Géométrie Algébrique 1

Return before Monday 6, November 2023

Write-up in LaTeX is strongly recommended, although not mandatory. All ressources at your disposal can be used, but only the results from the course need no proof.

All rings below are assumed commutative with unit.

1. Let R be a ring. Prove that the following sets of data are equivalent :
   (a) a morphism of R-schemes $f : \mathbb{A}^n_R \to \mathbb{A}^m_R$,
   (b) an m-tuple of polynomials $f_1, \dots, f_m \in R[x_1, \dots, x_n]$,
   (c) a set of mappings $f(B) : B^n \to B^m$, for all R-algebras B, functorial in B.
2. We call *infinitesimal* a tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in B^n$ such that $\epsilon_i\epsilon_j = 0$ for $1 \leqslant i, j \leqslant n$. Prove that if $f : \mathbb{A}^n_R \to \mathbb{A}^m_R$ is as above, there exists a matrix $J = J(x_1, \dots, x_n)$ of size (m, n) with entries in $R[x_1, \dots, x_n]$ such that for all R-algebras B, for all $b \in B^n$, and for all infinitesimals $\epsilon \in B^n$, we have $f(b + \epsilon) = f(b) + J(b)\epsilon$.

Let $R \to A$ be a ring map and M an A-module. A map $d : A \to M$ is called an R-*derivation* if it is R-linear and if $d(xy) = xd(y) + yd(x)$ for all $x, y \in A$. We denote by $\mathrm{Der}_R(A, M)$ the set of R-derivations ; it is an A-module.

3. Show that the functor $\mathrm{Mod}(A) \to \mathrm{Mod}(A)$, $M \mapsto \mathrm{Der}_R(A, M)$ is corepresentable, that is, there exists an A-module $\Omega_{A/R}$ together with a derivation $d : A \to \Omega_{A/R}$ such that the map
$$\mathrm{Hom}_A(\Omega_{A/R}, M) \longrightarrow \mathrm{Der}_R(A, M), \quad u \longmapsto u \circ d$$
is an isomorphism, fonctorially in M. The module $\Omega_{A/R}$ is called the *module of (Kähler) differential* 1*-forms* of A/R.
4. We assume that $A = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ for some integers $n, m \geqslant 1$. Prove that $X = \mathrm{Spec}(A)$ is the fibre at 0 of a morphism of R-schemes $f : \mathbb{A}^n_R \to \mathbb{A}^m_R$. If J is as in Question 2 above, with transpose J^*, and $R[x] := R[x_1, \dots, x_n]$, provide an isomorphism of A-modules :
$$\Omega_{A/R} \simeq \mathrm{coker}(J^* : R[x]^m \longrightarrow R[x]^n) \otimes_{R[x]} A.$$

We recall that the Krull dimension of an algebra of finite type E over a field k, denoted $\dim(E)$, is the supremum d of the lengths of the chains of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_d$ in E. This is stable by base change, that is $\dim(E \otimes_k k') = \dim(E)$ for all field extensions k'/k. We say that A *is a smooth* R*-algebra of relative dimension* d if $\dim(A \otimes_R k) = d$ for all residue fields k of R, and if moreover $\Omega_{A/R}$ is locally free of rank d.

5. Let $R \to R'$ be a ring map and $A' = A \otimes_R R'$. Show that if $R \to A$ is smooth, then $R' \to A'$ is smooth.
6. We consider the case where $n = m = 1$, that is $A = R[x]/(f)$, and we assume that f is monic.
   (a) Give an equivalent condition, phrased in terms of f, ensuring that A is a smooth R-algebra of relative dimension 0.
   (b) When $R = \mathbb{Z}$ and f has degree 2, prove that A is smooth over $\mathbb{Z}$ if and only if f is split with roots α, β satisfying $|\alpha - \beta| = 1$.
   (c) When $R = \mathbb{Z}$ and $f = X^3 + pX + q$, prove that A is not smooth over $\mathbb{Z}$.

Comment : a celebrated theorem of Minkowski claims that the discriminant of a number field K is never equal to ± 1. This implies that the ring of algebraic integers $\mathcal{O}_K$ is never a smooth $\mathbb{Z}$-algebra.

7. Let $h : X \to S$ be a morphism of schemes and $\mathscr{M}$ a quasi-coherent $\mathcal{O}_X$-module. We define the notion of $\mathcal{O}_S$-derivation $d : \mathcal{O}_X \to \mathscr{M}$ in the same way as in commutative algebra. Show that there exists a quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/S}$ endowed with a derivation $d : \mathcal{O}_X \to \Omega_{X/S}$ satisfying a suitable universal property, and such that for all affine opens $V = \mathrm{Spec}(R) \subset S$ and all affine opens $U = \mathrm{Spec}(A) \subset h^{-1}(V)$, we have $\Omega_{X/S|U} \simeq \widetilde{\Omega}_{A/R}$.

We now consider a field k of characteristic $p \neq 2$ and the plane curve $C \subset \mathbb{A}^2_k$ defined by the equation $y^2 = f(x)$ where $f \in k[x]$ is a polynomial of degree $d \geqslant 2$.

8. Give a necessary and sufficient condition on f for C to be smooth.
9. We assume that C is smooth. Is the module $\Omega_{C/k}$ (globally) free of rank 1 ?
10. Let $\widetilde{C}$ be the Zariski closure of C in $\mathbb{P}^2_k$, also called the *projective completion* of C. This is defined by the homogeneous equation $y^2 z^{d-2} = f(x,z)$ where $f(x,z) = z^d f(x/z)$. Describe an affine open containing all the points of $\widetilde{C} \setminus C$. Is the curve $\widetilde{C}$ smooth ?