A Remark on the Regularity of Solutions of Maxwell’s Equations on Lipschitz Domains

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Abstract

Let $\vec{u}$ be a vector field on a bounded Lipschitz domain in $\mathbb{R}^3$, and let $\vec{u}$ together with its divergence and curl be square integrable. If either the normal or the tangential component of $\vec{u}$ is square integrable over the boundary, then $\vec{u}$ belongs to the Sobolev space $H^{1/2}$ on the domain. This result gives a simple explanation for known results on the compact embedding of the space of solutions of Maxwell’s equations on Lipschitz domains into $L^2$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with connected Lipschitz boundary $\Gamma$. This means that $\Gamma$ can be represented locally as the graph of a Lipschitz function. For properties of Lipschitz domains, see [7], [3], [2]. In particular, $\Gamma$ has the strict cone property.

We consider real vector fields $\vec{u}$ on $\Omega$ satisfying in the distributional sense

$$\vec{u} \in L^2(\Omega) ; \quad \text{div} \, \vec{u} \in L^2(\Omega) ; \quad \text{curl} \, \vec{u} \in L^2(\Omega) . \quad (1)$$

We denote the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$.

It is well known that functions $\vec{u}$ satisfying (1) have boundary values $\vec{n} \times \vec{u}$ and $\vec{n} \cdot \vec{u}$ in the Sobolev space $H^{-1/2}(\Gamma)$ defined in the distributional sense by the natural extension of the Green formulas

$$(\text{curl} \, \vec{u}, \, \vec{v}) - (\vec{u}, \, \text{curl} \, \vec{v}) = <\vec{n} \times \vec{u}, \, \vec{v}> \quad (2)$$

$$(\text{div} \, \vec{u}, \, \varphi) + (\vec{u}, \, \text{grad} \, \varphi) = <\vec{n} \cdot \vec{u}, \, \varphi> \quad (3)$$

for all $\vec{v}, \varphi \in H^1(\Omega)$.

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Here \( \vec{n} \) denotes the exterior normal vector which exists almost everywhere on \( \Gamma \), and \(<\cdot, \cdot>\) is the natural duality in \( H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \) extending the \( L^2(\Gamma) \) inner product.

It is known that for smooth domains (e.g., \( \Gamma \in C^{1,1} \)), each one of the two boundary conditions

\[
\vec{n} \times \vec{u} \in H^{1/2}(\Gamma) \quad \text{or} \quad \vec{n} \cdot \vec{u} \in H^{1/2}(\Gamma)
\]

implies \( \vec{u} \in H^1(\Omega) \), see [2] and, for the case of homogeneous boundary conditions, [6], where one finds also a counterexample for a nonsmooth domain. Such counterexamples are derived from nonsmooth weak solutions \( v \in H^1(\Omega) \) of the Neumann problem (\( \partial_n := \vec{n} \cdot \text{grad} \) denotes the normal derivative)

\[
\Delta v = g \in L^2(\Omega) ; \quad \partial_n v = 0 \text{ on } \Gamma
\]

If \( \vec{u} = \text{grad} v \), then \( \vec{u} \) satisfies (1) and \( \vec{n} \cdot \vec{u} = 0 \) on \( \Gamma \), and \( \vec{u} \in H^s(\Omega) \) if and only if \( v \in H^{1+s}(\Omega) \). For smooth or convex domains, one knows that \( v \in H^2(\Omega) \). If \( \Omega \) has a nonconvex edge of opening angle \( \alpha \pi \), \( \alpha > 1 \), then, in general, the solution \( v \) of (5) is not in \( H^{1+s}(\Omega) \) for \( s = 1/\alpha \), hence \( \vec{u} \notin H^s(\Omega) \). This upper bound \( s \) for the smoothness of \( \vec{u} \) can be arbitrary close to 1/2.

Regularity theorems for (1), (4) have applications in the numerical approximation of the Stokes problem [2] and in the analysis of initial-boundary value problems for Maxwell’s equations [6]. The compact embedding into \( L^2(\Omega) \) of the space of solutions of the time-harmonic Maxwell equations is needed for the principle of limiting absorption. This compact embedding result was shown by Weck [10] for a class of piecewise smooth domains and by Weber [9] and Picard [8] for general Lipschitz domains. In these proofs, no regularity result for the solution \( \vec{u} \) was used or obtained. See Leis’ book [6] for a discussion.

In this note, we use the result by Dahlberg, Jerison, and Kenig [4], [5] on the \( H^{3/2} \) regularity for solutions of the Dirichlet and Neumann problems with \( L^2 \) data in potential theory (see Lemma 1 below). Together with arguments similar to those described by Girault and Raviart [2], this yields \( \vec{u} \in H^{1/2}(\Omega) \) (Theorem 2). The compact embedding in \( L^2 \) is an obvious consequence of this regularity. If instead of Lemma 1, one uses only the more elementary tools from [1], one obtains \( H^{3/2-\epsilon} \) regularity for solutions of the Dirichlet and Neumann problems in potential theory and, consequently \( \vec{u} \in H^{1/2-\epsilon}(\Omega) \) for any \( \epsilon > 0 \). This kind of regularity is also known for the case of an open manifold \( \Gamma \) (screen problem). It suffices, of course, for the compact embedding result.

The proof of the following result can be found in [4].

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Lemma 1  (Dahlberg-Jerison-Kenig) Let \( v \in H^1(\Omega) \) satisfy \( \Delta v = 0 \) in \( \Omega \). Then the two conditions

(i) \( v \mid _\Gamma \in H^1(\Gamma) \) and (ii) \( \partial_n v \mid _\Gamma \in L^2(\Gamma) \)

are equivalent. They imply \( v \in H^{3/2}(\Omega) \).

Remarks.

a.) The first assertion in the Lemma goes back to Nečas [7].

b.) There are accompanying norm estimates, viz.

There exist constants \( C_1, C_2, C_3 \), independent of \( v \) such that

\[
C_1 \| \partial_n v \|_{L^2(\Gamma)} \leq \| \vec{n} \times \text{grad} v \|_{L^2(\Gamma)} \leq C_2 \| \partial_n v \|_{L^2(\Gamma)},
\]

\[
\| v \|_{H^{3/2}(\Omega)} \leq C_3 \| v \mid _\Gamma \|_{H^1(\Gamma)}.
\]

c.) The boundary values are attained in a stronger sense than the distributional sense (2), (3), namely pointwise almost everywhere in the sense of nontangential maximal functions in \( L^2(\Gamma) \).

Theorem 2  Let \( \vec{u} \) satisfy the conditions (1) in \( \Omega \) and either

\[
\vec{n} \times \vec{u} \in L^2(\Gamma) \tag{6}
\]

or

\[
\vec{n} \cdot \vec{u} \in L^2(\Gamma). \tag{7}
\]

Then \( \vec{u} \in H^{1/2}(\Omega) \).

If (1) is satisfied, then the two conditions (6) and (7) are equivalent.

Proof. The proof follows the lines of [2]. It is presented in detail to make sure that it is valid for Lipschitz domains.

Let \( \vec{f} := \text{curl} \vec{u} \in L^2(\Omega) \). Then \( \text{div} \vec{f} = 0 \) in \( \Omega \).

According to [2, Ch. I, Thm 3.4] there exists \( \vec{w} \in H^1(\Omega) \) with

\[
\text{curl} \vec{w} = \vec{f}, \quad \text{div} \vec{w} = 0 \quad \text{in} \ \Omega. \tag{8}
\]

The construction of \( \vec{w} \) is as follows:

Choose a ball \( \mathcal{O} \) containing \( \overline{\Omega} \) in its interior and solve in \( \mathcal{O} \setminus \overline{\Omega} \) the Neumann problem: \( \chi \in H^1(\mathcal{O} \setminus \overline{\Omega}) \) with

\[
\Delta \chi = 0 \ \text{in} \ \mathcal{O} \setminus \overline{\Omega}; \ \partial_n \chi = \vec{n} \cdot \vec{f} \ \text{on} \ \Gamma; \ \partial_n \chi = 0 \ \text{on} \ \partial \mathcal{O}. \tag{9}
\]

Note that \( \vec{n} \cdot \vec{f} \in H^{-1/2}(\Gamma) \) satisfies the solvability condition \( \langle \vec{n} \cdot \vec{f}, 1 \rangle = 0 \) because \( \text{div} \vec{f} = 0 \) in \( \Omega \).
Define $\vec{f}_0 := \vec{f}$ in $\Omega$, $\vec{f}_0 := \text{grad } \chi$ in $\Omega \setminus \Omega$, $\vec{f}_0 := 0$ in $\mathbb{R}^3 \setminus \Omega$. Then $\vec{f}_0 \in L^2(\mathbb{R}^3)$ has compact support and satisfies $\text{div } \vec{f}_0 = 0$ in $\mathbb{R}^3$. Therefore $\vec{f}_0 = \text{curl } \vec{w}$ for some $\vec{w} \in H^1(\mathbb{R}^3)$ with $\text{div } \vec{w} = 0$ in $\mathbb{R}^3$. One obtains $\vec{w}$ for example by convolution of $\vec{f}_0$ with a fundamental solution of the Laplace operator in $\mathbb{R}^3$ and taking the curl.

Thus (8) is satisfied. The function $\vec{z} := \vec{u} - \vec{w}$ satisfies

$$\vec{z} \in L^2(\Omega) \quad \text{and} \quad \text{curl } \vec{z} = 0 \quad \text{in } \Omega.$$  

(10)

Since $\Omega$ is simply connected, there exists $v \in H^1(\Omega)$ with

$$\vec{z} = \text{grad } v.$$  

(11)

Then $v$ satisfies

$$\Delta v = \text{div } \vec{u} \in L^2(\Omega).$$  

(12)

We can apply Lemma 1 to $v$, because by subtraction of a suitable function in $H^2(\Omega)$, we obtain a homogeneous Laplace equation from (12).

Now, since $\vec{w} \mid_{\Gamma} \in H^{1/2}(\Gamma)$, condition (i) in the Lemma is equivalent to

$$\vec{n} \times \text{grad } v = \vec{n} \times \vec{z} = \vec{n} \times \vec{u} - \vec{n} \times \vec{w} \in L^2(\Gamma)$$

and hence to (6), and condition (ii) is equivalent to

$$\vec{n} \cdot \text{grad } v = \vec{n} \cdot \vec{z} = \vec{n} \cdot \vec{u} - \vec{n} \cdot \vec{w} \in L^2(\Gamma)$$

and hence to (7). Therefore the Lemma implies that (6) and (7) are equivalent.

Also, $v \in H^{3/2}(\Omega)$ is equivalent to $\text{grad } v \in H^{1/2}(\Omega)$, hence to

$$\vec{u} = \vec{z} + \vec{w} = \text{grad } v + \vec{w} \in H^{1/2}(\Omega).$$

\[\blacksquare\]

**Remark.** The accompanying norm estimates are:

There exist constants $C_1$, $C_2$, $C_3$, independent of $\vec{u}$ such that

$$\|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} \leq C_1 \left( \|\vec{u}\|_{L^2(\Omega)} + \|\text{div } \vec{u}\|_{L^2(\Omega)} + \|\text{curl } \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)} \right),$$

$$\|\vec{n} \cdot \vec{u}\|_{L^2(\Gamma)} \leq C_2 \left( \|\vec{u}\|_{L^2(\Omega)} + \|\text{div } \vec{u}\|_{L^2(\Omega)} + \|\text{curl } \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} \right),$$

$$\|\vec{u}\|_{H^{1/2}(\Omega)} \leq C_3 \left( \|\vec{u}\|_{L^2(\Omega)} + \|\text{div } \vec{u}\|_{L^2(\Omega)} + \|\text{curl } \vec{u}\|_{L^2(\Omega)} + \|\vec{n} \times \vec{u}\|_{L^2(\Gamma)} \right).$$
References


