_i$ is closed.

Now let S be a constructible set, stable under generization. Then S^c is a closed set, stable under specialization and therefore closed. So S is open.

(d)

$$f^{-1}\left(\coprod_{i=1}^n Z_i \cap U_i\right) = \coprod_{i=1}^n f^{-1}(Z_i \cap U_i) = \coprod_{i=1}^n f^{-1}(Z_i) \cap f^{-1}(U_i)$$

Since f is continuous, $f^{-1}Z_i$ is closed and $f^{-1}U_i$ is open. Hence the preimage of a constructible set is constructible.

19. [BLOG]

(a) If $S \subseteq X$ is a constructible set then we can restrict the morphism to $f|_S : S \to Y$. So it is enough to show that $f(X)$ itself is constructible. If $\{V_i\}$ is an affine cover of Y and $\{U_{ij}\}$ is an affine cover for each $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible for each i, j, then $f(X) = \bigcup f(U_{ij})$ is constructible, so we can can assume that X and Y are affine. Similarly, if $\{V_i\}$ are the irreducible components of Y and $\{U_{ij}\}$ are the irreducible components of $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible for each i, j, then $f(X) = \bigcup f(U_{ij})$ is constructible, so we can assume that X and Y are irreducible. Reducing a scheme doesn't change the topology, so we can assume that X and Y are reduced. Putting these last two together, we can assume that X and Y are integral.

Now show that we can assume f to be dominant. Suppose that $f(X)$ is constructible for every dominant morphism. We have an induced morphism $f' : X \to \overline{f(X)} = C$ from X into the closure of its image C.

Then f' is certainly dominant, so $f'(X)$ is constructible in C. This means that it can be written as $\coprod U_i \cap Z_i$, a disjoint union of locally closed subsets. Since C is closed in Y, each Z_i is still closed in Y. The subsets U_i on the other hand, can be obtained as $U = V_i \cap C$ for some open subsets V_i of Y by the definition of the induced topology on C. We now have $f(X) = \coprod U_i \cup Z_i = \coprod V_i \cap (C \cap Z_i)$, which is constructible.

(b) Let n be the number of generators of B as an A-algebra. We split the proof of the algebraic result into the cases $n = 1$ and $n > 1$. If $n = 1$, write $B = A[t]$, where $t \in B$ generates B as a A-algebra. Pick a non-zero $b \in B$ and write it as $b = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_0$, where $c_d \neq 0, c_i \in B$. If t has no relations, ie B is the polynomial ring in one variable over A, let $a = a_d$. Let K be an algebraically closed field, and let $\varphi : A \to K$ such that $\varphi(a) \neq 0$. The polynomial $\sum_{i=0}^{d} \varphi(a_i) x^i$ has d roots, and K is infinite, so there exists $r \in K$ such that $\sum_{i=0}^{d} \varphi(a_i) r^i \neq 0$. Extend φ to $\varphi' : A[t] \to K$ by mapping t to r.

Now suppose that $t \in K(B)$ is algebraic over $K(A)$, where $K(A)$ is the quotient field of A. Then there exists equations $\sum_{i=0}^{d} a_i t^i = 0$ and $\sum_{i=1}^{e} a'_i (b^{-1})^i = 0$, where $a_i, a'_i \in K(A)$ and $a_d \neq 0, a'_e \neq 0$. Let $a = a_d a'_e$. Let K be algebraically closed and let $\varphi : A \to K$ such that $\varphi(a) \neq 0$. First extend φ to $A_a \to K$ in the obvious way by sending $\frac{1}{a}$ to $\frac{1}{\varphi(a)}$. Next extend φ to some valuation ring $R \supseteq A_a$. From the equations, t, b^{-1} are both integral over A_a. Since the integral closure of $A_a = \bigcap\{\text{valuation rings of } K(A_a)\}$, $t, b^{-1} \in R$. Since $t \in R$, so is b, so $b \in R^\times$. Therefore the extension $R \to K$ maps b to $x \neq 0$. Since $t \in R$, $A \subseteq R$, restrict to B to get a map $\varphi' : B \to K$ that maps b to a non-zero element.

For $n > 1$, proceed by induction.

(c) By part b), there exists some $a \in A$ such that $D(a) \subseteq f(X)$. We will show that $f(X) \cap V(a)$ is constructible in Y. If this intersection is empty, we are done, so assume not. Note that $V(a) =$ Spec $(A/(a))$, so consider the map f' : Spec $B/aB \to$ Spec $A/(a)$ induced by f, whose image is $f(X) \cap V(a)$. Since $A \to B$ is injective, $A/(a) \to B/aB$ is injective, so f' is dominant. Also, both rings are Noetherian, so the ideal (a) has a primary decomposition $\bigcap \mathfrak{p}_i$, where $\mathfrak{p}_i$ are primary ideals. Furthermore, $\sqrt{\mathfrak{p}_i}$ are prime, so relabel these as $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Then $\sqrt{(a)} = \bigcap \mathfrak{p}_i$ so $V(a) = \bigcup V(\mathfrak{p}_1)$ as topological spaces since $V(a) = V(\sqrt{a})$ as topological spaces. For each $\mathfrak{p}_i B$, we can do the same since B is noetherian, so we have maps Spec $B/\mathfrak{q}_j \to$ Spec $A/\mathfrak{p}_i$ for primes $\mathfrak{a}_j \in$ Spec B, and the union of their images is $f(X) \cap V(a)$. While the scheme structure may be different, constructibility is a topological property and we are preserving the underlying topological space. These maps now involve integral domains, so each image contains a nonempty subset by part b), and hence is constructible in $V(\mathfrak{p}_i)$ by Noetherian induction. A locally closed subset of $V(\mathfrak{p}_i)$ is also a locally closed subset of Spec B, so in fact images of Spec $B/\mathfrak{q}_j \to$ Spec $A/\mathfrak{p}_i$ are constructible in Spec B. Since constructibility is closed under finite unions, $f(X) \cap V(a)$ is constructible. Thus $f(X)$ is constructible.

(d) Let $f : \mathbb{A}^1_k \to \mathbb{P}^2_k$ be a morphism given by $x \mapsto (x, 1, 0)$. Then $f(\mathbb{A}^1_k)$ is neither open nor closed since $(x, 1, 0)$ is not the zero set of any ideal of homogeneous polynomials, and neither is its complement.

20. Let X be an integral scheme of finite type over a field k.

(a) For any closed point $P \in X$, let Spec A be the affine scheme containing it. Let $\mathfrak{m}$ be the corresponding maximal ideal of P in Spec A. Then

$$\begin{aligned}\dim X &= \dim A \text{ \textit{(by 1.1)}} \\ &= \text{ht } \mathfrak{m} + \dim A/\mathfrak{m} \text{ \textit{(since $A/\mathfrak{m}$ is a field, has dim = 0)}} \\ &= \text{ht } \mathfrak{m}A_{\mathfrak{m}} \\ &= \dim \mathcal{O}_P\end{aligned}$$

(b) Let $K(X)$ be the function field of X. By Them 1.8A, since X is an integral domain of finite type, by part a) of the theorem, dim $X =$ tr.d $K(X)/k$.

(c) Let Y be a closed subset of X. Then codim $(Y, X) =$ codim (Spec B/b, Spec $B) = \inf_{p \supseteq b}$ codim (Spec B/b, Spec $B) = \inf_{p \supseteq b}$ ht $(p) = \inf_{p \in Y}$ dim $\mathcal{O}_{p,X}$

(d) If Y is irreducible, this is 1.8.A(b). If Y is reducible, let $Z \subseteq Y$ be an irreducible closet subset of largest dimension. Then dim $Y+$ codim$(Y, X) =$ dim $Z+$ codim$(Z, X) =$ dim X.

(e) This is prop 1.10

(f) if $k \subseteq k'$ is a field extension, dim $X' =$ dim $(X \times_k k') =$ dim $X+$ dim $k =$ dim X.

21. For (e), consider Spec $R[t]_u \subseteq$ Spec $R[t]$, where $\mathfrak{m}_R = (u)$. Then with $K = Q(R)$, dim $R[t]_u =$ dim $K[t] = 1 \neq 2 =$ dim $R[t]$ For (a) and (d) it suffices to find a maximal ideal of height 1. Consider $(ut - 1)$. In $R[t]/(ut - 1), t$ becomes an inverse for u and thus this ring is $Q(R)$. $R[t]$ is a UFD so every principal prime ideal has height one. $P = (ut - 1)$ for (a) and $Y = V(P)$ for (d).

22. Ingredients:

4 tablespoons mayonnaise, 2 tablespoons Creole mustard, 1/2 teaspoon Creole seasoning, 1/8 teaspoon freshly ground black pepper, 1 tablespoon finely chopped fresh parsley, 1 tablespoon finely chopped green onion, 2

teaspoons finely minced red bell pepper, optional, 1 pound jumbo lump crabmeat, 1 1/4 cup fresh fine bread crumbs, divided,

Preparation: Combine mayonnaise, mustard, parsley, and seasonings; set aside. Drain crabmeat; gently squeeze to get as much of the liquid out as possible. Put crabmeat in a bowl. With a spatula or wooden spoon, fold in mayonnaise mixture and 1 cup of the bread crumbs, just until blended. Shape into 8 crab cakes, about 2 1/2 inches in diameter. I use a biscuit or cookie cutter with an open top to shape the cakes and press the ingredients down to make them hold together. Press gently into reserved crumbs. Cover and chill for 1 to 2 hours. Heat clarified butter or oil over medium heat. Fry crab cakes for about 5 minutes on each side, carefully turning only once. Serve with lemon wedges and Remoulade or other sauce.

23. Let V, W be two varieties over an algebraically closed field and let $V \times W$ be their product. First show that $t(V) \times_k t(W)$ is an abstract variety (ie an integral separated scheme of finite type over an algebraically closed field k). By Corr 4.6(d), $t(V) \times_k t(W)$ is separated, and since k is algebraically closed, it is integral by prop 4.10 and of finite type. So $t(V) \times_k t(W) = t(Y)$ for some variety Y. But then Y clearly satisfies the universal property, so $Y = V \times W$ by uniqueness.

2.4 Separated and Proper Morphisms

1. Let $f : X \to Y$ be a finite morphism. Since properness is local and f is finite, we can take both X and Y to be affine, say Spec B and Spec A respectively. Let R be a valuation ring and K its quotient field. Consider the following commutative diagram:

$$\begin{tikzcd} \text{Spec } K \arrow[r, "v"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \text{Spec } R \arrow[r, "u"] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since everything is affine, we can turn this diagram into a commutative diagram of rings:

$$\begin{tikzcd} K & B \arrow[l, "v"'] \arrow[dl, dashed] \\ R \arrow[u, "i"] & A \arrow[l, "u"] \arrow[u, "f"'] \end{tikzcd}$$

Now, since $A \to B$ is finite, B is integral over A (AM Remark p 60). Then $u(A) \hookrightarrow v(B)$ is integral. But since R is a valuation ring, R is integrally closed. Since $u(A) \subseteq R$ and R is integrally closed, $v(B) \subseteq R$. Thus by the Valuative Criterion of Properness, $f : X \to Y$ is proper.

2. Let U be the dense open subset of X on which f and g agree. Let $Z = X \times_S Y$. Consider $f \times g : X \to Y \times_S Y$. Then $(f \times g)(U)$ is contained in $\Delta(Y)$ by assumption. Since Y is separated, $\Delta(Y)$ is closed. Thus $(f \times g)^{-1}(\Delta(Y))$ is a closed set containing the dense set U, ie all of X. So $(f \times g)(X) \subseteq \Delta(Y)$. Thus $f = g$ as maps of topological spaces. To prove equality of the sheaf maps, it suffices to show equality locally. So we can let $X = \text{Spec } B$ and $Y = \text{Spec } A$ and let $U = D(h)$. Then the associated map on rings $f : A \to B_h$ and $g : A \to B_h$ are the same by assumption. Thus for all $a \in A$, $\frac{f(a)}{1} = \frac{g(a)}{1}$, so there exists an integer n_a such that $h^{n_a}(f(a) - g(a)) = 0$. Thus $\text{Im}(f - g) \subseteq \bigcup \text{Ann}(h^n)$. A simple check shows that $D(h) \subseteq V(\text{Ann } (h))$. Because X is reduced, and $U = D(h)$ is dense by assumption, this forces $\text{Ann}(h) = 0$. Similarly, $\text{Ann}(h^n) = 0$ for all n. Thus $f : A \to B$ and $g : A \to B$ are equal, so the morphisms $f, g : \text{Spec } B \to \text{Spec } A$ are equal.

   a) Consider the case when $X = Y = \text{Spec } k[x, y]/(x^2, xy)$, the affine line with nilpotents at the origin, and consider the two morphisms $f, g : X \to Y$, one the identity and the other defined by $x \mapsto 0$, ie killing the nilpotents at the origin. These agree on the complement of the origin, which is a dense open subset, but the sheaf morphism disagrees at the origin.

   b) Consider the affine line with two origins. Let f and g be the two open inclusions of the regular affine line. They agree on the complement of the origin, but send the origin two different places.

3. Consider the commutative diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

   Since X is separated over S, Δ is a closed immersions. Closed immersions are stable under base extensions (ex II.3.11(a)) and so $U \cap V \to U \times_S V$ is a closed immersion. But $U \times_S V$ is affine since all of U, V, S are. So $U \cap V \to U \times_S V$ is a closed immersion into an affine scheme and so $U \cap V$ is affine (ex II.3.11(b)).

   For an example when X is not separated, consider the affine plane with two origins and the two copies U, V of the usual affine plane inside it as open affines. Then $U \cap V$ is $\mathbb{A}^2 - \{0\}$ which is not affine (ex I.3.6).

4. [BLOG] Since $Z \to S$ is proper and $Y \to S$ is separated, by Cor. II.4.8e, $Z \to Y$ is proper. Proper morphisms are closed by definition and so $f(Z)$ is closed in Y.

   Now show that $f(Z)$ is proper over S:

Finite type : This follows from it being a closed subscheme of a scheme Y of finite type over S. (ex II.3.13(a)

Separated : This follows from the change of base square and the fact that closed immersions are preserved under base extensions

$$\begin{tikzcd} f(Z) \arrow[r] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"'] \\ f(Z) \times_S f(Z) \arrow[r] & Y \times_S Y \end{tikzcd}$$

Universally closed: Let $T \to S$ be some other morphism and consider the following diagram:

$$\begin{tikzcd} T \times_S Z \arrow[r] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ T \times_S f(Z) \arrow[r] \arrow[d, "s'"] & f(Z) \arrow[d, "s"] \\ T \arrow[r] & S \end{tikzcd}$$

Show that $T \times_S Z \to T \times_S f(Z)$ is surjective: Suppose $x \in T \times_S f(Z)$ is a point with residue field $k(x)$. Following it horizontally we obtain a point $x' \in f(Z)$ with residue field $k(x') \subset k(x)$ and this lifts to a point $x'' \in Z$ with residue field $k(x'') \supset k(x')$. Let k be a field containing both $k(x)$ and $k(x'')$. The inclusions $k(x''), k(x) \subset k$ give morphisms Spec $k \to T \times_S f(Z)$ and Spec $k \to Z$ which agree on $f(Z)$ and therefore lift to a morphism Spec $k \to T \times_S Z$, giving a point in the preimage of x. So $T \times_S Z \to T \times_S f(Z)$ is surjective.

Now suppose that $W \subseteq T \times_S f(Z)$ is a closed subset of $T \times_S f(Z)$. Its vertical preimage $(f')^{-1}W$ is a closed subset of $T \times_S Z$ and since $Z \to S$ is universally closed, the image $s' \circ f'((f')^{-1}(W))$ in T is closed. As f' is surjective, $f'((f')^{-1}(W)) = W$ and so $s' \circ f'((f')^{-1}(W)) = s'(W)$. Hence, $s'(W)$ is closed in T.

5. [BLOG] Let X be an integral scheme of finite type over a field k, having a function field K.

   (a) Let R be the valuation ring of a valuation on K. Having center on some point $x \in X$ is equivalent to an inclusion $\mathcal{O}_{x,X} \subseteq R \subseteq K$ (such that $\mathfrak{m}_R \cap \mathcal{O}_{x,X} = \mathfrak{m}_x$) which is equivalent to a diagonal morphism in the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur] & \operatorname{Spec} k
\end{tikzcd}$$

But by the valuative criterion for separability, this diagonal morphism (if it exists) is unique. Therefore, the center, it is exists, is unique.

(b) Same argument as in a), except the valuative criterion now tells us that exactly one such diagonal morphism exists, so every valuation of K/k has a unique center.

(c) Ingredients: 2 eggs,1/2 cup milk,3 slices bread, crumbled, 2 pounds lean ground beef,1/2 cup finely chopped onion,2 tablespoons chopped parsley,1 clove garlic, smashed, minced,1 teaspoon salt,1/2 teaspoon pepper Preparation: In a medium bowl, beat eggs lightly; add milk and bread and let stand for about 5 minutes. Add ground beef, onion, parsley, garlic, salt, and pepper; mix gently until well blended. Shape into about 24 meatballs, about 1 1/2 inches in diameter. Place meatballs in a generously greased large shallow baking pan. Bake meatballs at 450 for 25 minutes. In a Dutch oven, in hot oil over medium heat, saut onion until tender and just begins to turn golden. Add remaining sauce ingredients; bring to a boil. Reduce heat, cover, and simmer for 30 minutes. Taste and adjust seasoning, adding more salt, if necessary. Add meatballs; cover and simmer 50 to 60 minutes longer, stirring from time to time. Cook spaghetti according to package directions; drain. Serve spaghetti topped with meatballs in sauce; sprinkle with grated Parmesan cheese.

(d) Suppose that there is some $a \in \Gamma(X, \mathcal{O}_X)$ such that $a \notin k$. Consider the image $a \in K$. Since k is algebraically closed, a is transcendental over k and so $k[a^{-1}]$ is a polynomial ring. Consider the localization $k[a^{-1}]_{(a^{-1})}$. This is a local ring contained in K and therefore there is a valuation ring for $R \subset K$ that dominates it. Since $\mathfrak{m}_R \cap k[a^{-1}]_{(a^{-1})} = (a^{-1})$ we see that $a^{-1} \in \mathfrak{m}_R$.

Now since X is proper, there exists a unique dashed morphism in the diagram on the left:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k
\end{tikzcd}
\qquad
\begin{tikzcd}
K & \Gamma(X, \mathcal{O}_X) \arrow[l] \arrow[dl, dashed] \\
R \arrow[u] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

Taking global sections gives the diagram on the right which implies that $a \in R$ and so $v_R(a) \geq 0$. But $a^{-1} \in \mathfrak{m}_R$ and so $v_R(a^{-1}) > 0$. This gives a contradiction since $0 = v_R(1) = v_R(\frac{a}{a}) = v_R(a) + v_R(\frac{1}{a}) > 0$.

6. Since X and Y are affine varieties, say Spec A and Spec B respectively, by definition they are integral and so $f : X \to Y$ comes from the ring homomorphism $B \to A$, where A and B are integral domains. Let $K = k(A)$. Then for the valuation ring R of K that contains $\varphi(B)$ we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since f is proper, the dashed arrow exists. From Thm II.4.11A, the integral closure of $\varphi(B)$ in K is the intersection of all valuation rings of K which contain $\varphi(B)$. As the dashed morphism exists for any valuation ring K containing $\varphi(B)$, it follows that A is contained in the integral closure of $\varphi(B)$ in K. Hence every element of A is integral over B, and this together with the hypothesis that f is of finite type implies that f is finite.

7. [BLOG] *Schemes over* $\mathbb{R}$.

   (a)

   (b) Since $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$ if X_0 is affine then certainly X is. Conversely, if $X = \operatorname{Spec} A$ is affine, then $X_0 = \operatorname{Spec} A^\sigma$

   (c) Given f_0, we get that f commutes with the involution. Conversely, suppose that we are given f that commutes with σ. In the case where $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, we get an induced morphism on σ invariants $A^\sigma \to B^\sigma$ and this gives us the morphism $X_0 \to Y_0$. If X and Y are not affine then take a cover of X by σ preserved open affines $\{U_i\}$ and for each i take a cover $\{V_{ij}\}$ of $f^{-1}U_i$ with each V_{ij} a σ-preserved open affine of Y. Let $\pi : Y \to Y_0$ be the projection and recall that it is affine by part b). In the affine case we get $\pi(V_{ij}) \to \pi(U_i)$ and we can glue these together to give a morphism $Y_0 \to X_0$.

   (d) See Case II of part (e)

   (e) *Case I: σ has no fixed points:* Let $x \in X \cong \mathbb{P}^1_{\mathbb{C}}$ be a closed point and consider the space $U = X \backslash \{x, \sigma x\}$. Since σ has no fixed points, and $PGL_{\mathbb{C}}(1)$ is transitive on pairs of distinct points, we can find a $\mathbb{C}$-automorphism f that sends $(x, \sigma x)$ to $(0, \infty)$. Therefore assume that x and σx are 0 and ∞ and so $U \cong \operatorname{Spec} \mathbb{C}[t, t^{-1}]$. Note that the lift of σ is still $\mathbb{C}$ semi-linear by the commutativity of the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X \arrow[r, "\sigma"] \arrow[d] & X \arrow[r, "f^{-1}"] \arrow[d] & X \arrow[d] \\ \mathbb{C} \arrow[r, "id"] & \mathbb{C} \arrow[r, "\alpha"] & \mathbb{C} \arrow[r, "id"] & \mathbb{C} \end{tikzcd}$$

Now σ induces an invertible semi-linear $\mathbb{C}$-algebra homomorphism on $C[t, t^{-1}]$. We will show that σ acts via $t \mapsto -t^{-1}$. The element t must get sent to something invertible and therefore gets sent to something of the form at^k for some $k \in \mathbb{Z}$. Since $\sigma^2 = id$, it follows that $k = \pm 1$. Furthermore, by considering σ on the function field $\mathbb{C}(t)$, it can be seen that $k = -1$ since otherwise the valuation ring $\mathbb{C}[t]_{(t)} \subset \mathbb{C}(t)$ would be fixed, implying that σ has a fixed point. Now $t\sigma t = a$ is fixed by σ and σ acts by conjugation on constants, thus $a \in \mathbb{R}$. If a is positive, the ideal $(t - \sqrt{a})$ is preserved contradicting the assumption of no fixed points, so $a \in \mathbb{R}_{\leq 0}$. Now replacing t with $\dfrac{1}{\sqrt{-a}}$ by a change of coordinates. With this new t, our involution is $t \mapsto -t^{-1}$.

Now rewrite $C[t, t^{-1}]$ as $\dfrac{\mathbb{C}[\frac{X}{Z}, \frac{Y}{Z}]}{(1 + \frac{XY}{Z^2})}$ via $\dfrac{X}{Z} \mapsto t^{-1}$ and $\dfrac{Y}{Z} \mapsto -t$, so the involution acts by switching $\frac{X}{Z}$ and $\frac{Y}{Z}$ (and conjugation on scalars). Now consider the two subrings $\mathbb{C}[-t]$ and $\mathbb{C}[t^{-1}]$ of the function field $\mathbb{C}(t)$. We have isomorphisms

$$\begin{array}{lr} \frac{\mathbb{C}[\frac{Y}{X}, \frac{Z}{X}]}{(\frac{Y}{X} + (\frac{Z}{X})^2)} \cong \mathbb{C}[-t] & t = \frac{Z}{X} \\ \frac{\mathbb{C}[\frac{X}{Y}, \frac{Z}{Y}]}{(\frac{X}{Y} + (\frac{Z}{Y})^2)} \cong \mathbb{C}[t^{-1}] & -t^{-1} = \frac{Z}{Y} \end{array}$$

and σ acts by switching these two rings and conjugation on scalars. These open affines patch together in a way compatible with σ to form an isomorphism

$$\mathrm{Proj}\ \frac{\mathbb{C}[X, Y, Z]}{(XY + Z^2)} \cong \mathbb{P}^1_{\mathbb{C}}$$

where σ acts on the quadric by switching X and Y, and conjugation on scalars. Making a last change of coordinates $U = \frac{1}{2}(X + Y)$ and $V = \frac{i}{2}(Y - X)$, we finally get the isomorphism

$$\mathcal{Q} := \mathrm{Proj}\ \frac{\mathbb{C}[X, Y, Z]}{(U^2 + V^2 + Z^2)} \cong \mathrm{Proj}\ \frac{C[X, Y, Z]}{(XY + z^2)} \cong \mathbb{P}^1_{\mathbb{C}} = X$$

where σ acts on $\mathcal{Q}$ by conjugation of scalars alone. Hence

$$X_0 \cong \mathcal{Q}_0 = \mathrm{Proj}\ \frac{\mathbb{R}[X, Y, Z]}{(U^2 + V^2 + Z^2)}$$

Case II: σ has at least one fixed point: Now suppose that σ fixes a closed point of x. This means that σ restricts to a semi-linear automorphism of the complement of the fixed point Spec $\mathbb{C}[t] \subset \mathbb{P}^1_{\mathbb{C}}$. Since σ is invertible, t gets sent to something of the form $at + b$. There exists a change of coordinates $s = ct + d$ such that $\sigma s = s$ and so in these new coordinates we get a σ invariant isomorphism $X \cong \mathbb{P}^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

8. [BLOG] Let $\mathcal{P}$ be a property of a morphism of schemes such that:

(a) a closed immersion has $\mathcal{P}$;

(b) a composition of two morphism having $\mathcal{P}$ has $\mathcal{P}$;

(c) $\mathcal{P}$ is stable under base extension.

(d) Let $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ be the morphisms. The morphism $f \times f'$ is a composition of base changes of f and f' as follows:

$$\begin{tikzcd}
 & X \arrow[dd] \\
X \times X' \arrow[ur] \arrow[dd] & \\
 & Y \\
Y \times X' \arrow[ur] \arrow[dr] \arrow[dd] & \\
 & X' \arrow[dd] \\
Y \times Y' \arrow[dr] & \\
 & Y'
\end{tikzcd}$$

Therefore $f \times f'$ has property $\mathcal{P}$.

(e) Same argument as above but note that since g is separated, the diagonal morphism $Y \to Y \times_Z Y$ is a closed embedding and therefore satisfies $\mathcal{P}$

$$\begin{tikzcd}
& Y \arrow[dd] \\
X \arrow[ur, "f"] \arrow[dd] & \\
& Y \times_Z Y \\
X \times_Z Y \arrow[ur] \arrow[dr] \arrow[dd] & \\
& X \arrow[dd] \\
Y \arrow[dr, "g"] & \\
& Z
\end{tikzcd}$$

(f) Consider the factorization:

$$\begin{tikzcd}
X_{red} \arrow[dr, "\Gamma_{f_{red}}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "f_{red}"'] & & \\
& Y_{red} \times_Y X_{red} \arrow[r] \arrow[d] & X_{red} \arrow[d] \\
& Y_{red} \arrow[r] & Y
\end{tikzcd}$$

The morphism $X_{red} \to X \to Y$ is a composition of a closed immersion and a morphism with the property $\mathcal{P}$ and therefore it has property $\mathcal{P}$. Therefore the vertical morphism from the fiber product is a base change of a morphism with property $\mathcal{P}$ and therefore by assumption has property $\mathcal{P}$. To see that f_{red} has property $\mathcal{P}$, it remains only to see that the graph $\Gamma_{f_{red}}$ has property $\mathcal{P}$. For then f_{red} will be a composition of morphisms with property $\mathcal{P}$. To see this recall that the graph is the following base change:

$$\begin{tikzcd}
X_{red} \arrow[r] \arrow[d, "\Gamma"] & Y_{red} \arrow[d, "\Delta"] \\
X_{red} \times_Y Y_{red} \arrow[r] & Y_{red} \times_Y Y_{red}
\end{tikzcd}$$

But $Y_{red} \times_Y Y_{red} = Y_{red}$ and $\Delta = id_{Y_{red}}$. So Δ is a closed immersion and Γ is a base change of a morphism with property $\mathcal{P}$.

9. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two projective morphisms. This gives rise to a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[dr, "f"'] & \mathbb{P}^r \times Y \arrow[r, "id\times g'"] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s \times Z \arrow[d] \\ & Y \arrow[r, "g'"] \arrow[dr, "g"'] & \mathbb{P}^s \times Z \arrow[d] \\ & & Z \end{tikzcd}$$

where f' and g', and therefore $id \times g'$ are closed immersions. Now using the Segre embedding, the projection $\mathbb{P}^r \times \mathbb{P}^s \times Z \to Z$ factors as

$$\mathbb{P}^r \times \mathbb{P}^s \times Z \to \mathbb{P}^{rs+r+s} \times Z \to Z$$

So since the Segre embedding is a closed immersion, we are done since we have a closed immersion $X \to \mathbb{P}^{rs+r+s}$ which factors $g \circ f$.

10. See Shaf, Bk 2, pg 69 for the statement about complete varieties.

11. (a)

 (b) The only reason we needed to consider arbitrary valuation rings was because Thm 6.1A only gives us that some valuation ring dominates the local ring of η_0 on $\overline{\{\eta_i\}}$ (see pg 99). But now by part a), we are allowed to consider only discrete valuation rings.

12. (a) Let $R \subset K$ be a valuation ring of K. We will show that $\mathfrak{m}_R$ is principal, which will imply that R is discrete. Let $t \in \mathfrak{m}_R$. If $(t) = \mathfrak{m}_R$, then we are done. If not, choose some $s \in \mathfrak{m}_R \backslash (t)$. Note that t is transcendental over k. To see this, suppose that it satisfies some polynomial $\sum_{i=0}^n a_i t^i = 0$ with $a_0 \neq 0$. Then $a_0 = t \sum a_i t^{i-1}$ and so $a_0 \in (t)$. But a_0 is a unit so we get a contradiction, hence there is no such polynomial. Now since K has dimension 1 and t is transcendental, K is a finite algebraic extension of $k(t)$. The element $s \notin (t)$ and so it is algebraic over k. Hence it satisfies some polynomial with coefficients in $k(t)$. Let $\sum_{i=0}^n a_i s^i = 0$, with $a_0 \neq 0$ be this polynomial. Then $a_0 = s \sum a_i s^{i-1}$. Write $a_0 = \frac{f(t)}{g(t)}$. Then we have $\frac{f(t)}{g(t)} = s \sum a_i s^{i-1}$ and so $f(t) = g(t) s \sum a_i s^{i-1}$ implying that $f(t) \in (s) \subseteq \mathfrak{m}_R$. Since $t \in \mathfrak{m}_R$, the polynomial $f(t)$ can not have any constant term, else this term would be in $\mathfrak{m}_R$ contradicting the fact that it is a proper ideal) and so $t \in (s)$ and hence $(s) \supset (t)$. If $(s) = \mathfrak{m}_R$ we are done, so assume not. Repeat the above process to obtain an increasing chain of ideals $(t) \subset (s) \subset (s_1) \subset \ldots$ all contained in $\mathfrak{m}_R$. Since R is noetherian, this chain must stabilize and so there is some s_i such that $(s_i) = \mathfrak{m}_R$. Hence $\mathfrak{m}_R$ is principal and therefore by Thm 1.6.2A, the valuation ring R is discrete.

(b) i. Consider an affine neighborhood Spec A of X. Let x_1 correspond to the prime ideal $\mathfrak{p} \subseteq A$ of height 1. Then $\mathcal{O}_{X,x_1} \cong A_{\mathfrak{p}}$, which is a Noetherian local domain of dimension 1. X is nonsingular so A is integrally closed and thus so is $A_{\mathfrak{p}}$. By Thm 1.6.2A, $A_{\mathfrak{p}}$ is a DVR. $R = \mathcal{O}_{X,x_1}$ clearly has center x_1.

ii. Assume X' is nonsingular. Then by the previous part, R is a DVR. f induces an inclusion $\mathcal{O}_{X,x} \hookrightarrow R$, so R dominates $\mathcal{O}_{X,x_0}$

iii. R is clearly a valuation ring which dominates $\mathcal{O}_{X,x_1}$.

2.5 Sheaves of Modules

1. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. Define $\mathscr{E}^*$ to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$

(a) We can cover X with open sets U_α with $\mathscr{E}|_{U_\alpha}$ free of rank n_α. First consider $X = U_\alpha$. An element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X^n)(X)$ is determined by where it takes the standard basis elements in $\mathcal{O}_X^n(X)$, and similarly for any subset U of X. So $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X^n)(X) \cong \mathcal{O}_X^n$. Taking the dual is equivalent to applying Hom again, which is again isomorphic to $\mathcal{O}_X^n$. But the isomorphism with the double dual is canonical, so we can patch these isomorphisms on each U_α together to get an isomorphism $\mathscr{E}^{**} \cong \mathscr{E}$

(b) Define a map on any open set U where $\mathscr{E}$ is free: $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}|_U, \mathcal{O}_X|_U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}|_U, \mathscr{F}|_U)$ by taking $\sum \check{e}_i \otimes a_i$ to the map sending $\check{e}_i$ to a_i from $\mathscr{E}|_U(U)$ to $\mathscr{F}|_U(U)$. This determines the whole morphism. It is injective and surjective, so thus an isomorphism. Now glue all the maps and take the sheafification to get the desired isomorphism.

(c) This follows immediately from the sheafification of the module isomorphism (AM p 28): $\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}\ (M, \mathscr{H}om(N, P))$, making the obvious module substitutions.

(d) If $\mathscr{E}$ is free of finite rank, write $\mathscr{E} \cong \mathcal{O}_Y^n$. Then

$$\begin{aligned} f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y^n)) \\ &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \\ &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X)^n \\ &\cong f_*(\mathscr{F})^n \\ &\cong f_*(\mathscr{F}) \otimes \mathcal{O}_Y^n \\ &\cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E} \end{aligned}$$

If $\mathscr{E}$ is locally free, then do the same argument as above on an open cover $\{U_i\}$ and glue on intersections.

2. Let $(R, \mathfrak{m})$ be a DVR and $K = R_0$ its field of fractions. Let $X = \mathrm{Spec}\ R$.

(a) $X = \{0, \mathfrak{m}\}$ and the nontrivial open sets of X are X, and $\{0\}$. Now $\mathcal{O}_X(X) \cong R$ and $\mathcal{O}_X(\{0\}) \cong K$, so to give an $\mathcal{O}_X$ module $\mathscr{F}$, it is equivalent to give an R-module M and a K-module L. The restriction map $f : \mathcal{O}_X(X) \to \mathcal{O}_X(\{0\})$, (equivalently $f : M \to L$) is an R-module homomorphism. We can then define an R-linear map $\rho : M \otimes_R K \to L$ such that $\rho(m \otimes k) = kf(m)$. Conversely, given ρ, define $f : M \to L$ by $f(m) = \rho(m \otimes 1)$. Them $f(rm) = \rho(rm \otimes 1) = r\rho(m \otimes 1) = rf(m)$.

(b) Let $\mathscr{F}$ be the $\mathcal{O}_X$ module. Since $K \cong R_0, M \otimes_R K \cong M \otimes_R R_0 \cong M_0$. $\mathscr{F}$ is quasicoherent iff $\mathscr{F} = \widetilde{M}$ iff $L \cong M_0$ iff $L \cong M \otimes_R K$ iff ρ is an isomorphism.

3. Let $X = \operatorname{Spec} A$ be an affine scheme. If $f : \widetilde{M} \to \mathscr{F}$ is a homomorphism, then we get a "global section" homomorphism $f(X) : \widetilde{M}(X) \to \mathscr{F}(X)$, which is equivalent to $f(X) : M \to \Gamma(X, \mathscr{F})$. Conversely, if we are given a map $f : M \to \Gamma(X, \mathscr{F})$, define a map $f^{\#}$ locally on $\mathcal{D}(f)$, $f^{\#}|_{\mathcal{D}(f)}(\frac{m}{g}) \mapsto \frac{f(m)}{g}$. Then globally, $f^{\#}_X = f$, so the map $f \mapsto f^{\#}$ is injective. However, if $f^{\#}$ induces f it is also clear that f induces $f^{\#}$, so $f \mapsto f^{\#}$ is surjective. Thus $\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$.

4. Let X be a scheme and $\mathscr{F}$ an $\mathcal{O}_X$-module. Assume that $\mathscr{F}$ is quasicoherent. Then for every open neighborhood U, $\mathscr{F}|_U \cong \widetilde{M}$. If $\{m_i\}_{i \in I}$ is a set of generators of M, then the A-homomorphism $A^I \xrightarrow{\phi} M$ defined by $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i M_i$ is surjective. Constructing a free A-module A^J similarly with $\ker \phi$, we have an exact sequence

$$A^J \to A^I \twoheadrightarrow M \to 0$$

Since $\tilde{}$ is an exact functor, we have

$$\widetilde{A^J} \to \widetilde{A^I} \twoheadrightarrow \widetilde{M} \to 0$$

and thus $\widetilde{M} \cong \mathscr{F}|_U$ is the cokernel of free sheaves on U.

Conversely, let $\mathscr{F}$ be a sheaf such that for every neighborhood U $\mathscr{F}$ is the cokernel of a morphism of free sheaves on U. Then we have the exact sequence

$$\mathscr{F}' \to \mathscr{F}'' \twoheadrightarrow \mathscr{F}|_U \to 0$$

Then since $\mathscr{F}$ and $\mathscr{F}''$ are free and thus quasicoherent, by Prop 5.7, $\mathscr{F}|_U$ is quasicoherent as well.

The proof for X noetherian is similar and uses the fact that a submodule of a finite module over a Noetherian ring is finite.

5. Let $f : X \to Y$ be a morphism of schemes.

(a) Let $f : \mathbb{A}^2_k \to A^1_k$ be the projection to the x-axis. Then $\Gamma(\mathbb{A}^2_k, f_*\mathcal{O}_{\mathbb{A}^2}) = k[x, y]$, which is not a finite $k[x]$-module, so $f_*\mathcal{O}_{A^2}$ is not coherent.

(b) Let $f : X \to Y$ be a closed immersion. Let $X = \bigcup U_i$ be an affine open cover of X, where $U_i = \text{Spec } A_i$. Then $f : f^{-1}(U_i) \to U_i$ is a closed immersion. By Ex II.3.11(b) these are of the form $\text{Spec}(A_i/I_i) \to \text{Spec } A_i$ for some ideal I_i. Since A_i/I_i is a finite A_i-module, f is finite.

(c) Let $f : X \to Y$ be a finite morphism of noetherian schemes and let $\mathscr{F}$ be coherent on X. Pick an affine open cover for $X = \bigcup$ Spec A_i. It is enough to show this locally by restricting f to one of these covers. We get a map f: Spec $B \to$ Spec A, where B is a finite A-module and $\mathscr{F}|_{\text{Spec } B} = \widetilde{M}$ for some A-module M. Then $f_*\mathscr{F}$(Spec $A) \cong B \otimes_A M$ is just the extension of scalars. Since both B and M are finite A-modules, so is their tensor product. Thus $f_*(\mathscr{F})|_{\text{Spec B}}$ is coherent.

6. (a) Let A be a ring, M an A-module. Let $X = \text{Spec } A$, and $\mathscr{F} = \widetilde{M}$. Let $\mathfrak{p} \in V(\text{Ann } m)$. Then $\mathfrak{p} \supseteq \text{Ann } m$, so localizing at $\mathfrak{p}$ means everything in Ann m is localized as well. So $m_\mathfrak{p} \neq 0$ and thus $\mathfrak{p} \in$ Supp m. Conversely, let $\mathfrak{p} \in$ Supp m. Then $m_\mathfrak{p} \neq 0$ is equivalent to $am \neq 0$ for $a \notin \mathfrak{p}$. Thus $a \notin$ Ann m, so Ann m must have been localized as well. So Ann $m \subseteq \mathfrak{p}$, which is equivalent to $\mathfrak{p} \in V$(Ann m). Thus $V(\text{Ann } m) = \text{Supp } m$.

(b) Let A be noetherian and M finitely generated, say $M = Am_1 + \ldots + Am_n$. Then Ann $M = \bigcap$ Ann m_i. Also, Supp $\mathscr{F}$ = Supp $\widetilde{M}$ = $\{\mathfrak{p} \in \text{Spec } A \mid M_\mathfrak{p} \neq 0\}$. Since $M_\mathfrak{p}$ is generated by the images of the generators m_i, $M_\mathfrak{p} \neq 0$ iff some $m_i \neq 0$ in $M_\mathfrak{p}$ iff some $\text{Ann}(m_i) \subseteq \mathfrak{p}$ iff Ann $M = \bigcap$ Ann $m_i \subseteq \mathfrak{p}$ iff $\mathfrak{p} \in V(\text{Ann } M)$.

(c) The support of a coherent sheaf is locally closed by part b), so is closed on all of X. (Closed is a local property)

(d) Let $U = X - Z$ and $j : U \hookrightarrow X$ be the inclusion. Let $U = V(\mathfrak{a})^c$. From I.1.20(b), we get an exact sequence

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*\mathscr{F}$$

By prop I.5.8(c), $j_*\mathscr{F}$ is quasi-coherent, and since the sheaf $\mathscr{H}^0_Z(\mathscr{F})$ is the kernel of quasi-coherent sheaves, $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. Then $\Gamma_\mathfrak{a}(M)^\sim \cong \mathscr{H}^0_Z(\mathscr{F})$ iff $\Gamma_\mathfrak{a}(M) \cong \Gamma_Z(\mathscr{F})$. $m \in \Gamma_Z(\mathscr{F})$ iff Supp $m \subset V(\mathfrak{a})$ iff $V(\text{Ann } m) \subseteq V(\mathfrak{a})$ iff $\sqrt{\mathfrak{a}} \subseteq \sqrt{\text{Ann } m}$ iff $\mathfrak{a}^n \subseteq$ Ann m (by Noetherian assumption) iff $m \in \Gamma_\mathfrak{a}(m)$. Thus $\Gamma_\mathfrak{a}(M)^\sim \cong \mathscr{H}^0_Z(\mathscr{F})$ as desired.

(e) Let X be noetherian and Z be closed. The question is local so we may assume $X = \text{Spec } A$ and $Z = V(\mathfrak{a})$ and $\mathscr{F} = \widetilde{M}$. By the argument of (d), $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent, and if M is finite, so is $\Gamma_\mathfrak{a}(M)$. So $\mathscr{H}^0_Z(\mathscr{F})$ is coherent if $\mathscr{F}$ is.

7. Let X be a noetherian scheme and let $\mathscr{F}$ be a coherent sheaf.

(a) Let $X = \text{Spec } A$ and let $\mathscr{F} = \widetilde{M}$. Then M is a finite A-module, generated by $m_1, \ldots, m_n$. The stalk for $x \in X$ is $\mathscr{F}_x \cong M_\mathfrak{p}$ for some $\mathfrak{p} \in \text{Spec } A$. Then $M_\mathfrak{p} \cong A_\mathfrak{p} x_1 + \ldots + A_\mathfrak{p} x_n$, where the x_i can be taken to be sections on some principal open set $\mathcal{D}(f)$. In $M_\mathfrak{p}$, let the images of each generator m_i be $\frac{a_{i,1}}{g_{i,1}} x_1 + \ldots + \frac{a_{i,n}}{g_{i,n}} x_n$. Set $g = \prod_{i,j} g_{i,j}$. Then the m_i are in the span of the x_i in the open set $\mathcal{D}(fg)$. Set $h = fg$. Then $M_h = A_h x_1 + \ldots + A_h x_n$. The x_i are linearly independent in M_h since they are linearly independent in $M_\mathfrak{p}$, so the sum is in fact a direct sum. Thus $\mathscr{F}|_{\mathcal{D}(h)} \cong \widetilde{M_h}$ is free.

(b) Let $\mathscr{F}$ be locally free. Then by definition, the stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$. The converse follows immediately by part a).

(c) If $\mathscr{F}$ is invertible, then by part 1.b, $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{F}^* \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F}) \cong \mathcal{O}_X$.

Conversely, suppose there exists a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$. It is enough to show this statement locally. Pick a point $x \in X$ in an open affine neighborhood $U = \text{Spec } A$ such that $\mathscr{F}|_U \cong \widetilde{M}$ and $\mathscr{G}|_U \cong \widetilde{N}$ for finite A-modules M and N. By assumption, $\mathscr{F}_x \otimes_{\mathcal{O}_{x,X}} \mathscr{G}_x \cong \mathcal{O}_{x,X}$. Let x correspond to the prime ideal $\mathfrak{p}$ in A. Then the assumption is equivalent to $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \cong A_\mathfrak{p}$. By assumptions, the following isomorphisms hold:

$$\begin{aligned} k(x) &\cong k(x) \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X} \\ &\cong k(x) \otimes_{\mathcal{O}_{x,X}} \mathscr{F}_x \otimes_{\mathcal{O}_{x,X}} \mathscr{G}_x \\ &\cong k(x) \otimes_{\mathcal{O}_{x,X}} \mathscr{F}_x \otimes_{\mathcal{O}_{x,X}} \mathscr{G}_x \otimes_{k(x)} k(x) \\ &\cong (\mathscr{F}_x \otimes_{\mathcal{O}_{x,X}} k(x)) \otimes_{k(x)} (\mathscr{G}_x \otimes_{\mathcal{O}_{x,X}} k(x)) \end{aligned}$$

These are equivalent to:

$$\begin{aligned} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p} \\ &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} (M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}) \\ &\cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} (M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}) \otimes_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \\ &\cong (M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \otimes_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} (N_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \end{aligned}.$$

In particular, $(\mathscr{F}_x \otimes_{O_{x,X}} k(x))$ and $(\mathscr{G}_x \otimes_{O_{x,X}} k(x))$ are 1 dimensional $k(x)$-vector spaces. Equivalently, $(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$ and $(N_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$ are 1-dimension free $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-modules. Since M and N are finite, by Nakayama's lemma, the generator of $(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$ lifts to a generator m of $M_\mathfrak{p}$. Similarly, let n be the generator of $N_\mathfrak{p}$. Then $n \otimes m$ generates $M_\mathfrak{p} \otimes N_\mathfrak{p}$, which by assumption is isomorphic to $A_\mathfrak{p}$. Then the map $\varphi : A_\mathfrak{p} \to M_\mathfrak{p}$ defined by $\frac{a}{b} \mapsto \frac{a}{b} m$, and the map $\varphi^{-1} : M_\mathfrak{p} \to M_\mathfrak{p} \otimes N_\mathfrak{p} \cong A_\mathfrak{p}$ defined by $\frac{am'}{b} \mapsto \frac{m'}{s} \otimes n \mapsto \frac{a}{b}$ are easily checked to be inverses. Thus $M_\mathfrak{p} \cong A_\mathfrak{p}$ and thus $\mathscr{F}_x \cong \mathcal{O}_{x,X}$. So $\mathscr{F}$ is invertible as desired.

8. Let X be a noetherian scheme and let $\mathscr{F}$ be a coherent sheaf on X. Let $\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$, where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x.

   (a) To show that the set $S := \{x \in X \mid \varphi(x) \geq n\}$ is closed, we will show that its compliment $S^c = \{x \in X \mid \varphi(x) < n\}$ is open. Since these are all local properties, we can assume that $X = \text{Spec } A$ is affine, $\mathscr{F} \cong \widetilde{N}$ for some finite A-module N, generated by $n_1, \ldots, n_r$. Let $\mathfrak{p}$ be the prime ideal corresponding to $x \in X$. Let $n \in \mathbb{Z}$. Then $\varphi(\mathfrak{p})$ $=\dim_{k(\mathfrak{p})} N_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} = \dim_{k(\mathfrak{p})} N_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} N_\mathfrak{p}$. By Nakayama's Lemma, this number is equal to the minimal number of generators of $N_\mathfrak{p}$ as an $A_\mathfrak{p}$ module. Let $N_\mathfrak{p}$ be minimally generated by $m_1, \ldots, m_r$, with $r < n$. We argue the same way as in $7(a)$. In $N_\mathfrak{p}$, write $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. Define $s := \prod s_{ij}$. Then $sn_j = \sum b_{ij} m_j$, where $b_{ij} \notin \mathfrak{p}$. Therefore $s \notin \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{D}(s)$. For an arbitrary prime ideal $\mathfrak{q} \in \mathcal{D}(s)$, it is easy to see that $N_\mathfrak{q}$ is generated by $n_1, \ldots, n_r$, so $\mathfrak{q} \in S^c$. Thus $\mathcal{D}(s) \subseteq S^c$. So every point in S^c has an open neighborhood contained in S^c. Since S^c is a union of open sets, it is open and thus S is closed.

   (b) Since X is connected, the rank of $\mathscr{F}$ is the same everywhere, say n. Then for all $x \in X, \mathscr{F}_x \cong \mathcal{O}_x^{\oplus n}$. Thus $\varphi(x) = \dim_{k(x)} \mathcal{O}_x^{\oplus n} \otimes_{\mathcal{O}_x} k(x) = \dim_{k(x)} k(x)^{\oplus n} = n$. So φ is constant.

   (c) Since the criterion is local, we can let $X = \text{Spec } A$ and $\mathscr{F} \cong \widetilde{M}$ with M a finite A-module, with A reduced. Since the nilradical commutes with localization, $\eta(A_\mathfrak{p}) = \eta(A_f) = \eta(A) = 0$ for all $f \in A$ and $\mathfrak{p} \in \text{Spec } A$. Choose $\mathfrak{p} \in X$. As in the previous parts, use Nakayama's lemma to lift a basis for the $k(\mathfrak{p})-$vector space $M_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} M_\mathfrak{p}$ to a set of generators $m_1, \ldots, m_n \in M_\mathfrak{p}$. By 5.7(b), it is enough to show that $\mathscr{F}_x \cong M_\mathfrak{p}$ is a free $\mathcal{O}_x \cong A_\mathfrak{p}$-module. To show this, it is enough to show that the m_i are linearly independent. Suppose $\sum \frac{a_i}{b_i} m_i = 0$ with $a_i \in A, b_i \notin \mathfrak{p}$ Set $b = \prod b_i$ and clear denominators so that $\sum a'_i m_i = 0$. Since the images of the m_i are a basis for $M_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} M_\mathfrak{p}$ over $A_\mathfrak{p}/\mathfrak{p} A_\mathfrak{p}$, each $a'_i \in \mathfrak{p}$ for all i. Choose $e \in A$ such that if $\mathfrak{q} \in \mathcal{D}(e)$, then $M_\mathfrak{q}/\mathfrak{m}_\mathfrak{q} M_\mathfrak{q}$ is generated by the images of the m_i. Now let $f = aeb$. From our choice of e, if $\mathfrak{q} \in \mathcal{D}(f)$, then the images of the m_i in $M_\mathfrak{q}/\mathfrak{m}_\mathfrak{q} M_\mathfrak{q}$ are generators. Since φ is locally constant, their images are in fact basis. In particular, they are linearly independent. Then $\sum \frac{a_i}{b_i} m_i = 0$ holds in $M_\mathfrak{q}$, and thus a_i is in the intersection of all prime ideals not containing f. This is just the nilradical of A_f, which is 0 by assumption. Thus $a_i = 0$ and $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for all $x \in X$ and by 5.7(b), $\mathscr{F}$ is locally free.

9. Let S be a graded ring, generated by S_1 as an S_0-algebra. Let M be a graded S-module. Let $X = \text{Proj } S$.

   (a) $\Gamma_*(\widetilde{M}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \widetilde{M}(d)) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \widetilde{M(d)})$. Any $m \in M_d$ can be thought of as a section of $\widetilde{M(d)}$. Viewing m_d as a section, we can define $\alpha = \bigoplus_{d\in\mathbb{Z}} \alpha_d : M_d \to \Gamma(X, \widetilde{M(d)})$ which is a homomorphism on abelian groups. If $s \in S_d, m \in M_{d'}$, then $s\alpha(m)$ is defined as the image of $m \otimes s$ in $\Gamma(X, \widetilde{M(d')} \otimes \mathcal{O}_X(d))$ under the isomorphism $\widetilde{M(d')} \otimes \mathcal{O}_X(d) \cong \widetilde{M(d+d')}$. So $s\alpha(m) = \alpha(sm)$ and thus $\alpha : M \to \Gamma_*(\widetilde{M})$ is a graded-module homomorphism.

(b) Now let $S_0 = A$ be a finitely generated k-algebra for some field k, where S_1 is a finitely generated A-module, and let M be a finitely generated S-module. Let's show $\alpha : M \to \Gamma_*(\widetilde{M})$ is an isomorphism for $d \gg 0$ in the case $M = S$. By (5.19), $S' := \Gamma_*(\widetilde{S}) = Sy_1 + \ldots + Sy_m$ and then there exists some $n > 0$ such that $x_i^n y_j \in S$ for all i, j. So everything in S' of high enough degree will be in S and α is an isomorphism for $d \gg 0$. For the general case...

(c) [BLOG] By part (b), $M \approx \Gamma_*(\widetilde{M})$ if M is finitely generated. By Prop II.5.15, $\widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$ if $\mathscr{F}$ is quasi-coherent. So we have to show that for a quasi-finitely generated graded S-module M, $\widetilde{M}$ is coherent and for a coherent sheaf $\mathscr{F}$ that $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Let M be a quasi-finitely generated graded S-module. Then there is a finitely generated graded S-module M' such that $M_{\geq d} \cong M'_{\geq d}$ for $d \gg 0$. This implies that for every element $f \in S_1$, $M_{(f)} \cong M'_{(f)}$ since $\frac{m}{f^n} = \frac{mf^d}{f^{n+d}}$. Since M' is finitely generated, $M'_{(f)}$ is finitely generated. S is generated by S_1 as an S_0-algebra so open subsets of the form $M_{(f)}$ cover $X = \text{Proj } S$ and so there is a cover of X on which $\widetilde{M}$ is locally equivalent to a coherent sheaf. Hence $\widetilde{M}$ is coherent.

Now consider a coherent $\mathcal{O}_X$-module $\mathscr{F}$. Then by Theorem II.5.17, $\mathscr{F}(n)$ is generated by a finite number of global sections for $n \gg 0$. Let M' be the submodule of $\Gamma_*(\mathscr{F})$ generated by these sections. The inclusion $M' \hookrightarrow \Gamma_*(\mathscr{F})$ induces an inclusion of sheaves $\widetilde{M}" \hookrightarrow \widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$, where the last isomorphism comes from Prop II.5.15. Tensoring with $\mathcal{O}(n)$ we have an inclusion $\widetilde{M(n)}' \hookrightarrow \mathscr{F}(n)$ that is actually an isomorphism since $\mathscr{F}(n)$ is generated by global sections in M'. Tensoring again with $\mathcal{O}(-n)$ we then find that $\widetilde{M'} \cong \mathscr{F}$. Now M' is finitely-generated and so by part (b), $M_d \cong \Gamma(X, \widetilde{M'}(d)) \cong \Gamma(X, \mathscr{F}(d)) = \Gamma_*(\mathscr{F})_d$ for $d \gg 0$. Hence, $M_d \cong \Gamma_*(\mathscr{F})$ for $d \gg 0$ and $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

10. Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \text{Proj } S$.

(a) First show that $\overline{I}$ is in fact an ideal. For any $s, t \in \overline{I}$, by definition there exists an $n, m > 0$ for each i such that $x_i^n s \in I$ and $x_i^m t \in I$. Then $\overline{I}$ is closed under multiplication since $x_i^{n+m} st \in I$. $\overline{I}$ is closed with respect to addition since $x_i^{n+m}(s+t) \in I$. Lastly, $\overline{I}$ is closed under multiplication by S since for any $a \in S$, $ax^n s \in I$. So $\overline{I}$ is an ideal. To show it is homogeneous, we will show that is $s \in \overline{I}$, then each homogeneous component of s is in $\overline{I}$. Write $s \in \overline{I}$ as $s = s_0 + s_1 + \ldots + s_r$, a sum of its homogeneous components. Then there exists some n for each i such that $x_i^n s = x_i^n(s_0 + s_1 + \ldots + s_r) \in I$. Since I is a homogeneous ideal, $x_i^n s_i \in I$ for all j. Thus $s_j \in \overline{I}$ and $\overline{I}$ is a homogeneous ideal.

(b) First show that the closed subschemes determined by I and $\overline{I}$ are the same, ie Proj $(S/I) \cong$ Proj $(S/\overline{I})$. As $I \subset \overline{I}$, we know that $V(\overline{I}) \subset V(I)$. Conversely, if $P = (x_0, \ldots, x_n) \in V(I)$, then some $x_i \neq 0$, say x_0. For $f \in V(\overline{I})$, there exists some n such that $x_0^n f \in I$, and thus $x_0^n f(P) = 0$. So $f(P) = 0$ and $P \in V(\overline{I})$. Thus $V(I) = V(\overline{I})$. To prove equality of sheaves, consider the canonical surjection $S/I \to S/\overline{I}$ given by $\overline{a} \mapsto \hat{a}$. This induces a surjection of local rings $(S/I)_{(f)} \to (S/\overline{I})_{(f)}$ which associates $\hat{a}/f^r$ to $\overline{a}/f^r$ for homogeneous a, f with def $f > 0$ and deg $a = r$deg f. it will be enough to show that this map is also injective. If $\hat{a}/f^r = 0$, then $f^m a \in \overline{I}$. There is therefore an n such that $x_i^n f^m a \in I$ for all i. For $k \gg 0$, $f^k a \in I$,so $\overline{a}/f^r = 0$ is in $(S/I)_{(f)}$. Thus Proj $(S/I) \cong$ Proj $(S/\overline{I})$. So if $\overline{I_1} = \overline{I_2}$, Proj $(S/\overline{I_1}) \cong$ Proj $(S/\overline{I_2})$ implies that Proj $S/I_1 \cong$ Proj S/I_2. Thus if I_1 and I_2 have the same saturation iff they define the same closed subscheme of I.

(c) Let $s \in \Gamma(X, \mathcal{O}_X(n))$ such that $x_i^{k_i} s \in \Gamma(X, \mathscr{I}_Y(n+k_i))$. Restricting, we get $x_i^{k_i} s \in \Gamma(D_+(x_i), \mathscr{I}_Y(n+k_i))$. Tensoring by $x_i^{-k_i}$, we get $s \in \Gamma(D_+(x_i), \mathscr{I}_Y(n))$. The $D_+(x_i)$ cover X and $\mathscr{I}_Y(n)$ is a sheaf, so $s \in \Gamma(X, \mathscr{I}_Y(n))$. Thus $\Gamma_*(J_Y)$ is saturated.

(d) Clear from Prop II.5.9, Cor 5.16 and c).

11. Let S and T be two graded rings with $S_0 = T_0 = A$. Define the Cartesian product $S \times_A T$ to be the graded ring $\bigotimes_{d \geq 0} S_d \otimes_A T_d$. Let $X =$ Proj S and $Y =$ Proj T.

First show that Proj$(S \times_A T) \cong X \times_A Y$. Let $\alpha_0, \ldots, \alpha_r$ and $\beta_0, \ldots, \beta_s$ be the generators of the A-modules S and T, respectively. Then $\alpha_i \otimes \beta_j$ become the generators of $S_1 \otimes_A T_1$ and $S \times_A T = A[\alpha_i \otimes \beta_j]$. It is easily checked that $S \times_A T_{(\alpha_i \otimes \beta_j)} \cong S_{(\alpha_i)} \otimes_A T_{(\beta_j)}$ for all $0 \leq i \leq r, 0 \leq j \leq s$. Thus $D_+(\alpha_i \otimes \beta_j) \cong$ Spec $S_{(\alpha_i)} \times_A$ Spec $T_{(\beta_j)} \cong D_+(\alpha_i) \times D_+(\beta_j)$. Thus Proj $S \times_A T \cong X \times_A Y$.

The sheaf $\mathcal{O}(1)$ on Proj $(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes \mathfrak{p}_2^*(\mathcal{O}_Y(1))$ on $X \times Y$ follows immediately from previous result that Proj $S \times_A T \cong X \times_A Y$, Prop 5.12(c), and the universal property of the Cartesian product.

12. (a) Let X be a scheme over a scheme Y, and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Letting i_1 be the closed immersion induced by $\mathscr{L}$, and i_2 be the closed immersion induced by $\mathscr{M}$. We have the following diagram:

$$\begin{tikzcd}
 & \mathbb{P}^r & & \\
X \arrow[ur, "i_1"] \arrow[rr, dashed, "i_{1,2}"] \arrow[d] \arrow[dr, "i_2"'] & & \mathbb{P}^r \times \mathbb{P}^s \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \arrow[r] & \mathbb{P}^{rs+r+s} \\
Y & \mathbb{P}^s & &
\end{tikzcd}$$

Then $\mathcal{O}(1)$ on $\mathbb{P}^{rs+r+s}$ from the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ is isomorphic to $\mathcal{O}(1)$ of $\mathbb{P}^r \times \mathbb{P}^s$. By the previous example, this is isomorphic to $p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^s}(1))$. Then since $\mathscr{L} \cong i_1^*(\mathcal{O}_{\mathbb{P}^r}(1))$ and $\mathscr{M} \cong i_2^*(\mathcal{O}_{\mathbb{P}^s}(1)), \mathscr{L} \otimes \mathscr{M} \cong i_{1,2}^*(p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^s}(1))) \cong i_{1,2}^*(\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(1))$. Thus $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(b) To show that $\mathscr{L}$ and $f^*\mathscr{M}$ are very ample relative to Z, we need to exhibit a morphism from $X \to \mathbb{P}^N_Z$. We have the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & \mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times Y \arrow[r, "p_1"] & \mathbb{P}^r_{\mathbb{Z}} & \\
Y \arrow[r] \arrow[d, "g"] & \mathbb{P}^s_Z = \mathbb{P}^s_{\mathbb{Z}} \times Z & \mathbb{P}^s_{\mathbb{Z}} \times \mathbb{P}^r_Z \times Z \arrow[l] \arrow[u] \arrow[r] & \mathbb{P}^{rs+s+r}_{\mathbb{Z}} \times Z = \mathbb{P}^N_Z \\
Z & & &
\end{tikzcd}$$

Thus we get a map into $\mathbb{P}^N_{\mathbb{Z}} \times Z$ and by tensoring the pullbacks of $\mathcal{O}(1)$ with respect to the correct maps as above, we see that $\mathscr{L} \otimes f^*\mathscr{M}$ is very ample relative to Z.

13. Let S be a graded ring, generated by S_1 as an S_0-algebra. Let $d > 0$ and let $S^{(d)} := \bigoplus_{n \geq 0} S_n^{(d)}$, where $S_n^{(d)} = S_{nd}$. Let $X = \text{Proj } S$. Since S is generated by S_1 over S_0, $S^{(d)}$ is generated by $S_1^{(d)} = S_d$ over S_0. So the sets $D_+(f)$, with $f \in S_d$ cover both Proj S and Proj $S^{(d)}$. Via the identity map $\frac{s}{f^n} \mapsto \frac{s}{f^n}$, we get $S_{(f)} \cong S^{(d)}_{(f)}$. Thus Spec $S_{(f)} \cong$ Spec $S^{(d)}_{(f)}$. Glue these isomorphisms together to get that Proj $S \cong$ Proj $S^{(d)}$. Use these same maps to find that $S(d)_{(f)} \cong S^{(d)}(1)_{(f)}$ for $f \in S_n$. So $\mathcal{O}(1)$ and $\mathcal{O}_X(d)$ correspond under these isomorphisms.

14. Let k be algebraically closed. Let X be a connected normal closed subscheme of $\mathbb{P}^r_k$.

(a) Let S be the homogeneous coordinate ring of X and let $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$. To show S is a domain, it suffices to show that I_X is prime, which is equivalent to showing that X is irreducible. Note that X is reduced. Else then some local ring $\mathcal{O}_{x,X}$ contains nilpotents and then $\mathcal{O}_{x,X}$ is not integrally closed since it is not an integral domain. If X were reducible, then some point x would be contained in two irreducible components, so the local ring at the point would have zero divisors. So since X is normal, X is irreducible and S is a domain.

Consider the sheaf $\mathscr{L} = \bigoplus_{n\geq 0} \mathcal{O}_X(n)$. Then $\mathscr{L}_\mathfrak{p} = \bigoplus_{n\geq 0} S(n)_{(\mathfrak{p})} = \{\frac{s}{f} \in S_\mathfrak{p} \mid \deg s \geq \deg f\}$. Any element integral over $\mathscr{L}_\mathfrak{p}$ is of course integral over $S_\mathfrak{p}$, and thus is in $S_\mathfrak{p}$ since X is normal. However, nothing with total negative degree can be integral over $\mathscr{L}_\mathfrak{p}$, so $\mathscr{L}_\mathfrak{p}$ is integrally closed. Thus $\Gamma(X, \mathscr{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n)) = S'$ is integrally closed. S' is contained in the integral closure of S by pg 122-123, so S' is the integral closure of S.

(b) This follows exactly from ex 5.9(b) since $\widetilde{S} \cong \mathcal{O}_X$.

(c) Choose $d \gg 0$ such that by part c), $S_{nd} = S'_{nd}$ for all $n > 0$. Then if $s \in K(S^{(d)})$ is integral over $S^{(d)}$, it lies in $S'^{(d)} = S^{(d)}$. Thus the d-uple embedding of X is projectively normal.

(d) If X is projectively normal, then S is integrally closed so $S = S'$. Thus $S_n = \Gamma(X, \mathcal{O}_X(n))$ for all n. Let $T = A[x_0, \dots, x_r]$. Then $T \to S$ is surjective and $T_n = \Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A(n)})$ by part (a) or 5.13. So $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \twoheadrightarrow \Gamma(X, \mathcal{O}_X(n))$ is surjective. Conversely, the map is surjective implies $S = S'$ when S is normal by part (a).

15. Let X be a noetherian scheme, U an open subset, and $\mathscr{F}$ be a coherent sheaf on U.

(a) Let $X = \operatorname{Spec} A$ be a noetherian affine scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $\mathscr{F} = \widetilde{M}$ for some A-module M. Then $M = \bigcup M_\alpha$, where each M_α is a finite submodule of M. Then applying $\sim$ to both sides, we see that $\mathscr{F}$ is the union of its coherent subsheaves.

(b) Let $X = \operatorname{Spec} A$ be a noetherian affine scheme, U an open subset, and $\mathscr{F}$ coherent on U. Let $\mathscr{F} \cong \widetilde{M}$. Let $i\colon U \hookrightarrow X$ be the inclusion. U is noetherian so $i_*\mathscr{F}$ is quasi-coherent on X by Prop 5.8(c). By part (a), we can write $i_*\mathscr{F} = \bigcup \mathscr{G}_\alpha$, where each $\mathscr{G}_\alpha$ is a coherent subsheaf, say isomorphic to $\widetilde{N_\alpha}$. Since X is noetherian, every directed set of submodules has a maximal element, which is the union of all the N_α. Thus the maximal sheaf is $\widetilde{\bigcup N_\alpha} := \mathscr{F}'$, which is coherent by construction. Then $\mathscr{F}"|_U \cong i^*\mathscr{F}' \cong i^*i_*\mathscr{F} \cong \mathscr{F}$. There exists a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_U \cong \mathscr{F}$.

(c) Let $\mathscr{G}$ be a quasi-coherent sheaf on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Consider $\rho^{-1}(i_*\mathscr{F}) \subseteq \mathscr{G}$. $\rho^{-1}(i_*\mathscr{F})$ is the pullback of a quasi-coherent sheaf under a map of quasi-coherent sheaves so is thus quasi-coherent. Since $\rho^{-1}(i_*\mathscr{F})|_U = \mathscr{F}$, we can apply the same argument as in part (b).

(d) Cover X with finitely many open affine sets $U_1, \ldots, U_n$. Extend $\mathscr{F}|_{U_1 \cap U}$ to a coherent sheaf $\mathscr{F}' \subseteq \mathscr{G}|_{U_1}$ and glue $\mathscr{F}$ and $\mathscr{F}'$ together via $U_1 \cap U$ to get a coherent sheaf $\mathscr{F}$ on $U \cap U_1$. Now repeat with $U_2, \ldots, U_n$ to get the desired result.

(e) Let $s \in \mathscr{F}(U)$ and $\mathscr{G}$ be the sheaf on U generated by s. $\mathscr{G}$ is coherent since on any affine open set in $U, \mathscr{G}|_U = \widetilde{M}$ with M generated by the image of s. So $\mathscr{G}$ extends to a coherent sheaf G' on X and $s \in \mathscr{G}'(U)$.

16. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(a) Let $\mathscr{F}$ locally free of rank n. Then it is clear from the construction of $T^r(\mathscr{F}), S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ they each is again locally free. The rank of $T^r(\mathscr{F}) = \mathcal{O}_X^n \otimes \ldots \otimes \mathcal{O}_X^n \cong \mathcal{O}_X^{\oplus nr}$ is nr. The rank of $S^r(\mathscr{F})$ is equal to the number of homogeneous polynomial of degree r in n variables, which is $\binom{n+r-1}{n-1}$. Lastly, the rank of $\bigwedge^r(\mathscr{F})$ is equal to the number of tuples $(i_1, \ldots, i_r)$, with $1 \leq i_1 < \ldots < i_r \leq n = \binom{n}{r}$.

(b) Let $\mathscr{F}$ be locally free of rank n. Let the basis elements be $x_1, \ldots, x_n$. Then $\bigwedge^n \mathscr{F} \cong \mathcal{O}_X$. The multiplication map $\bigwedge^n \mathscr{F} \otimes \bigwedge^{n-r} \mathscr{F} \to \bigwedge^n \mathscr{F}$ is given by $f x_n \wedge \ldots x_{n-r} \otimes g x_1 \wedge \ldots \wedge x_r \mapsto fg x_1 \wedge \ldots \wedge x_n$. Every global section f of $\bigwedge^{n-r} \mathscr{F}$ defines a morphism $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$ defined by $g \mapsto f \wedge g$. Conversely, given a morphism $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$, it induces a morphism of global sections $\varphi : \Gamma(X, \bigwedge^r \mathscr{F}) \to \Gamma(X, \bigwedge^n \mathscr{F}) \cong \Gamma(X, \mathcal{O}_X)$. Thus we can define a global section of $\bigwedge^{n-r} \mathscr{F}$ by $\sum (-1)^{ki} \varphi(x_{i_1} \wedge \ldots \wedge x_{i_r}) x_{j_1} \wedge \ldots \wedge e_{j_{n-r}}$, where the j_k are the elements that do not appear as i_l for some l. These operations are inverses and we get the isomorphism $\bigwedge^r \mathscr{F} \cong (\bigwedge^{n-r} \mathscr{F})^* \otimes \bigwedge^n \mathscr{F}$.

(c) [SAM] Let $U \subseteq X$ be an open set on which $\mathscr{F}|_U, \mathscr{F}'|_U, \mathscr{F}''|_U$ are free. It is enough to find a basis independent filtration on U and then glue them together. First, we can pick any splitting $\mathscr{F}|_U \cong \mathscr{F}'|_U \oplus \mathscr{F}''|_U$. Then from this we see that

$$S^r(\mathscr{F}|_U) \cong \bigoplus_{i=0}^{r} (S^i(\mathscr{F}'|_U) \otimes S^{r-i}(\mathscr{F}''|_U)).$$

Set $F^{r+1} = 0$ and assume by induction that we have chosen $F^j, F^{j+1}, \ldots, F^{r+1}$ such that $F^i / F^{i+1} \cong S^i(\mathscr{F}'|_U) \otimes S^{r-i}(\mathscr{F}''|_U)$. Consider the image of

$$S^{j-1}(\mathscr{F}'|_U) \otimes S^{r-j+1}(\mathscr{F}''|_U) \to S^r(\mathscr{F}|_U)/F^j$$

Its preimage under the projection $S^r(\mathscr{F}|_U) \to S^r(\mathscr{F}|_U)/F^j$ is independent of the chosen splitting. To see this, suppose that $x_1, \ldots, x_p$ is a basis of $\mathscr{F}'|_U$ and that $y_1, \ldots, y_q$ is a basis for $\mathscr{F}''|_U$. Then picking another basis $y_1 + c_1, \dots, y_q + c_q$, where $c_1, \dots, c_q \in \mathscr{F}'|_U$, we have $x_i \otimes (y_j + c_j) \mapsto x_i y_j + x_i c_j$, which is equal to $x_i y_j$ in $S^r(\mathscr{F}|_U)/F^j$ because $x_i c_j \in S^r(\mathscr{F}'|_U) = F^r \subseteq F^j$. So choose F^{j-1} to be this preimage. When we are done, the filtration is independent of the chosen splitting.

(d) [SAM] The filtration is obtained in exactly the same way as in part (c). The isomorphism is obtained by setting $r = n$ and noting that

$$F^p/F^{p+1} \cong \bigwedge^{p}(\mathscr{F}') \otimes \bigwedge^{n-p}(\mathscr{F}'')$$

is zero unless $p = n'$ and $n - p = n''$, and hence $F^{n'} = F^n = \bigwedge^n(\mathscr{F})$.

(e) [SAM] We proceed by induction on n, the case $n = 0$ being clear. For $n > 0$,

$$\begin{aligned} T^n(f^*(\mathscr{F})) &= f^*(\mathscr{F}) \otimes_{\mathcal{O}_X} T^{n-1}(f^*(\mathscr{F})) \\ &= (f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} f^*(T^{n-1}(\mathscr{F})) \\ &\cong f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} f^*(T^{n-1}(\mathscr{F})) \\ &= f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}(\mathscr{F}^{\otimes n-1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &= f^*(T^n(\mathscr{F})), \end{aligned}$$

where the last isomorphism follows because f^{-1} is defined as a colimit, which commutes with left adjoints (in this case $\otimes$).

Let $\mathscr{I}$ be the degree n part of the sheaf ideal such that $T(\mathscr{F})/\mathscr{I} = S(\mathscr{F})$. Since f^* is a left adjoint, it is right exact, so

$$f^*\mathscr{I} \to f^*(T^n(\mathscr{F})) \to f^*(S^n(\mathscr{F})) \to 0$$

is exact. In fact, for sections x, y of $\mathscr{I}$, one has $f^*(x \otimes y) = f^*x \otimes f^*y$ since tensor commutes with f^*, so we can write an exact sequence

$$0 \to f^*\mathscr{I} \to T^n(f^*(\mathscr{F})) \to S^n(f^*(\mathscr{F})) \to 0$$

We have already shown that $T^n(f^*(\mathscr{F})) = f^*(T^n(\mathscr{F}))$, so we deduce that $S^n(f^*(\mathscr{F})) = f^*(S^n(\mathscr{F}))$. Showing that $\bigwedge$ commutes with f^* proceeds in the same way.

17. A morphism $f : X \to Y$ of schemes is affine if there is an open affine cover $\{V_i\}$ such that $f^{-1}(V_i)$ is affine for each i.

(a) By definition, there exists an affine open cover $\{V_\lambda \mid \lambda \in \Lambda\}$ such that all $f^{-1}(V_\lambda)$ are affine. The intersection $V \cap V_\lambda$ has an open cover of principal open sets $D(f_{ij})$. Then the restriction $f|_{f^{-1}(V_i)} : f^{-1}V_i \to V_i$ is induced by $\varphi_i : A_i \to B_i$, where $A_i = \Gamma(V_i, \mathcal{O}_Y)$ and $B_i = \Gamma(f^{-1}(V_i), \mathcal{O}_X)$. Therefore $f^{-1}(D(f_{ij})) = D(\varphi(f_{ij}))$ and we have an affine open cover of $f^{-1}(V_i)$. Thus $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V$ is an affine morphism.

What remains to be shown is that $f^{-1}(V_i)$ is actually affine. This follows from the next lemma:

Lemma: If $f : X \to Y$ is an affine morphism, and Y is affine, then so is X. For any $\alpha \in \Gamma(Y, \mathcal{O}_Y)$, with $D(\alpha) \subseteq V_i$, $f^{-1}(D(\alpha)) = f^{-1}(V_i) \times_{V_i} D(\alpha)$ is also affine. Hence, let $\Phi := \{\alpha \in \Gamma(Y, \mathcal{O}_Y) \mid D(\alpha) \subseteq V_i \text{ for some } i\}$. Since Y is quasi-compact, there exists $\alpha_i, \ldots, \alpha_r \in \Phi$ such that $Y = \bigcup_{i=1}^r D(\alpha_i)$ and the $f^{-1}(D(\alpha_i))$ are affine. We set $Y_i = D(\alpha_i)$ and $X_i = f^{-1}(D(\alpha_i))$. We have the commutative diagram:

$$\begin{tikzcd} X_i \arrow[d, "f_i"] & \subseteq & X \arrow[r, "\Psi"] \arrow[d, "f"] & \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \arrow[dl, "g"] \\ Y_i & \subseteq & Y & \end{tikzcd}$$

Thus $X_i = \Psi^{-1}(g^{-1}(Y_i))$ and so $\Psi|_{X_i} : X_i \to g^{-1}(Y_i)$. Then by 5.8(c), $f_*(\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)^\sim$ as a $\Gamma(X, \mathcal{O}_X)^\sim$-module. Hence $\Gamma(X_i, \mathcal{O}_{X_i}) = \Gamma(Y_i, f_*(\mathcal{O}_X)) = \Gamma(Y_i, \Gamma(X, \mathcal{O}_X)^\sim) = \Gamma(X, \mathcal{O}_X)_{\alpha_i}$. Letting $Z = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$, we have that $\Gamma(g^{-1}(Y_i), \mathcal{O}_{g^{-1}(Y_i)}) = \Gamma(Y_i, g_*(\mathcal{O}_Z)) \cong \Gamma(X, \mathcal{O}_X)_{\alpha_i}$. Hence $\Psi|_{X_i}$ is an isomorphism for each i and this Ψ is an isomorphism. So X is affine.

(b) If $f : X \to Y$ is an affine morphism, take an open cover V_i of Y. By part a), each f^{-1} is affine and thus quasi-compact. So there is a cover of Y with quasi-compact images and thus f is quasi-compact.

To show f is separated, it is enough to show that each restriction $f^{-1}(V_i) \to V_i$ is separated. But since this is an affine morphism, this result follows by Prop 4.1.

Any finite morphism is affine by definition. (finite iff proper and affine)

(c) [SAM] We wish to glue together the schemes Spec $\mathscr{A}(U)$ as U ranges over all open affines of Y. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ be two open affines. If $U \cap V = \emptyset$, there is nothing to do. Otherwise, cover $U \cap V$ with open sets that are distinguished in both U and V. Let $W = \operatorname{Spec} C$ be a distinguished open in $U \cap V$. Also, let $A' = \mathscr{A}(U), B' = \mathscr{A}(V)$ and $C' = \mathscr{A}(W)$. Since $\mathscr{A}$ is an $\mathcal{O}_Y$-module,

$$\begin{tikzcd} \mathscr{A}(U) \arrow[r, "\rho_{UW}"] & \mathscr{A}(W) \\ \mathcal{O}_Y(U) \arrow[u] \arrow[r] & \mathcal{O}_Y(W) \arrow[u] \end{tikzcd}$$

is an $\mathcal{O}_Y(U)$-module homomorphism where ρ_{UW} is the restriction map given by $\mathscr{A}$. As C is a localization of both A and B, we also have that C' is a localization of both A' and B' since $\mathscr{A}$ is quasi-coherent, and hence we can identify A' and B' along C'. There are maps $A \to A'$ and $B \to B'$ given by the $\mathcal{O}_Y$-algebra structure of $\mathscr{A}$, and they induce morphisms $g : \text{Spec } \mathscr{A}(U) \to U$ and $h : \text{Spec } \mathscr{A}(V) \to V$.

In fact, the isomorphisms given by the distinguished covering of $U \cap V$ patch together to give an isomorphism $g^{-1}(U \cap V) \to h^{-1}(U \cap V)$. Since these isomorphisms come from restriction maps of a sheaf, it is clear that they agree on triple overlaps, so this gives a gluing. Call this scheme X. The maps $\mathscr{A}(U) \to U$ for all open affines are compatible on overlaps, so glue these together to give a morphism $f : X \to Y$. For an inclusion $U \subseteq V$ of open affine of Y, the morphism $f^{-1}(U) \to f^{-1}(V)$ is given by the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$ by construction above.

If there is an X' and $f' : X' \to Y$ with the same properties of X, then we can define a morphism $X \to X'$ by gluing together morphisms on open affine Spec $\mathscr{A}(U)$, where U is an open affine of Y. Then this morphism will be an isomorphism, so we see that X is unique.

(d) [SAM] By construction, for every open affine $U \subset Y$, $f^{-1}(U) \cong \text{Spec } \mathscr{A}(U)$, so f is affine. Also, for every open set $U \subseteq Y$, we have $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)) \cong \mathscr{A}(U)$. The isomorphism is clear is U is affine, or if U is contained in some open affine. In the general case, cover Y with open affines U_i, and for each $U \cap U_i$, we have $\mathcal{O}_X(f^{-1}(U \cap U_i)) \cong \mathscr{A}(U \cap U_i)$, which follows from the construction. Since these isomorphisms are canonical, they patch together to give the isomorphism for U.

Conversely, suppose that $f : X \to Y$ is an affine morphism and set $\mathscr{A} = f_*(\mathcal{O}_X)$. For every open set $U \subseteq Y$, $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U))$, so there is a morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, which gives $\mathscr{A}(U)$ the structure of an $\mathcal{O}_Y(U)$-module. For an inclusion $V \subseteq U$, it is clear that the restriction map $\mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(V))$ is an $\mathcal{O}_X(U)$-module homomorphism. So $\mathscr{A}$ is an $\mathcal{O}_Y$-module.

In particular, for every open affine $U = \text{Spec } A \subseteq Y$, $f^{-1}(U) = \text{Spec } B$ is affine by (a). Considering B as an A-module, $\mathscr{A}|_U \cong \widetilde{B}$, so $\mathscr{A}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Now if $V \subseteq U$ is an open affine, the morphism on spectra $f^{-1}(V) \to f^{-1}(U)$ is induced by the map of rings $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(V)) = \mathscr{A}(V)$. From the uniqueness of **Spec** $\mathscr{A}$ in (c), we conclude $X \cong \textbf{Spec } \mathscr{A}$.

(e) [SAM] Let $\mathscr{M}$ be a quasi-coherent $\mathscr{A}$-module. We glue together the $\mathcal{O}_X(f^{-1}(U))$-modules $(\mathscr{M}(U))^\sim$ as U ranges over all open affines of Y. Given two open affines U and V of Y, we can cover their intersection with open sets that are distinguished in both. The sections of these distinguished open sets are given by localizing modules, and since they are the same in both $\mathscr{M}(U)$ and $\mathscr{M}(V)$, there is an isomorphism on their intersection. These isomorphisms are compatible with triple overlaps because they are given by localization. So we can glue these sheaves (Ex. 1.22) to get an $\mathcal{O}_X$-module which we call $\mathscr{M}^\sim$.

We claim that $\sim$ and f_* give an equivalence of categories between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathscr{A}$-modules. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $(f_*\mathscr{F})^\sim$ is naturally isomorphism to $\mathscr{F}$ because they are isomorphic on open affines and using Corollary 5.5. Similarly, if $\mathscr{M}$ is a quasi-coherent $\mathscr{A}$-module, then $f_*\widetilde{M}$ is naturally isomorphic to $\mathscr{M}$.

18. Vector Bundles. The fact that there is a one-to-one correspondence between linear classes of divisors, isomorphism classes of locally free sheaves of rank n, and isomorphism classes of rank n vector bundles is well documented. See Shaf II p 64 for the correspondence, among others. So when your analyst friends start asking you questions about vector bundles, you stop them immediately and say, "Hey man, I study algebra. Can you call them locally free sheaves?" Really stress this point. It tends to get the analysts really frustrated.

2.6 Divisors

1. Let X be a SINR scheme. (Thats my notation for a scheme satisfying (*), where SINR is separated, integral, noetherian, regular in codimension 1). Then $X \times \mathbb{P}^n$ is integral, noetherian and regular in codimension 1 by Prop II.6.6 and its proof, since these properties correspond to the properties of the dense open sets isomorphic to $X \times \mathbb{A}^n$. To show separatedness, We have the following diagram

$$\begin{tikzcd} & X \times \mathbb{P}^n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr] & & \mathbb{P}^n \arrow[dl] \\ & \mathrm{Spec}\ \mathbb{Z} & \end{tikzcd}$$

Then p_2 is a base extension of $X \to \mathrm{Spec}\ Z$, which is separated, and $\mathbb{P}^n \to \mathrm{Spec}\ Z$ is projective, therefore separated. Then the composition $X \times \mathbb{P}^n$ is separated as well by Cor II.4.6

[BLOG] After Proposition II.6.5 we have an exact sequence

$$\mathbb{Z} \overset{i}{\hookrightarrow} \mathrm{Cl}\ (X \times \mathbb{P}^1) \xrightarrow{j} \mathrm{Cl}\ X \to 0$$

The first map sends $n \mapsto nZ$, where Z is the closed subscheme $p_2^{-1}\infty \subset X \times \mathbb{P}^1$, and the second is the composition of $\mathrm{Cl}\ (X \times \mathbb{P}^1) \to \mathrm{Cl}\ (X \times \mathbb{A}^1) \cong \mathrm{Cl}\ X$. Consider the map $\mathrm{Cl}\ X \to \mathrm{Cl}\ (X \times \mathbb{P}^1)$ that sends $\sum n_i Z_i$ to $\sum n_i p_1^{-1} Z_i$. The composition $\mathrm{Cl}\ (X) \to \mathrm{Cl}\ (X \times \mathbb{P}^1) \to \mathrm{Cl}\ (X \times \mathbb{A}^1) \cong \mathrm{Cl}\ (X)$ sends a prime divisor Z to $p_1^{-1}Z$, then $(X \times \mathbb{A}^1) \cap p_1 Z$, and then back to Z since $(X \times \mathbb{A}^1) \cap p_1^{-1} Z$ is the preimage of Z under the projection $X \times \mathbb{A}^1 \to X$. Hence the morphism in the exact sequence above is split.

We now show that the morphism $\mathbb{Z} \to \mathrm{Cl}\ (X \times \mathbb{P}^1)$ is split as well, by defining a morphism $\mathrm{Cl}\ (X \times \mathbb{P}^1) \to Z$, which splits i. Let $k : \mathrm{Cl}\ (X) \to \mathrm{Cl}\ (X \times \mathbb{P}^1)$ denote the morphism we used to split j. Then we send a divisor ξ to $\xi - kj\xi$. This is in the kernel of j (since $jk = id$) and therefore in the image of i. So it remains only to see that i is injective.

Suppose $nZ \sim 0$ for some integer n. Taking the "other" $X \times \mathbb{A}^1$ we have Z as $p_2^{-1}(0)$. In the open subset $X \times \mathbb{A}^1$ we have Z as X embedding at the origin. So the local ring of Z in the function field $K(t)$, where K is the function field of X) is $K[t]_{(t)}$. Since $nZ \sim 0$ there is a function $f \in K(t)$ such that $\nu_Z(f) = n$ and $\nu_Y(f) = 0$ for every other prime divisor Y. So f is of the form $t^n \frac{g(t)}{h(t)}$, where $g \in K[t]$ and $t \nmid g(t), h(t)$. If the degree of g and h is 0, then changing coordinates back $t \mapsto t^{-1}$, we see that $\nu_Y(f) = -n$, where Y is another copy of X embedded at the origin or infinity, depending on which coordinates we are using; the one opposite to

Z at any rate. If one of g or h has degree higher than zero then, it will have an irreducible factor in $K[t]$, which will correspond to a prime divisor of the form $p_2^{-1}x$ for some $x \in \mathbb{P}^1$, and the value of f will not be zero at this prime divisor. Hence there is no rational function with $(f) = nZ$ and so i is injective. Hence $\mathrm{Cl}\,(X \times \mathbb{P}^1) \cong \mathrm{Cl}\,(X) \times \mathbb{Z}$.

2.

3.

4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a square-free nonconstant polynomial. Let $A = k[x_1, \ldots, x_n, z]/(z^2 - f)$. Following the hint, in the quotient field K of A, $\frac{1}{g+zh}\frac{g-zh}{g-zh} = \frac{g-zh}{g^2-fh^2}$ since $z^2 = f$ in A, so every element can be written in the form $g' + zh'$, where g' and h' are in the $k(x_1, \ldots, x_n)$. Hence $K = k(x_1, \ldots, x_n)[z]/(z^2 - f)$. This is a degree 2 extension, and thus Galois with automorphism $z \mapsto -z$. Let $\alpha = g + hz \in K$, where $g, h \in k(x_1, \ldots, x_n)$. The minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Then α is integral over $k[x_1, \ldots, x_n]$ iff $2g, g^2 - h^2 f \in k[x_1, \ldots, x_n]$ iff $2g, h^2 f \in k[x_1, \ldots, x_n]$.

   Assume α is integral over $k[x_1, \ldots, x_n]$ Then $g \in k[x_1, \ldots, x_n]$ and thus $h^2 f \in k[x_1, \ldots, x_n]$. If h had a nontrivial denominator, then $h^2 f \notin k[x_1, \ldots, x_n]$ since f is square-free. Thus $h \in k[x_1, \ldots, x_n]$ so $\alpha \in A$.

   Conversely, if $\alpha \in A$, then $2g, h^2 f \in k[x_1, \ldots, x_n]$ so α is integral over $k[x_1, \ldots, x_n]$. Thus A is the integral closure of $k[x_1, \ldots, x_n]$ and is thus integrally closed.

5. Let char $k \neq 2$ and let X be the affine quadric hypersurface Spec $k[x_0, \ldots, x_n]/(x_0^2 + x_1^2 + \ldots + x_r^2)$

   (a) Let $r \geq 2$. This follows from the previous example with $f = -(x_0^2 + \ldots + x_r^2)$, which is square-free. Since the localization of an integrally closed ring is again integrally closed, X is normal.

   (b) [BLOG] Assume that -1 has a root i in k. Consider the change of coordinates

   $$x_0 \mapsto \frac{y_0 + y_1}{2} \qquad x_1 \mapsto \frac{y_0 - y_1}{2i}$$

   Then $x_0^2 + x_1^2 = y_0 y_1$.
   Let $A = \text{Spec } k[x_0, \ldots, x_n]/(x_0 x_1 + x_2^2 + \ldots + x_r^2)$. Now we imitate Example II.6.5.2. We take the closed subscheme $\mathbb{A}^{n+1}$ with ideal $(x_1, x_2^2 + \ldots + x_r^2)$. This is a subscheme of X and is fact $V(x_1)$ considering $x_1 \in A$. We have an exact sequence

   $$\mathbb{Z} \to \mathrm{Cl}\,(X) \to \mathrm{Cl}\,(X - Z) \to 0$$

   Now since $V(x_1) \cap X = X - Z$, the coordinate ring of $X - Z$ is

   $$k[x_0, x_1, x_1^{-1}, x_2, \ldots, x_n]/(x_0 x_1 + x_2^2 + \ldots x_r^2)$$

As in Example II.6.5.2, since $x_0 = -x_1^{-1}(x_2^2 + \ldots + x_r^2)$ in this ring we can eliminate x_0 and since every element of the ideal $(x_0x_1 + x_2^2 + \ldots + x_r^2)$ has an x_0 term, we have an isomorphism between the coordinate ring of $X - Z$ and $k[x_1, x_1^{-1}, x_2, \ldots, x_n]$. This is a unique factorization domain so by Proposition II.6.2 Cl $(X - Z) = 0$. So we have a surjection $\mathbb{Z} \to$ Cl (X) which sends n to $n \cdot \mathbb{Z}$.

$r = 2$:
In this case the same reasoning as in Ex II.6.5.2 works. Let $\mathfrak{p} \subset A$ be the prime associated to the generic point of Z. Then $\mathfrak{m}_\mathfrak{p}$ is generated by x_2 and $x_1 = x_0^{-1}x_2^2$ so $v_Z(x_1) = 2$. Since Z is cut out by x_1, there can be no other prime divisors Y with $v_Y(x_1) \neq 0$. It remains to see that Z is not a principal divisor. If it were then Cl (X) would be zero and by Prop II.6.2, this would imply that A is a UFD (since A is normal by part a), which would imply that every height one prime ideal is principle. Consider the prime idea (x_1, x_2) of A which defines Z. Let $\mathfrak{m} = (x_0, x_1, \ldots, x_n)$. We have $\mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension n over k with a basis $\{\overline{x}_i\}$. The ideal $\mathfrak{m}$ contains $\mathfrak{p}$ and its image in $\mathfrak{m}/\mathfrak{m}^2$ is a subspace of dimension at least 2. Hence $\mathfrak{p}$ cannot be principle.

$r = 3$
We use Example II.6.6.1 and Exercise II.6.3(b). Using a similar change of coordinates as the beginning of this exercise, we see that X is the affine cone of the projective quadric of Example II.6.6.1. Thus, by Exercise II.6.3(b), we have an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to$ Cl $(X) \to 0$. We already know that Cl (X) is $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, or 0. Tensoring with $\mathbb{Q}$ gives an exact sequence $\mathbb{Q} \to \mathbb{Q}^2 \to$ Cl $(X) \otimes \mathbb{Q} \to 0$ of $\mathbb{Q}$ vector spaces. Hence Cl $(X) = \mathbb{Z}$ as the other two cases contradict the exactness of the sequence of $\mathbb{Q}$-vector spaces.

$r \geq 4$
In this case we claim that Z is principle. Consider the ideal (x_1) in A. Its corresponding closed subset is Z and so if we can show that (x_1) is prime, then Z will be the principle divisor associated to the rational function x_1. Showing that (x_1) is prime is the same as showing that $A/(x_1)$ is integral, which is the same as showing that $\frac{k[x_0,\ldots,x_n]}{(x_1,x_2^2+\ldots,x_r^2)}$ is integral since $(x_1, x_0x_1 + x_2^2 + \ldots + x_r^2) = (x_1, x_2^2 + \ldots + x_r^2)$. This is the same as showing that $\frac{k[x_0,x_2,\ldots,x_n]}{(x_2^2+\ldots+x_r^2)}$ is integral, (where the variable x_1 is missing from the top) which is the same as showing that $f = x_2^2 + \ldots x_r^2$ is irreducible. Suppose f is a product of more than one nonconstant polynomial. Since it has degree two, it is the product of at most two linear polynomials, say $a_0x_0 + a_2x_2 + \ldots + a_nx_n$ and $b_0x_0 + b_2x_2 + \ldots b_nx_n$. Expanding the product of these two linear polynomials and comparing the coefficients with f we find that (I)

$a_ib_i = 1$ for $2 \le i \le r$, and (II) $a_ib_j + a_jb_i = 0$ for $2 \le i, j \le r$ and $i \ne j$. WOLOG we can assume that $a_2 = 1$. The relation (I) implies that $b_2 = 1$, and in general, $a_i = b_i^{-1}$ for $2 \le i \le r$. Putting this in the second relation gives (III) $a_i^2 + a_j^2 = 0$ for $2 \le i \ne j \le r$ and this together with the assumption that $a_2 = 1$ implies that (IV) $a_j^2 = -1$ for each $2 < j \le r$. But if $r \ge 4$ then we have from (III) that $a_3^2 + a_4^2 = 0$ which contradicts (IV). Hence $x_2^2 + \ldots + x_r^2$ is irreducible, so $\frac{k[x_0, x_2, \ldots, x_n]}{(x_2^2 + \ldots x_r^2)}$ is integral, so $A/(x_1)$ is integral, so (x_1) is prime and hence Z is the principle divisor corresponding to x_1. So Cl $(X) = 0$

(c) For each of these we use the exact sequence of Ex II.6.3(b)

$r = 2$

We have the exact sequence $0 \to \mathbb{Z} \to$ Cl $(Q) \to \mathbb{Z}/2 \to 0$ where the first morphism sends 1 to the class of $H \cdot Q$ a hyperplane section. Tensoring with $\mathbb{Q}$ we get an exact sequence $\mathbb{Q} \xrightarrow{2}$ Cl $(Q) \otimes \mathbb{Q} \to 0 \to 0$ and so since Cl (Q) is an abelian group we see that it is $\mathbb{Z} \otimes T$ where T is some torsion group. Tensoring with $\mathbb{Z}/p$ for a prime p we get either $\mathbb{Z}/2 \xrightarrow{0}$ Cl $(Q) \otimes (\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$ if $p = 2$ or $\mathbb{Z}/p \xrightarrow{2}$ Cl $(Q) \otimes (\mathbb{Z}/p) \to 0 \to 0$ if $p \ne 2$. Hence $T = 0$, and so Cl $(Q) \cong \mathbb{Z}$ and the class of a hyperplane section is twice the generator.

$r = 3$

This is example II.6.6.1

$r \ge 4$

We have an exact sequence $0 \to \mathbb{Z} \to$ Cl $(Q) \to 0 \to 0$, hence Cl $(Q) = \mathbb{Z}$ and it is generated by $Q \cdot H$.

(d)

6. Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$.

   (a) Let P, Q, R be collinear points on X. Let the line they lie on be l. By Bezout's Theorem, P, Q and R are the only points on $l \cap X$. Then $P + Q + R \sim 3P_0$ as divisors and thus $(P - P_0) + (Q - P_0) + (R - P_0) \sim 0$. Thus $P + Q + R = 0$ in the group law.

   Conversely, let $P + Q + R = 0$ in the group law on X. Let l be a line through P and Q. Again, by Bezout's Theorem, this line must intersect X in another point T. This is equivalent to $P + Q + T \sim 3P_0$. Then by the uniqueness of inverses in the group law, $R = T$ and P, Q, and R are collinear.

   (b) Let $P \in X$ have order 2 in the group law. Then $P + P + P_0 = 0$. By part a), $2P$ and P_0 are collinear counting multiplicity. So this line passes through P with multiplicity 2, which is in fact the tangent line. Thus $T_P(X)$ passes through P_0.

   Conversely, let the tangent line P pass through P_0. By exercise 1.7.3, the intersection multiplicity with X is ≥ 2. Then by Bezout's

Theorem, $T_P(X)$ intersects X in 3 points counting multiplicity, of which at least 2 are P. Since $P_0 \neq P$, the three points are P, P and P_0. Then $P + P + P_0 = 0$ and since $P_0 = 0$, we get that $2P = 0$ and P has order 2.

(c) If P is an inflection point, the line $T_P(X)$ passes through X at P with multiplicity ≥ 3. By Bezout's Theorem, this multiplicity is exactly 3. So in the group law, $P + P + P = 0$. So we see that P has order 3.

Conversely, let P have order 3. Then $P + P + P = 0$ and by part a), the three points are collinear. So there is an line l such that $l \cap X$ in the point P with intersection multiplicity 3. This l is then $T_P(X)$ and P is therefore an infection point.

(d) The rational points on any elliptic curve form a finitely-generated abelian subgroup by the famous Mordell-Weil Theorem. The 4 obvious rational points on X are $(0,1,0), (1,0,1), (-1,0,1)$, and $(0,0,1)$. Each non-identity point has order 2 and this group is $Z_2 \times Z_2$. By some simple, but tedious calculations on $\mathbb{A}^2$, (see [SAM]) we can show that these are the only rational points.

7. See Joseph H. Silverman, "The Arithmetic of Elliptic Curves" Prop 2.5 on page 61.

8. (a) Let $f : X \to Y$ be a morphism of schemes. Lets show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphism of Picard groups $f^* : \text{Pic } Y \to \text{Pic } X$. By Prop II.5.2(e), we see that f^* takes locally free sheaves of rank n to locally free sheaves of rank n. Restricting locally to $X = \text{Spec } A$ and $Y = \text{Spec } B$, we consider $\mathscr{L}$ and $\mathscr{M}$ in Pic Y, where $\mathscr{L} \cong \widetilde{M}$ and $\mathscr{M} \cong \widetilde{N}$. Then we have

$$\begin{aligned} f^*(\mathscr{L} \otimes M) \quad &\cong f^*(\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N}) \\ &\cong f^*(\widetilde{M \otimes_B N}) \\ &\cong (M \otimes_B N \otimes_B A)^\sim \\ &\cong ((M \otimes_B A) \otimes_A (N \otimes_B A))^\sim \\ &\cong \widetilde{M \otimes_B A} \otimes_{\mathcal{O}_X} \widetilde{N \otimes_B A} \\ &\cong f^*(\widetilde{M}) \otimes_{\mathcal{O}_X} f^*(\widetilde{N}) \\ &\cong f^*(\mathscr{L}) \otimes f^*(\mathscr{M}) \end{aligned}$$

Thus f^* is a homomorphism. (Also, $f^*(\mathcal{O}_Y) = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$)

(b) It is enough to show equivalence for the images of points. Let $Q \in$ Cl X and let t be a local parameter at Q. Let U_Q be a neighborhood of Q in which $t = 0$ only at Q. Then $\{U_Q, t), (X - Q, 1)\}$ is a Cartier divisor corresponding to Q. The associated sheaf $\mathscr{L}(Q)$ satisfies $\mathscr{L}(Q)(U_Q) = \frac{1}{t}\mathcal{O}_Y(U_Q)$ and $\mathscr{L}(Q)(X - Q) = \mathcal{O}_Y(X - Q)$. $f^*\mathscr{L}(Q) = f^{-1}\mathscr{L}(Q) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ satisfies $f^*\mathscr{L}(Q)|_{f^{-1}(U_Q)} =$

$f^*(\frac{1}{t}\mathcal{O}_X|_{f^{-1}(U_Q)}$ and $f^*\mathscr{L}(Q)|_{X-Q} = \mathcal{O}_X|_{X-Q}$. The associated Cartier divisor of $f^*\mathscr{L}(Q)$ is $\{(f^{-1}(U_Q), f^*(t)), (f^{-1}(X-Q), 1)\}$ and the corresponding Weil divisor is $\sum_{P\in X} v_P(t)P$, which is exactly the image of Q under the map Cl $(Y) \to$ Cl (X). Note that $f^*t = t$ since f^* is an inclusion of function fields.

(c) We just need to show that image of the hyperplane divisor is the same. Assume that X is not contained in the hyperplane $x_0 = 0$ whose Cartier Divisor is $H = \{D_+(x_i), \frac{x_0}{x_i})\}$. The associated sheaf $\mathscr{L}(H)$ satisfies $\mathscr{L}(H)|_{D_+(x_i)} = \frac{x_i}{x_0}\mathcal{O}_{\mathbb{P}^n}|_{\mathcal{D}_+(x_i)}$. The pullback sheaf $f^*\mathscr{L}(H) = f^{-1}\mathscr{L}(H) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_X$ satisfies $f^*(\mathscr{L}(H))|_{f^{-1}D_+(x_i)} = \frac{x_i}{x_0}\mathcal{O}_X$, with associated Cartier Divisor $\{(f^{-1}D_+(x_i), \frac{x_0}{x_i})\} = \{(D_+(x_i)\cap X, \frac{x_o}{x_i})\}$. The corresponding Weil Divisor is obtained by taking the valuations of the $\frac{x_0}{x_i}$ considered as functions on X at codimension 1 subvarieties. The result is the same Weil divisor as in example 2(a).

9.

10. Let X be a noetherian scheme. Define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all the expression $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherence sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) Let $X = \mathbb{A}^1_k$ and let $\mathscr{F}$ be a coherent sheaf on X. If $\mathscr{F} \cong \mathscr{G}$ for some coherent sheaf $\mathscr{G}$, then there is an exact sequence $0 \to 0 \to \mathscr{F} \to \mathscr{G} \to 0$, so $\gamma(\mathscr{F}) \cong \gamma(\mathscr{G})$ in $K(X)$ and we only need to consider coherent sheaves up to isomorphism. Since $X =$ Spec $k[t]$ is affine, we only need to consider $\mathscr{F} \cong \widetilde{M}$, where M is a finite $k[x]$ module, which are of the form $k[t]^m \oplus k^m$. The image of $k[t]^m \oplus k^n$ in $K(X)$ is equal to $mk[t] + nk$, the sum of the components. Also, from the short exact sequence $0 \to k[t] \xrightarrow{t} k[t] \to k \to 0$, we see that $\gamma(k) = 0$ in $K(X)$. in any short exact sequence, the alternating sum of the ranks is 0, we can never get the equality $\gamma(k[t]) = 0$ if $m \neq 0$. Thus $K(X) \cong \mathbb{Z}$, generated by $\gamma(k[t])$.

(b) Let X be any integral scheme, $\mathscr{F}$ a coherent sheaf. Define the rank of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\xi$, where ξ is the generic point of X and $K = \mathcal{O}_\xi$ is the function field of X. If we have a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, then the sequence $0 \to \mathscr{F}'_\xi \to \mathscr{F}_\xi \to \mathscr{F}''_\xi \to 0$ is exact. Each term is a finite dimensional vector space, so $\dim_K \mathscr{F}_\xi = \dim_K \mathscr{F}'_\xi + \dim_K \mathscr{F}''_\xi$. Thus the rank homomorphism is well defined.

If $U = \text{Spec } A$ is an affine neighborhood of ξ, then an extension of $\widetilde{A}$ to X (ex 5.15) will have rank 1. Thus the rank map is surjective.

(c) [BLOG] *Surjectivity on the right*: Every coherent sheaf $\mathscr{F}$ on $X - Y$ can be extended to a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}|_{X-Y} = \mathscr{F}$ by ex 5.15, so the morphism on the right is surjective.

Exactness in the middle: Suppose that $\mathscr{F}$ is a coherent sheaf on X with support in Y. We will show (below) that there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \dots \supseteq \mathscr{F}_n = 0$ such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y. Assuming we have such a finite filtration, we have $\gamma(\mathscr{F}_i) = \gamma(\mathscr{F}_{i+1}) + \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$ in $K(X)$ and so $\gamma(\mathscr{F}) = \sum_{i=0}^{n-1} \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$. Hence, the class represented by $\mathscr{F}$ is in the image of $K(Y) \to K(X)$. Now if $\sum n_i \gamma(\mathscr{F}_i)$ is in the kernel of $K(X) \to K(X - Y)$, the

Proof of claim Let $i : Y \to X$ be the closed embedding of Y into X and consider the two functors $i_* : Coh(X) \to Coh(Y)$ (ex II.5.5) and $i^* : Coh(Y) \to Coh(X)$. These functors are adjoint (pg 110) and so we have a natural morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ for any coherent sheaf $\mathscr{F}$ on X. Let Spec A be an open affine subschemes of X on which $\mathscr{F}$ has the form $\widetilde{M}$. Closed subschemes of affine schemes correspond to ideals bijectively and so $\text{Spec } A \cap Y = \text{Spec } A/I$ for some ideal $I \subset A$ and the morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ restricted to Spec A has the form $M \to M/IM$. Thus we see that η is surjective. Let $\mathscr{F}_0 = \mathscr{F}$ and define $\mathscr{F}_j$ inductively as $\mathscr{F}_j = \ker(\mathscr{F}_{j-1} \to i_* i^* \mathscr{F}_j)$. It follows from our definition that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y so we just need to show that the filtration $\mathscr{F} \supseteq \mathscr{F}_1 \supseteq \dots$ is finite.

On our open affine we have $\mathscr{F}_j|_{\text{Spec } A} I^j M$. Now the support of $\widetilde{M}$ contained in the closed subscheme $\text{Spec } A/I = V(I)$ so by Ex II.5.6(b) we have $\sqrt{\text{Ann } M} \supseteq \sqrt{I} \supseteq I$. Since A is noetherian, every ideal is finitely generated. In particular, I is finitely generated. So there exists some N such that $\text{Ann } M \supseteq I^N$ (see the proof of Exercise II.5.6(d) for details). Hence $0 = I^N M$ and so the filtration is finite when restricted to an open affine. Since X is noetherian, there is a cover by finitely many affine opens $\{U_i\}$ and so if n_i is the point at which $\mathscr{F}_i|_{U_i} = 0$, then $\mathscr{F}_{\max\{n_i\}} = 0$. So the filtration is finite.

11. The Grothendieck Group of a Nonsingular Curve.
See [BLOG].

12. Let X be a complete nonsingular curve. [SAM]

(a) Let D be a divisor. Consider $K(X) \to \text{Pic } X \to \mathbb{Z}$, where the first map is projection via the isomorphism $K(X) \cong \text{Pic } X \oplus \mathbb{Z}$ from the previous problem. For the second map, we write an invertible sheaf as a Weil divisor $\sum n_i P_i$ and map it to $\sum n_i$. Let deg be the composition $K(X) \to \mathbb{Z}$ where $\deg \mathscr{F} = \deg \gamma(\mathscr{F})$. It is immediately clear from the definition of $K(X)$ that condition (3) is satisfied. From the definition of degree of a divisor, it is also clear that condition (1) is satisfied.

If F is a torsion sheaf, then $\gamma(\mathscr{F}) = \gamma(\mathcal{O}_D)$ for some effective divisor $D = \sum n_i P_i$. The stalk of $\mathcal{O}_D$ at P_i is k^{n_i}, whose length as a k-module is n_i. We claim that this is also the length of k^{n_i} as an $\mathcal{O}_{P_i}$-module. Since k is algebraically closed, we have an embedding $k \hookrightarrow \mathcal{O}_{P_i}$ and the residue field of $\mathcal{O}_{P_i}$ is k. So a filtration of k^{n_i} as an $\mathcal{O}_{P_i}$-module can be extended to a k-filtration. On the other hand, a maximal k-filtration of k^{n_i} has simple quotients, and we claim that such a filtration remains simple over $\mathcal{O}_{P_i}$. To see this, let $M \cong < a >$ be a simple nonzero module. Then it is isomorphic to $\mathcal{O}_{P_i} =$ Ann a. Since $\mathcal{O}_{P_i}$ is local, Ann $a \subseteq \mathfrak{m}_P$, which means that $\mathfrak{m}_P/$ Ann a must be 0 since it is a submodule of M. Hence, $M \cong \mathfrak{m}_P/$ Ann $a \cong k$, which gives the claim. Thus, $\deg(\mathscr{F}) = \sum n_i = \sum_{P \in X}$ $\operatorname{len}(\mathscr{F}_P)$, so this function also satisfies condition (2).

Finally, the degree function must be unique. To see why, we can check by induction on the rank of a sheaf. If a sheaf has rank 0, then it is a torsion sheaf, and condition (2) forces uniqueness of degree. For invertible sheaves of rank 1, condition (1) forces uniqueness. For all other sheaves, we can find an exact sequence as in (Ex. 6.11(c)) and then condition (3) forces uniqueness by induction.

2.7 Projective Morphisms

1. Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Then to show that f is an isomorphism, it is enough to show that at stalks, $f_x{:}\mathscr{L}_x \to \mathscr{M}_x$ is an isomorphism, which is equivalent to showing the surjective map of A-modules $f : A_\mathfrak{p} \to A_\mathfrak{p}$ is an isomorphism, where $X = \text{Spec } A$ and $\mathfrak{p}$ is the prime ideal corresponding to $x \in X$. Since f is a module homomorphism, scalars pop-out and $f(a) = f(1 \cdot a) = a \cdot f(1)$, so f is completely determined by where 1 gets mapped to. Since f is surjective, there is some element b that gets mapped to 1, so f is just multiplication by b, which is a unit since $b \cdot f(1) = 1$, and thus f is invertible and thus an isomorphism.

2. Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheave on X and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_n\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$ and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leq m$. Let $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ be the corresponding morphisms. Consider the map $\pi : \mathbb{P}^m_k \to \mathbb{P}^n_k$, which is given by $\mathcal{O}(1)$ with sections $x_0, x_1, \ldots, x_n$. Letting $L = \mathcal{Z}(x_0, x_1, \ldots, x_n)$, we get that $\pi : \mathbb{P}^m_k \backslash L \to \mathbb{P}^n_k$ is a morphism. All sections pullback to each other in the commutative diagram, so they define the same map and thus differ by the linear projection. Lastly, the automorphism of $\mathbb{P}^n$ comes from changing the basis $\{s_0, \ldots, s_n\}$ to $\{t_0, \ldots, t_n\}$.

3. Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then if φ is induced by the structure sheaf $\mathcal{O}_{\mathbb{P}^n_k}$ and global section $a_0, \ldots, a_m \in k = \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$, then $\varphi(\mathbb{P}^n_k)$ is the point $(a_0, \ldots, a_m) \in \mathbb{P}^m_k$. If φ is induced by $\mathcal{O}(r)$ for some $r > 0$, then φ is defined by $m+1$ homogeneous degree r polynomials with no common zeros in $\mathbb{P}^n_k$. Thus there are at least $m+1$ of them so $m \geq n$. φ is finite by Thm 8, p 65 in Shaf I, so dim $\varphi(\mathbb{P}^n) = n$.

   By first using the r-uple embedding and then project using the homogeneous polynomials $x_0, \ldots, x_m$, we obtain φ. Lastly, apply an automorphism of $\mathbb{P}^n_k$ corresponding to changing the basis of the linear space in $\Gamma(\mathbb{P}^n_k, \mathcal{O}(r))$ used in part (a).

4. (a) Let X be a scheme of finite type over a noetherian ring A and let $\mathscr{L}$ be an ample invertible sheaf. Then by Thm 7.6, $\mathscr{L}^n$ is very ample for some $n > 0$. Thus we have an immersion $X \to \mathbb{P}^N_A$, which is separated since both open and closed immersions are separated. $\mathbb{P}^N_A \to \text{Spec } A$ is separated so the composition $X \to \mathbb{P}^N_A \to$Spec A is separated.

   (b) Let X be the affine line over a field k with the origin doubled. Invertible sheaves on X are given by pairs of invertible sheaves on $\mathbb{A}^1$ whose restrictions to $\mathbb{A}^1 \backslash 0$ are equal. Any pair $(\mathscr{L}, \mathscr{L}')$ is isomorphic to $(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1})$, since Pic $\mathbb{A}^1 = 0$, So $(\mathscr{L}_1, \mathscr{L}_2) \equiv (\mathscr{L}_1, \mathscr{L}_2) \otimes (\mathscr{L}_1^{-1}, \mathscr{L}_1^{-1}) = (\mathcal{O}_{\mathbb{A}^1}, \mathscr{L})$ with $\mathscr{L}|_{\mathbb{A}^1 \backslash 0} = \mathcal{O}_{\mathbb{A}^1 \backslash 0}$. So $\mathscr{L}$ is the sheaf corresponding to a divisor $n \cdot 0$ for some integer n. It follows that Pic $X = \mathbb{Z}$, with every invertible sheaf isomorphic to $(\mathcal{O}_{\mathbb{A}^1}, \mathscr{L}(n \cdot 0))$ for a unique n. Global sections of $(\mathcal{O}_{\mathbb{A}^1}, \mathscr{L}(n\cdot 0))$ are pairs (f,g) with $f\in\Gamma(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}), g\in\Gamma(\mathbb{A}^1,\mathscr{L}(n\cdot 0))$ and $f|_{\mathbb{A}^1\backslash 0} = g|_{\mathbb{A}^1\backslash 0}$. It follows that $f=g$ so $\Gamma(X,(\mathcal{O}_{\mathbb{A}^1},\mathscr{L}(n\cdot 0)) = \Gamma(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1})\cap\Gamma(\mathbb{A}^1,\mathscr{L}(n\cdot 0))$, which is $k[t]$ if $n\geq 0$ and $(t^{-n})\subseteq k[t]$ if $n<0$. If $n<0$, then clearly no local ring of a point in $\mathbb{A}^1\backslash 0$ is generated by the images of global section. If $n>0$, then the local ring at the second origin is $\frac{1}{t^n}k[t]_{(t)}$, which is not generated by images of a global sections. And if $n=0$, then clearly images of global sections generate each local ring. Let $\mathscr{L}_n = (\mathcal{O}_{\mathbb{A}^1},\mathscr{L}(n\cdot 0))$. Then $\mathscr{L}_n\otimes\mathscr{L}_m = \mathscr{L}_{n+m}$ so no power of $\mathscr{L}_n$ is generated by global sections if $n\neq 0$. And $\mathscr{L}_1\otimes\mathscr{L}_0^{\otimes n}\cong\mathscr{L}_1$ is not generated by global sections for all n so X has no ample sheaf.

5. Let X be a noetherian scheme and let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves.

   (a) Let $\mathscr{L}$ be ample and $\mathscr{M}$ generated by global sections (gbgs). Then $\mathscr{M}^n$ is gbgs as well. Let $\mathscr{F}\in$ Coh (X). Then since $\mathscr{L}$ is ample, $\mathscr{F}\otimes\mathscr{L}^n$ is gbgs for $n\gg 0$. Then $\mathscr{F}\otimes(\mathscr{L}\otimes\mathscr{M})^n\cong(\mathscr{F}\otimes\mathscr{L}^n)\otimes\mathscr{M}^n$ is gbgs for $n\gg 0$ and thus $\mathscr{L}\otimes M$ is ample since gbgs $\otimes$ gbgs is gbgs.

   (b) Let $\mathscr{L}$ be ample. Then since $\mathscr{M}$ is coherent, $\mathscr{L}^{n_1}\otimes\mathscr{M}$ is gbgs. For any $\mathscr{F}\in\mathrm{Coh}(X)$, $\mathscr{F}\otimes\mathscr{L}^{n_2}$ is gbgs. Thus $\mathscr{F}\otimes(\mathscr{M}\otimes\mathscr{L}^n)\cong(\mathscr{F}\otimes\mathscr{L}^{n_1})\otimes(\mathscr{M}\otimes\mathscr{L}^{n_2})\otimes\mathscr{L}^{n-n_1-n_2}$ for $n\gg 0$. Since each term is gbgs, so is the entire tensor product and thus $\mathscr{M}\otimes\mathscr{L}^n$ is ample.

   (c) Let $\mathscr{L}$ and $\mathscr{M}$ be ample. Then for any coherent sheaf $\mathscr{F}$, $\mathscr{F}\otimes(\mathscr{L}\otimes\mathscr{M})^n\cong(\mathscr{F}\otimes\mathscr{L}^n)\otimes\mathscr{M}^n$, which is the tensor of sheaves gbgs since $\mathscr{L}$ and $\mathscr{M}$ are both ample, so is thus gbgs. Therefore $\mathscr{L}\otimes\mathscr{M}$ is ample.

   (d) $\mathscr{L}$ and $\mathscr{M}$ are finitely generated by global sections so there are corresponding morphisms to $\mathbb{P}^n_A$ and $\mathbb{P}^m_A$, say $\varphi_{\mathscr{L}}$ and $\varphi_{\mathscr{M}}$ such that $\varphi_{\mathscr{L}}$ is an immersion and $\varphi^*_{\mathscr{L}}(\mathcal{O}(1)) = \mathscr{L}$ and $\varphi^*_{\mathscr{M}}(\mathcal{O}(1)) = \mathscr{M}$. Let φ be the product of $\varphi_{\mathscr{L}}$ and $\varphi_{\mathscr{M}}$ corresponding to the Segre embedding. Then $\varphi^*(\mathcal{O}(1)) = \mathscr{L}\otimes\mathscr{M}$ and φ is an immersion since $\varphi_{\mathscr{L}}$ is. Let $\mathscr{L}\otimes\mathscr{M}$ is very ample.

   (e) Suppose that $\mathscr{L}^m$ is very ample and $\mathscr{L}^r$ is gbgs for $f\geq r_0$. Then by part (d), $\mathscr{L}^n$ is very ample for $n\geq m+r$.

6. *The Riemann-Roch Problem.* Let X be a non-singular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n>0$, we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine dim $|nD|$ as a function of n, and, in particular, its behavior for large n.

   (a) Let D be very ample and $\varphi_D : X\hookrightarrow\mathbb{P}^n_k$ the corresponding embedding in projective space. We may consider X as a subvariety of $\mathbb{P}^n_k$ with $D = \mathcal{O}_X(1)$. Let $S(X)$ be the homogeneous coordinate ring of X. The comment after the proof on pg 123 says that $\Gamma(X,\mathcal{O}_X(n))\cong\Gamma(X,\mathcal{O}_X(1)^n) = S_n$ for $n\gg 0$. Taking $n\gg 0$, we have that dim $|nD| = \dim_k\Gamma(X,\mathcal{O}_X(1)^n) - 1 = \dim_k S_n - 1 = P_X(n)-1$.

(b) Let D correspond to a torsion element of Pic X of order r. Then if $r \mid n$, $nD = 0$ and thus nD corresponds to $\mathcal{O}_X$. Thus $h^0(nD) = \dim \Gamma(X, \mathcal{O}_X) = 1$ and so $\dim |nD| = h^0(nD) - 1 = 0$.

If $r \nmid n$, then since r is the smallest positive integer such that $rD = 0$, we get that $nD \neq 0$. Now the fact that $\dim |nD| = -1$ will follow if we can show that $h^0(nD) = 0$, which is equivalent to showing that nD is not effective. So assume nD is effective. Since $nD \neq 0$,, $nD \sim E > 0$, where E is some effective divisor. Then multiplying both sides by r we get $0 \sim rnD \sim rE > 0$ which is a contradiction. Thus nD is not effective and we are done.

7. *Some Rational Surfaces.* Let $X = \mathbb{P}^2_k$ and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}_X(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) By ex 7.6.1, $\mathcal{O}(2)$ is very ample on $\mathbb{P}^2_k$ and thus gives an embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. To show that the image corresponds to the 2-uple embedding, fix a conic $D \in |\mathcal{O}(2)|$, where D is the zero locus of x^2. Then $\Gamma(\mathbb{P}^2_k, \mathscr{L}(D)) = \text{span } \{\frac{x^2}{x^2}, \frac{y^2}{x^2}, \frac{z^2}{x^2}, \frac{xy}{x^2}, \frac{xz}{x^2}, \frac{yz}{x^2}\}$. Thus the embedding corresponding to $|D|$ is $\varphi_{|D|}(x : y : z) = (\frac{x^2}{x^2} : \frac{y^2}{x^2} : \frac{z^2}{x^2} : \frac{xy}{x^2} : \frac{xz}{x^2} : \frac{yz}{x^2})$. Since we have homogeneous coordinates, we can clear denominators to get exactly the Veronese surface.

(b) To show that points are separated, consider the points $(a_0 : b_0 : c_0)$ and $(a_1 : b_1 : c_1)$. If $a_0 = 0$ and $a_1 \neq 0$, then the function x^2 separates points. If $a_0 = a_1 = 0$, then our sections are y^2, z^2 and yz. These are just the sections of the very ample sheaf $\mathcal{O}(2)$ on $\mathbb{P}^1$, so these sections separate points. This argument is similar for the other coordinate hyperplanes. Thus we can assume that our distinct points are off the coordinate hyperplanes and thus in any of the standard affine open sets we want. Picking the affine set $x = 1$ and our points $(\alpha, \beta), (\gamma, \delta)$, the functions $y^2 - \alpha^2(1)$ and $z^2 - \beta^2(1)$ separate all points except the case that $(\gamma, \delta) = (\pm\alpha, \pm\beta)$. For the case $(-\alpha, -\beta)$, use $y - yz - (\alpha - \alpha\beta)(1)$. The other cases are similar.

Now show tangent lines are separated. In the affine piece $z = 1$, we have $1, x^2, y^2, xy - y, x - y$. Let our point be (α, β). If $\alpha \neq 0$, the curves $x - y - (\alpha - \beta)(1)$ and $x^2 - \alpha^2(1)$ have no tangent lines in common. If $\beta \neq 0$, then $x - y - (\alpha - \beta)(1)$ and $y^2 - \beta^2(1)$ have no tangent lines in common. For the last case, if $\alpha = \beta = 0$, then $xy - y$ and $x - y$ have different tangent lines at the origin. So tangent lines are separated. The affine piece $y = 1$ is similar. In the piece $x = 1$, we have $1, y^2, z^2, y - yz, z - yz$, and $(0, 0)$ is the only point not dealt with. $y - yz$ and $z - yz$ have different tangent lines at $(0, 0)$, so all good.

(c) Let $Q, R \in \mathbb{P}^2$. If P, Q and R are not collinear, then the space of a line through P and Q does not go through R. If they are collinear, then by Bezout's Them, any conic through P and Q can not pass through R. Let $P = (0, 0, 1)$ Then $\delta = \text{span } \{x^2, y^2, xy, xz, yz\}$. In the affine pieces $x = 1$ and $y = 1$, the separation of tangent vectors is obvious as above. So δ gives an immersion $U \to \mathbb{P}^4$. To see $\widetilde{X} \to \mathbb{P}^4$ is a closed immersion, see (V, 4.1). The hyperplane divisors on $\widetilde{X} \subseteq \mathbb{P}^4$ are the strict transforms of conics in $\mathbb{P}^2$ through P. They intersect in three places if we choose two conics in $\mathbb{P}^2$ through P intersecting in four points transversally. So deg $\widetilde{X} = 3$. A line through P and a conic through P intersect in two places if chosen in general position. After blowing up P, they intersect in one point. So lines in $\mathbb{P}^2$ are sent to lines in $\mathbb{P}^4$. Lastly, separate lines through P separate after blowing up.

8. Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X and let $\pi : \mathbb{P}(E) \to X$ be the corresponding projective space bundle. Then by letting $Y = X$ in Proposition 7.12, we get that there is a natural 1-1 correspondence between sections of π and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

9. Let X be a regular noetherian scheme and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X.

   (a) [BLOG] There is a natural morphism $\alpha : \text{Pic } X \times \mathbb{Z} \to \mathbb{P}(\mathscr{E})$ defined by $(\mathscr{L}, n) \mapsto (\pi^* \mathscr{L}) \otimes \mathcal{O}(n)$. We claim that this gives the desired isomorphism. Let r be the rank of $\mathscr{E}$. Pick a point $i : x \hookrightarrow X$ and an open affine neighborhood U of x such that $\mathscr{E}$ is free. Let $k(x)$ be the residue field. On U we have $\pi^{-1}U = \mathbb{P}^{r-1}_U$ and so we obtain an embedding $\mathbb{P}^{r-1}_{k(x)} \to \mathbb{P}^{r-1}_U \to \mathbb{P}(E)$. Clearly, $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(n)|_U \cong \mathcal{O}_U(n)$ and we know that Pic $\mathbb{P}^{r-1}_{k(x)} = \mathbb{Z}$ so we have obtained a left inverse to $\mathbb{Z} \to \text{Pic } \mathbb{P}(E)$. So it remains to show that α is surjective, and that Pic $X \to$ Pic $\mathbb{P}(\mathscr{E})$ is injective.

   Injectivity: Suppose that $\pi^* \mathscr{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. Then by Proposition II.7.11 we see that $\pi_*(\pi^* \mathscr{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$ and by the Projection Formula we have $\mathscr{L} \otimes \pi_* \mathcal{O}(n) \cong \mathcal{O}_X$. Again by Prop II.7.II we know that $\pi_* \mathcal{O}(n)$ is the degree n part of the symmetric algebra on $\mathscr{E}$ and since rank $\mathscr{E} \geq 2$ this implies that $n = 0$ and $\mathscr{L} \cong \mathcal{O}_X$. Hence α is injective.

   Surjectivity: Let $\{U_i\}$ be an open cover of X for which $\mathscr{E}$ is locally trivial, and such that each U_i is integral and separated. We can find such a cover since every affine scheme is separated, and X is regular implies that the local rings are reduced. The subschemes $V_i := \mathbb{P}(\mathscr{E}|_{U_i}) \cong U_i \times \mathbb{P}^{r-1}$ form an open cover of $\mathbb{P}(\mathscr{E})$ and since X is regular, each U_i is regular, and in particular, regular in codimension one, and hence satisfies (*), so we can apply Ex II.6.1 to find that Pic $V_i \cong$ Pic $U_i \times \mathbb{Z}$.

Now if $\mathscr{L} \in \text{Pic } \mathbb{P}(\mathscr{E})$ then for each i, by restricting we get an element $\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i \in \text{Pic } V_i \cong \text{Pic } U_i \times \mathbb{Z}$ together with transition isomorphisms

$$\alpha_{ij} : (\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i)|_{V_{ij}} \to (\mathcal{O}_j(n_j) \otimes \pi_j^* \mathscr{L}_j)|_{V_j}$$

that satisfy the cocycle condition. These isomorphisms push forward to give isomorphisms

$$\alpha_{ij} : \pi_*(\mathcal{O}_i(n_i)|_{V_{ij}}) \otimes \mathscr{L}_i \to \pi_*(\mathcal{O}_j(n_j)|_{V_{ji}}) \otimes \mathscr{L}_j$$

via the projection formula. By Prop II.7.11 and considering ranks, we see that $n_i = n_j$. Furthermore, it can be seen from the definition of $\mathbb{P}(\mathscr{E})$ that $\mathcal{O}_j(n)|_{V_{ij}} = \mathcal{O}_{ij}(n)$ and so our isomorphism α_{ij} is $\mathcal{O}_{ij}(n) \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij}(n) \otimes \pi_j^* \mathscr{L}_j|_{V_{ij}}$. Tensoring this with $\mathcal{O}_{ij}(-n)$ we get isomorphisms $\mathcal{O}_{ij} \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij} \otimes \pi_j^* \mathscr{L}_j$ and the projection formula together with II.7.11 again tells us that we have isomorphisms $\beta_{ij} : \mathscr{L}_i|_{U_{ij}} \cong \mathscr{L}_j|_{U_{ij}}$, and it can be shown that these satisfy the cocycle condition as a consequence of the α_{ij} satisfying the condition. Hence we can glue the $\mathscr{L}_i$, together to obtain a sheaf $\mathscr{M}$ on X such that $\pi * M \otimes \mathcal{O}(n)$ is isomorphic to $\mathscr{L}$ on each connected component of X (where n depends on the component.)

(b) Suppose first that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$. Let $f : \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ be an isomorphism. By (a) we may write $f^*(\mathcal{O}'(1)) = \mathcal{O}(1) \otimes \pi^* \mathscr{L}$ for some $\mathscr{L} \in \text{Pic } X$. By Ex 5.1.d and II.7.11, $\mathscr{E}' = \pi_*^{'}(\mathcal{O}'(1)) = \pi_*(\mathcal{O}(1) \otimes \pi^* \mathscr{L}) = \pi_* \mathcal{O}(1) \otimes \mathscr{L} = \mathscr{E} \otimes \mathscr{L}$.

Now Suppose $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$. By (7.11b) we get a surjection $\pi^* \mathscr{E}' \cong \pi^* \mathscr{E} \otimes \pi^* \mathscr{L} \twoheadrightarrow \mathcal{O}(1) \otimes \pi^* \mathscr{L}$, which gives a map $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ by (7.12). Writing $\mathscr{E} \cong \mathscr{E}' \otimes \mathscr{L}^{-1}$, we similarly get a map in the opposite direction inverse to the first.

10. $\mathbb{P}^n$*-bundles Over a Scheme* Let X be a noetherian scheme.

(a) Super

(b) Let $\mathscr{E}$ be a locally free sheaf of rank $n+1$ on X. Then on an open affine set $U = \text{Spec } A$ on X, we get that $\mathscr{E} \cong \mathcal{O}|_U^{\oplus(n+1)}$. If $\pi : \mathbb{P}(\mathscr{E}) \to X$ is the natural morphism, $\pi^{-1}(U) \cong \text{Proj } \mathscr{S}(\mathscr{E})(U) \cong \text{Proj } \mathscr{S}(\mathcal{O}|_U^{\oplus(n+1)})(U) = \text{Proj } A[x_0, \ldots, x_n] = \mathbb{P}_U^n$. These constructions glues to give a $\mathbb{P}^n$-bundle over X.

(c)

(d) We want to show the following 1:1 equivalence for X regular:

$$\{\mathbb{P}^n\text{-bundles over } X\} \overset{1:1}{\leftrightarrow} \{\text{Locally free sheaves } \mathscr{E}/\sim \text{ of rank } n+1\}$$

where $\mathscr{E} \sim \mathscr{E}'$ iff $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X. But this follows immediately from parts (b)(c) and ex II.7.9.

11. On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown-up schemes

    (a) Let $\mathscr{I}$ be a coherent sheaf of ideals on X. Let $U \subseteq X$ be an open affine set. Then locally, the blow-up of $\mathscr{I}$ is Proj $(\bigoplus_{n\geq 0} \mathscr{I}(U)^n)$ and locally the blow-up of $\mathscr{I}^d$ is Proj $(\bigoplus_{n\geq 0} \mathscr{I}(U)^{nd})$. These are isomorphic by Ex II.5.13. Gluing gives the global isomorphism $\mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n) \cong \mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^{nd})$ as desired.

    (b) This is exactly Lemma II.7.9

    (c)

12. [BLOG] Let X be a noetherian scheme and let Y, Z be two closed subschemes, neither one containing the other. Let $\widetilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $I_Y + I_J$). Suppose they do meet at some point $P \in \widetilde{Y} \cap \widetilde{Z} \subset \widetilde{X}$. Then $\pi(P)$ is contained in some open affine set $U =$ Spec A, and the preimage of this open is $\pi^{-1}U =$ Proj $(\bigoplus_{d\geq 0}(I_Y(U) + I_Z(U))^d)$. Then $Y \cap U =$ Spec A/I_Y and $Z \cap U =$ Spec A/I_Z. Then $\pi^{-1}(U \cap Y) =$ Proj $(\bigoplus_{d\geq 0}((I_Y + I_Z)(A/I_Y)(U)^d)) \subset \widetilde{Y}$ and similar for Z. The closed embedding $\pi^{-1}(U \cap Y) \to \pi^{-1}(U)$ is given by a homomorphism of homogeneous rings $\bigoplus_{d\geq 0}(I_Y + I_Z)^d \to \bigoplus_{d\geq 0}((I_Y + I_Z)(A/I_Y))^d$ and similarly for Z. Clearly the kernel of this ring homomorphism is the homogeneous ideal $\bigoplus_{d\geq 0} I_Y^d$ and similarly for Z. Now if the two closed subschemes intersect as assumed, then there exists a homogeneous prime ideal of $\bigoplus_{d\geq 0}(I_Y + I_Z)^d$ that contains both of these homogeneous ideals. But $\bigoplus_{d\geq 0} I_Y^d$ and $\oplus_{d\geq 0} I_Z^d$ generate $\bigoplus_{d\geq 0} I_Z^d$ generate $\bigoplus_{d\geq 0}(I_Y + I_Z)^d$ so there can be no proper homogenous prime ideal containing them both. Hence the intersection is empty.

13. *A Complete Non-projective Variety*

    (a)

    (b)

    (c)

    (d)

14. (a) Consider $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1_k}(-1)$ on $\mathbb{P}^1$. Since $\mathcal{O}(-1)$ is invertible, $\mathbb{P}(\mathcal{O}(-1)) \cong \mathbb{P}^1$ and the natural morphism is $\pi : \mathbb{P}^1 \cong \mathbb{P}^1$. If the sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{O}(\mathbb{P}^1)) \cong \mathbb{P}^1$ were very ample, it would give rise to a projective immersion $\varphi_{|\mathcal{O}(1)|}\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n = \mathbb{P}^n_{\mathbb{P}^1}$. Then $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_P(1) = \mathcal{O}_{\mathbb{P}^1}(-1)$, which is a contradiction, since the pullback of effective divisors under an immersion is effective.

    (b) Let $f : X \to Y$ be a morphism of finite type and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $\mathscr{L}$ is ample relative to Y and for some $n > 0$, $\mathscr{L}^n$ is very ample on X relative to Y. If $\pi : P \to X$ is the projection, then by Prop 7.10, $\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^m$ is very ample on P relative to X for $m \gg 0$. Thus by Ex 5.12, for n fixed and $m \gg 0$, $\mathcal{O}_P(1) \otimes \mathscr{L}^{m+n}$ is very ample on P relative to Y.

2.8 Differentials

1. Let X be a scheme.

   (a) Let $(B, \mathfrak{m})$ be a local ring containing a field k, and assume that the reside field $k(B) = B/\mathfrak{m}$ of B is a separable generated extension of k. To show the exact sequence

   $$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

   is exact on the left is equivalent to showing that $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B)$ is injective. This in turn is equivalent to showing that the dual map

   $$\delta^* : \mathrm{Hom}_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \to \mathrm{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

   is surjective. The term on the left is isomorphic to $\mathrm{Hom}_B(\Omega_{B/k}, k(B)) \cong \mathrm{Der}_k(B, k(B))$. If $d : B \to k(B)$ is a derivation, then $\delta^*(d)$ is obtained by restricting to $\mathfrak{m}$ and noting that $d(\mathfrak{m}^2) = d(\sum a_i c_i) = \sum (a_i d(c_i) + d(a_i) c_i) = 0$ for $a_i, c_i \in \mathfrak{m}$. Now to show that δ^* is surjective, let $h \in \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k(B))$. For any $b \in B$, write $b = c + \lambda$ with $\lambda \in k(B), c \in \mathfrak{m}$ in the unique way using the section $k(B) \to B \to k(B)$ from Thm 8.25A. Define $db = h(\bar{c})$, where $\bar{c} \in \mathfrak{m}/\mathfrak{m}^2$ is the image of c. Then one verifies immediately that d is a $k(B)$-derivation and that $\delta^*(d) = h$. Thus δ^* is surjective as required.

   (b) With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Assume that $\Omega_{B/k}$ is free of rank = dim B+ tr.d. $k(B)/k$. By part a) we have the short exact sequence

   $$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

   Thus

   $$\begin{array}{ccc} \dim \Omega_{B/k} \otimes k(B) & = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \Omega_{k(B)/k} \\ \shortparallel \textit{ (by assumption)} & \qquad \shortparallel \textit{ (Thm 8.6A since k perfect)} \\ \dim B + \text{tr.d. } k(B)/k & = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \text{tr.d. } k(B)/k \end{array}$$

   Thus $\dim B = \dim \mathfrak{m}/\mathfrak{m}^2$ and B is regular.

   Conversely, assume that $(B, \mathfrak{m})$ is a regular local ring, where now B is a localization of an algebra of finite type over k. Let $B = A_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Let K be the quotient field of B. Then by part a),

$$\begin{aligned}\dim_{k(B)} \Omega_{B/k} \otimes k(B) &= \dim \mathfrak{m}/\mathfrak{m}^2 + \Omega_{k(B)/k} \\ &= \dim B + \text{tr.d. } k(B)/k \text{ by Thm 8.6A} \\ &= \dim_K \Omega_{B/k} \otimes_B K \text{ by claim below}\end{aligned}$$

Then by Lemma 8.9, $\Omega_{B/k}$ is free of rank dim B + tr.d $k(B)/k$.
Proof of claim: By (II.8.2A), $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$ and since k is perfect, K is a separately generated extension by Thm 1.4.8A. Then $\dim \Omega_{K/k} =$ tr.d. K/k by Thm 8.6A. Thus:

$$\begin{aligned}\dim_K \Omega_{B/k} \otimes_B K &= \text{tr.d. } K/k \\ &= \dim A \\ &= ht\,\mathfrak{p} + \dim A/\mathfrak{p} \\ &= \dim B + \dim A/\mathfrak{p} \\ &= \dim B + \text{tr.d. Frac}(A/\mathfrak{p})/k \\ &= \dim B + \text{tr.d. } k(B)/k\end{aligned}$$

(c) Let X be an irreducible scheme of finite type over a perfect field k and let dim $X = n$. Let $x \in X$ be a point not necessarily closed. Let Spec A be an open affine neighborhood of x and define $B = A_{\mathfrak{p}} = \mathcal{O}_{X,x}$. By part b), $\mathcal{O}_{X,x}$ is a regular local ring if and only if $\Omega_{B/k} \cong \Omega_{A/k} \cong (\Omega_{X/k})_x$ is free of rank dim B+ tr.d $k(B)/k =$ dim $A =$ dim $X = n$.

(d) Let X be a variety over an algebraically closed field k. Let $U = \{x \in X \,|\, \mathcal{O}_x$ is a regular local ring $\}$. U is dense since it contains the open dense set of Cor 8.16. If $x \in U$, then $(\Omega_{X,x})_x$ is free by part c) so there exists an open neighborhood V of x such that $\Omega_{X/k}|_V$ is free of rank dim X by ex II.5.7(a). Using c) again, $V \subseteq U$ and thus U is open.

2. Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Define $Z \subseteq X \times V$ by $\{(x, s) \,|\, s_x \in m_x \mathscr{E}_x\}$. Let $p_1 : X \times V \to X$ and $p_2 : X \times V \to V$ be the projections restricted to Z. Then for all $x \in X$, the fiber of the first projection $p_1^{-1}(x_0) = \{(x_0, s) \,|\, s_{x_0} \in \mathfrak{m}_{x_0} \mathscr{E}_{x_0}\}$. This is the set of sections that vanish at x_0, which is the kernel of the $k(x_0)$-vector space map $V \otimes_k k(x_0) \to \mathscr{E}_{x_0} \otimes_{\mathcal{O}_{x_0}} k(x_0) \cong \mathscr{E}_{x_0} \otimes_{\mathcal{O}_{x_0}} \mathcal{O}_{x_0}/\mathfrak{m}_{x_0} \cong \mathscr{E}_{x_0}/\mathfrak{m}_{x_0}\mathscr{E}_{x_0}$. Since $\mathscr{E}$ is generated by global sections, this map is surjective, so since $\mathscr{E}$ is locally free of rank r, dim $V -$ dim ker $=$ rk $\mathscr{E}_{x_0} = r$. Thus $\dim \ker = \dim V - r$. Therefore $\dim Z = \dim X + \dim V - r$. Since we are assuming that $r > n$, $\dim Z = n + \dim V - r < \dim V$. Thus the second projection $p_2|_Z : Z \to V$ can not be surjective. Any $s \in V$ not in the image then has the desired property.

The morphism $\mathcal{O}_x \to \mathscr{E}$ is then defined by multiplication by this s as above. By looking at stalks, we see that the cokernel $\mathscr{E}'$ is locally free using (Ex II.5.7(b)) with rank $\mathscr{E}' =$ rk $\mathscr{E} - 1$.

3. *Product Schemes*

   (a) Let X and Y be schemes over another scheme S. By (8.10) and (8.11) we get exact sequences

$$p_2^*\Omega_{Y/S} \to \Omega_{X\times Y/S} \to p_1^*\Omega_{X/S} \to 0$$
$$p_1^*\Omega_{X/S} \to \Omega_{X\times Y/S} \to p_2^*\Omega_{Y/S} \to 0$$

   The existence of the second map in the second sequence gives injectivity of the first map in the first sequence. The first map in the second sequence gives a section of the second map in the first sequence. So the first sequence is short exact and splits, giving $\Omega_{X\times Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$ as desired.

   (b) Let X and Y be nonsingular varieties over a field k. Then starting from the short exact sequence of part a),

$$0 \to p_1^*\Omega_{X/k} \to \Omega_{X\times Y/k} \to p_2^*\Omega_{Y/k} \to 0$$

   take the highest exterior power of each term to get

$$\bigwedge^{\dim X \dim Y} \Omega_{X\times Y/k} \cong \bigwedge^{\dim X} p_1^*\Omega_{X/k} \otimes \bigwedge^{\dim Y} p_2^*\Omega_{Y/k}$$

   Then by (Ex I.5.16(e)), exterior powers commute with pullbacks, and we get that $\omega_{X\times Y} \cong p_1^*(\omega_X) \otimes p_2^*(\omega_Y)$.

   (c) Let Y be a nonsingular plane cubic curve and let X be the surface $Y \times Y$. By (8.20.3), $\omega_Y \cong \mathcal{O}_Y$, so $\omega_{Y\times Y} = p_1^*\mathcal{O}_Y \otimes p_2^*\mathcal{O}_Y \cong \mathcal{O}_{Y\times Y}$. Then $\dim_k \Gamma(Y \times Y, \mathcal{O}_{Y\times Y}) = 1$ so $p_g(Y \times Y) = 1$. By (Ex 1.7.2), $p_a(Y) = \frac{1}{2}(3-2)(3-1) = 1$. Then by part e) of the same exercise, $p_a(Y \times Y) = 1 \cdot 1 - 1 - 1 = -1$.

4. *Complete Intersections* A closed subscheme Y of $\mathbb{P}^n_k$ is called a (*strict, global) complete intersection)* if the homogeneous ideal I_Y of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n)$ elements.

   (a) Let Y be a closed subscheme of codimension r in $\mathbb{P}^n$. If $I_Y = (f_1, \ldots, f_r)$, then it is obvious that $Y = \bigcap_{i=1}^r H_i$, where $H_i = \mathcal{Z}(f_i)$. Conversely, Let $Y = \bigcap^r H_i$, where we can assume that each H_i is irreducible and reduced. Now, since the homogeneous coordinate ring $S = k[x_0, \ldots, x_n]$ of $\mathbb{P}^n$ is factorial, the irreducibility of each H_i implies that (I_{H_i}) is a prime ideal. Thus $I_{H_{i+1}}$ is a non zero-divisor mod I_{H_i}; that is $(I_{H_1}, I_{H_2}, \ldots, I_{H_r})$ is a regular sequence. Now $S/(I_{H_1}, \ldots, I_{H_r})$ has degree $\sum \deg H_i$ by Bezout's Theorem. Since $(I_{H_1}, \ldots, I_{H_r})$ is contained in I_Y, we must have $(I_{H_1}, \ldots, I_{H_r}) = I \cap J$, where codim $J > 2$. But by the Unmixedness Theorem, the ideal $(I_{H_1}, \ldots, I_{H_r})$ has no primary components of codimension > 2, so $J = \emptyset$ and thus $I_Y = (I_{H_1}, \ldots, I_{H_r})$.

(b) Let Y be a complete intersection of dimension ≥ 1 in $\mathbb{P}^n$ and let Y be normal. Then the singular locus Sing Y has codimension ≥ 2 and thus the singular locus of the affine cone $C(Y)$ over Y has codimension ≥ 2. Thus the homogeneous coordinate ring of $S(C(Y))$ is integrally closed by (8.23b) and thus so is $S(Y)$. So Y is projectively normal.

(c) Since Y is projectively normal, by (Ex 5.14b), $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y(l))$. In particular, taking $l = 0$ gives that $k \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y)$, so $\Gamma(Y, \mathcal{O}_Y) = k$ and thus Y is connected.

(d) Now let $d_1, \ldots, d_r \geq 1$ be integers, with $r < n$. Then by applying Ex 8.20.2 r times, we ge the existence of nonsingular hypersurfaces $H_1, \ldots, H_r \subset \mathbb{P}^n$ with deg $H_i = d_i$ such that $Y = \bigcap^r H_i$. Y is irreducible since by part c) it is connected and nonsingular.

(e) Let Y be a nonsingular complete intersection as in (d). Then by the adjunction formula, we immediately get that $\omega_Y = \mathcal{O}_Y(\sum d_i - n - 1)$. For example, if $Y = H_1 \cap H_2$, then $K_Y \sim (K_{P^n} + Y)|_Y = (-n - 1)H + H_1|_{H_2} + H_2|_{H_1} = (-n-1)H + \deg H_1 + \deg H_2$.

(f) Let Y be a nonsingular hypersurface of degree d in $\mathbb{P}^n$. Then by adjunction, $K_Y = (K_{\mathbb{P}^n} + Y)|_Y$ which gives that $K_Y \sim (-n-1+d)H$. Let $I_Y = (f)$, where f has degree d. Then we have the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$$

Twisting by $(-n-1+d)$ and applying the functor Γ we get the short exact sequence:

$$0 \to \Gamma(Y, I_Y(d-n-1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) \to \Gamma(Y, \mathcal{O}_Y(d-n-1) \to 0$$

Note that the sequence is exact on the right by part c). Comparing dimensions we get:

$$\dim_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) = \dim_k \Gamma(Y, I_Y(d-n-1)) + \Gamma(Y, \mathcal{O}_Y(d-n-1))$$

which is equivalent to

$$\binom{d-1}{n} = 0 + p_g(Y)$$

(g) Let Y be a nonsingular curve in $\mathbb{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e. Then by e), we have $K_Y \sim (d+e-4)$ and by a) we have $I_Y = (f, g)$, where $Y = \mathcal{Z}(f) \cap \mathcal{Z}(g)$. By similar arguments as in part f), we get that

$$p_g(Y) = \dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) - \dim_k \Gamma(Y, I_Y(d+e-4))$$

Now, $\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) = \binom{d+e-1}{3} = \frac{(d+e-1)(d+e-2)(d+e-3)}{6}$. Since $I_Y = (f, g)$, then any element of $I_Y(d+e-4)$ is of the form

$h_1 f + h_2 g$ where the degree of h_1 is $e-4$ and the degree of h_2 is $d-4$. Thus the dimension of the global section of $I_Y(d+e-4) = \binom{e-1}{3} + \binom{d-1}{3}$. So

$$\begin{aligned} p_g(Y) &= \dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) - \dim_k \Gamma(Y, I_Y(d+e-4)) \\ &= \tbinom{d+e-1}{3} - \tbinom{e-1}{e} - \tbinom{d-1}{3} \\ &= \tfrac{(d+e-1)(d+e-2)(d+e-3)}{6} - \tfrac{(e-1)(e-2)(e-3)}{6} - \tfrac{(d-1)(d-2)(d-3)}{6} \\ &= \tfrac{3d^2e+ede^2-12de+6}{6} \\ &= \tfrac{1}{2}de(d+e-4)+1 \end{aligned}$$

Note, for a nonsingular curve, $p_g = p_a$ always by Serre Duality (Ch 4).

5. *Blowing up a Nonsingular Subvariety* As in (8.24), let X be a nonsingular subvariety and let Y be a nonsingular subvariety of codimension $r \geq 2$. Let $\pi : \widetilde{X} \to X$ be the blowing-up of X along Y. Let $Y' = \pi^{-1}(Y)$.

   (a) By (6.5c), we get a sequence

   $$\mathbb{Z} \to \mathrm{Cl}\ \widetilde{X} \to \mathrm{Cl}\ \widetilde{X} - Y' \to 0$$

   Now, since π is an isomorphism outside of Y', Cl $\widetilde{X} - Y' \cong$ Cl $X - Y \cong$ Cl X since $\operatorname{codim}(Y, X) \geq 2$. Then the map $\pi^*\mathrm{Cl}\ X \to \mathrm{Cl}\ \widetilde{X}$ gives a section of the above sequence, so we only need to verify that $\mathbb{Z} \to \mathrm{Cl}\ \widetilde{X}$ is injective. If $nY' \sim 0$ for some $n > 0$, then there exists some $f' \in k(\widetilde{X})$ with a zero of order n along Y'. But $\widetilde{X} \to X$ is surjective and birational, so f' corresponds to a regular function f on X with zeros only along Y. Since $\operatorname{codim}_X Y \geq 2$, this is a contradiction. Thus the sequence is split short exact and Cl $\widetilde{X} \cong$ Cl $X \oplus \mathbb{Z}$.

   (b) Following the hint, by part a) write $\omega_{\widetilde{X}}$ as $f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M}$ on X and some integer q. Now, $X - Y \cong \widetilde{X} - Y'$, so $\omega_{\widetilde{X}}|_{\widetilde{X}-Y'} \cong \omega_X|_{X-Y}$. Pic $X \cong$ Pic U (II.6.5) so $\mathscr{M} \cong \omega_X$. By adjunction:

   $$\begin{aligned} \omega_{Y'} &\cong \omega_{\widetilde{X}} \otimes \mathscr{L}(Y') \otimes \mathcal{O}_{Y'} \\ &\cong f^*\omega_X \otimes \mathscr{L}((q+1)Y') \otimes \mathcal{O}_{Y'} \\ &\cong f^*\omega_X \otimes \mathcal{I}_{Y'}^{-q-1} \otimes \mathcal{O}_{Y'} \ \textit{(Prop II.6.18)} \\ &\cong f^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(1)^{-q-1} \otimes \mathcal{O}_{Y'} \ \textit{(7.13)} \\ &\cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1) \end{aligned}$$

   Now take a closed point $y \in Y$ and let Z be the fiber of Y' over y, ie $Z = y \times_Y Y'$. By (Ex II.8.3b),

   $$\begin{aligned} \omega_Z &\cong \pi_1^*\omega_y \otimes \pi_2^*\omega_{Y'} \\ &\cong \pi_1^*\mathcal{O}_y \otimes \pi_2^*(f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) \\ &\cong \pi_2^*(f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) \\ &\cong \mathcal{O}_y \otimes \pi_2^*\mathcal{O}_{Y'}(-q-1)) \\ &\cong \mathcal{O}_Z(-q-1) \end{aligned}$$

Z is just $\mathbb{P}^{r-1}$, so $\omega_Z \cong \mathcal{O}_Z(-r)$. Thus $q = r - 1$.

6. *The Infinitesimal Lifting Property* [BLOG]

   (a) Since g and g' both lift f, the difference $g - g'$ is a lift of 0, and therefore the image lands in the submodule I of B'. The homomorphisms g and g' are algebra homomorphisms and so they both send 1 to 1. Hence the difference sends 1 to 0 and so for any $c \in k$, we have $\theta(k) = k\theta(1) = 0$. For the Leibnitz rule we have

$$\begin{aligned}\theta(ab) &= g(ab) - g'(ab)\\ &= g(a)g(b) - g'(a)g'(b)\\ &= g(a)g(b) - g'(a)g'(b) + (g'(a)g(b) - g'(a)g(b))\\ &= g(b)\theta(a) + g'(a)\theta(b)\end{aligned}$$

   We can consider it as an element of $\mathrm{Hom}_A(\Omega_{A/k,I}, I)$ by the universal property of the module of relative differentials.

   Conversely, for any $\theta \in \mathrm{Hom}_A(\Omega_{A/k,I}, I)$, we obtain a derivation $\theta \circ d : A \to I$ which we can compose with the inclusion $I \hookrightarrow B'$ to get a k-linear morphism from A into B'. Since the sequence is exact, this θ vanishes on composition with $B' \to B$ and so $g + \theta$ is another k-linear homomorphism lifting f. We just need to show that it is actually a morphism of k-algebras; that is, that is preserves multiplication:

$$\begin{aligned}g(ab) + \theta(ab) &= g(ab) + \theta(a)g(b) + g(a)\theta(b)\\ &= g(ab) + \theta(a)g(b) + g(a)\theta(b) + \theta(a)\theta(b) \text{ since } I^2 = 0 \text{ and } \theta(a), \theta(b) \in I\\ &= (g(a) + \theta(a))(g(b) + \theta(b))\end{aligned}$$

   (b) A k-homomorphism out of P is uniquely determined by the images of the x_i, which can be anything. So for each i, choose a lift b_i of $f(x_i)$ in B' and we obtain a morphism h by sending x_i to b_i and extending to a k-algebra homomorphism. if $a \in P$ is in J, then by commutivity, the image of $h(a)$ in B will be 0, implying that $h(a) \in I$ so we have at least a k-linear map $J \to I$. If $a \in J^2$ then $h(a) \in I^2 = 0$ so this map descends to $\overline{h} : J/J^2 \to I$. The last thing to check is that the map $\overline{h}$ is A-linear, which follows from h preserving multiplication.

   (c) Applying the global section functor to the exact sequence of (8.17) with $X = \mathrm{Spec}\ P$, $Y = \mathrm{Spec}\ A$ gives an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

   which is exact on the right as well by (8.3A). Now, since A is nonsingular, $\Omega_{A/k}$ is locally free and therefore projective so $\mathrm{Ext}^i(\Omega_{A/k}, I) = 0$ for all $i > 0$. So the exact sequence

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) \to \mathrm{Hom}_A(J/J^2, I) \to \mathrm{Ext}^1_A(\Omega_{A/k}, I) \to \dots$$

shows that $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) \to \mathrm{Hom}_A(J/J^2, I)$ is surjective. So we can find a P-morphism $\theta : \Omega_{P/k} \to I$ whose image is $\overline{h}$ from part (b). We then define θ' as the composition $P \xrightarrow{d} \Omega_{P/k} \to I \to B'$ to obtain a k-derivation $P \to B'$. Let $h' = h - \theta$. For any element $b \in J$, we have $h'(b) = h(b) - \theta(b) = \overline{h}(b) - \overline{h}(b) = 0$, so h' descends to a morphism $g : A \to B'$ which lifts f.

7. [BLOG] Let X be affine and nonsingular. Let $\mathscr{F}$ be a coherent sheaf on X. This problem is then equivalent to the following: Given a ring A', an ideal $I \subset A'$ such that $I^2 = 0$ and $A'/I \cong A$, such that $I \cong M$ as an A-module (where M is the finitely generated A-module corresponding to $\mathscr{F}$), show that $A' \cong A \oplus M$ as an abelian group, with multiplication defined by $(a, m)(a', m') = (aa', am' + a'm)$.

   Using the infinitesimal lifting property, we obtain a morphism $A \to A'$ that lifts the given isomorphism $A'/I \cong A$. This together with the given data provides the isomorphism $A \oplus M \cong A'$ of abelian groups where we use the isomorphism $M \cong I$ to associate M with I as an A-module. If $a \in A$, then $(a, 0)(a', m') = (aa', am')$ using the A-module structure on A and $M \cong I$. If $m \in M \cong I$, then $(0, m)(a', m') = (0, a'm)$ since $mm' \in I^2$. So we have the required isomorphism.

8. This follows exactly as the proof of (8.19).

2.9 Formal Schemes

(*skip*)

3 Chapter 3: Cohomology

3.1 Derived Functors

3.2 Cohomology of Sheaves

1. (a) Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X. Let $U = X - \{P, Q\}$. Then $\mathbb{Z}_U$ is a subsheaf of Z_X so we have a short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_{\{P,Q\}} \to 0$$

   Taking cohomology gives a long exact sequence

$$0 \to \Gamma(X, \mathbb{Z}_U) \to \Gamma(X, \mathbb{Z}) \to \Gamma(X, \mathbb{Z}_{\{P,Q\}}) \to H^1(X, \mathbb{Z}_U) \to \dots$$

   If we assume $H^1(X, \mathbb{Z}_U) = 0$, we have the equivalent long exact sequence:

$$0 \to \Gamma(X, \mathbb{Z}_U) \to \mathbb{Z} \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z} \to 0 \to \dots$$

   But this would imply that $\mathbb{Z}$ surjects onto $\mathbb{Z} \oplus \mathbb{Z}$ which is a contradiction. So $H^1(X, \mathbb{Z}_U) \neq 0$.

   (b)

2. Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Then since $\mathbb{P}^1$ is connected (simply connected in fact), the constant sheaf $\mathcal{K}$ is flasque. From (II, Ex. 1.21d), we can write the quotient sheaf $\mathcal{K}/\mathcal{O}$ as the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$. Since skyscraper sheaves are trivially flasque, we have a flasque resolution of $\mathcal{O}_{\mathbb{P}^1}$ as desired.

   To show that $\mathcal{O}_{\mathbb{P}^1}$ is acyclic, apply Γ to the flasque resolutions. The resulting sequence is exact by (II,Ex 1.21e) so all higher cohomology vanishes and so $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for all $i \geq 0$. Note, for $i \geq 2$ this result follows immediately from Grothendieck vanishing and for $i = 1$, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ by either looking at the long exact sequence or from Serre Duality.

3. *Cohomology with Supports:* Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y.

   (a) Let

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

   be a short exact sequence of sheaves. Clearly $\Gamma_Y(X, \mathscr{F}') \subseteq \Gamma_Y(X, \mathscr{F})$, so the functor $\Gamma_Y(X, \cdot)$ preserves injections.

   Now let $s \in \Gamma_Y(X, \mathscr{F})$ be sent to 0 in $\Gamma_Y(X, \mathscr{F}'')$. We can view s as an element of $\Gamma(X, \mathscr{F})$ that gets sent to zero in $\Gamma(X, \mathscr{F}'')$. Since $\Gamma(X, \cdot)$ is left exact, s is the image of some $s' \in \Gamma(X, \mathscr{F}')$. To show that $\Gamma_Y(X,\cdot)$ is left exact, we have to show that $s'_x = 0$ for all $x \notin Y$. Let $x \in X \backslash Y$. Considering the short exact sequence of stalks

$$0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$$

we see that $s_x = 0$ since $s \in \Gamma_Y(X, \mathscr{F})$. Thus $s'_x = 0$ and so $s' \in \Gamma_Y(X, \mathscr{F}')$ as desired and the functor $\Gamma_Y(X,\cdot)$ is left exact.

(b) Let
$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$
be a short exact sequence of sheaves with $\mathscr{F}'$ flasque. By part (a), $\Gamma_Y(X,\cdot)$ is left exact, so we just need to show the map $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(X,\mathscr{F}'')$ is surjective. Let $s'' \in \Gamma_Y(X,\mathscr{F}'')$ and view s'' as an element of $\Gamma(X,\mathscr{F}'')$. Since $\mathscr{F}'$ is flasque, the map $\Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}'')$ is surjective and we can lift s'' to a section $s \in \Gamma(X,\mathscr{F})$. Thus for all $p \in U := X\backslash Y, s_p \in \mathscr{F}'_p$. Therefore $s|_U \in \Gamma(U,\mathscr{F}')$. Since $\mathscr{F}'$ is flasque, similar as before we can lift $s|_U$ to a section $s' \in \Gamma(X,\mathscr{F}')$. Clearly $s'_p = s_p$ for all $p \in U$. Therefore $s - s' \in \Gamma_Y(X,\mathscr{F})$ and $s - s'$ is mapped to $s'' - 0 = s''$. Thus $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(X,\mathscr{F}'')$ is surjective

(c) Copy the proof of Prop III.2.5 and use part b).

(d) Obvious

(e) Using the maps of (d), we get a short exact sequence of chain complexes $\Gamma_Y(X, I^\bullet), \Gamma(X, I^\bullet)$ and $\Gamma(X - Y, I^\bullet)$, where $I^\bullet$ is an injective resolution of $\mathscr{F}$. This gives the long exact sequence of cohomology.

(f) For any sheaf $\mathscr{F}, \Gamma_Y(X,\mathscr{F}) = \Gamma_Y(V,\mathscr{F}|_V)$, where V is an open subset of X containing Y. Therefore, applying the functors $\Gamma_Y(X,\cdot)$ and $\Gamma_Y(V,\cdot|_V)$ to an injective resolution of a sheaf gives the same complex and thus the same cohomology group.

4. *Mayer-Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Given $\mathscr{F} \in \mathfrak{Ab}(X)$, let $0 \to \mathscr{F} \to I_0 \to I_1 \to \dots$ be an injective resolution of $\mathscr{F}$ where each I_i is constructed using the method of Prop 2.2. That is, each I_i is a direct product of sheaves with support a single point. Then

$$0 \to \Gamma_{Y_1 \cap Y_2}(X, I_i) \to \Gamma_{Y_1}(X, I_i) \oplus \Gamma_{Y_2}(X, I_i) \to \Gamma_{Y_1 \cup Y_2}(X, I_i) \to 0$$

is a short exact sequence. The only hard part is to show surjectivity, which follows from the structure of the I_i. Thus we get the long exact sequence of cohomology from the above short exact sequence by applying the Snake Lemma.

5. Let X be a Zariski space. Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the local space of X at P and give it the induced topology. Let $j : X_P \hookrightarrow X$ be the inclusion and for any sheaf $\mathscr{F}$, let $\mathscr{F}_P = j^*\mathscr{F}$. The

claim is that $\Gamma_P(X,\mathscr{F}) = \Gamma_P(X,\mathscr{F}_P)$. Any open set containing P contains X_P, so the gluing property of sheaves does not affect $\Gamma(X_P,\mathscr{F}_P) = \overset{\lim}{\rightarrow}_{U\ni P}$ $\Gamma(U,\mathscr{F})$. Given $s \in \Gamma_P(X,\mathscr{F})$ we clearly get a section $\overline{s} \in \Gamma_P(X_P,\mathscr{F}_P)$. Given $\overline{s} \in \Gamma_P(X_P,\mathscr{F}_P)$, let $s \in \Gamma(U,\mathscr{F})$ represent it. By taking a smaller U, we may assume the support of s is P. Then glue S and $0 \in \Gamma(X\backslash P,\mathscr{F})$ together to get a global section. So we have a bijection $\Gamma_P(X,\mathscr{F}) \leftrightarrow \Gamma_P(X_P,\mathscr{F}_P)$. If $0 \to \mathscr{F} \to I_0 \to I_1 \to \ldots$ is a flasque resolution of $\mathscr{F}$, then $0 \to \mathscr{F}_P \to I_{0,P} \to I_{1,P} \to \ldots$ is an injective resolution of $\mathscr{F}_P$ and we can repeat the same argument to show $\Gamma_P(X,I_i) \cong \Gamma_P(X_P,I_{i,p})$. So the cohomology groups are equal.

6. Let X be a noetherian topological space and let $\{\mathcal{I}_\alpha\}_{\alpha\in A}$ be a direct systems of injective sheaves of abelian groups on X. For the first claim in the hint, "$\Rightarrow$" is the definition of injective. Suppose the second condition holds and $\mathscr{F}$ is a subsheaf of $\mathscr{G}$. Let $f : \mathscr{F} \to \mathcal{I}$ be a morphism of sheaves and H a subsheaf of $\mathscr{G}$ maximal with respect to the existence of a sheaf morphism $h : H \to \mathcal{I}$ extending f. Let s be a section of $\mathscr{G}$ not in H and let $< s >$ be the subsheaf of G generated by s. If $s \in \Gamma(U,\mathscr{G})$, then $< s > \cong \mathbb{Z}_U$, $< s > \cap H$ is a subsheaf of $< s >$ with a map to $\mathcal{I}$ so by assumption that map extends to a map $< s > \to \mathcal{I}$. So there is a map from the sheaf generated by s and H to $\mathcal{I}$ extending f, contradicting the maximality of H. Thus f has an extension to $\mathscr{G}$, so $\mathcal{I}$ is injective.

   For the second claim, we just need to show that any $\mathscr{R} \subseteq \mathbb{Z}$ is finitely generated. $\mathscr{R}(U)$ is a direct sum of groups $r_i\mathbb{Z}$, one for each component of U. Since the restriction maps of $\mathscr{R}$ are the identity (at least on a connected U), the maximum r occurs in $\mathscr{R}(X)$. For a fixed $r' \leq r$ take finite open cover of connected sets of the union of sets U with $\Gamma(U,\mathscr{R}) = r'\mathbb{Z}$. Do this for each $r' \leq r$ to get a finite collection of open sets $\{U_i\}$. Then the set $\{r_i\}$, where $\Gamma(U_i,\mathscr{R}) = r_i\mathbb{Z}$ generates $\mathscr{R}$. So any map $\mathscr{R} \to \overset{\lim}{\rightarrow} \mathcal{I}_\alpha$ is determined by the images of the r_i, which by taking equivalent elements we can assume all lie in some $\mathcal{I}_\alpha$, then $\mathscr{R} \to \mathcal{I}_\alpha$ has an extension $\mathbb{Z}_U \to \mathcal{I}_\alpha$ which gives an extension $\mathbb{Z}_U \to \overset{\lim}{\rightarrow} \mathcal{I}_\alpha$, so $\overset{\lim}{\rightarrow} \mathcal{I}_\alpha$ is injective.

7. Let S^1 be the circle (with its usual topology) and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.

   (a) Using the construction of Prop 2.2, build an injective resolution of $\mathbb{Z}$. Let $I_0 = \prod_{p\in S^1} i_P(\mathbb{Z})$, where i_P is the skyscraper sheaf. $I_1 = \prod_{P\in S^1} i_P(I_{0,P}/\mathbb{Z})$ and $I_2 = \prod_{P\in S^1} i_P(I_{1,P}/I_{0,P})$. $I_0 \to I_1 \to I_2$ induces $\Gamma(S^1,I_0) \overset{d_1}{\to} \Gamma(S^1,I_1) \overset{d_2}{\to} \Gamma(S^1,I_2)$. $\ker d_2 = \{f : S^1 \to \coprod I_{0,P}/\mathbb{Z} \,|\, f$ locally looks like a $\mathbb{Z}$- valued function modulo constant functions $\}$ and $\mathrm{Im}\, d_1 = \{f : S^1 \to \coprod I_{0,P}/\mathbb{Z} \,|\, f$ is a $\mathbb{Z}$ valued function modulo constant functions $\}$. Any $f \in \ker d_2$ locally looks like a $\mathbb{Z}$-valued function but as you wrap around S^1 the values may jump by some integer. So $\ker d_2/\mathrm{Im}\ d_1 \cong \mathbb{Z}$.

Notes:
1. see also p 220 ex 4.0.4
2. $H^1(S^1, \mathbb{Z})$ is the abelianization of $\pi(S^1)$ which is $\mathbb{Z}$.

(b) Let $\mathscr{R}$ be the sheaf of continuous real-valued functions and let $\mathscr{D}$ be the sheaf of all real-valued functions. Then we have a short exact sequence

$$0 \to \mathscr{R} \to \mathscr{D} \to \mathscr{D}/\mathscr{R} \to 0$$

This gives a long exact sequence

$$0 \to H^0(S^1, \mathscr{R}) \to H^0(S^1, \mathscr{D}) \xrightarrow{\alpha} H^0(S^1, \mathscr{D}/\mathscr{R}) \to H^1(S^1, \mathscr{R}) \to 0$$

where the last term is 0 since $\mathscr{D}$ is flasque. To show that $H^1(S^1, \mathscr{R}) = 0$ is equivalent to show that $H^0(S^1, \mathscr{D}) \xrightarrow{\alpha} H^0(S^1, \mathscr{D}/\mathscr{R})$ is surjective.

Let $s \in H^0(S^1, \mathscr{D}/\mathscr{R})$. Then $s = \{(U_i, s_i)\}$ where on $U_i \cap U_j \neq \emptyset, s_i - s_j$ is continuous (ie in $\mathscr{R}$). Since S^1 is compact, we can choose a finite subcover $\{U_i\}_{i=0}^N$. Choose these $\{U_i\}_{i=0}^N$ such that for an consecutive sets U_0, U_1, U_2,

$$(U_0 \cap U_1) \cap (U_1 \cap U_2) = \emptyset \quad (*)$$

shrinking the U_i if necessary.

Define $r_i = s_{i+1} - s_i$ and extend by zero so r_i is defined on all of U_i. Set $r = \{(U_i, r_i)\}$. On $U_i \cap U_j$, $r_i - r_{i+1} = r_i$ (since $r_{i+1} \equiv 0$ on $U_i \cap U_{i+1}$) which is continuous by (*). Therefore $r \in H^0(S^1, \mathscr{D}/\mathscr{R})$.

Define $t = \{(U_i, t_i)\}$, where $t_i = s_i + r_i$. Then $t_i|_{U_i \cap U_{i+1}} = s_i + s_{i+1} - s_i = s_{i+1} = t_{i+1}|_{U_i \cap U_{i+1}}$. Thus t is a function $t : S^1 \to \mathbb{R}$ and $t \in H^0(S^1, \mathscr{D})$ gets mapped to itself in $H^0(S^1, \mathscr{D}/\mathscr{R})$. Thus t is in the image of α.

Define $r' \in H^0(S^1, \mathscr{D})$ by $r'|_{U_i \cap U_{i+1}} = \begin{cases} r_i & \text{on } U_i \cap U_{i+1} \\ 0 & \text{else} \end{cases}$. Then $r' \overset{\alpha}{\mapsto} r$ so r is in the image of α. Therefore $s = t - r$ is in the image of α, so α is surjective and $H^1(S^1, \mathscr{R}) = 0$.

3.3 Cohomology of a Noetherian Affine Scheme

1. Let X be a noetherian scheme. If $X = \text{Spec } R$ is affine, then $X_{\text{red}} = \text{Spec } R/\eta(R)$ is affine, where $\eta(R)$ is the nilradical or R.

   [BLOG]Conversely, let X_{red} be affine. We want to show that X is affine by using Theorem 3.7 and induction on the dimension of X. If X has dimension 0, then affineness follows from the noetherian hypothesis since it must have finitely many points and each of these is contained in an affine neighborhood. So suppose the result is true for noetherian schemes of dimension $< n$. Let X have dimension n. Let $\mathscr{N}$ be the sheaf of nilpotent elements on X and consider a coherent sheaf $\mathscr{F}$. For every integer i we have a short exact sequence

$$0 \to \mathscr{N}^{d+1} \cdot \mathscr{F} \to \mathscr{N}^{d} \cdot \mathscr{F} \to \mathscr{G}_d \to 0$$

where $\mathscr{G}_d$ is the quotient. This short exact sequence gives rise to a long exact sequence

$$\ldots \to H^0(X, \mathscr{G}_d) \to H^1(X, \mathscr{N}^{d+1} \cdot \mathscr{F}) \to H^1(X, \mathscr{N}^{d} \cdot \mathscr{F}) \to H^1(X, \mathscr{G}_d) \to \ldots$$

Since X is noetherian, there is some m for which $\mathscr{N}^d = 0$ for all $d \geq m$, so if we can show that $H^1(X, \mathscr{G}_d)$ is zero for each d, then the statement $H^1(X, \mathscr{F}) = 0$ will follow by induction and the long exact sequence above. Since the sheaf $\mathscr{G}_d = \mathscr{N}^d \cdot \mathscr{F} / \mathscr{N}^{d+1} \cdot \mathscr{F}$ on X and X_{red} has the same underlying topological space as X but with the sheaf of rings $\mathcal{O}_{X_{\text{red}}} = \mathcal{O}_X / \mathscr{N}$, we see that $\mathscr{G}_d$ is also a sheaf of $\mathcal{O}_{X_{\text{red}}}$-modules. Since cohomology is defined as cohomology of sheaves of abelian groups, we have $H^1(X, \mathscr{G}_d) = H^1(X_{\text{red}}, \mathscr{G}_d)$ and so it follows from Them 3.7 that $H^1(X, \mathscr{G}_d) = 0$ and thus X is affine.

2. Let X be a reduced noetherian scheme. If X is affine, then each irreducible component is a closed subscheme of X and thus affine by Corr II.5.10.

   Conversely, let $X = X_1 \cup X_2 \cup \ldots \cup X_n$ where each X_i is irreducible and affine. Let $\mathscr{I}_j$ be the ideal sheaf of X_j. Let I be a coherent ideal sheaf on X. Then we have the filtration

$$I \supseteq \mathscr{I}_1 \cdot I \supseteq \mathscr{I}_1 \cdot \mathscr{I}_2 \cdot I \supseteq \ldots \supseteq \mathscr{I}_1 \cdot \ldots \cdot \mathscr{I}_n \cdot I$$

   Rename each element in the filtration so that we have

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq \mathcal{I}_n$$

   Now, $I_n = 0$ since anything in $\mathscr{I}_1 \cdot \ldots \mathscr{I}_n$ vanishes on all of X and thus is in the nilradical of $\mathcal{O}_X$. Since X is reduced, $I_n = 0$. For all $j = 0, \ldots, n-1$, the quotient I_j / I_{j+1} is a coherent sheaf on the irreducible component X_{j+1}. Therefore $0 = H^1(X_{j+1}, I_j/I_{j+1}) = H^1(X, I_j/I_{j+1})$ by Serre's theorem. Then from the taking the cohomology of the short exact sequence

$$0 \to I_{j+1} \to I_j \to I_j / I_{j+1} \to 0$$

   we see that

$$H^1(X, I_n) = 0 \Rightarrow H^1(X, I_{n-1}) = 0 \Rightarrow \ldots \Rightarrow H^1(X, I) = 0$$

   and thus X is affine again by Serre's Theorem.

3. Let A be a noetherian ring and let $\mathfrak{a}$ be an ideal in A.

(a) Let
$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$
be a short exact sequence of A-modules. Clearly $\Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M)$ is injective. Now let $m \in \ker g$ with $\mathfrak{a}^n m = 0$ for some n. By the left-exactness of Γ, there exists $m' \in M'$ such that $f(m') = m$. Then $\mathfrak{a}^n m' \subseteq \ker f = 0$. Thus $m' \in \Gamma_{\mathfrak{a}}(M')$ and $\Gamma_{\mathfrak{a}}$ is left exact.

(b) Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Let
$$0 \to M \to I_0 \to I_1 \to \dots$$
be an injective resolution of M. Then
$$0 \to \widetilde{M} \to \widetilde{I_0} \to \widetilde{I_1} \to \dots$$
is a flasque resolution of $\widetilde{M}$. $H^i_{\mathfrak{a}}(M)$ is the cohomology of $O \to \Gamma_{\mathfrak{a}}(I_0) \to \Gamma_{\mathfrak{a}}(I_1) \to \dots$ and $H^i_Y(X, \widetilde{M})$ is the cohomology of $0 \to \Gamma_Y(X, \widetilde{I_0}) \to \Gamma_Y(X, \widetilde{I_1}) \to \dots$. By by Ex II.5.6, $\Gamma_{\mathfrak{a}}(I_i) \cong \Gamma_Y(X, \widetilde{I_i})$. Thus $H^i_{\mathfrak{a}}(M) \cong H^i_Y(X, \widetilde{M})$.

(c) $H^i_{\mathfrak{a}}$ is a quotient of $\Gamma_{\mathfrak{a}}(I_i)$ and therefore every element of $H^i_{\mathfrak{a}}(M)$ is annihilated by some power of $\mathfrak{a}$.

4. *Cohomological Interpretation of Depth*

(a) Let A be noetherian. If $\operatorname{depth}_{\mathfrak{a}}(M) \geq 1$, then there exists $x \in a$ such that x is not a zero-divisor for M. But then neither is x^n for any n. Thus $\mathfrak{a}^n$ can not annihilate any element and thus $\Gamma_{\mathfrak{a}}(M) = 0$.

[BLOG] Now suppose $\Gamma_{\mathfrak{a}}(M) = 0$ and M is finitely generated. So for any nonzero $m \in M$ and $n \geq 0$, there is an $x \in \mathfrak{a}^n$ such that $xm \neq 0$. This means that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for any associated prime $\mathfrak{p}$ of M (i.e. primes $\mathfrak{p}$ such that $\mathfrak{p} = Ann(m)$ for some $m \in M$). So $\mathfrak{a} \not\subseteq U_{\mathfrak{p} \in Ass(M)} \mathfrak{p}$ [Eisenbud, Lemma 3.3, Thm 3.1(a)]. The latter set is the set of zero divisors of M (including zero) [Eisenbud, Thm 3.1(b)] and so we find that there is an element $x \in \mathfrak{a}$ that is not a zero divisor in M. Hence $\operatorname{depth}_{\mathfrak{a}} M \geq 1$.

(b) [SAM] Let T_n be the statement that $\operatorname{depth}_{\mathfrak{a}} M \geq n$ if and only if $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$. We prove by induction on n that T_n is true for all n. The case $n = 0$ is (a), so suppose it true for n and choose M with $\operatorname{depth}_{\mathfrak{a}} M \geq n + 1$. Let $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an M-regular sequence; we get a short exact sequence
$$0 \to M \xrightarrow{\cdot x1} M \to M/x_1 M \to 0$$
which gives rise to a long exact sequence on cohomology
$$\dots \to H^{n-1}_{\mathfrak{a}}(M/x_1 M) \to H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M) \to \dots$$

The first term vanishes since $\operatorname{depth}_{\mathfrak{a}} M/x_1M \geq n$. Also, the map $H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$ is multiplication by x_1, which is not injective (Ex. 3.3(c)) if $H^n_{\mathfrak{a}}(M) \neq 0$, so we conclude that $H^n_{\mathfrak{a}}(M) = 0$. So $\operatorname{depth}_{\mathfrak{a}} M \geq n+1$ implies that $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n+1$.

Conversely, suppose that $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n+1$. Then the long exact sequence on cohomology gives that $H^i_{\mathfrak{a}}(M/x_1M) = 0$ for all $i < n$. By induction, $\operatorname{depth}_{\mathfrak{a}} M/x_1M \geq n-1$, so $\operatorname{depth}_{\mathfrak{a}} M \geq n$. Hence T_{n+1} is also true.

5. Let X be a Noetherian scheme and let P be a closed point of X. Let U be any open neighborhood of P.

   Then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely over a section $\mathcal{O}_X$ of U
   $\Leftrightarrow \Gamma(U, \mathcal{O}_X) \to \Gamma(U-P, \mathcal{O}_X)$ is bijective
   $\Leftrightarrow H^0_P(U, \mathcal{O}_X|_U) = H^1_P(U, \mathcal{O}_X|_U) = 0$ (by Ex 2.3(e))
   $\Leftrightarrow H^0_P(\operatorname{Spec} \mathcal{O}_P, \mathcal{O}_{\operatorname{Spec} \mathcal{O}_P}) = H^1_P(\operatorname{Spec} \mathcal{O}_P, \mathcal{O}_{\operatorname{Spec} \mathcal{O}_P}) = 0$ (by Ex 2.5)
   $\Leftrightarrow H^0_{\mathfrak{m}}(\mathcal{O}_P) = H^1_{\mathfrak{m}}(\mathcal{O}_P) = 0$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$ (by Ex 3.3(b))
   $\Leftrightarrow \operatorname{depth}_{\mathfrak{m}} \mathcal{O}_P \geq 2$ (by Ex 3.4(b))

6. Let X be a noetherian scheme.

   (a) If X is affine, then $\sim$ gives an equivalence of categories $\operatorname{Mod}(A) \cong \operatorname{Qco}(X)$, where $X = \operatorname{Spec} A$. So an injective A-module I induces an injective object $\widetilde{I} \in \operatorname{Qco}(X)$. In the general case, we need to show that if $f : U \hookrightarrow X$ is an inclusion of $U \cong \operatorname{Spec} A$ in X, then $f_*(\widetilde{I})$ is injective. By pg 110, $\operatorname{Hom}_{\mathcal{O}_X}(\,\cdot\,, f_*(\widetilde{I})) \cong \operatorname{Hom}_{\mathcal{O}_U}(f^*\cdot, \widetilde{I}) = \operatorname{Hom}_{\mathcal{O}_U}(\,\cdot|_U, \widetilde{I})$, which is a composition of the exact functors $\cdot|_U$ and $\operatorname{Hom}_{\mathcal{O}_U}(\,\cdot, \widetilde{I})$ and thus is exact. Therefore $f_*(\widetilde{I})$, and thus $\mathscr{G}$ in (3.6) is injective.

   (b)

   (c) By part (b), an injective resolution in $\operatorname{Qco}(X)$ is a flasque resolution and hence can be used to compute cohomology.

7.

4 Chapter 4: Curves

4.1 Riemann-Roch Theorem

1. Let X be a curve and let P be a point. To show that there exists a non-constant rational function $f \in K(X)$ which is regular everywhere except at P is equivalent to showing that $h^0(nP) \neq 0$ for $n \gg 0$. Using the title of the section as a hint, let's use the Riemann-Roch Theorem for the divisor nP. Then $h^0(nP) - h^1(nP) = \deg nP + 1 - g$. By Serre Duality, $H^1(X, nP) = H^0(X, K_X - nP)$. The degree of $K_X - nP = 2g - 2 - n$ so for $n \gg 0$, the degree of $K_X - nP < 0$ and thus by Serre Duality, $H^1(X, nP) = 0$. Thus $h^0(X, nP) = n - 1 + g$ and again for $n \gg 0$, this is non-zero and we are done.

2. Let X be a curve and let $P_1, \dots, P_r \in X$ be points. Then for each P_i, apply the previous exercise to obtain an f_i regular everywhere except at P_i. Then let $f = \sum f_i$ be the desired function.

3. Let X be an integral, separated, regular, 1 dimensional scheme of finite type over k which is not proper over k. Following the hint, embed X in a proper curve $\overline{X}$ over k. By remark II.4.10.2(e), X can be embedded as an open subset of a complete variety. Then Proposition I.6.7 and Proposition I.6.9 show that X can be embedded as an open subset of a complete curve, which we call $\overline{X}$. The complement of $\overline{X}$ in X is closed, and hence a finite set of points. Say $\overline{X} = X \cup \{P_1, \dots, P_r\}$. Let f be as in the previous exercise. Then by (II,6.8), f defines a finite morphism $\overline{X} \to \mathbb{P}^1$. Thus $f^{-1}(\mathbb{A}^1) = X$ is affine.

4. Using (III Ex, 3.1, Ex, 3.2), we reduce to the case X is integral. Let $\widetilde{X}$ be the normalization of X. Then $\widetilde{X}$ is not proper since by (II,Ex 4.4), X would be proper. Thus by the previous exercise, $\widetilde{X}$ is affine and by (III, Ex 4.2), X is affine.

5. $$\begin{aligned} \dim|D| &= h^0(D) - 1 \\ &= \deg D - g + h^0(K - D) \\ &\leq \deg D - g + h^0(K) \textit{ (since D effective)} \\ &\leq \deg D - g + g \\ &\leq \deg D \end{aligned}$$

   Equality occurs iff $h^0(K-D) = h^0(K) = g$. If $D = 0$, then certainly equality holds. If $g = 0$, then since $D \geq 0$, equality holds as well. Conversely, suppose that $h^0(K-D) = h^0(K) = g$ and $g > 0$. Then $h^0(K-D) = h^0(K)$ so $D \sim 0$. Since $D \geq 0$, $D = 0$.

6. Let X be a curve of genus g. Let $D = \sum^{g+1} P_i$ for $g + 1$ points P_i on X.

By Riemann-Roch,

$$\begin{aligned} h^0(D) &= \deg D + 1 - g + h^1(D) \\ &= g + 1 + 1 - g + h^1(D) \\ &= 2 + h^1(D) \end{aligned}$$

Thus $h^0(D) \geq 2$ so there exists a nonconstant rational function $f \in k(X)$ with poles at a nonempty subset of the P_i and regular elsewhere. This f gives a finite morphism $X \to \mathbb{P}^1$ by (II.6.8) with $f^{-1}(x_\infty)$ at most these $g+1$ points P_i. Thus deg $f \leq g+1$.

7. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

   (a) Let X be a curve of genus 2. Then deg $K = 2g-2 = 2$ and $\dim |K| = h^0(K) - 1 = g - 1 = 1$. To show $|K|$ is base point free, cheat a little and skip ahead to Prop IV.3.1. Lets show that $\dim |K-P| = \dim |K| - 1$. This is equivalent to showing that $h^0(K-P) = h^0(K) - 1 = g - 1 = 1$. By Riemann-Roch,

   $$\begin{aligned} h^0(K-P) &= \deg K - P + 1 - g + h^1(K-P) \\ &= 2g - 2 - 1 + 1 - 2 + h^0(P) \\ &= h^0(P) \\ &= 1 \end{aligned}$$

   So by the proposition, $|K|$ is base point free. Note that $h^0(P) = 1$ since $h^0(P) \geq 1$ since P is effective and $h^0(P) \leq 1$ else X would be rational. Since $g \neq 0$, this is clearly not the case. Thus we get a morphism $\varphi_{|K|} : X \to \mathbb{P}^1$, which is finite by (II.6.8) of deg $K = 2$ and thus X is hyperelliptic.

   (b) Let $X \subset Q$ be a curves of genus g corresponding to a divisor of type $(g+1, 2)$. We have to give a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. Viewing $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, consider the second projection restricted to the curve X: $p_2|_X : X \to \mathbb{P}^1$. This is non-constant and thus finite by (II.6.8). Let $P \in \mathbb{P}^1$ be a point. Then by (II.6.9), deg $p_2^*|_X(P) = (\deg p_2)(\deg P)$ which gives $2 = \deg p_2$. Thus X is hyperelliptic and there exist hyperelliptic curves of any genus $g \geq 2$.

8. <u>p_a *of a Singular Curve*</u> Let X be an integral projective scheme of dimension 1 over k, and let $f : \widetilde{X} \to X$ be its normalization. Then there is an exact sequence of sheaves of X,

   $$0 \to \mathcal{O}_X \to f_* \mathcal{O}_X \to \sum_{P \in X} \widetilde{\mathcal{O}}_P / \mathcal{O}_P \to 0$$

   (a) Since $\widetilde{X}$ is a nonsingular projective curve, $f_* \mathcal{O}_{\widetilde{X}}$ has no nonconstant global sections. Since $\sum_{P \in X} \widetilde{\mathcal{O}}_P / \mathcal{O}_P$ is flasque, by (III, Ex 4.1),

$H^1(X, f_*\mathcal{O}_{\widetilde{X}}) \cong H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ so we get an exact sequence

$$0 \to H^0(X, \sum_{P \in X} \widetilde{\mathcal{O}}_p/\mathcal{O}_p) \to H^1(X, \mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to 0$$

Using (III, Ex 5.3), we get $p_a(X) = p_a(\widetilde{X}) + \sum_{P \in X} \dim_k \widetilde{\mathcal{O}}_P/\mathcal{O}_P = p_a(\widetilde{X}) = \sum \delta_p$

(b) If $p_a(X) = 0$, then $\delta_P = 0$ for all $P \in X$. That is, every local ring of X is integrally closed, hence regular. Then $X \cong \mathbb{P}^1$ by (1.3.5).

(c)

9. Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of $X/$

   (a) Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}. Then $\xi(\mathcal{O}_X) = 1 - p_a$ so by using the exact sequence

   $$0 \to \mathscr{L}(D) \to \mathscr{L}(D + P) \to k(P) \to 0$$

   as in the proof of the Riemann-Roch Theorem, the result follows immediately

   (b) Let D be a Cartier divisor, $\mathscr{M} = \mathscr{L}(D)$, and let L be very ample. Choose $n > 0$ such that $\mathscr{M} \otimes \mathscr{L}^n$ is generated by global sections. Then by Exercise II.7.5(d), $\mathscr{M} \otimes \mathscr{L}^{n+1}$ and $\mathscr{L}^{n+1}$ are very ample. By (II.6.15), we may write $\mathscr{M} \otimes \mathscr{L}^{n+1} \cong \mathscr{L}(D')$ and $\mathscr{L}^{n+1} = \mathscr{L}(D'')$. Then $D' - D'' \sim D$. By replacing D' with a linearly equivalent Cartier Divisor, we may assume that $D = D' - D''$.

   (c) By (b), we only need to prove this in the case $\mathscr{L} \cong \mathscr{L}(D)$ with D an effective very ample Cartier divisor. D is the pullback of a hyperplane, which we may choose to miss the singular locus of X. In that case, Supp $D \subseteq X_{\text{reg}}$.

   (d) X is Cohen-Macauley so by (III.7.6) $H^1(X, \mathscr{L}(D)) \cong \text{Ext}^0(\mathscr{L}(D), \omega_X^\circ) \cong \text{Ext}^1(\mathcal{O}_X, \omega_X^\circ \otimes \mathscr{L}(-D)) \cong H^0(X, \omega_X^\circ \otimes \mathscr{L}(-D))$. So $\dim H^1(X, \mathscr{L}(D)) = l(K - D)$. We get the formula from part (a).

10. Let X be an integral projective scheme of dimension 1 over k, which is locally a complete intersection and has $p_a = 1$. Fix a point $P_0 \in X_{\text{reg}}$. Use Ex 1.9c to write any invertible sheaf as a Weil divisor in X_{reg}. By Ex 1.9d applied to K we get deg $K = l(K) - 1 = \dim H^0(X, \omega_X^\circ) - 1 = \dim_k H^1(X, \mathcal{O}_X) - 1 = p_a - 1 = 0$. Where we used (III.7.7) since X is a local complete intersection. Now we show that for any divisor D of degree 0 there is a unique $P \in X_{\text{reg}}$such that $D \sim P - P_0$. Apply Ex 1.9 to $D + P_0$. Since deg $(K - D - P_0) = -1$, we get $l(D + P_0) = 1 + 1 - 1$, so there is a unique P such that $D + P_0 \sim P$.

4.2 Hurwitz's Theorem

1. To show $\mathbb{P}^n$ is simply connected, I know of three ways. One is the way Hartshorne wants you to do it which is done in other solutions. Another is to compute the fundamental group $\pi_1(\mathbb{P}^n)$ and to show it is 0. The third way is to use the Fulton-Hansen Theorem, which states:

   Let X be a complete irreducible variety, and $f : X \to \mathbb{P}^n \times \mathbb{P}^n$ a morphism with the property that dim $f(X) > n$. Then $f^{-1}(\Delta)$ is connected, where Δ denotes the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$.

   Now, we claim that if X is an irreducible variety and $f : X \to \mathbb{P}^n$ is a finite, unramified morphism, then if $2 \dim X > n$, then f is a closed immersion. Indeed, saying that f is unramified means that the diagonal $\Delta_X \subset X \times_{\mathbb{P}^n} X = (f \times f)^{-1}(\Delta_{\mathbb{P}^n})$ is open and closed. The diagonal is connected by the Fulton-Hansen theorem, so $\Delta_X = X \times_{\mathbb{P}^n} X$ and f is injective. Thus f is closed and we get as a corollary that every subvariety of $\mathbb{P}^n$ with dimension $> n/2$ is simply connected. In particular, $\mathbb{P}^n$ is simply connected.

2. *Classification of Curves of Genus 2:* Fix an algebraically closed field k of characteristic $\neq 2$.

   (a) Let X be a curve of $g = 2$ over k. Then the canonical linear system $|K|$ determines a finite morphism $f{:}X \to \mathbb{P}^1$ of degree $2g-2 = 4-2 = 2$. By Hurwitz's theorem, we get

   $$2(2) - 2 = 2(-2) + \deg R$$

   Thus $\deg R = 6$. If $Q \in \mathbb{P}^1$ is a closed branch point, then $\deg f^*(Q) = 2$, so there must be six ramification points, each with ramification index 2.

   (b) Let $\alpha_1, \ldots, \alpha_6 \in k$ be distinct points. Let K be the extension of $k(x)$ determined by the equation $z^2 = \prod^6(x - \alpha_i)$. Then X is the projective closure of the affine plane curve defined by this equation and f is the projection onto the $x-$coordinate. Away from the α_i, $x - \alpha_i$ is a local parameter so there is no ramification. At the α_i, z is a local parameter. Thus there is ramification at each α_i. Again by Hurwitz's formula, with $n = 2$ and $\deg R = 6$, we get that $g_X = 2$.

   (c) Let P_1, P_2, P_3 be three distinct points in $\mathbb{P}^1$. By (I, Ex 6.6) we just need to find the correct linear fractional transformation $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ which will send $P_1 \mapsto 0, P_2 \mapsto 1, P_3 \mapsto \infty$. The following does just that:

   $$\varphi(z) = \begin{cases} \frac{z-P_1}{z-P_3} \cdot \frac{P_2-P_3}{P_2-P_1} & \text{if } P_1, P_2, P_3 \neq \infty \\ \frac{P_2-P_3}{z-P_3} & \text{if } P_1 = \infty \\ \frac{z-P_1}{z-P_3} & \text{if } P_2 = \infty \\ \frac{z-P_1}{P_2-P_1} & \text{if } P_3 = \infty \end{cases}$$

(d) Ok

(e) This follows immediately from (a) - (d)

3. *Plane Curves:* Let X be a curve of degree d in $\mathbb{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P. Considering $T_P(X)$ as a point of the dual projective plane $(\mathbb{P}^2)^*$, the map $P \to T_P(X)$ gives a morphism of X to its dual curve X^* in $(\mathbb{P}^2)^*$. Note that even though X is nonsingular, X^* in general will have singularities. Assume that char $k = 0$.

   (a) Fix a line $L \subset \mathbb{P}^2$ which is not tangent to X. Define $\varphi : X \to L$ by $P \mapsto T_P(X) \cap L$. Let's consider the case when $P \in L$. Change coordinates such that $P = \{(0,0)\}$, the origin in $\mathbb{A}^2$ and such that L is defined by $\{y = 0\}$ and T_P is defined by $\{x = 0\}$. Then for any point $Q = (Q_x, Q_y) \in X$, the tangent line at Q, T_Q, is defined by $\{\frac{\partial f}{\partial x}|_Q(x - Q_x) + \frac{\partial f}{\partial y}|_Q(y - Q_y) = 0\}$. Then $\varphi(Q)$ can be found by setting $y = 0$ and solving for x, which gives

$$\varphi(Q) = \frac{\frac{\partial f}{\partial y}|_Q Q_y}{\frac{\partial f}{\partial x}|_Q} + Q_x$$

   Note that $\varphi(0) = 0$. Let t be a local parameter at $0 \in \mathbb{A}^1$. Then $\varphi^*(t) = \frac{\frac{\partial f}{\partial y} \cdot y}{\frac{\partial f}{\partial x}} + x$. Since $T_P = \{x = 0\}$, $\frac{\partial f}{\partial y}(0) = 0$ and x vanishes at 0 to order ≥ 2. Since $\frac{\partial f}{\partial y} \cdot y \in \mathfrak{m}_0^2$ and $\frac{\partial f}{\partial x} \neq 0$, $\varphi^*(t) \in \mathfrak{m}_0^2$. So φ is ramified at 0.

   Now consider the case that $P \notin L$. Change coordinates such that $P = \{(0,0)\}$ in $\mathbb{A}^2$, L is the line at infinity, and $T_P = \{x = 0\}$. Then for any $Q \in X$, the projective tangent line at $Q = \{(Q_x, Q_y)\}$ is $\frac{\partial f}{\partial x}|_Q(x - Q_x z) + \frac{\partial f}{\partial y}|_Q(y - Q_y z) = 0$. The line at infinity is found by setting $z = 1$. Then a point is mapped to the intersection of the tangent line and the line at infinity. So Q gets mapped to the slope of its tangent line. So $\varphi : X \to \mathbb{P}^1$ maps $Q \mapsto (-\frac{\partial f}{\partial y}|_Q : \frac{\partial f}{\partial x}|_Q)$. Since $\frac{\partial f}{\partial x}|_{\{(0,0)\}} \neq 0$, near P we have $\varphi : X \to \mathbb{A}^1$ where $Q \mapsto -\frac{\partial f}{\partial y}|_Q / \frac{\partial f}{\partial x}|_Q$. Since $\varphi(0) = 0$, the equation of X is then $f(x,y) = ax + by + cx^2 + dxy + ey^2 + higher\,order\,terms$. Let t be the local coordinate at 0. Then:

$$\begin{aligned}
\varphi^*(t) \in \mathfrak{m}_0^2 \quad &\Leftrightarrow \frac{\partial f}{\partial y} \in \mathfrak{m}_0^2 \\
&\Leftrightarrow \frac{\partial}{\partial y}(ax + by + cx^2 + dxy + ey^2) \in \mathfrak{m}_0^2 \\
&\Leftrightarrow b + dx + 2ey \in \mathfrak{m}_0^2 \\
&\Leftrightarrow b + 2ey \in \mathfrak{m}_0^2 \ \ (T_0 = \{x = 0\} \Rightarrow y \notin \mathfrak{m}_0^2) \\
&\Leftrightarrow f|_{T_0} \text{has degree } \geq 3 \text{ in } y \\
&\Leftrightarrow \text{intersection multiplicity of } f \text{ with } T_0 \text{ is } \geq 3 \\
&\Leftrightarrow 0 \text{ is an inflection point}
\end{aligned}$$

   This φ is ramified at P if and only if either $P \in L$ or P is an inflection point of X. By Hurwitz's formula, the degree of the ramification divisor is finite, so X has only a finite number of inflection points.

(b)

(c) Let $O = (0,0)$ in $\mathbb{A}^2$ and change coordinates such that $P = (0,1) \in \mathbb{A}^2$. Let L be the line at infinity. Let $\varphi : X \to \mathbb{P}^1$ be the projection from O. Then $(x,y) \mapsto (x : y)$. Near P, $\varphi : U \to \mathbb{A}^1_{y \neq 0}$ is defined by $(x,y) \mapsto x/y$. Then $\varphi(P) = 0$. Now, φ is ramified at P iff $\varphi^*(t) = \frac{x}{y} \in \mathfrak{m}_P^2$, where t is a local parameter of 0. Since $y \neq 0$, $\frac{x}{y} \in \mathfrak{m}_P^2$ iff $x \in \mathfrak{m}_P^2$ iff $\{x = 0\}$ tangent to X at P.
Applying Hurwitz's theorem, we get

$$(d-1)(d-2) - 2 = d(-2) + \deg R$$

So $\deg R = d^2 - d = d(d-1)$. R is reduced since 0 is not on any inflection or tangent line, so the number of tangent lines to X is thus $\deg R = d(d-1)$.

(d) Choose $O \in X$ not containing any inflectional or multiple tangents and consider the projection $\varphi : X \to \mathbb{P}^1$ from O. Then $\deg \varphi = d-1$. By Hurwitz's theorem:

$$\begin{aligned} 2g_X - 2 &= n(2g_Y - 2) + \deg R \\ &= (d-1)(-2) + \deg R \\ &= 2d + 2 + \deg R \end{aligned}$$

Rearranging gives $\deg R = (d-1)(d-2)$. The map is unramified at O since O is not an inflection point and thus O lies on $(d+1)(d-2)$ tangents of X, not counting the tangents at O.

(e) $\varphi^{-1}(P) = \{Q \in X \mid P \in T_Q(X)\}$. If P does not lie on any inflection tangent or multiple tangents, then by part (c), $|\varphi^{-1}(P)| = d(d-1)$ Thus $\deg \varphi = d(d-1)$. By Hurwitz's theorem, $\deg R = 3d^2 - 5d$. Ignoring the ramification of type 1 in part (a), we get the desired result.

(f) Let X be a plane curve of degree $d \geq 2$ and assume that the dual curve X^* has only nodes and ordinary cusps as singularities. Since the map $\varphi : X \to X^*$ is finite and birational and X is already normal, by the universal property of normalization, X is the normalization of X^*. Following the hint we find that

$$p_a(X^*) = \frac{1}{2}(d(d-1) - 1)(d(d-1) - 2)$$

and

$$\begin{aligned} p_a(X^*) &= p_a(X) + \text{no. of sing pts} \\ &= p_a(X) + \text{no. of inflection pts of } X + \text{ no bitangents of } X \\ &= \tfrac{1}{2}(d-1)(d-2) + 3d(d-2) + \text{ no. of bitangents} \end{aligned}$$

Equating the two and solving for the number of bitangents gives the desired result.

(g) A plane cubic has degree 3, so plugging in from the equation in part (e), we get that there are $3 \cdot 3(3-2) = 9$ inflection points, all ordinary since $r = 3$. The fact that a line joining 2 inflection points meets at at third inflection point will follow from Ch 4, Ex 4.4b or from Shaf I p 184.

(h) A plane quartic has deg 4 so, by part (f) the number of bitangents is $\frac{1}{2}4(4-2)(4-3)(4+3) = 28$.

4. *A Funny Curve in Characteristic* p: Let X be the plane quartic curve $x^3y + y^3z + z^3x = 0$ over a field of characteristic 3. Then looking in the affine piece $z = 1$, the partials of X are:

$$\begin{aligned} f'_x &= 3x^2y + 1 = 1 \\ f'_y &= x^3 + 3y^2 = x^3 \end{aligned}$$

Thus for no point are the partials all zero, so X is nonsingular. Similar calculations for the other affine pieces.

To show that every point is an inflection point, we compute the Hessian form (Shaf I p 18):

$$\begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix} = \begin{pmatrix} 6xy & 3x^2 & 0 \\ 0 & 6yz & 3y^2 \\ 3z^2 & 0 & 6zx \end{pmatrix}$$

Since this matrix is the zero matrix 0_3 in characteristic 3, every point $P \in X$ satisfies the equation $\det 0_3 = 0$ and thus every point is an inflection point.
The tangent line at a point $P = (x_0, y_0, z_0)$ is $f'_x(x - x_0) + f'_y(y - y_0) + f'_z(z - z_0) = 0$ which is equivalent to $z_0^3(x - x_0) + x_0^3(y - y_0) + y_0^3(z - z_0) = 0$. This is equivalent to $z_0^3x + x_0^3y + y_0^3z = 0$ since $z_0^3x_0 + x_0^3y_0 + y_0^3z_0 = 0$ since it lies on X. Thus the natural map of $X \to X^*$ given by $P \mapsto T_P(X)$ is $(x_0, y_0, z_0) \mapsto (x_0^3, y_0^3, z_0^3)$, the Frobenius map. The corresponding morphism on the function fields is then purely inseparable and finite, so by Prop 2.5, $X \cong X^*$.

5. *Automorphisms of a Curve of Genus* ≥ 2. Let X be a curve of genus ≥ 2 over a field of characteristic 0. Let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f : X \to Y$ of degree n.

(a) Let $P \in X$ be a ramification point and $e_p = r$. Let $y \in Y$ be a branch point. Let $x_1, \ldots, x_s$ be the points of X lying above y. They form a single orbit for the action of G on X. Since the x_i's are all in the same orbit, they all have conjugate stabilizer subgroups, and in particular, each stabilizer subgroup is of the same order r. Moreover, the number s of points in this orbit is the index of the stabilizer, and so is equal to $|G|/r$. Thus for every branch point $y \in Y$, there is an integer $r \geq 2$ such that $f^{-1}y$ consists of exactly $|G|/r$ points of X, and at each of these preimages, f has multiplicity r.

We therefore have the following, applying Hurwitz's formula:

$$\begin{aligned} 2g_X - 2 &= |G|(2g_Y - 2) + \textstyle\sum_{i=1}^s \frac{|G|}{r_i}(r_i - 1) \\ &= |G|(2g_Y - 2 + \textstyle\sum_{i=1}^s (1 - \frac{1}{r_i})) \end{aligned}$$

which rearranging gives the desired form:

$$(2g_X - 2)/n = 2g_Y - 2 + \sum_{i=1}^s (1 - \frac{1}{r_i})$$

(b) Suppose first that $g_Y \geq 1$. If the ramification $R = \sum_{i=1}^s (1 - \frac{1}{r_i}) = 0$, then $g_Y \geq 2$, which implies that $|G| \leq g_X - 1$. If $R \neq 0$, this forces $R \geq 1/2$. Then $2g_Y - 2 + R \geq 1/2$, so we have $|G| \leq 4(g_X - 1)$. This finishes the case $g_Y \geq 1$.
Now assume that $g_Y = 0$. Then the equation from part (a) reduces to

$$2g_X - 2 = |G|(-2 + R)$$

which forces $R > 2$. It is elementary then to check that if $R = \sum_{i=1}^s (1 - \frac{1}{r_i}) > 2$, then in fact $R \geq 2\frac{1}{42}$. Therefore $R - 2 \geq 1/42$. Therefore $|G| \leq 84(g - 1)$ as claimed.

6. f_* *for Divisors*: Let $f : X \to Y$ be a finite morphism of curves of degree n. We define a homomorphism $f_* : \text{Div } X \to \text{Div } Y$ by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$ for any divisor $D = \sum n_i P_i$ on X.

(a) For any locally free sheaf $\mathscr{E}$ on Y of rank r, define $\det \mathscr{E} = \wedge^r \mathscr{E} \in \text{Pic } Y$. In particular, for any invertible sheaf $\mathscr{M}$ on X, $f_*\mathscr{M}$ is locally free of rank n on Y, so we can consider $\det f_*\mathscr{M} \in \text{Pic } Y$. Let D be a divisor on X. Since $f : X \to Y$ is finite, we can assume that X and Y are affine. Then $\mathscr{L}(-D)$ is quasicoherent and by Prop III.8.1, $R^1 f_*\mathscr{L}(-D) = 0$. Then from the short exact sequence

$$0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

we get the short exact sequence

$$0 \to f_*\mathscr{L}(-D) \to f_*\mathcal{O}_X \to f_*\mathcal{O}_D \to 0$$

Assume that D is effective. Then by Prop. II.6.11b, we get that

$$\det f_*\mathscr{L}(-D) \cong \det f_*\mathcal{O}_X \otimes (\det f_*\mathcal{O}_D)^{-1}$$

Since $f_*\mathcal{O}_D \cong \bigoplus_{i=1}^n \mathcal{O}_{f_*D}, \det f_*\mathcal{O}_D = \det \mathcal{O}_{f_*D} = \mathscr{L}(f_*D)$. Therefore $\det f_*\mathcal{O}_D^{-1} = \mathscr{L}(-f_*D)$. For an arbitrary divisor D, write $D = D_1 - D_2$ as the difference of two effective divisors. Then tensoring

$$0 \to \mathscr{L}(D_1) \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$$

with $\mathscr{L}(D_2)^{-1}$ we get

$$0 \to \mathscr{L}(D) \to \mathscr{L}(D_2)^{-1} \to \mathcal{O}_{D_1} \to 0$$

Applying f_* and taking determinants of this short exact sequence we get $f_*\mathscr{L}(D)$ as above.

(b) Since $\mathscr{L}(D)$ only depends on the linear equivalence class of D, so does f_*D. Since def $f = n, f^*$ of a point is a degree n divisor. Thus f_*f^* is multiplication by n.

(c) [SAM] Since X and Y are nonsingular curves, Ω_X and Ω_Y are their respective dualizing sheaves. From (Ex. III.7.2(a)), we have $f^!\Omega_Y = \Omega_X$. By (Ex. III.6.10(a)), this means that $f_*\Omega_X = \mathrm{Hom}_Y(f_*\mathcal{O}_X, \Omega_Y) = (f_*\mathcal{O}_X)^* \otimes \Omega_Y$. The determinant of the RHS is $\det(f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}$ because $(f_*\mathcal{O}_X)^*$ is locally free of rank n, and Ω_Y is a line bundle, so we are done.

(d) By Prop 2.3, $K_X \sim f^*K_Y + R$. Therefore $f_*K_X \sim nK_Y + B$. Thus $\mathscr{L}(-B) \cong \Omega_Y^{\otimes n} \otimes \mathscr{L}(f_*K_X)^{-1}$. By parts (a) and (b), we get that $\mathscr{L}(-B) \cong \Omega_Y^{\otimes n} \otimes \det f_*\mathcal{O}_X \otimes \det(f_*\Omega_X)^{-1} \cong (\det f_*\mathcal{O}_X)^2$.

7. *Étale Covers of degree 2*. Let Y be a curve over a field k of characteristic $\neq 2$.

(a) Each stalk of $f_*\mathcal{O}_X$ is a rank 2 free module over the corresponding stalk of $\mathcal{O}_Y$. So each stalk of $f_*\mathcal{O}_X$ is isomorphic to the corresponding stalk of $\mathcal{O}_Y$. Thus $\mathscr{L}$ is invertible. Then taking determinants of terms in

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathscr{L} \to 0$$

as in Ex II.6.11, we get that $\mathscr{L} \cong \det \mathscr{L} \cong \det f_*\mathcal{O}_X \otimes (\det \mathcal{O}_Y)^{-1} \cong \det f_*\mathcal{O}_X$. Thus $\mathscr{L}^2 = \mathscr{L}(-B) = \mathcal{O}_Y$ since there is no ramification.

(b) $f : X \to Y$ is an affine morphism and if Spec A pulls back to Spec B, then clearly B is integral over A so f is finite. Thus X is integral, separated, of finite type over k and dim $X = 1$. Thus X is a curve. Since the integral closure of a Dedekind Domain and a localization of a Dedekind Domain at a maximal ideal is a DVR, we see that X is smooth. The function field of X is clearly a degree 2 extension of $k(Y)$ so deg $f = 2$. Thus by Ex III.10.3, f is étale.

(c) The map $\sigma \mapsto (\sigma + \tau\sigma)/2$ from $f_*\mathcal{O}_Y \to \mathcal{O}_Y$ is a section of the short exact sequence

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathscr{L} \to 0$$

Thus the sequence splits and $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathscr{L}$. So by Ex III.5.17, $X \cong \textbf{Spec}\ (\mathcal{O}_Y \oplus \mathscr{L})$. So starting with X we get $\mathscr{L}$ which gives back X. Starting with $\mathscr{L}$, we get $\textbf{Spec}\ (\mathcal{O}_Y \oplus \mathscr{L})$ which gives $\mathscr{L}$ back. Thus the processes are inverses.

4.3 Embeddings in Projective Space

1. Let X be a curve of genus 2. Let D be a divisor on X such that deg $D \geq 5$. Then D is very amply by Cor 3.2(b).

   Conversely, let D be very ample. If deg $D = 1$ or 2, then $\varphi_{|D|} : X \hookrightarrow \mathbb{P}^N \to \mathbb{P}^3$ and the image is either a line or a conic in $\mathbb{P}^3$. However, both are contained in some $\mathbb{P}^2$ and thus X is a line or a plane conic. Since $g_X = 0$, this can not happen.

   Now let deg D be 3 or 4. Then D is non-special and thus by Riemann-Roch:
   $$\begin{aligned} h^0(D) &= \deg D + 1 - g \\ &\{3,4\} + 1 - 2 \\ &\{2,3\} \end{aligned}$$

   Thus dim $|D| = 1$ or 2 Since X can not be embedded into $\mathbb{P}^1$, deg $D \neq 3$. If D were very ample of degree 4, then $\varphi_{|D|} : X \hookrightarrow \mathbb{P}^2$ embeds X as a deg 4 plane curve of genus 2. But Plüker's formula gives that $g = (d-1)(d-2)/2$ and $2 \neq 3$. Thus deg $D \neq 4$. Thus deg $D \geq 5$

2. Let X be a plane curve of degree 4 (and thus $g_X = 3$)

   (a) By Ex II.8.20.3, $\omega_X \cong \mathcal{O}_X(1)$ so the effective canonical divisors are just the hyperplane sections.

   (b) Let D be an effective divisor of degree 2 on X. Since K is very ample, we have an embedding $\varphi_{|K|} : X \hookrightarrow \mathbb{P}^2$. Let $L = P + Q$. Let l be the line through P and Q. If $P = Q$, then l is the tangent line through P. Thus we can assume that $K = P + Q + R + S$. Then dim $|D| = \dim |K| - 2 = 2 - 2 = 0$, where the first equality comes from Prop 3.1(b).

   (c) A degree 2 morphism $\varphi : X \to \mathbb{P}^1$ is induced by a deg 2 divisor D with dim $|D| > 0$, By part (b), this can not happen and thus X is not hyperelliptic.

3. Let X be a curve of genus ≥ 2 which is a complete intersection in some $\mathbb{P}^n$. Assume that $X = \bigcap H_i$ where each H_i is a hypersurface. By (II, Ex 8.4(d)), K is a multiple of the hyperplane divisor. Therefore $\mathscr{L}(K) \cong \mathcal{O}_X(n)$ for some $n > 0$ since $2g - 2 > 0$. Then $|K|$ induces the d -uple embedding and thus K is very ample. Thus by ex. 3.1, if $g = 2$, deg $K = 2$ is not very ample and thus X is not a complete intersection.

4. Let X be the d-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$ for any $d \geq 1$. We call X the *rational normal curve of degree d* in $\mathbb{P}^d$.

   (a) By (II,Ex 5.14), the d-uple embedding is projectively normal since $\mathbb{P}^1$ is already projectively normal. We know the image of the d-uple embedding is $\mathbb{Z}(\ker\theta)$, where θ is the corresponding ring homomorphism. Then it is easy to check that $\ker\theta$ is generated by

   $$x_{i+2}^2 - x_i x_{i+2} \text{and } x_0 x_d - x_1 x_{d-1}$$

   for $i = 0, \ldots, d-2$

   (b) Let $X_d \subset \mathbb{P}^n$, $d \leq n$, and $X \not\subset \mathbb{P}^{n-1}$. By (I, Ex 7.7), if $d < n, X \subset \mathbb{P}^{n-1}$. Therefore $d = n$.

   Another way to do this is to notice that if H if the hyperplane divisor, then deg $H = d$ and dim $|H| = n$ since $X \not\subset \mathbb{P}^{n-1}$. Pick a point $P \in X$ not in Bs $|H|$. Then deg $(H - P) = d - 1$ and dim $|H - P| = n - 1$. Continue to get a divisor D with deg $D = 0$ and dim $|D| = n - d$. This is only possible if $d = n$. By Riemann-Roch, $h^0(H) = n + 1 - g + h^0(K - H)$. Therefore $h^0(K - H) = g$. But $h^0(L) = g$ and we can pick a hyperplane through any point so for all $P \in X$, $h^0(K - H) = g$. We can not have every point $P \in X$ a base point of $|K|$ so thus $g = 0$. Thus $X \subseteq \mathbb{P}^1$ and $\mathscr{L}(H) \cong \mathcal{O}_X(n)$. The embedding $X \hookrightarrow \mathbb{P}^n$ is induced by the complete linear system $|H|$.

   (c) Take n small enough such that the curve is in $\mathbb{P}^n$ but not in $\mathbb{P}^{n-1}$. Then by part (b), $n = 2$

   (d) Obvious.

5. Let X be a curve in $\mathbb{P}^3$ not contained in any plane.

   (a) Let $O \notin X$ be a point such that the projection from O induces a birational morphism φ from X to its image in $\mathbb{P}^2$. If the image $\varphi(X)$ were non-singular, then φ is an isomorphism and $X \cong \varphi(X)$. Since X is not contained in a hyperplane, $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to \Gamma(X, \mathcal{O}_X(1))$ is injective and thus dim $H^0(X, \mathcal{O}_X(1)) \geq 4$. $\varphi(X)$ is a complete intersection so by Ex II.5.5(a), dim $H^0(\varphi(X), \mathcal{O}_{\varphi(X)}(1)) \leq \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$. The pull back of a hyperplane section under the projective map is a hyperplane section so we see that $X \ncong \varphi(X)$.

   (b) Let X have degree d and genus g. Then project from a point (which is degree preserving), so we have $\varphi(X)$ has degree d as well. X is the normalization of $\varphi(X)$ so by Ex 1.8 X has a lower genus. Thus $g_X < g_{\varphi(X)} = \frac{1}{2}(d-1)(d-2)$.

   (c) Now let $\{X_t\}$ be the flat family of curves induced by the projection whose fiber over $t = 1$ is X, and whose fiber X_0 over $t = 0$ is a scheme with support $\varphi(X)$. Assume that X_0 does not have any nilpotents.

Then X_0 would be the curve $\varphi(X)$. But the genus of $\varphi(X)$ is larger then the genus of X which would contradict the fact that all the fibers of a flat family have the same Hilbert Polynomial.

6. *Curves of Degree 4*

   (a) Let X be a curve of degree 4 in some $\mathbb{P}^n$. If $n \geq 4$, by ex 3.4(b), $n = 4$ and $g = 0$ thus X is the rational quartic. If $X \subseteq \mathbb{P}^3$ not contained in any hyperplane, then by ex 3.5(b), $g < 3$. If $g = 0, X$ is a rational quartic curve. If $g = 2, \deg K = 2, \deg H = 4$ and so by Riemann-Roch, $h^0(H) = 3$. But $h^0(H) \geq 4$ since $X \not\subseteq \mathbb{P}^2$. Thus the other possibility is that $g = 1$. If $g = 1$ and $X \subseteq \mathbb{P}^2$, $g = (4-1)(4-2)/2 = 3$.

   (b) From the short exact sequence

   $$0 \to I_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

   we can twist to get the short exact sequence

   $$0 \to I_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$$

   Now $\dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \binom{2+3}{2} = 10$ and $\dim H^0(X, \mathcal{O}_X(2)) = h^0(2H) = 8+1-1 = 8$ by Riemann Roch. Therefore $\dim H^0(\mathbb{P}^3, I_X(2)) \geq 2$. Thus X is contained in two quadric hypersurfaces which are necessarily irreducible since X is not contained in a hyperplane. The intersection of these 2 quadric hypersurfaces has degree 4 by Bezout's Theorem and thus must be all of X.

7. The curve X defined by $xy + x^4 + y^4 = 0$ has a single node. A curve projecting to this curve would have degree 4 and genus 2 by Plüker's formula. By Ex 3.6, no such curve exists.

8. We say a (singular) integral curve in $\mathbb{P}^n$ is *strange* if there is a point which likes on all the tangent lines at nonsingular points of the curve.

   (a) The tangent line at (t, t^p, t^2p) points in the direction of $(1, pt^{p-1}, 2pt^{2p-1}) = (1, 0, 0)$ and thus contains the point of infinity on the x-axis. $(0, 0, 1, 0)$ is the other point on the curve. In x, y, w coordinates, the parametrization is (t^{2p-1}, t^p, t^{2p}). The tangent at $(0, 0, 0)$ points in the $(1, 0, 0)$ direction so it still contains $(1, 0, 0, 0)$. Thus $(1, 0, 0, 0)$ is contained in all tangent lines of X.

   (b) When $\mathrm{char}(k) = 0$, X has finitely many singular points. By choosing a point in general position, we can still project X into $\mathbb{P}^3$. Let $P \in X$ be a strange point. Choose an affine open set such that P is the point at infinity on the x-axis as well as the other necessary conditions as in the proof of Thm 3.9. The resulting morphism is ramified at all but finitely many points of X. The image is thus a point, else the map would be inseparable which would contradict the fact that $\mathrm{char}(k) = 0$. Thus $X = \mathbb{P}^1$.

9. Let X be a curve of degree d in $\mathbb{P}^3$ not contained in any plane. Then 3 points are collinear iff there is a multisecant line passing through them. A hyperplane in $\mathbb{P}^3$ intersects X at exactly d points iff the hyperplane does not pass through any tangent lines of X. By Prop 3.5, the dimension of the tangent space of X is ≤ 2. By similar arguments, we can show that the dimension of the space of multisecant lines is ≤ 1. Thus the union of these spaces is a proper closed subset of $(\mathbb{P}^3)^*$ which has dimension 3. Thus almost all hyperplanes intersect X in exactly d points.

10.

11. (a) Let X be a nonsingular variety of dimension r in $\mathbb{P}^n$ with $n > 2r = 1$. Then to show that there is a point $O \notin X$ such that the projection from O induces a closed immersion of X into $\mathbb{P}^{n-1}$, we need to find a point not lying on any tangent or multisecant line. This is done in Shaf I, page 136.

    (b)

12.

4.4 Elliptic Curves

1.

References

[BLOG] Solutions to Hartshorne, Algebraic Geometry Blog, available at http://algebraicgeometry.blogspot.com/.

[SAM] Solutions to Hartshorne, Algebraic Geometry, by Steven V Sam, available at http://math.mit.edu/~ssam/soln/.