Invertibility of the biharmonic single layer potential operator

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Abstract. *The* 2×2 *system of integral equations corresponding to the biharmonic single layer potential in* $\mathbb{R}^2$ *is known to be strongly elliptic. It is also known to be positive definite on a space of functions orthogonal to polynomials of degree one. We study the question of its unique solvability without this orthogonality condition. To each curve* Γ*, we associate a* 4×4 *matrix* B_Γ *such that this problem for the family of all curves obtained from* Γ *by scale transformations is equivalent to the eigenvalue problem for* B_Γ*. We present numerical approximations for this eigenvalue problem for several classes of curves.*

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1 INTRODUCTION

Let $\Omega^- \subset \mathbb{R}^2$ be a bounded domain with boundary Γ and exterior $\Omega^+ = \mathbb{R}^2 \setminus \overline{\Omega^-}$. The biharmonic Dirichlet problem

$$\left\{ \begin{array}{rcll} \Delta^2 u & = & 0 & \text{in } \Omega^- \quad (\text{or in } \Omega^+) \\ u & = & g_0 & \text{on } \Gamma \\ \partial_n u & = & g_1 & \text{on } \Gamma, \end{array} \right. \tag{1.1}$$

can be solved using the system of integral equations on Γ

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Gamma \Big\{ G(x,y)\, \varphi_0(y) + \partial_{n(y)} G(x,y)\, \varphi_1(y) \Big\}\, ds(y) & = & g_0(x) \\ \displaystyle\int_\Gamma \Big\{ \partial_{n(x)} G(x,y)\, \varphi_0(y) + \partial_{n(x)} \partial_{n(y)} G(x,y)\, \varphi_1(y) \Big\}\, ds(y) & = & g_1(x), \end{array} \right. \tag{1.2}$$

whose solution (φ_0, φ_1) provides a couple of densities allowing the representation of u as the following "single layer potential":

$$u(x) = \int_\Gamma \Big\{ G(x,y)\, \varphi_0(y) + \partial_{n(y)} G(x,y)\, \varphi_1(y) \Big\}\, ds(y), \quad x \in \Omega^- \cup \Omega^+. \tag{1.3}$$

In (1.2) and (1.3), G is a fundamental solution for Δ^2 in $\mathbb{R}^2$. Here, we take the standard fundamental solution

$$G(x,y) = \frac{1}{8\pi}\, |x-y|^2 \log|x-y|.$$

We call the integral operator of system (1.2) *"the biharmonic single layer potential operator"* on the curve Γ. This operator appears not only in the solution of the clamped plate problem by a single layer potential ansatz, as in (1.2), but also in the solution of that problem by a direct method and also in discretizations of the Poincaré-Steklov operator for the bilaplacian, used for example in coupling methods of finite and boundary elements or in other domain decomposition methods (see [10]).

In this paper, we study conditions on the curve Γ that guarantee existence and uniqueness of solutions of this system of integral equations.

The system (1.2) is a direct analogue of the first-kind integral equation

$$-\frac{1}{2\pi} \int_\Gamma \log|x-y|\, \varphi(y)\, ds(y) = g(x) \tag{1.4}$$

whose solutions yield the densities φ allowing the representation of the solutions of the Dirichlet problem for the Laplace operator, by a single layer potential in Ω^- or in Ω^+.

Many properties of the single layer potential operator of equation (1.4) are known to have straightforward generalizations to the 2×2 system of integral operator of system (1.2).

Thus it is known (supposing for the moment that Γ is smooth) that this is a strongly elliptic system of pseudodifferential operators of orders $\begin{pmatrix} -3 & -2 \\ -2 & -1 \end{pmatrix}$ which defines a self-adjoint operator on the Sobolev space $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$. It also defines a bounded positive definite bilinear form on the subspace of codimension 3:

$$\Big\{(\varphi_0, \varphi_1) \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \mid \quad \langle \varphi_0, P\rangle + \langle \varphi_1, \partial_n P\rangle = 0, \ \ \forall P \in \mathbb{P}_1\Big\} \tag{1.5}$$

where $\mathbb{P}_1$ is the space of first degree polynomials. On this subspace, the system (1.2) has therefore a nice variational formulation which lends itself to numerical approximations by Galerkin methods (see [7], [2]).

Instead of restricting the function space, one can also augment the system (1.2) by adding 3 scalar unknowns and 3 orthogonality conditions in order to obtain a uniquely solvable system.

These procedures of either restricting or augmenting the function space are also well known for equation (1.4) whose operator is positive definite on the subspace of $H^{-1/2}(\Gamma)$ orthogonal to constants. But in this case, more is known: One has existence and uniqueness of solutions of equation (1.4) itself if and only if

$$\operatorname{cap}\Gamma \neq 1,$$

where $\operatorname{cap}\Gamma$ denotes the logarithmic capacity of Γ (which also coincides with its transfinite diameter and its conformal radius, *cf* [9]). If $\operatorname{cap}\Gamma < 1$, the integral operator of (1.4) is positive definite on $H^{-1/2}(\Gamma)$, and if $\operatorname{cap}\Gamma > 1$, it has one negative eigenvalue. These results have been known for a long time. A systematic presentation can be found in [11].

For another 2×2 system of integral equations associated to the same biharmonic Dirichlet problem (1.1), consisting of the first line of (1.2) joint with the new equation

$$-\frac{1}{2\pi}\int_\Gamma \Big\{ \log|x-y|\,\varphi_0(y) - \partial_{n(y)}\log|x-y|\,\varphi_1(y)\Big\}\,ds(y) - \frac{1}{2}\varphi_1(x) = 0,$$

FUGLEDE [5] solved the question of invertibility completely: He obtained existence and uniqueness if and only if $\operatorname{cap}\Gamma \notin \{1, \frac{1}{e}\}$.

For our system (1.2), the result is known if Γ is a circle of radius R [4]: One has existence and uniqueness if and only if $R \neq \frac{1}{e}$ — let us recall that the logarithmic capacity of a circle is equal to its radius! This can be shown by an explicit representation of (1.2) in terms of Fourier series (see also the book [3], where many detailed calculations related to system (1.2) can be found).

Our result (Theorem 5.2) is the following: For any given curve Γ the system (1.2) considered on the scaled curve

$$\rho\Gamma := \{ \rho x \in \mathbb{R}^2 \mid x \in \Gamma \}$$

is uniquely solvable for any (φ_0, φ_1) if and only if the scale factor $\rho > 0$ satisfies

$$\rho \notin S_\Gamma$$

where the set S_Γ of exceptional scale factors has between 1 and 4 elements.

No regularity of the curve Γ is required. Thus polygons, open arcs and curves with several connected components are allowed. In fact, we consider a generalized form of system (1.2) where Γ can be any compact set in $\mathbb{R}^2$.

We describe the exceptional set S_Γ numerically for several examples of curves Γ. Besides the unit circle for which $S_\Gamma = \{\frac{1}{e}\}$, the only other example of a curve where the result is analytically known to us, is the interval where the two exceptional lengths $4\,e^{-1/2} = 2.426...$ and $4\,e^{-3/2} = 0.8925...$ occur.

Another simple example is a set Γ consisting of 4 points. Here the existence of 4 exceptional scale factors can be shown analytically in many cases.

A single exceptional scale factor (with a kernel of dimension 2) is found if the curve Γ is sufficiently symmetric. For the square, we find an exceptional side length of 0.601970... corresponding to a capacity of $0.355265... = e^{-1.03489...}$. For connected curves, we never found more than two exceptional scale factors. In fact, if the curve Γ consists of 2 connected components (rectangles or intervals for example), these 2 connected components have to be sufficiently far apart to produce 4 exceptional scale factors. We also present numerical results for curves with 4 connected components.

Note that similar results for other choices of fundamental solutions of Δ^2 can be obtained directly from our results. Let, for example,

$$G_c(x,y) = \frac{1}{8\pi}\,|x-y|^2\Big(\log|x-y| - c\Big).$$

Then the exceptional scale factors for the system corresponding to (1.2) by replacing G by G_c can be obtained from those for (1.2) by multiplication by e^c. This can easily be seen from the relation

$$G_c(x,y) = e^{2c}\,G(e^{-c}x, e^{-c}y).$$

2 THE HARMONIC SINGLE LAYER POTENTIAL

As a motivation for our analysis, let us briefly recall the situation for the harmonic single layer potential operator.

Let V_Γ be the integral operator defined in (1.4). The curve Γ needs only to be Lipschitz continuous, and it can be open or closed. Let φ be a function on Γ and

$$u(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,\varphi(y)\,ds(y) \tag{2.1}$$

its single layer potential in $\mathbb{R}^2$. If $\int_\Gamma \varphi\,ds = 0$, then $u(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$, and the jump relations together with Green's formula shows that $\mathbf{grad}\,u \in L^2(\mathbb{R}^2)$ and

$$\langle \varphi, V_\Gamma \varphi\rangle = \int_{\mathbb{R}^2} |\mathbf{grad}\,u(x)|^2\,dx. \tag{2.2}$$

This formula shows that $V_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is Fredholm of index zero, that V_Γ has at most one negative eigenvalue, that $\dim\ker V_\Gamma \le 1$ and that the augmented integral equation

$$\left\{\begin{array}{rcl} V_\Gamma\psi - \omega &=& g \\ \langle\psi, 1\rangle &=& \xi, \end{array}\right. \tag{2.3}$$

has for any given $g \in H^{1/2}(\Gamma)$, $\xi \in \mathbb{R}$, a unique solution $\psi \in H^{-1/2}(\Gamma)$, $\omega \in \mathbb{R}$. Choosing $g = 0$, the map $\xi \mapsto \omega$ being linear, there is therefore a well-defined constant c_Γ such that

$$\omega = c_\Gamma \xi. \tag{2.4}$$

By definition, the *logarithmic capacity* $\operatorname{cap}\Gamma$ satisfies

$$-\frac{1}{2\pi}\log\operatorname{cap}\Gamma = c_\Gamma.$$

The operator V_Γ is invertible if and only of $c_\Gamma \neq 0$ and positive definite on $H^{-1/2}(\Gamma)$ if and only of $c_\Gamma > 0$.

Now c_Γ has a simple behavior with respect to scaling: For $\rho > 0$, let $\rho\Gamma = \{\rho x \mid x \in \Gamma\}$. Then a simple change of variables in the definition of V_Γ shows that

$$c_{\rho\Gamma} = c_\Gamma - \frac{\log\rho}{2\pi}. \tag{2.5}$$

It follows immediately from this equation that $V_{\rho\Gamma}$ is positive definite for $\log\rho < 2\pi c_\Gamma$, and that it is *non invertible* if and only if $\rho = e^{2\pi c_\Gamma}$, i.e.

$$\rho\operatorname{cap}\Gamma = 1. \tag{2.6}$$

Since the interior Dirichlet problem is always uniquely solvable, the representation formula (2.1) shows that the kernel of V_Γ is related to the following exterior Dirichlet problem with radiation condition at infinity:

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \Omega^+, \quad u\Big|_\Gamma = 0, \quad u \in H^1_{\text{loc}}(\Omega^+), \\ \exists\, c \in \mathbb{R} : u(x) = c \log|x| + \mathscr{O}\left(|x|^{-1}\right) \ \text{ as } |x| \to \infty. \end{cases} \tag{2.7}$$

V_Γ is non-invertible if and only if the problem (2.7) has a non-trivial solution.

This observation shows the connection with conformal mappings ϕ of Ω^+ to the exterior D^+ of a disk D: the *conformal radius* of Γ is defined as the radius of the disk D if one imposes the restriction on ϕ:

$$\phi(x) = x + \mathscr{O}(1) \quad \text{as} \quad x \to \infty. \tag{2.8}$$

Considering the function

$$u(x) = \log|\phi(x)|$$

we see that u is harmonic and satisfies the radiation condition of (2.7); u satisfies the boundary condition on Γ if and only if the conformal radius is equal to 1. Thus V_Γ is non-invertible if and only if the conformal radius is equal to 1. Comparing with (2.6) we obtain the equality between the logarithmic capacity and the conformal radius.

Concrete examples are also obtained in this way. If Γ is a circle of radius R, its conformal radius is equal to R and V_Γ is non-invertible if and only if $R = 1$.

If Γ is the interval $[-R, +R]$, then

$$x \mapsto \phi(x) = \frac{1}{2}\left(x + \sqrt{x^2 - R^2}\right)$$

is a conformal mapping (2.8) of $\mathbb{C} \setminus \Gamma$ to the exterior of the disk of radius $\frac{R}{2}$. Thus $\operatorname{cap}\Gamma = \frac{R}{2}$.

Let us finally mention the unit square $\Gamma = \partial[0,1]^2$. Here the Schwarz-Christoffel formulas [6] show that the conformal mapping is an elliptic integral and the capacity is given by

$$\begin{aligned} \operatorname{cap}\Gamma &= \frac{1}{2}\left(1 + \int_{-\infty}^{1} (\sqrt{1 + w^{-4}} - 1)\, dw\right)^{-1} \\ &= \frac{1}{2}\left(\sqrt{2} - 2\int_0^1 \frac{t^2\, dt}{\sqrt{1+t^4}}\right)^{-1} = \frac{1}{2}\left(\sqrt{2} - \frac{2}{3}\, {}_2\mathrm{F}_1(\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{4}; -1)\right)^{-1} \\ &= \frac{\Gamma(1/4)^2}{4\pi^{3/2}} = 0.590\ 170\ 299\ 5... \end{aligned}$$

3 REDUCTION TO A FINITE-DIMENSIONAL PROBLEM

In this section, we generalize the situation of the harmonic single layer previously described in such a way that also the biharmonic single layer potential operator (1.2) will be covered. In fact, any strongly elliptic operator, being the sum of a positive and a compact operator, is also the sum of a positive operator and of an operator of finite rank; it satisfies therefore the following hypothesis.

Hypothesis 3.1 *Let X be a real Hilbert space whose dual we denote by X'. Let $A : X \to X'$ be a linear bounded operator. We assume that there is a finite-dimensional subspace $\mathcal{P}$ of X' such that A is positive on the subspace $X_0 = \mathcal{P}^\perp$ of X, i.e. :*

$$\forall x \in X \setminus \{0\}, \qquad \big(\forall p \in \mathcal{P} \quad \langle p, x\rangle = 0\big) \quad \Longrightarrow \quad \langle Ax, x\rangle > 0.$$

Lemma 3.2 *Under Hypothesis* 3.1*, let $(p_j)_{1\le j\le d}$ be a basis of $\mathcal{P}$. The extension $\hat{A}$ of A is defined as:*

$$\begin{pmatrix} X \\ \times \\ \mathbb{R}^d \end{pmatrix} \ni \begin{pmatrix} x \\ \\ \omega \end{pmatrix} \quad \xrightarrow{\hat{A}} \quad \begin{pmatrix} Ax - \sum_j \omega_j p_j \\ \\ \langle p_k, x\rangle_{k=1,\dots,d} \end{pmatrix} \in \begin{pmatrix} X' \\ \times \\ \mathbb{R}^d \end{pmatrix}.$$

Then $\hat{A}$ is an isomorphism.

PROOF.
(i) As a straightforward consequence of the assumptions, we obtain that $\ker \hat{A} = \{0\}$.
(ii) Let us prove that

$$AX_0 \oplus \mathcal{P} = X'. \tag{3.1}$$

- Let $y \in AX_0 \cap \mathcal{P}$. Then there exist $x_0 \in X_0$ and $p \in \mathcal{P}$ such that $y = Ax_0 = p$. The scalar product with x_0 yields that $\langle Ax_0, x_0\rangle = 0$, whence $x_0 = 0$ and $y = 0$.
- Let $x \in (AX_0 \oplus \mathcal{P})^\perp$. Since x belongs to $\mathcal{P}^\perp$, x is in X_0. Then, since x belongs to $AX_0^\perp$, x is orthogonal to Ax, whence $x = 0$. We have just proved (3.1).
(iii) We prove that $\hat{A}$ is onto: Let $y \in X'$, $\xi \in \mathbb{R}^d$ and let $(q_k)_{1\le k\le d}$ be a dual basis of $(p_j)_{1\le j\le d}$. With (3.1) we have the existence of $x_0 \in X_0$ and $\omega \in \mathbb{R}^d$ such that

$$Ax_0 - \sum_j \omega_j p_j = y - \sum_k \xi_k A q_k.$$

Then $x = x_0 + \sum_k \xi_k q_k$ and $\omega = (\omega_j)$ solves $\hat{A}(x, \omega) = (y, \xi)$. ∎

Definition 3.3 Under Hypothesis 3.1, let $B : \mathbb{R}^d \to \mathbb{R}^d$ be the operator $\xi \mapsto \omega$ with ω defined by

$$\hat{A}^{-1}(0, \xi) = (x, \omega).$$

■

Thus for any given ξ, ω is the unique element of $\mathbb{R}^d$ such that there exists a solution $x \in X$ to the problem

$$\begin{cases} Ax = \sum_j \omega_j p_j \\ \langle p_k, x \rangle = \xi_k, \quad k = 1, \dots, d. \end{cases}$$

The operator B is a finite-dimensional reduction of A which inherits several important properties of A, and the reciprocity also holds:

Theorem 3.4 *Under Hypothesis* 3.1*, the following assertions hold :*
(i) *A is an isomorphism if and only if B is an isomorphism.*
(ii) *If A is symmetric, then B is symmetric.*
(iii) *If A is symmetric, then A is positive definite if and only if B is positive definite.*

For the proof of this theorem, the following lemma is useful:

Lemma 3.5 *For ξ and $\tilde{\xi}$ in $\mathbb{R}^d$, we denote*

$$(x, \omega) = \hat{A}^{-1}(0, \xi) \quad \textit{and} \quad (\tilde{x}, \tilde{\omega}) = \hat{A}^{-1}(0, \tilde{\xi}).$$

Then

$$\langle B\xi, \tilde{\xi} \rangle = \langle Ax, \tilde{x} \rangle. \tag{3.2}$$

Proof. We have $\langle B\xi, \tilde{\xi} \rangle = \sum_j \omega_j \tilde{\xi}_j$.
But $Ax = \sum_j \omega_j p_j$ and $\langle p_k, \tilde{x} \rangle = \tilde{\xi}_k$. Therefore:

$$\langle Ax, \tilde{x} \rangle = \sum_j \omega_j \langle p_j, \tilde{x} \rangle = \sum_j \omega_j \tilde{\xi}_j = \langle B\xi, \tilde{\xi} \rangle.$$

■

Proof of Theorem 3.4.
(i) Since A and B have index 0, it is sufficient to prove that

$$\ker A = \{0\} \quad \Longleftrightarrow \quad \ker B = \{0\}.$$

- If $Ax = 0$, then ξ defined by $\xi_k = \langle p_k, x \rangle$ satisfies $B\xi = 0$. If $\ker B = \{0\}$, $\xi = 0$, then $x \in X_0$, therefore $x = 0$.
- If $B\xi = 0$, then $\hat{A}^{-1}(0, \xi) = (x, 0)$, i.e. $Ax = 0$ and $\xi_k = \langle p_k, x \rangle$. If $\ker A = \{0\}$, then $x = 0$, therefore $\xi = 0$.

(ii) If A is symmetric, then

$$\forall x, \tilde{x} \in X, \qquad \langle Ax, \tilde{x} \rangle = \langle A\tilde{x}, x \rangle.$$

Let ξ and $\tilde{\xi}$ in $\mathbb{R}^d$. Defining x and $\tilde{x}$ as in Lemma 3.5, we end the proof with the equality (3.2).

(iii) - If $A > 0$: Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and $(x, \omega) = \hat{A}^{-1}(0, \xi)$. By (3.2) we have

$$\langle B\xi, \xi \rangle = \langle Ax, x \rangle.$$

Since $\xi_k = \langle p_k, x \rangle$ and $\xi \neq 0$, x is not 0. Therefore $\langle Ax, x \rangle > 0$, whence $\langle B\xi, \xi \rangle > 0$.

- If $B > 0$, by *(i)* A is invertible. Let $x \in X$; by (3.1) there exists $x_0 \in X_0$ and $\omega \in \mathbb{R}^d$ such that

$$Ax = Ax_0 + \sum_j \omega_j p_j.$$

Since A is invertible, there exist $x_j \in X$, $Ax_j = p_j$. Thus $x = x_0 + \sum_j \omega_j x_j$ and we have

$$\begin{aligned} \langle Ax, x \rangle &= \langle Ax_0 + \sum_j \omega_j p_j, x_0 + \sum_k \omega_k x_k \rangle \\ &= \langle Ax_0, x_0 \rangle + \sum_j \omega_j \langle p_j, x_0 \rangle + \sum_k \omega_k \langle Ax_0, x_k \rangle + \langle \sum_j \omega_j p_j, \sum_k \omega_k x_k \rangle. \end{aligned} \tag{3.3}$$

The second term of the right hand side is zero, and the third too by the symmetry of A. Finally, setting $\tilde{x} = \sum_k \omega_k x_k$, we have

$$\langle \sum_j \omega_j p_j, \sum_k \omega_k x_k \rangle = \langle A\tilde{x}, \tilde{x} \rangle. \tag{3.4}$$

Since $A\tilde{x} = \sum_j \omega_j p_j$, for $\xi_k = \langle p_k, \tilde{x} \rangle$ we have $\hat{A}^{-1}(0, \xi) = (\tilde{x}, \omega)$. Therefore, by (3.2) we have

$$\langle A\tilde{x}, \tilde{x} \rangle = \langle B\xi, \xi \rangle \tag{3.5}$$

By (3.3) - (3.5) we obtain that

$$\langle Ax, x \rangle = \langle Ax_0, x_0 \rangle + \langle B\xi, \xi \rangle. \tag{3.6}$$

We conclude by remarking that $\xi_k = \langle p_k, x \rangle$; thus if $\xi = 0$ and $x_0 = 0$ then $x = 0$.

■

4 THE BIHARMONIC SINGLE LAYER POTENTIAL OPERATOR

We first give a different definition of the integral operator of the system (1.2) which is completely equivalent to the representation in (1.2) if Γ is a smooth curve. There are two reasons for this reformulation:

Firstly, we shall have a scalar problem instead of a 2×2 system. This leads to a greatly simplified notation.

Secondly, we can drop any regularity assumption for Γ. Note that already in the case where Γ has a finite number of corners, the single layer representation (1.2) has to be taken with a grain of salt: The densities (φ_0, φ_1) belong to the dual space of the space of traces of H^2, the energy space for the bilaplacian. On smooth parts of Γ, the first and second traces belong to $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, and thus for Γ smooth, the natural domain of definition of our operator is $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$. At corners, however, the traces of H^2 satisfy compatibility conditions [8], [2], therefore this dual space is a certain quotient space, and the two densities (φ_0, φ_1) cannot be separated in a natural way. If Γ is only Lipschitz, then there is no neat separation of φ_0 and φ_1 at all and even $H^{-3/2}(\Gamma)$ has no longer an intrinsic meaning.

We consider therefore now the case where Γ is an arbitrary compact subset of $\mathbb{R}^2$. We define two Sobolev spaces on Γ: Let $\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)$ be the closure of $\mathscr{C}_0^\infty(\mathbb{R}^2 \setminus \Gamma)$ in $H^2(\mathbb{R}^2)$. Then we define

$$H^2_\gamma(\Gamma) := H^2(\mathbb{R}^2) \big/_{\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)} \tag{4.1}$$

and we denote by

$$\gamma : H^2(\mathbb{R}^2) \longrightarrow H^2_\gamma(\Gamma) \tag{4.2}$$

the natural projection on this quotient space. It is clear that γ is also defined on $H^2_{\mathrm{loc}}(\mathbb{R}^2)$. We further define

$$H^{-2}_\Gamma = \{\varphi \in H^{-2}(\mathbb{R}^2) \mid \operatorname{supp} \varphi \subset \Gamma\}. \tag{4.3}$$

Lemma 4.1 (i) *The duality between $H^2(\mathbb{R}^2)$ and $H^{-2}(\mathbb{R}^2)$ induces a natural duality between $H^2_\gamma(\Gamma)$ and H^{-2}_Γ.*
(ii) *If Γ is the closure of an open set Ω with Lipschitz boundary, then $H^2_\gamma(\Gamma) = H^2(\Omega)$.*
(iii) *If Γ is the boundary of an open set and is smooth ($\mathscr{C}^2$ will suffice), then $H^2_\gamma(\Gamma)$ is in a natural way isomorphic to $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. H^{-2}_Γ is isomorphic to $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$. This isomorphism is given by the association of $\varphi \in H^2_\gamma(\Gamma)$ with $(\varphi_0, \varphi_1) \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that for any $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$:*

$$\langle \varphi, \chi \rangle = \int_\Gamma (\varphi_0\, \chi + \varphi_1\, \partial_n \chi)\; ds,$$

where the latter integrals are interpreted in the obvious generalized sense.
(iv) *If* Γ *is a finite set of points* $x_1, \dots, x_n$, *then* $H^2_\gamma(\Gamma)$ *is isomorphic to* $\mathbb{R}^n$ *and a basis of* H^{-2}_Γ *is given by the* n *Dirac masses* δ_{x_i}, $i = 1 \dots, n$.

PROOF. *(i)* is clear if one notes that $\operatorname{supp}\varphi \subset \Gamma$ is equivalent to $\langle \varphi, \chi \rangle = 0$ for all $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2 \setminus \Gamma)$.
(ii) is just the definition of $H^2(\Omega)$ as space of restrictions to Ω of $H^2(\mathbb{R}^2)$.
For *(iii)*, we note that the trace operator $u \mapsto (u\big|_\Gamma, \partial_n u\big|_\Gamma)$ maps $H^2(\mathbb{R}^2)$ onto $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ and that $\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)$ is its kernel.
(iv) comes from the fact that $\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)$ coincides with the subspace of the $u \in H^2(\mathbb{R}^2)$ such that $u(x_i) = 0$ for $i = 1, \dots, n$. ■

Let now
$$G(x,y) = \frac{1}{8\pi} |x-y|^2 \log|x-y|$$
be the fundamental solution of the biharmonic equation. By G we will also denote the convolution operator with this locally integrable function. Thus for any function or distribution φ with compact support, $G\varphi$ is defined, is $\mathscr{C}^\infty$ outside the support of φ, and satisfies
$$\Delta^2 G\varphi = \varphi. \tag{4.4}$$
By taking Fourier transforms, one sees that for all $s \in \mathbb{R}$,
$$G : H^s_{\text{comp}}(\mathbb{R}^2) \longrightarrow H^{s+4}_{\text{loc}}(\mathbb{R}^2)$$
is continuous. In particular,
$$G : H^{-2}_\Gamma \longrightarrow H^2_{\text{loc}}(\mathbb{R}^2)$$
is continuous and the following definition makes sense.

Definition 4.2 $V_\Gamma := \gamma G : H^{-2}_\Gamma \to H^2_\gamma(\Gamma)$. ■

It is clear that for a smooth curve Γ, the operator V_Γ is just given by the 2×2 matrix of integral operators in (1.2) and that for piecewise smooth curves this is the natural interpretation of the "single layer potential operator".

In order to characterize the functions having a representation $u = Gf$ with $\operatorname{supp} f$ compact, we consider the following behavior at infinity:
$$\begin{aligned} u(x) = a|x|^2 \log|x| &+ (b_1 x_1 + b_2 x_2)(2\log|x| + 1) \\ &+ c(\log|x| + 1) + d_1 \frac{x_1^2 - x_2^2}{|x|^2} + d_2 \frac{x_1 x_2}{|x|^2} + \mathscr{O}\left(|x|^{-1}\right). \end{aligned} \tag{$\mathscr{R}$}$$
Here a, b_1, b_2, c, d_1, d_2 are real constants.

Proposition 4.3 *Let $u \in \mathscr{D}'(\mathbb{R}^2)$ such that $f = \Delta^2 u$ has compact support. Then $u = Gf$ if and only if u satisfies condition $(\mathscr{R})$ at infinity. If $(\mathscr{R})$ is satisfied, then*

$$\begin{array}{lll} a = \frac{1}{8\pi}\langle f, 1\rangle\ ; & b_1 = -\frac{1}{8\pi}\langle f, x_1\rangle\ ; & b_2 = -\frac{1}{8\pi}\langle f, x_2\rangle\ ; \\ c = \frac{1}{8\pi}\langle f, |x|^2\rangle\ ; & d_1 = \frac{1}{16\pi}\langle f, x_1^2 - x_2^2\rangle\ ; & d_2 = \frac{1}{4\pi}\langle f, x_1 x_2\rangle\ . \end{array} \tag{4.5}$$

Proof. Let $u = Gf$ with $f \in \mathscr{E}'(\mathbb{R}^2)$. As $|x| \to \infty$, we have

$$\begin{aligned} |x-y|^2 \log|x-y| &= |x|^2 \log|x| - (x_1 y_1 + x_2 y_2)(2\log|x| + 1) \\ &\quad + (\log|x| + 1)|y|^2 + \frac{1}{2|x|^2}\Big((x_1^2 - x_2^2)(y_1^2 - y_2^2) + 4x_1x_2y_1y_2\Big) + \mathscr{O}\left(|x|^{-1}\right), \end{aligned}$$

uniformly in $y \in \operatorname{supp} f$. This shows that u satisfies $(\mathscr{R})$ with coefficients given in (4.5).

Conversely, let $u \in \mathscr{D}'(\mathbb{R}^2)$ be such that $\Delta^2 u$ has compact support and u satisfies $(\mathscr{R})$. For the given coefficients $a, \dots, d_2$, we can find $f_0 \in \mathscr{E}'(\mathbb{R}^2)$ such that (4.5) holds with $f = f_0$. Let $u_1 = u - Gf_0$. Then u_1 satisfies $u_1(x) = \mathscr{O}(|x|^{-1})$ as $|x| \to \infty$. It suffices to show that $u_1 = Gf_1$ where $f_1 = \Delta^2 u_1$, because then $u = u_1 + Gf_0 = G(f_0 + f_1)$ and $\Delta^2 u = f_0 + f_1$.

Let B_R be a ball of radius R containing $\operatorname{supp} f_1$. On $\overline{B}_R$ we can approximate u_1 by $\mathscr{C}^\infty$ functions $(u^n)_{n\in\mathbb{N}}$ in such a way that $u^n \to u_1$ in $\mathscr{C}^\infty$ in a neighborhood of the boundary Γ_R of B_R, and $f^n = \Delta^2 u^n \to f_1$ in $\mathscr{E}'(\mathbb{R}^2)$. For u^n we can use Green's formula in B_R to obtain:

$$\begin{aligned} u^n(x) &= \int_{B_R} G(x-y)\, f^n(y)\, dy \\ &\qquad + \int_{\Gamma_R} \Big(G(x-y)\, \partial_n \Delta u^n(y) - \partial_{n(y)} G(x-y)\, \Delta u^n(y) \\ &\qquad\qquad + \Delta G(x-y)\, \partial_n u^n(y) - \partial_{n(y)} \Delta G(x-y)\, u^n(y) \Big)\, ds(y) \\ &=: Gf^n + \mathscr{J}_R(u^n; x). \end{aligned}$$

In the limit $n \to \infty$, we obtain in B_R:

$$u_1 = Gf_1 + \mathscr{J}_R(u_1; \cdot).$$

Now we use

$$u_1 = \mathscr{O}\left(R^{-1}\right)\ ; \quad \partial_n u_1 = \mathscr{O}\left(R^{-2}\right)\ ; \quad \Delta u_1 = \mathscr{O}\left(R^{-3}\right)\ ; \quad \partial_n \Delta u_1 = \mathscr{O}\left(R^{-4}\right)$$

on Γ_R as $R \to \infty$ to conclude that $\lim_{R\to\infty} \mathscr{J}_R(u_1; x) = 0$. This shows $u_1 = Gf_1$ in $\mathbb{R}^2$ and hence $u = Gf$ with $f = f_0 + f_1$. ∎

Whereas condition $(\mathscr{R})$ corresponds to the representability by G in terms of (volume and surface) potentials, positivity will follow from finiteness of energy in the exterior domain. For this, the constants a, b_1, b_2 in $(\mathscr{R})$ have to vanish. We consider therefore the following behavior at infinity

$$u(x) = c \log|x| + \mathscr{O}(1) \quad \text{as} \quad x \to \infty. \tag{$\mathscr{R}_0$}$$

Proposition 4.4 *Let $f \in H^{-2}(\mathbb{R}^2)$ with compact support and $u = Gf$. Then the following are equivalent:*
(i) $\Delta u \in L^2(\mathbb{R}^2)$
(ii) $\int_{\mathbb{R}^2} |\Delta u|^2\, dx = \langle f, u\rangle$
(iii) u *satisfies* $(\mathscr{R}_0)$
(iv) $\langle f, 1\rangle = \langle f, x_1\rangle = \langle f, x_2\rangle = 0$.

Proof. Let the ball B_R contain $\operatorname{supp} f$. Approximating f by smooth functions as above, we see that Green's formula in B_R holds in the form

$$\int_{B_R} (\Delta u)^2\, dx - \langle f, u\rangle = \int_{\Gamma_R} \Big(\partial_n \Delta u\, u - \Delta u\, \partial_n u\Big)\, ds. \tag{4.6}$$

Using the asymptotics $(\mathscr{R})$, we see that the integral over Γ_R tends to a finite limit as $R \to \infty$ if and only if $a = b_1 = b_2 = 0$. This limit is then 0. This limit is also 0 if only condition $(\mathscr{R}_0)$ is satisfied. ∎

Proposition 4.5 *Let Γ be any compact subset of $\mathbb{R}^2$. Then there exists $\lambda > 0$ such that for any $\varphi \in H_\Gamma^{-2}$ satisfying $\langle\varphi, 1\rangle = \langle\varphi, x_1\rangle = \langle\varphi, x_2\rangle = 0$ there holds*

$$\langle\varphi, V_\Gamma\varphi\rangle \geq \lambda \|\varphi\|^2_{H^{-2}}. \tag{4.7}$$

Proof. Let $u = G\varphi$. We can use *(ii)* of Proposition 4.4 to obtain

$$\langle\varphi, V_\Gamma\varphi\rangle = \int_{\mathbb{R}^2} |\Delta u|^2\, dx. \tag{4.8}$$

Now the operator $\Delta : L^2(\mathbb{R}^2) \to H^{-2}(\mathbb{R}^2)$ is continuous, and we have an estimate

$$\|\varphi\|^2_{H^{-2}} = \|\Delta^2 u\|^2_{H^{-2}} \leq \frac{1}{\lambda}\|\Delta u\|^2_{L^2}.$$

∎

We are now in a situation where we can apply the results of section 3 to $A = V_\Gamma$ and $X = H_\Gamma^{-2}$. For the space X_0 we could take the subspace of codimension 3 defined by Proposition 4.4 *(iv)*. Thus $\mathcal{P}$ could be the space of traces of polynomials of degree ≤ 1 spanned by $p_0 = \gamma 1$, $p_1 = \gamma x_1$, $p_2 = \gamma x_2$, where γ is the projection onto $H^2_\gamma(\Gamma)$, and 1, x_1, and x_2 denote the respective polynomial functions. In this way, the question of invertibility of V_Γ could be reduced to that of a 3×3 matrix B_Γ. The problem with this approach is that this 3×3 matrix does not have a simple behavior under scale transformations.

The problem disappears if we add a second degree polynomial $p_3 = \gamma |x|^2$ to the space $\mathcal{P}$. Our question is then reduced to the investigation of a 4×4 matrix B_Γ, and we will show now that this matrix has nice scaling behavior. The problem of invertibility of V_Γ for a given Γ and all $\rho > 0$ will thus be reduced to a single eigenvalue problem for a 4×4 matrix.

5 SCALING

We apply now the results of section 3 to the situation

$$A = V_\Gamma\ ; \quad X = H_\Gamma^{-2}\ ; \quad X' = H^2_\gamma(\Gamma)\ ; \quad \mathcal{P} = \operatorname{span}\{p_0, p_1, p_2, p_3\},$$

where p_0, p_1, p_2, p_3 are the equivalence classes (i.e. the traces, cf Lemma 4.1) in $H^2_\gamma(\Gamma) := H^2_{\mathrm{loc}}(\mathbb{R}^2)/\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)$ of the four polynomials indicated above

$$p_0 = \gamma 1\ ; \quad p_1 = \gamma x_1\ ; \quad p_2 = \gamma x_2\ ; \quad p_3 = \gamma |x|^2. \tag{5.1}$$

According to Proposition 4.5, A is positive on the subspace

$$X_0 = \{\varphi \in X \mid \quad \langle p_j, \varphi \rangle = 0; \quad j = 0, \ldots, 3\}.$$

Hence according to Theorem 3.4, the invertibility and positivity of V_Γ on H_Γ^{-2} are equivalent to those of the 4×4 matrix B_Γ defined as follows.

The system

$$\begin{cases} V_\Gamma \varphi = \sum_{j=0}^{3} \omega_j p_j \\ \langle \varphi,\, p_k \rangle = \xi_k \quad (k = 0, \ldots, 3) \end{cases} \tag{5.2}$$

has for any given vector $\xi \in \mathbb{R}^4$, a unique solution $(\varphi, \omega) \in H_\Gamma^{-2} \times \mathbb{R}^4$. The linear map $\xi \mapsto \omega$ is given by

$$\omega = B_\Gamma \xi. \tag{5.3}$$

Besides Proposition 4.5, for the application of Theorem 3.4, we need that p_0, p_1, p_2, p_3 is indeed a basis of $\mathcal{P}$. Thus the compact set Γ has to satisfy the following condition:

$$\begin{cases} \textit{The traces of the 4 polynomials } 1,\ x_1,\ x_2,\ |x|^2 \textit{ in } H^2_\gamma(\Gamma) \\ \textit{are linearly independent.} \end{cases} \tag{$\mathscr{P}$}$$

Note that if Γ consists only of a finite number of points, then $\tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)$ is the space of H^2 functions vanishing at these points. The trace γu in $H^2(\Gamma)$ is therefore just the restriction of u to Γ. Thus $(\mathscr{P})$ is satisfied for a finite set Γ if and only if Γ is not a subset of a line or a circle.

If Γ is a curve, or contains a curve, it is not hard to see that $(\mathscr{P})$ is always satisfied. Thus, if Γ has a connected component which is not reduced to one point, $(\mathscr{P})$ is still satisfied (if not, Γ were contained in a circle or a line, where connected sets are intervals).

From now on, we will assume that $(\mathscr{P})$ holds. We define the following 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D_\rho = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \rho^{-1} \end{pmatrix} \tag{5.4}$$

Proposition 5.1 *Let* $\rho > 0$. *Then*

$$B_{\rho\Gamma} = D_\rho \left(B_\Gamma + \frac{\log \rho}{8\pi} C \right) D_\rho .$$

PROOF. The change of variables $x' = \frac{x}{\rho}$, $y' = \frac{y}{\rho}$, gives for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$

$$\int |x-y|^2 \log |x-y| \, \varphi(\frac{y}{\rho}) \, dy =$$
$$= \rho^4 \left(\int |x'-y'|^2 \log |x'-y'| \, \varphi(y') \, dy' + \log \rho \int |x'-y'|^2 \, \varphi(y') \, dy' \right).$$

By introducing the dilation operator T_ρ

$$T_\rho \varphi(x) = \varphi(\frac{x}{\rho}), \tag{5.5}$$

this can be written for $x \in \rho\Gamma$ as:

$$GT_\rho\varphi(x) = \rho^4 \left(T_\rho G\varphi(x) + \frac{\log \rho}{8\pi} \Big(\tfrac{1}{\rho^2} \, \xi_0 |x|^2 - \tfrac{2}{\rho} (\xi_1 x_1 + \xi_2 x_2) + \xi_3 \Big) \right), \tag{5.6}$$

where $\xi_k = \langle \varphi, p_k \rangle$. By continuity, (5.6) holds for $\varphi \in \mathscr{E}'(\mathbb{R}^2)$. Now T_ρ maps H_Γ^{-2} to $H_{\rho\Gamma}^{-2}$. We use (5.6) for the solution φ of (5.2) and find

$$V_{\rho\Gamma} T_\rho \varphi = \rho^4 \left(\sum_{j=0}^{3} \omega_j p_j + \frac{\log \rho}{8\pi} \Big(\tfrac{1}{\rho^2} \, \xi_0 p_3^\rho - \tfrac{2}{\rho} (\xi_1 p_1^\rho + \xi_2 p_2^\rho) + \xi_3 p_0^\rho \Big) \right). \tag{5.7}$$

Here we wrote p_j^ρ for the projection in $H^2_\gamma(\rho\Gamma)$ of the respective polynomial. Thus

$$\Big(T_\rho p_0, T_\rho p_1, T_\rho p_2, T_\rho p_3\Big) = \Big(p_0^\rho, \tfrac{1}{\rho}\, p_1^\rho, \tfrac{1}{\rho}\, p_2^\rho, \tfrac{1}{\rho^2}\, p_3^\rho\Big) \tag{5.8}$$

and if $\xi'_k = \langle T_\rho \varphi, p_k^\rho \rangle$, then

$$\xi' = \begin{pmatrix} \rho^2 & 0 & 0 & 0 \\ 0 & \rho^3 & 0 & 0 \\ 0 & 0 & \rho^3 & 0 \\ 0 & 0 & 0 & \rho^4 \end{pmatrix} \xi.$$

Therefore, if (φ, ω) is the solution of (5.2), then $(T_\rho\varphi, \omega')$ is the solution of

$$\begin{cases} V_{\rho\Gamma} T_\rho \varphi = \displaystyle\sum_{j=0}^{3} \omega'_j p_j^\rho \\ \langle T_\rho \varphi,\ p_k^\rho \rangle = \xi', \end{cases} \tag{5.9}$$

where $\omega' = B_{\rho\Gamma}\xi'$, and we obtain from (5.7)-(5.9):

$$\begin{aligned} B_{\rho\Gamma} &= \rho^4 \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & \frac{1}{\rho^2} \end{pmatrix} + \frac{\log \rho}{8\pi} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{2}{\rho} & 0 & 0 \\ 0 & 0 & -\frac{2}{\rho} & 0 \\ \frac{1}{\rho^2} & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \rho^2 & 0 & 0 & 0 \\ 0 & \rho^3 & 0 & 0 \\ 0 & 0 & \rho^3 & 0 \\ 0 & 0 & 0 & \rho^4 \end{pmatrix}^{-1} \\ &= D_\rho B_\Gamma D_\rho + D_\rho \frac{\log \rho}{8\pi} C D_\rho. \end{aligned}$$

■

The following theorem is the main result of this paper.

Theorem 5.2 *Let Γ be a compact subset of $\mathbb{R}^2$ satisfying condition $(\mathscr{P})$. Let B_Γ and C be defined by* (5.3) *and* (5.4)*. For $\rho > 0$ the operator on $\rho\Gamma$,*

$$V_{\rho\Gamma} : H^{-2}_{\rho\Gamma} \longrightarrow H^2_\gamma(\rho\Gamma)$$

is not an isomorphism if and only if

$$\rho = e^{-8\pi\lambda},$$

where λ is a real eigenvalue of the matrix $C^{-1}B_\Gamma$. There is at least one such eigenvalue.

Proof. By Theorem 3.4, $V_{\rho\Gamma}$ is invertible if and only if $B_{\rho\Gamma}$, and thus $B_\Gamma + \frac{\log \rho}{8\pi} C$, is invertible. To see that there is at least one ρ such that $B_{\rho\Gamma}$ is not invertible, we look at the spectrum of C, which is $\{+1, -1, -2, -2\}$. Thus the spectrum of $B_\Gamma + \frac{\log \rho}{8\pi} C$ has, counting multiplicities, 1 positive and 3 negative elements as $\log\rho \to +\infty$, and 1 negative and 3 positive elements as $\log\rho \to -\infty$. From the selfadjointness of V_Γ we find that B_Γ is symmetric. The spectrum of $B_\Gamma + \frac{\log\rho}{8\pi}C$ is therefore always real, and from the continuous dependence on ρ we see that at least two eigenvalues have to change sign and therefore to vanish as $\log\rho$ runs from $-\infty$ to $+\infty$. ∎

Corollary 5.3 *For ρ sufficiently small or sufficiently large, $V_{\rho\Gamma}$ is not positive definite on $H^{-2}_{\rho\Gamma}$. For small ρ, $V_{\rho\Gamma}$ is negative on a subspace of dimension 1, and for large ρ, on a subspace of dimension 3.*

For small ρ, $V_{\rho\Gamma}$ is positive definite not only on the subspace of $H^{-2}_{\rho\Gamma}$ of codimension 3 characterized by the conditions $\xi_0 = \xi_1 = \xi_2 = 0$ (see Proposition 4.5), but also on a subspace of codimension 1, characterized by one equation $\xi_0 + \alpha\xi_3 = 0$, i.e.

$$\langle\varphi, 1 + \alpha|x|^2\rangle = 0, \quad \textit{with some } \alpha > 0.$$

Remark 5.4 If for a given ρ less than the first exceptional value for Γ, one wants to use this subspace of codimension 1 where $V_{\rho\Gamma}$ is positive, then the admissible values of α can be determined numerically from the matrix $B_{\rho\Gamma}$, hence from B_Γ. As ρ tends to 0, they behave like $\mathscr{O}(\rho^{-2})$. ∎

Generalizations : Along the lines of the preceding paragraphs, it is easy to treat the case of *higher dimensions*.

In $\mathbb{R}^3$, if we use the standard fundamental solution of the bilaplacian

$$G(x,y) = -\frac{1}{8\pi}\,|x-y|,$$

we see that one orthogonality condition is sufficient to render V_Γ positive definite. The space $\mathcal{P}$ consists of constants. Now if φ is a positive distribution in H^{-2}_Γ, then due to the negativity of G, $\langle\varphi, V_\Gamma\varphi\rangle < 0$. It follows that H^{-2}_Γ decomposes into a subspace X_0 where V_Γ defines a positive definite quadratic form, and a one-dimensional space where it is negative definite. Hence V_Γ is invertible for any Γ.

In $\mathbb{R}^4$, the standard fundamental solution of the bilaplacian is

$$G(x,y) = -\frac{1}{4\pi^2}\,\log|x-y|.$$

One can see that, once again, orthogonality to constants implies positivity of the bilinear form associated to V_Γ. Since the scaling behavior of G is the same as for the fundamental solution of the Laplacian in $\mathbb{R}^2$, one obtains the same type of results: To any Γ, there exists exactly one ρ such that $V_{\rho\Gamma}$ is not invertible.

Thus there is a logarithmic capacity in $\mathbb{R}^4$ associated to the bilaplacian just as the one in $\mathbb{R}^2$ associated to the Laplace operator.

In $\mathbb{R}^n$, $n \geq 5$, the standard fundamental solution of the bilaplacian is

$$G(x,y) = -\frac{\Gamma(n/2)}{4\pi^{n/2}(n-2)(n-4)}\,|x-y|^{4-n},$$

and V_Γ is always positive definite.

6 EXAMPLES

We have seen in §4 that the non-invertibility of V_Γ is equivalent to the existence of a non trivial solution of the exterior Dirichlet problem

$$\Delta^2 u = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma\,; \quad u \in \tilde{H}^2(\mathbb{R}^2 \setminus \Gamma)\,; \quad u \text{ satisfies } (\mathscr{R}) \text{ at } \infty. \tag{6.1}$$

Let Γ_R be a **circle** of radius R and Ω_R the domain exterior to Γ_R. For any α, β, the function

$$u_1(x) = \alpha\, x_1(2\log|x| + 1) + \beta\,\frac{x_1}{|x|^2}$$

is biharmonic in Ω_R and satisfies condition $(\mathscr{R})$ at infinity. The two Dirichlet conditions $u_1\Big|_{\Gamma_R} = 0$, $\partial_n u_1\Big|_{\Gamma_R} = 0$ lead to the vanishing of

$$\det\begin{pmatrix} 2\log R + 1 & R^{-2} \\ R^{-1}(2\log R + 3) & -R^{-3} \end{pmatrix},$$

hence to $R = e^{-1}$, $\beta = e^{-2}\alpha$, and

$$u_1(x) = x_1\Big(2 + \log|x| + 1 + \frac{1}{e^2|x|^2}\Big).$$

Likewise, we find a function u_2 by exchanging x_1 and x_2. We see that $\boxed{R = e^{-1}}$ is an exceptional radius of multiplicity 2.

Let now Γ be the **interval** $\{(x_1,x_2) \mid x_1 \in [-R,+R],\ x_2 = 0\}$. The function

$$u_1(x) = x_2 \log\frac{1}{R}\left|x + \sqrt{x^2 - R^2}\right| \qquad (x = x_1 + ix_2)$$

satisfies $\Delta^2 u_1 = 0$ in $\mathbb{R}^2 \setminus \Gamma$, $u_1 = \partial_n u_1 = 0$ on Γ. Its behavior at infinity is

$$u_1(x) = x_2\Big(\log|x| + \log\frac{2}{R}\Big) + \mathscr{O}\left(|x|^{-1}\right).$$

Therefore it satisfies $(\mathscr{R})$ if $\log\frac{2}{R} = \frac{1}{2}$. The exceptional length is $\boxed{2R = 4e^{-1/2}}$.

The function

$$u_2(x) = x_1 \log\frac{1}{R}\left|x + \sqrt{x^2 - R^2}\right| - \operatorname{Re}\sqrt{x^2 - R^2}$$

also satisfies $\Delta^2 u_2 = 0$ in $\mathbb{R}^2 \setminus \Gamma$, $u_2 = \partial_n u_2 = 0$ on Γ. Its asymptotics at infinity is

$$u_2(x) = x_1\Big(\log|x| + \log\frac{2}{R} - 1\Big) + \mathscr{O}\left(|x|^{-1}\right).$$

which satisfies $(\mathscr{R})$ if $\log\frac{2}{R} = \frac{3}{2}$. This corresponds to a second exceptional length $\boxed{2R = 4e^{-3/2}}$.

Our last analytic example is a case where we are able to exhibit 4 exceptional ratios: we take as Γ the set of the **four points**

$$X_1 = (0,0), \quad X_2 = \varepsilon(\cos\theta, \sin\theta), \quad X_3 = (1,0), \quad X_4 = (1,0) + \varepsilon(\cos\theta, \sin\theta), \tag{6.2}$$

which are the corners of a parallelogram. The length $\varepsilon \in (0,1)$ and the angle $\theta \in [0, \pi/2]$ are considered as parameters. In the basis $\{\delta_{X_1}, \delta_{X_2}, \delta_{X_3}, \delta_{X_4}\}$, the matrix of V_Γ is $(G(X_i - X_j))_{1\le i,j\le 4}$, hence

$$V_{\rho\Gamma} = \frac{\rho^2}{8\pi}\begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & \delta & \beta \\ \beta & \delta & 0 & \alpha \\ \gamma & \beta & \alpha & 0 \end{pmatrix}$$

where

$$\begin{aligned}
\alpha &= \varepsilon^2 \log\varepsilon + \varepsilon^2 \log\rho \\
\beta &= \log\rho \\
\gamma &= (1 + 2\varepsilon\cos\theta + \varepsilon^2)\Big(\log(1 + 2\varepsilon\cos\theta + \varepsilon^2) + \log\rho\Big) \\
\delta &= (1 - 2\varepsilon\cos\theta + \varepsilon^2)\Big(\log(1 - 2\varepsilon\cos\theta + \varepsilon^2) + \log\rho\Big).
\end{aligned}$$

Then we have the following factorization for the determinant of $V_{\rho\Gamma}$:

$$\det V_{\rho\Gamma} = \Big(\frac{\rho^2}{8\pi}\Big)^4 \Big(\gamma\delta - (\alpha^2 - \beta^2)\Big)\Big(\gamma\delta - (\alpha^2 + \beta^2)\Big).$$

When $\theta \neq 0, \pi/2$, this determinant is a polynomial of degree 4 in $\log\rho$, but when $\theta = 0$ (case when the points are aligned) or when $\theta = \pi/2$ (case when the points are on the same circle), the determinant degenerates into a polynomial of degree 3.

It is possible to obtain explicit formulas for the roots of the determinant. When ε is small enough, there are 4 real roots corresponding to 4 exceptional ratios ρ_1, ρ_2, ρ_3, ρ_4. We only give here their asymptotic behavior as $\varepsilon \to 0$, abbreviating $\cos\theta$ by c and $\sin\theta$ by s:

$$
\begin{aligned}
\rho_1 &= \varepsilon^{-\frac{1}{2c^2}}\, e^{-\frac{1}{2}+\frac{1}{4c^2}}\Big(1 - \frac{c^2}{2\log\varepsilon} + \mathscr{O}\left(\log^{-2}\varepsilon\right)\Big)\\
\rho_2 &= \varepsilon^{-\frac{1}{2s^2}}\, e^{-\frac{1}{2}+\frac{1}{4s^2}}\Big(1 + \frac{c^2}{2\log\varepsilon} + \mathscr{O}\left(\log^{-2}\varepsilon\right)\Big)\\
\rho_3 &= 1 + \frac{c^2}{2\log\varepsilon} + \mathscr{O}\left(\log^{-2}\varepsilon\right)\\
\rho_4 &= 1 - \frac{c^2}{2\log\varepsilon} + \mathscr{O}\left(\log^{-2}\varepsilon\right).
\end{aligned}
$$

In the case when $\theta = 0$ or $\pi/2$ for ε small enough, we find 3 exceptional ratios.

In the following we present the results of some numerical computations. For a given curve Γ, we solve the integral equation

$$
\left\{
\begin{array}{l}
V_\Gamma\varphi - \displaystyle\sum_{j=0}^{3}\omega_j p_j = 0\\
\langle\varphi,\, p_k\rangle = \delta_{kl} \quad (k = 0,\ldots,3).
\end{array}
\right.
\tag{6.3}
$$

For $l = 0,\ldots,3$, the vectors $\omega \in \mathbb{R}^4$ constitute the columns of the matrix B_Γ. We show the exceptional scale factors ρ for Γ which are given by $\rho = e^{-8\pi\lambda}$, where λ is a real eigenvalue of the 4×4 matrix $C^{-1}B_\Gamma$, see Theorem 5.2.

For the numerical approximation of (6.3), we use a very simple method: for the approximation of $\varphi = (\varphi_0, \varphi_1)$ we choose for φ_0 Dirac delta functions in the mesh points of some partition of Γ, and for φ_1 piecewise constant functions with break points in the same mesh points. This is suitable for the approximation of the space $H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)$. If Γ has corners, we use a mesh refinement at the corners. The first integral equation in (6.3) (this corresponds to the first line of the system (1.2)), is discretized by a Galerkin method using as test functions the same delta functions as for the approximation of φ_0. This corresponds therefore to nodal collocation. For the second integral equation we do not test by the piecewise constant functions, but we use the simpler mid-point collocation.

For smooth curves, the convergence of such an approximation scheme should not be hard to prove along the lines of [1]. We observe (see Figure 1 for the case of the unit square) that our computed exceptional scale factors ρ show a convergence rate $O(N^{-\alpha})$ with $\alpha > 2$, where N is the number of mesh points.

N	ρ	rate	extrapolation
20	0.60256859		
40	0.60211505		
80	0.60200206	2.005	0.60196458
160	0.60197682	2.161	0.60196955
320	0.60197129	2.191	0.60196974

Figure 1: Convergence rates (square)

In the third column, we show the convergence rate obtained from the 3 previous approximations and in the fourth column, the extrapolation from these values. The last value given, 0.60196974, should have an error of less than 10^{-8} .

Figure 2 shows the exceptional scale factors ρ_1 and ρ_2 for rectangles (solid lines) and for rectangular triangles (broken lines). The rectangles have side lengths 1 and a with a varying from 0 to 1 . The triangles are obtained by cutting these rectangles in half.

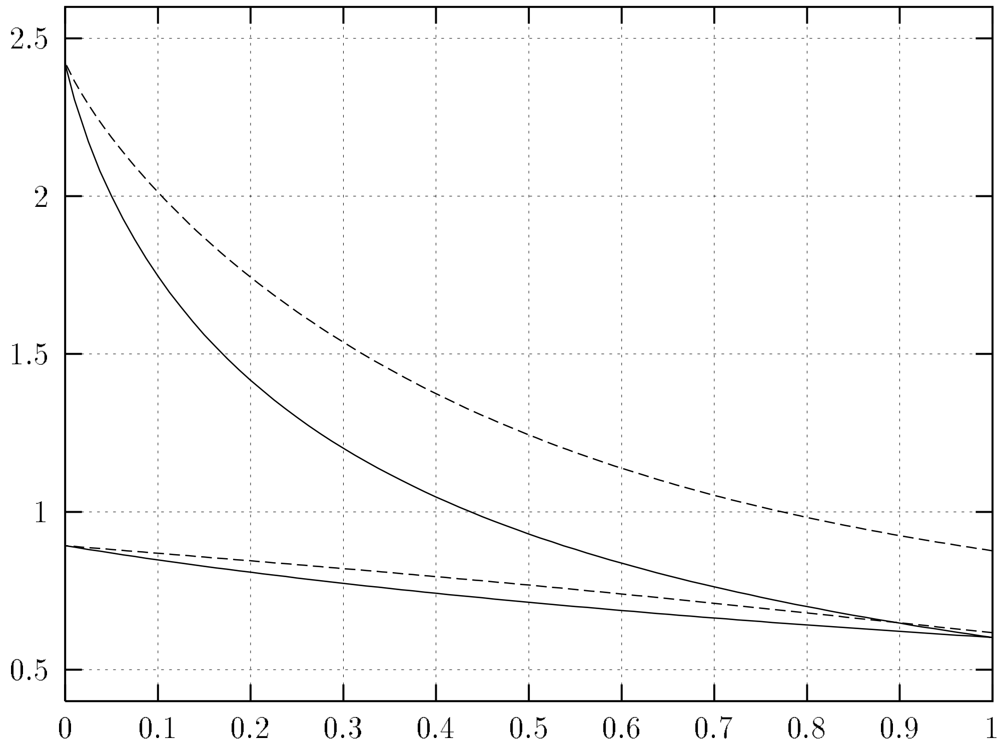


Figure 2: Rectangles (———) and Triangles (---)

One can see that as a tends to 0, ρ_1 and ρ_2 tend to the values $4e^{-1/2}$ and $4e^{-3/2}$ that we obtained above for the unit interval. For $a = 1$, the two values for the rectangle coincide and are equal to that of the unit square. Note that the values for aspect ratios a greater than one can also be read off this diagram: One has the obvious relation

$$\rho(\frac{1}{a}) = a\rho(a)\,.$$

It is not hard to prove that if the rectangle Γ has its center in $(x_1, x_2) = 0$ and its sides parallel to the axes, then for reasons of symmetry, the vectors $(\xi_0, \xi_1, \xi_2, \xi_3)$ equal to $(0,1,0,0)$ and $(0,0,1,0)$ are eigenvectors of B_Γ, each one corresponding to an exceptional scale factor. Our computations give back this fact.

For a curve Γ which is invariant by a transformation group of order ≥ 3, we always have a double exceptional scale factor.

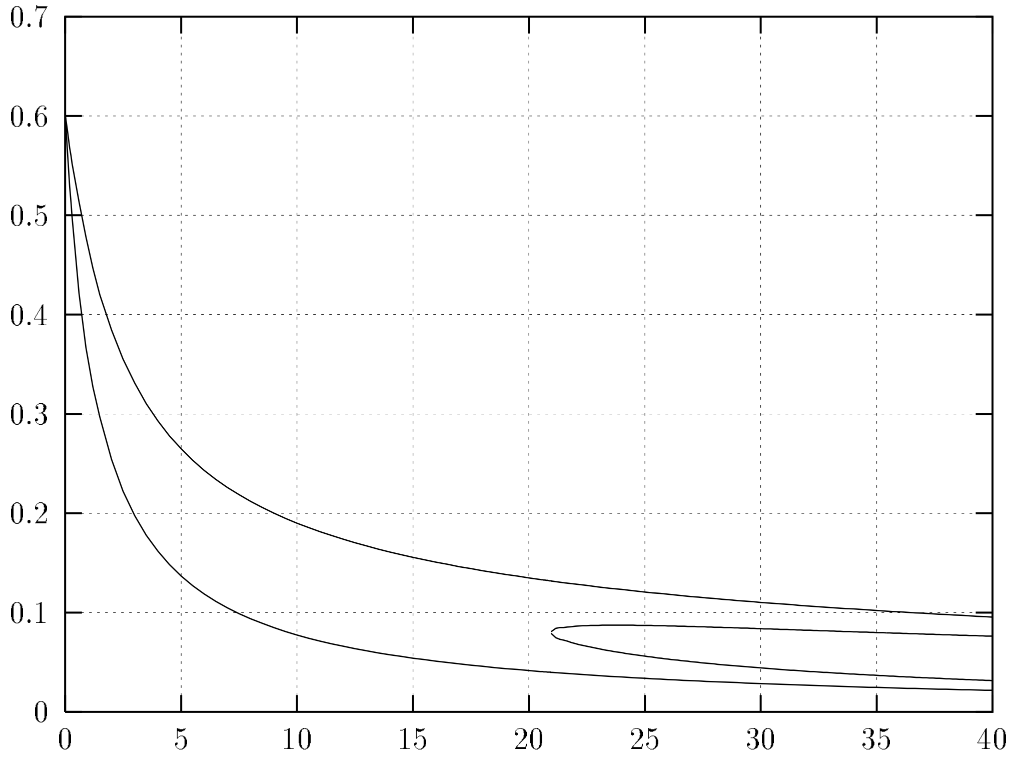


Figure 3: Two squares

In Figure 3 we show the exceptional scale factors for curves composed of two unit squares whose centers have a distance d with d varying from 0 to 40. For

$0 \leq d \leq 1$, one finds the same values as for the rectangles with $a = 1 + d$. For $d \geq 20.97$, there exist 4 exceptional scale factors.

The behavior of these scale factors as $d \to +\infty$ is also interesting. We can see that two of them behave like d^{-1} and the two other ones like $\mathcal{O}(d^{-1/2})$. The behavior in d^{-1} corresponds to the situation when Γ is reduced to two points: if $\Gamma = \{X_1, X_2\}$ with $|X_1 - X_2| = d$, then d^{-1} is a double exceptional scale factor. In the limit when $d \to +\infty$, the two squares behave (partly) as two points.

We observed the same behavior of the four exceptional scale factors for many examples of two remote connected components.

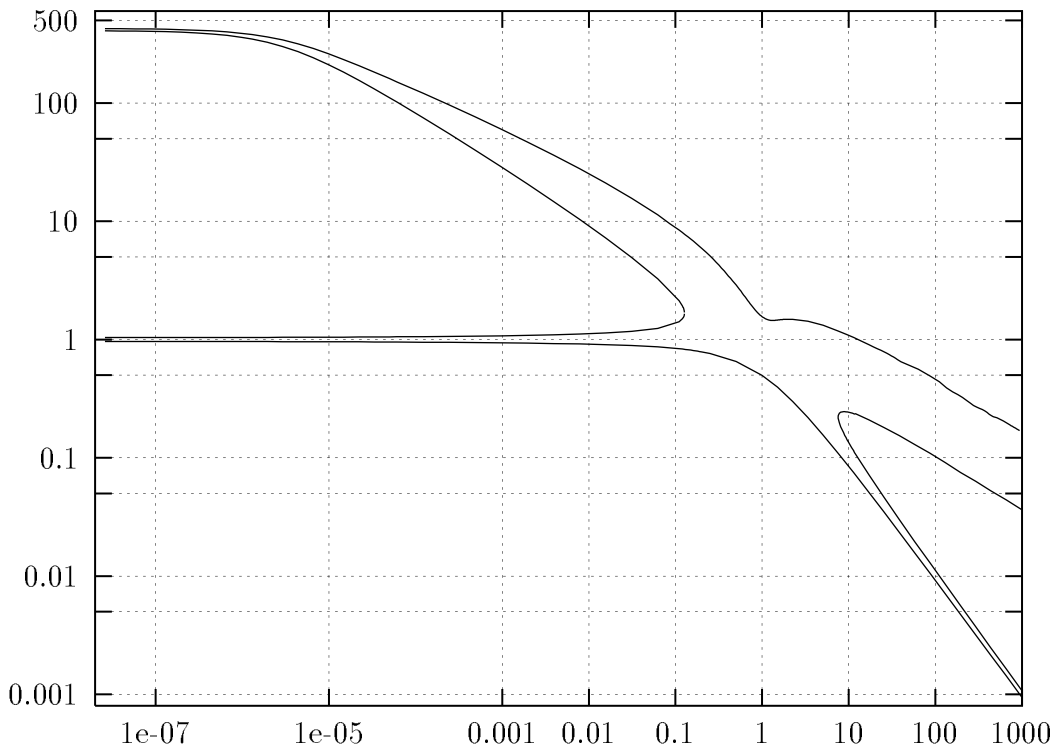
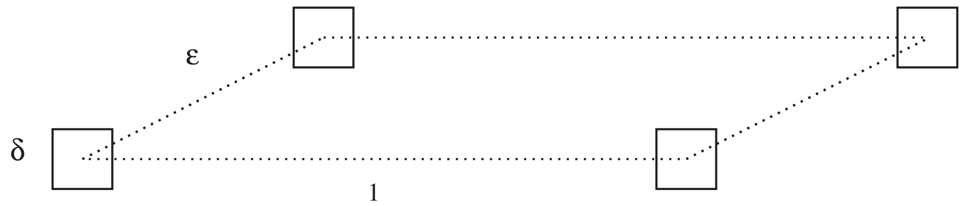


Figure 4: Four small squares

In Figure 4 we show the exceptional scale factors for curves composed of four squares with side lengths $\delta = 2 \cdot 10^{-6}$, with their centers located at the four corners of a parallelogram like in (6.2), with ε varying from 10^{-8} to 10^{3} and θ fixed to 30° .



The two lower scale factors behave like 1 as $\varepsilon \to 0$ and like ε^{-1} as $\varepsilon \to +\infty$. This corresponds to the situation of two points distant of 1 and ε^{-1}, when $\varepsilon < 1$ and $\varepsilon > 1$ respectively. This observation is in concordance with the situation of *two* remote connected components.

Concerning the two higher scale factors, we observe a change of behavior as ε becomes less than δ: then each pair of squares tends to *one* square of side δ.

The convergence towards the exceptional scale factors of the set (6.2) of four points as $\delta \to 0$ is very slow (in $\log^{-1}\delta$?) and when the ratio ε/δ becomes $\geq 10^8$ we encounter numerical fluctuations.

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