5. Cohomology and base change. At this point, we have to digress to prove a theorem of Grothendieck on the behavior of cohomology groups of a family of vector bundles E_y on a family of varieties X_y, parametrized by points $y \in Y$ where X_y is assumed to be a flat family of varieties. An important consequence of this result is the semicontinuity of the dimensions of the cohomology groups of the E_y.

We assume the following basic result.

THEOREM. *If $f: X \to Y$ is a proper morphism of locally noetherian preschemes and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X, for all $p \geqslant 0$ the direct image sheaves $R^p f_*(\mathscr{F})$ are coherent sheaves of $\mathcal{O}_Y$-modules.*

We recall the following definition. If $f: X \to Y$ is a morphism of preschemes and $\mathscr{F}$ a quasicoherent sheaf on X, $\mathscr{F}$ is said to be *flat* over Y or *f-flat* if for each $x \in X$, $\mathscr{F}_x$(for its natural structure of $\mathcal{O}_{Y,f(x)}$-module) is $\mathcal{O}_{Y,f(x)}$-flat. It is easily shown that this condition is equivalent to requiring that for $U \subset X$, $V \subset Y$ with $U$and V affine open, and $f(U) \subset V$, $\mathscr{F}(U)$ is a flat $\mathcal{O}_Y(V)$-module.

The main result of this section is the following

THEOREM. *Let $f: X \to Y$ be a proper morphism of noetherian schemes with $Y = \mathrm{Spec}\ A$ affine, and $\mathscr{F}$ a coherent sheaf on X, flat over Y. There is a finite complex $K^\bullet: 0 \to K^0 \to K^1 \to \dots \to K^n \to 0$ of finitely generated projective A-modules and an isomorphism of functors*

$$H^p(X \times_Y \mathrm{Spec}\ B, \mathscr{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B), \quad (p \geqslant 0)$$

on the category of A-algebras B.

PROOF. Choose a finite affine covering $\mathfrak{A} = \{U_i\}_{i \in I}$ of X by affine open subsets. Then the Čech complex $C^\bullet = C^\bullet(\mathfrak{A}, \mathscr{F}) = \oplus\, C^p(\mathfrak{A}, \mathscr{F})$ of alternating Čech cochains on $\mathfrak{A}$ with coefficients in $\mathscr{F}$ is a finite complex of A-flat modules, whose cohomologies are isomorphic to the cohomology groups $H^p(X, \mathscr{F})$.

Moreover, for all A-algebras B, $\{U_i \times_Y \operatorname{Spec} B\}$ is an affine covering of $X \times_Y \operatorname{Spec}(B)$, and $C^p(\mathfrak{U}, \mathscr{F}) \otimes_A B$ is the module of Čech p-cochains of $\mathscr{F} \otimes_A B$ for this covering. Therefore

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(C^\bullet \otimes_A B)$$

for all B, and, in fact, functorially in B.

We need the following basic lemma

LEMMA 1. *Let $C^\bullet$ be a complex of A-modules (A any noetherian ring) such that the $H^i(C^\bullet)$ are finitely generated A-modules and such that $C^p \neq (0)$ only if $0 \leqslant p \leqslant n$. Then there exists a complex $K^\bullet$ of finitely generated A-modules such that $K^p \neq (0)$ only if $0 \leqslant p \leqslant n$ and K^p is free if $1 \leqslant p \leqslant n$ and a homomorphism of complexes $\phi\colon K^\bullet \to C^\bullet$ such that ϕ induces isomorphisms $H^i(K^\bullet) \xrightarrow{\sim} H^i(C^\bullet)$, all i. Moreover if the C^p are A-flat, then K^0 will be A-flat too.*

PROOF. We define, by descending induction on m, diagrams:

$$\begin{array}{ccccccccc}
 & & K^m & \xrightarrow{\partial^m} & K^{m+1} & \xrightarrow{\partial^{m+1}} & K^{m+2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \phi_m} & & \downarrow{\scriptstyle \phi_{m+1}} & & \downarrow{\scriptstyle \phi_{m+2}} & & \\
\cdots \longrightarrow & C^{m-1} \longrightarrow & C^m & \xrightarrow[\partial^m]{} & C^{m+1} & \xrightarrow[\partial^{m+1}]{} & C^{m+2} & \longrightarrow & \cdots .
\end{array}$$

Put $K^p = 0$ for $p > n$. Suppose we have defined $(K^p, \phi_p, \partial^p)$ for $p \geqslant m + 1$ such that the following conditions hold:

(i) $\partial^p \phi_p = \phi_{p+1} \partial^p$, $(p \geqslant m + 1)$.

(ii) $\partial^{p+1} \circ \partial^p = 0$, $(p \geqslant m + 1)$.

(iii) The ϕ^p induces isomorphisms in cohomology $H^q(K^\bullet) \xrightarrow{\sim} H^q(C^\bullet)$ for $q \geqslant m + 2$, and a surjection $\ker \partial^{m+1} \to H^{m+1}(C^\bullet)$.

(iv) The K^p are A-free and finitely generated, $(p \geqslant m + 1)$.

We then construct K^m, ∂^m and ϕ_m so as to satisfy (i)-(iii) with $m + 1$ replaced by m.

Suppose first that $m \geqslant 0$. Let B^{m+1} be the kernel of the homomorphism $\ker \partial^{m+1} \to H^{m+1}(C^\bullet)$. Since B^{m+1} is finitely generated over A (A being noetherian), we can find a finitely generated free module K'^m and a surjection $\partial': K'^m \to B^{m+1}$. Further, since $H^m(C^\cdot)$ is a finitely generated A-module, we can find a surjection $K''^m \xrightarrow{\lambda} H^m(C^\cdot)$ with K''^m finitely generated and free. Let $\mu: K''^m \to Z^m(C^\cdot)$ be any lift of λ, and $\phi''_m K''^m \to C^m$ the composite of μ with the inclusion $Z^m(C^\cdot) \to C^m$. We then put $K^m = K'^m \oplus K''^m$, and define ∂^m: $K^m \to K^{m+1}$ by putting it equal to zero on K''^m and equal to ∂' on K'^m. Since $\phi_{m+1} \circ \partial'(K'^m) \subset \partial C^m$, we can find ϕ'_m: $K'^m \to C^m$ such that $\partial \circ \phi'_m = \phi_{m+1} \circ \partial'$. We then define ϕ_m: $K^m \to C^m$ as being equal to ϕ'_m on K'^m and ϕ''_m on K''^m. The conditions (i)-(iii) are evidently fulfilled with m instead of $m+1$.

Suppose then that $m = -1$, that is, that $\{K^p, \phi_p, \partial^p\}$ have been defined for $p \geqslant 0$ satisfying (i)-(iii). We then replace K^0 by $K^0/\ker \partial^0 \cap \operatorname{Ker} \phi_0$, and we take ϕ_0: $K^0 \to C^0$ and ∂^0: $K^0 \to K^1$ to be the induced mappings. Putting $K^p = 0$ for $p < 0$, we get a complex

$$0 \to K^0 \to K^1 \to K^2 \to K^3 \to \ldots \to K^n \to 0$$

and a homomorphism ϕ: $K^\cdot \to C^\cdot$ which by construction induces isomorphisms in cohomology. We have only to check that K^0 is A-flat when all the C^p are A-flat. Consider the 'mapping cylinder' complex L defined by $L^p = K^p \oplus C^{p-1}$ for $p \in \mathbf{Z}$, and ∂: $L^p \to L^{p+1}$ defined by $\partial(x, 0) = (\partial x, \phi(x))$, $\partial(0, y) = (0, -\partial y)$. If $C'^\cdot$ is the complex obtained from $C^\cdot$ by shifting degrees by one (and making a sign change in ∂), $C'^p = C^{p-1}$, we have an exact sequence of complexes $0 \to C'^\cdot \to L^\cdot \to K^\cdot \to 0$, and hence an exact cohomology sequence

$$\begin{array}{ccccccccc} & & H^p(C^\cdot) & & & & H^{p+1}(C^\cdot) & & \\ & & \| & & & & \| & & \\ H^p(K^\cdot) & \longrightarrow & H^{p+1}(C'^\cdot) & \longrightarrow & H^{p+1}(L^\cdot) & \longrightarrow & H^{p+1}(K^\cdot) & \longrightarrow & H^{p+2}(C'^\cdot) \end{array}$$

and one sees from the definition that the cohomology maps $H^p(K^\cdot) \to H^{p+1}(C'^\cdot) \cong H^p(C^\cdot)$ are the ones induced by ϕ: $K^\cdot \to C^\cdot$. Since these are all isomorphisms, $H^p(L^\cdot) = (0)$ for all $p \in \mathbf{Z}$. But then $0 \to K^0 = L^0 \to L^1 \to L^2 \to \ldots \to L^{n+1} \to 0$ is exact and the modules L^i are flat for $i \geqslant 1$, hence K^0 is A-flat.

Applying the lemma to our case, we have a complex $K^{\bullet}$, and a homomorphism $K^{\bullet} \to C^{\bullet}$ such that

$$H^p(K^{\bullet}) \xrightarrow{\sim} H^p(C^{\bullet}) \simeq H^p(X, \mathscr{F}), \text{ all } p.$$

Note that K^0 is A-projective, since it is A-flat and finitely generated over a noetherian A. It remains to check that for all A-algebras B, $H^p(K^{\bullet} \otimes_A B) \to H^p(C^{\bullet} \otimes_A B)$ is an isomorphism too. This is a consequence of

Lemma 2. *Let $C^{\bullet}$, $K^{\bullet}$ be any finite complexes of flat A-modules, and let $C^{\bullet} \to K^{\bullet}$ be a homomorphism of complexes inducing isomorphisms $H^p(C^{\bullet}) \xrightarrow{\sim} H^p(K^{\bullet})$ for all p. Then for every A-algebra B, the maps $H^p(C^{\bullet} \otimes_A B) \xrightarrow{\sim} H^p(K^{\bullet} \otimes_A B)$ are isomorphisms.*

Proof. Construct the 'mapping cylinder' $L^{\bullet}$ exactly as in the proof of Lemma 1. As before, we see that $L^{\bullet}$ is an exact finite complex of flat A-modules. Then it is easy to see that all the modules $Z^p = \operatorname{Ker}(L^p \xrightarrow{\partial^p} L^{p+1})$ are flat too, hence

$$0 \longrightarrow Z^p \longrightarrow L^p \longrightarrow Z^{p+1} \longrightarrow 0$$

is a short exact sequence of flat A-modules. Therefore

$$0 \longrightarrow Z^p \otimes_A B \longrightarrow L^p \otimes_A B \longrightarrow Z^{p+1} \otimes_A B \longrightarrow 0$$

is exact, from which it follows that $L^{\bullet} \otimes_A B$ is exact. But now $L^{\bullet} \otimes_A B$ is the mapping cylinder of the map $K^{\bullet} \otimes_A B \to C^{\bullet} \otimes_A B$. So using the cohomology sequence in reverse, it follows that $H^p(K^{\bullet} \otimes_A B) \to H^p(C^{\bullet} \otimes_A B)$ are isomorphisms.

For any morphism $f: X \to Y$ and $y \in Y$, we denote by X_y the fiber over y of f (i. e., the fiber product $X \times_Y \operatorname{Spec} k(y)$, considered as a scheme over $k(y)$), and for $\mathscr{F}$ quasi-coherent on X, we denote by $\mathscr{F}_y$ the sheaf $\mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$ on X_y.

We have then the following important corollary.

Corollary. *Let X, Y, f and $\mathscr{F}$ be as in the theorem (except that Y need not be affine). Then we have:*

(a) *For each $p \geqslant 0$, the function $Y \to \mathbf{Z}$ defined by*

$y \longmapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ *is upper semicontinuous on Y.*

(b) *The function $Y \to \mathbf{Z}$ defined by*

$$y \to \chi(\mathscr{F}_y) = \sum_{p=0}^{\infty} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is locally constant on Y.

Proof. The problem being local on Y, we may assume Y affine. Let $K^\bullet$ be a complex as in the proposition; by further localization, we may assume $K^\bullet$ to be a free complex. Denote by $d^p: K^p \to K^{p+1}$ the coboundary of K. We then have

$$\begin{aligned}\dim_{k(y)} H^p(X_y, \mathscr{F}_y) &= \dim_{k(y)}[\ker(d^p \otimes_A k(y))] - \\ &\qquad - \dim_{k(y)}[\operatorname{Im}(d^{p-1} \otimes_A k(y))] \\ &= \dim_{k(y)}[K^p \otimes k(y)] - \dim_{k(y)}[\operatorname{Im}(d^p \otimes k(y))] - \\ &\qquad - \dim_{k(y)}[\operatorname{Im}(d^{p-1} \otimes k(y))]. \quad (*)\end{aligned}$$

The first term being constant on Y, (b) follows on taking alternating sum of (*) over all p. We assert that for any $p \geqslant 0$, the function $\rho_p(y) = \dim_{k(y)} [\operatorname{Im}(d^p \otimes k(y))]$ is lower semi-continuous on Y. In fact, if r is any integer $\geqslant 0$, and $d^p_r: \Lambda^r K^p \to \Lambda^r K^{p+1}$ is the map induced by d^p,

$$\{y \in Y \mid \rho_p(y) < r\} = \{y \in Y \mid d^p_r \otimes k(y) = 0\},$$

and this set is closed since d^p_r is a homomorphism of free finitely generated modules, and hence is described by a matrix in A, and the above set is the set of common zeros of all entries of the matrix. This proves (a).

Moreover, the theorem gives the following criterion for putting together the cohomology groups of $\mathscr{F}$ along the fibres of f into a vector bundle on Y.

Corollary 2. *Let X, Y, f and $\mathscr{F}$ be as above. Assume Y is reduced and connected. Then for all p the following are equivalent:*

(i) $y \longmapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ *is a constant function,*

(ii) $R^pf_*(\mathscr{F})$ *is a locally free sheaf* $\mathscr{E}$ *on* Y, *and for all* $y \in Y$, *the natural map*

$$\mathscr{E} \otimes_{\mathscr{O}_Y} k(y) \longrightarrow H^p(X_y, \mathscr{F}_y)$$

is an isomorphism.

If these conditions are fulfilled, we have further that

$$R^{p-1} f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} k(y) \longrightarrow H^{p-1}(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

PROOF. Again assume Y affine, $K^\bullet$ as in the proposition. (ii) $\Rightarrow$ (i) is obvious. To prove (i) $\Rightarrow$ (ii), we need two lemmas.

LEMMA 1. *If* Y *is reduced and* $\mathscr{F}$ *a coherent sheaf on* Y *such that* $\dim_{k(y)}[\mathscr{F} \otimes_{\mathscr{O}_Y} k(y)] = r$, *all* $y \in Y$, *then* $\mathscr{F}$ *is locally free of rank* r *on* Y.

PROOF. For any $y \in Y$, let $\sigma_1, \ldots, \sigma_r \in \mathscr{F}_y$ lift generators of $\mathscr{F}_y \otimes k(y)$. Since $\sigma_1, \ldots, \sigma_r$ are extendable to sections in a neighborhood of y, we have a homomorphism $\sigma\colon \mathscr{O}_Y{}^r|_V \to \mathscr{F}|_V$ defined in a neighborhood V of y. Then σ is surjective on the stalks at y, by Nakayama's lemma, so coker(σ) is zero at y and hence in a neighborhood of y. Thus, we may assume σ to be surjective. Then by assumption, for every $y' \in V$, the map

$$\sigma \otimes k(y')\colon k(y')^r \to \mathscr{F}_{y'} \otimes_{\mathscr{O}_{y'}} k(y')$$

is an isomorphism. Thus, if $\mathfrak{O}$ is the kernel of σ, we have $\mathfrak{O}_{y'} \subset \mathfrak{M}_{y'}\, \mathscr{O}^r_{y'}$ for each $y' \in V$. Since Y is reduced, this means that $\mathfrak{O} = (0)$. Thus σ is an isomorphism.

We apply this in the following

LEMMA 2. *Let* Y *be a reduced, noetherian affine scheme, and let*

$$\mathscr{F} \xrightarrow{\ \phi\ } \mathfrak{O}$$

be a homomorphism of coherent locally free $\mathcal{O}_Y$-sheaves. If $\dim_{k(y)}[\mathrm{Im}(\phi \otimes k(y))]$ *is locally constant, then there are splittings:*

$$\mathscr{F} \simeq \mathscr{F}_1 \oplus \mathscr{F}_2$$
$$\mathfrak{O} \simeq \mathfrak{O}_1 \oplus \mathfrak{O}_2$$

such that $\phi \mid_{\mathscr{F}_1} = (0)$, $\mathrm{Im}(\phi) \subset \mathfrak{O}_1$, *and* $\phi\colon \mathscr{F}_2 \to \mathfrak{O}_1$ *is an isomorphism, i.e.*

$$\phi = \begin{bmatrix} 0 & \text{isom.} \\ 0 & 0 \end{bmatrix}$$

Proof. By Lemma 1, $\mathfrak{O}/\phi(\mathscr{F})$ is locally free. If $Y = \mathrm{Spec}\,(A)$, $M = \Gamma(Y, \mathscr{F})$, $N = \Gamma(Y, \mathfrak{O})$, then this means that $N/\phi(M)$ is A-projective. Therefore N splits into the direct sum of $\phi(M)$ and a second submodule isomorphic to $N/\phi(M)$. Or, in sheaves, $\mathfrak{O} \simeq \mathfrak{O}_1 \oplus \mathfrak{O}_2$, where $\mathfrak{O}_1 = \mathrm{Im}(\phi)$. Moreover, this shows that $\phi(M)$ is A-projective, too, so M splits into the direct sum of $\mathrm{Ker}(\phi)$ and a second submodule isomorphic to $\phi(M)$. Or, in sheaves, $\mathscr{F} \simeq \mathscr{F}_1 \oplus \mathscr{F}_2$, where $\phi(\mathscr{F}_1) = (0)$, $\phi\colon \mathscr{F}_2 \xrightarrow{\sim} \mathfrak{O}_1$.

Now assume (i) holds. Let $K^\bullet$ be the complex given by the theorem. As in the proof of Corollary 1, $\dim[\mathrm{Im}(d^{p-1} \otimes k(y))]$ and $\dim[\mathrm{Im}(d^p \otimes k(y))]$ are locally constant. By Lemma 2, applied first to $d_p\colon K^p \to K^{p+1}$, and second to $d_{p-1}\colon K^{p-1} \to \mathrm{Ker}(d_p)$, we get splittings into projective modules:

$$\begin{array}{ccccc} Z_{p-1} \oplus K'_{p-1} & & B_p \oplus H_p \oplus K'_p & & B_{p+1} \oplus K'_{p+1} \\ \| & & \| & & \| \\ K_{p-1} & \longrightarrow & K_p & \longrightarrow & K_{p+1} \end{array}$$

where $Z_{p-1} = \mathrm{Ker}(d_{p-1})$, $d_{p-1}\colon K'_{p-1} \to B_p$ is an isomorphism, $B_p \oplus H_p = \mathrm{Ker}(d_p)$, and $d_p\colon K'_p \to B_{p+1}$ is an isomorphism. It follows immediately that

$$H^p(K^\bullet \otimes_A B) \simeq H_p \otimes_A B \simeq H^p(K^\bullet) \otimes_A B, \text{ all } B$$

and $H^{p-1}(K^\bullet \otimes_A B) \simeq Z_{p-1} \otimes_A B/\mathrm{Im}(d_{p-2} \otimes B) \simeq H^{p-1}(K^\bullet) \otimes_A B$, all B. This proves (ii).

COROLLARY 3. *Let X, Y, f and $\mathscr{F}$ be as above (unlike Corollary 2, Y need not be reduced). Assume for some p that $H^p(X_y, \mathscr{F}_y) = (0)$, all $y \in Y$. Then the natural map*

$$R^{p-1} f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^{p-1}(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

PROOF. Again assume $Y = \operatorname{Spec}(A)$, $K^\bullet$ as in the theorem. For all $y \in Y$, we know that

$$K^{p-1} \otimes k(y) \xrightarrow{d^{p-1}} K^p \otimes k(y) \xrightarrow{d^p} K^{p+1} \otimes k(y)$$

is exact. Split the vector space $K^p \otimes k(y)$ into $\overline{W}_1 \oplus \overline{W}_2$, where $\overline{W}_1 = $ Image of $K^{p-1} \otimes k(y)$, and $\overline{W}_2$ is mapped injectively to $K^{p+1} \otimes k(y)$. To prove the corollary at y, we can replace A by any localization A_f, ($f \in A$, $f(y) \neq 0$). If we do this for a suitable f, we may assume that K^p itself splits into a direct sum of free modules $W_1 \oplus W_2$ such that (a) $\overline{W}_i = W_i \otimes k(y)$, and (b) $W_1 \subset \operatorname{Im}(d^{p-1})$. To do this, just lift a basis of $\overline{W}_1$ to any elements in the image of d^{p-1}, and lift a basis of $\overline{W}_2$ arbitrarily. But then since $W_2 \otimes k(y) \to K^{p+1} \otimes k(y)$ is injective, it follows that $W_2 \to K^{p+1}$ is also injective if A is replaced again by a suitable localization A_f. But then $\operatorname{Im}(d^{p-1}) \cap W_2 = (0)$, hence $W_1 = \operatorname{Im}(d^{p-1})$. Since W_1 is a projective module the surjection $K^{p-1} \to W_1 \to 0$ splits, and $K^{p-1} \simeq \operatorname{Ker}(d^{p-1}) \oplus W_1$. It follows that we have exact sequences

$$K^{p-2} \longrightarrow \operatorname{Ker}(d^{p-1}) \longrightarrow H^{p-1}(X, \mathscr{F}) \longrightarrow 0$$

$$K^{p-2} \otimes k(y) \longrightarrow \operatorname{Ker}(d^{p-1}) \otimes k(y) \longrightarrow H^{p-1}(X_y, \mathscr{F}_y) \longrightarrow 0.$$

Therefore $H^{p-1}(X_y, \mathscr{F}_y) \simeq H^{p-1}(X, \mathscr{F}) \otimes k(y)$ as required.

COROLLARY 4. *Let X, Y, and $\mathscr{F}$ be as above. If $R^k f_*(\mathscr{F}) = (0)$ for $k > k_0$, then $H^k(X_y, \mathscr{F}_y) = (0)$ for all $y \in Y$, and for $k > k_0$.*

PROOF. Use Corollary 3 and decreasing induction on k_0.

COROLLARY 5. *Let X, Y, f and $\mathscr{F}$ be as above. Then if B is a flat A-algebra,*

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \simeq H^p(X, \mathscr{F}) \otimes_A B.$$

PROOF. This follows immediately, from the fact that for B flat over A, and *any* complex $K^\bullet$,

$$H^p(K^\bullet \otimes_A B) \simeq H^p(K^\bullet) \otimes_A B.$$

COROLLARY 6. (Seesaw Theorem—provisional form). *Let X be a complete variety, T any variety and L a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T \mid L|_{X \times \{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and if on $X \times T_1$, $p_2 \colon X \times T_1 \to T_1$ is the projection, then $L|_{X \times T_1} \simeq p_2^* M$ for some line bundle M on T_1.

PROOF. We first make the remark that a line bundle M on a complete variety X is trivial if and only if $\dim H^0(X, \underline{M}) > 0$ and $\dim H^0(X, \underline{M}^{-1}) > 0$ where $\underline{M}$ denotes the sheaf of sections of M. In fact, the necessity of these conditions is clear. Suppose conversely that they hold. The first implies the existence of a non-zero homomorphism $\mathcal{O}_X \xrightarrow{\sigma} \underline{M}$, and the second implies a non-zero homomorphism $\mathcal{O}_X \to \underline{M}^{-1}$, hence on dualizing, a non-zero homomorphism $\underline{M} \xrightarrow{\tau} \mathcal{O}_X$. Hence $\tau(\sigma(1))$ is a non-zero section of $\mathcal{O}_X$, and since X is complete and connected, $\tau(\sigma(1))$ is a non-zero scalar. This implies that $\tau \circ \sigma$ is an isomorphism, hence σ and τ are isomorphisms.

It follows that T_1 is the set of points t of T such that $\dim H^0(X \times \{t\}, \underline{L} \mid X \times \{t\}) > 0$ *and* $\dim H^0(X \times \{t\}, \underline{L}^{-1} \mid X \times \{t\}) > 0$, and it follows from Corollary 1 that T_1 is closed. Replacing T by T_1 (so T is now merely a reduced scheme of finite type over k) and L by its restriction to $X \times T_1$, we may assume that $L \mid X \times \{t\}$ is trivial for each $t \in T$. Hence $\dim H^0(X \times \{t\}, L \mid X \times \{t\}) = 1$ for all $t \in T$, so that by Corollary 2, $p_{2*}(\underline{L}) = \underline{M}$ is an invertible sheaf on T and

$$\underline{M} \otimes_{\mathcal{O}_T} (kt) \leftarrow H^0(X \times \{t\}, \underline{L} \mid X \times \{t\})$$

is an isomorphism. It clearly follows from the triviality of $L \mid X \times \{t\}$ that the natural map $p_2^*(\underline{M}) \to \underline{L}$ is an isomorphism. Since $\underline{M}$ is the sheaf of sections of M, then $p_2^* M \simeq L$.

6. The theorem of the cube : I

THEOREM. *Let X, Y be complete varieties, Z any variety and x_0, y_0 and z_0 base points on X, Y, and Z, respectively. If L is any line bundle on $X \times Y \times Z$ whose restrictions to each of $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial, L is trivial.*

REMARK. Let T be a contravariant functor on the category of complete varieties into the category $\underline{\text{Ab}}$ of abelian groups. Let $X_0, \ldots, X_n$ be any system of complete varieties, x_i^0 a base point of X_i, and let π_i: $X_0 \times \ldots \times X_n \to X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n$ ($\hat{X}_i$ indicating the omission of the i-th factor X_i) be the projection map, and σ_i: $X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n \to X_0 \times \ldots \times X_n$ the 'inclusion' defined by

$$\sigma_i(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots, x_n).$$

Consider the homomorphisms

$$\alpha_T^n: \prod_{i=0}^{n} T(X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n) \to T(X_0 \times \ldots \times X_n),$$

$$\beta_T^n: T(X_0 \times \ldots \times X_n) \to \prod_{i=1}^{n} T(X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n)$$

defined by

$$\alpha_T^n(\xi_0, \ldots, \xi_n) = \sum_0^n \pi_i^*(\xi_i),\ \beta_T^n(\eta) = (\sigma_0^*(\eta), \sigma_2^*(\eta), \ldots, \sigma_n^*(\eta)).$$

One then proves by an easy induction on n that we have a natural splitting $T(X_0 \times \ldots \times X_n) = \text{Im}\,\alpha \oplus \text{Ker}\,\beta$. The functor T is said to be of *order n* (*linear* if $n=1$, *quadratic* if $n=2$, etc.) if α is surjective, or equivalently β is injective. (Note that the definition of α is independent of base points.)

Thus, the above theorem (when Z is also assumed complete) may be paraphrased as saying that the functor $\text{Pic}\,X$ is a quadratic functor on the category of complete varieties.