The notions of $\mathcal{O}_X(D)$ *for an effective Cartier divisor* D *on a scheme* X and $\mathcal{O}_{\mathbb{P}^n_k}(1)$ *on projective space* have some different features, but on projective space $\mathcal{O}_{\mathbb{P}^n_k}(H)$ (for a hyperplane H) coincides with $\mathcal{O}_{\mathbb{P}^n_k}(1)$, so it may get confusing.

I'll describe the 2 seperately, and then see how it goes to the same thing on $\mathbb{P}^n$.

(Remarks : Hartshorne may give more details than Voisin if you want very accurate definitions. In Hartshorne $\mathcal{O}_X(D)$ is denoted $\mathscr{L}(D)$. Also note that $\mathcal{O}_X(D)$ is defined for arbitrary Cartier divisors, not only the effective ones. We do not care about noneffective divisors here.)

$\mathcal{O}_X(D)$ (or $\mathcal{O}(D)$) for a Cartier divisor D on a scheme X

Recall that an effective Cartier divisor D is defined, locally on some open affine $U = \mathrm{Spec}(A)$, by a nonzerodivisor equation $f \in A$ up to units in A (cf Hartshorne p. 141 and 145). To it, is associated the invertible sheaf $\mathcal{O}_X(D)$ defined on U as the module $\frac{1}{f}A$.

Remark 1 : in this general definition of $\mathcal{O}_X(D)$, there is NO notion of degree involved.

Remark 2 : there is an injective map $\mathcal{O}_X \to \mathcal{O}_X(D)$, which locally is the map $A \to \frac{1}{f}A$, $a \mapsto a = \frac{1}{f}fa$.

In our example $X = \mathbb{P}^n_k$ and the divisor D is a hyperplane H with equation a linear form ℓ. (Let us not assume that $\ell = x_n$, since it is actually useless.) Let's look at the open affine $U_i = \{x_i \neq 0\}$, its function ring is $A_i = k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$, the ring of degree 0 elements inside the localization $B_i = k[x_0, ..., x_n, \frac{1}{x_i}]$. (Note : A_i is not graded.) Let $\ell_i = \ell/x_i$ be the image of ℓ in A_i. Then on U_i the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(H)$ is $\frac{1}{\ell_i}A_i$ and we have the map $A_i \to \frac{1}{\ell_i}A_i$ defined above.

$\mathcal{O}_{\mathbb{P}^n_k}(1)$ on projective space

The sheaf $\mathcal{O}_{\mathbb{P}^n_k}(1)$ on $X = \mathbb{P}^n_k$ is defined as the sheaf of rational functions inside $\mathcal{K}$ that have degree 1 ($\mathcal{K}$ is the constant sheaf on X defined by the fraction field K of $B = k[x_0, ..., x_n]$). More precisely, on U_i it is the sheaf associated to the module M_i of degree 1 elements inside $K = \mathrm{Frac}(B_i)$. Obviously $M_i = x_iA_i$, since x_i is invertible in B_i.

The isomorphisms $\varphi_{ij} \colon (\mathcal{O}_{\mathbb{P}^n_k}(1)_{|U_i})_{|U_j} \to (\mathcal{O}_{\mathbb{P}^n_k}(1)_{|U_j})_{|U_i}$ are just the identity mappings. More precisely, the function ring of $U_{ij} = U_i \cap U_j$ may be seen by localizing A_i at $\frac{x_j}{x_i}$, we get $A_{ij} = k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}, \frac{x_i}{x_j}]$, or by localizing A_j at $\frac{x_i}{x_j}$, we get $A_{ji} = k[\frac{x_0}{x_j}, ..., \frac{x_n}{x_j}, \frac{x_j}{x_i}]$ which is of course the same thing. Then φ_{ij} above is the identity map $x_iA_{ij} \to x_jA_{ji} = x_jA_{ij}$:

$$x_ia \mapsto x_ia = x_j\frac{x_i}{x_j}a \, .$$

The construction of $\mathcal{O}(1)$ actually makes sense starting from any graded ring (so one defines $\mathcal{O}(1)$ not just for projective space), and what's essential is the notion of degree. But in full generality, I think that there is no map $\mathcal{O}_X \to \mathcal{O}(1)$. See the differences with $\mathcal{O}_X(D)$ above.

$\mathcal{O}_{\mathbb{P}^n_k}(H) \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$ on projective space

Let $X = \mathbb{P}^n_k$ again. We define an isomorphism $\mathcal{O}_{\mathbb{P}^n_k}(H) \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$ which is just multiplication by ℓ. On the open affine U_i, this translates as multiplication by ℓ_ix_i since $\ell = \ell_ix_i$:

$$\frac{1}{\ell_i}A_i \xrightarrow{\ell_ix_i} x_iA_i$$

On U_i we have the following commutative square : the top row is the map $\mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}(H)$, the right vertical arrow is the isomorphism $\mathcal{O}_{\mathbb{P}^n_k}(H) \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$, and the bottom row is the map that comes as a consequence of this isomorphism :

$$\begin{tikzcd} A_i \arrow[r, hook] \arrow[d, equal] & \frac{1}{\ell_i}A_i \arrow[d, "\ell_ix_i"] \\ A_i \arrow[r, "\ell_ix_i"] & x_iA_i \end{tikzcd}$$

So we have a morphism $\mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}(1)$, but we see that it owes its existence to the hyperplane H... In other words there is no canonical morphism $\mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}(1)$, we have to choose a hyperplane. (This is consistent with the fact that there is no map $\mathcal{O}_X \to \mathcal{O}(1)$ in general.)