Recent progress in the analysis of the convergence of FEM for Maxwell eigenvalue problems

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Outline

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1. The Maxwell eigenvalue problem
2. A New tool of vector analysis: Regularized Poincaré integral operators
3. Application to the eigenvalue problem

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Maxwell eigenvalue problem (PEC boundary, homogeneous isotropic material)

Find $\omega \neq 0, \quad (\boldsymbol{E}, \boldsymbol{H}) \neq 0$ such that

(Maxwell EVP) $$\begin{cases} \mathbf{curl}\,\boldsymbol{E} - i\omega\boldsymbol{H} = 0 \quad \& \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\boldsymbol{E} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{E}\times\boldsymbol{n} = 0 \quad \& \quad \boldsymbol{H}\cdot\boldsymbol{n} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Variational formulation

Find $\omega \neq 0, \quad \boldsymbol{E} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \setminus \{0\}$ such that

$$\forall \widetilde{\boldsymbol{E}} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \ : \quad \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{E}} = \omega^2 \int_\Omega \boldsymbol{E} \cdot \widetilde{\boldsymbol{E}}$$

Energy space: $\mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \mid \mathbf{curl}\,\boldsymbol{u} \in L^2(\Omega)^3; \boldsymbol{u}\times\boldsymbol{n} = 0\}$

Galerkin discretization:

Restriction to finite-dimensional subspace $\mathscr{V}_N$, $N \to \infty$.

Eigenfrequencies are non-negative, discrete.

Problem: $\omega = 0$ has infinite multiplicity

Kernel: Electrostatic fields: gradients of all $\phi \in \mathring{H}^1(\Omega)$ (+ harmonic forms).

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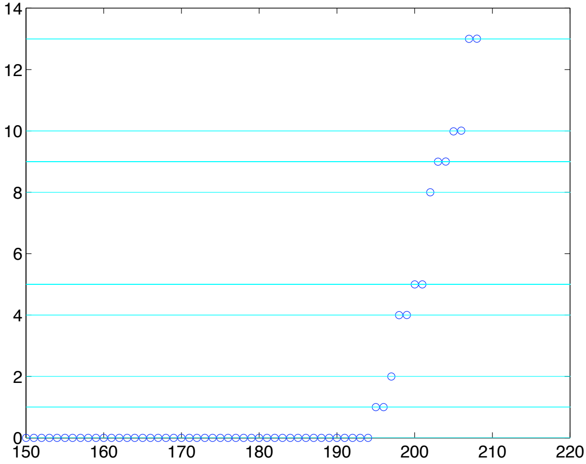
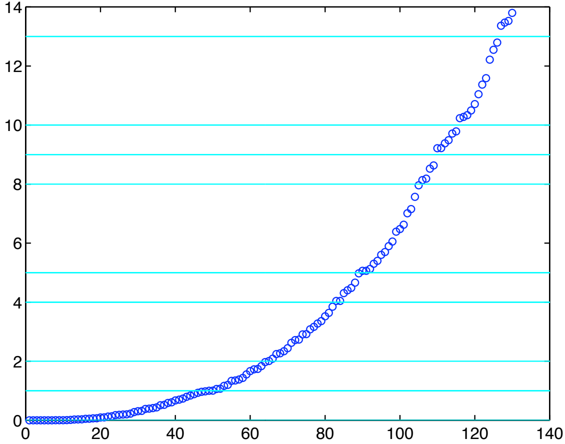
Good approximation: Triangular edge elements (15 nodes per side, $\mathbb{P}_1$)



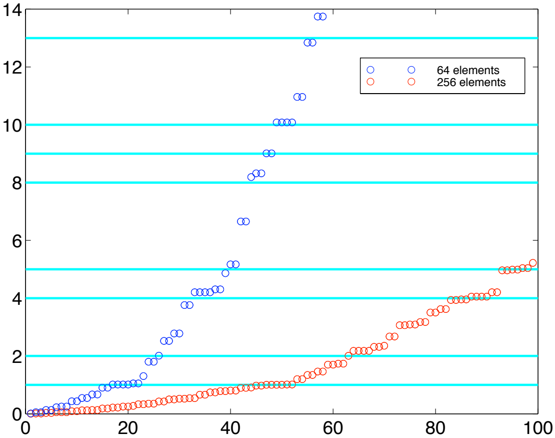
Eigenvalue ω_k^2 vs. rank k

Bad approximation: Nodal triangular elements (15 nodes per side, $\mathbb{P}_1$)



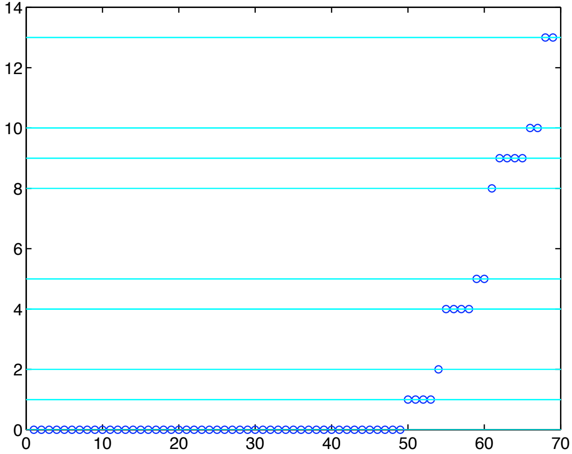
Eigenvalue ω_k^2 vs. rank k

Bad approximation: Nodal square elements ($\mathbb{Q}_1$)



All eigenvalues converge to 0 !

Yet another bad approximation: One square element ($\mathbb{Q}_8$)



Wrong multiplicities !

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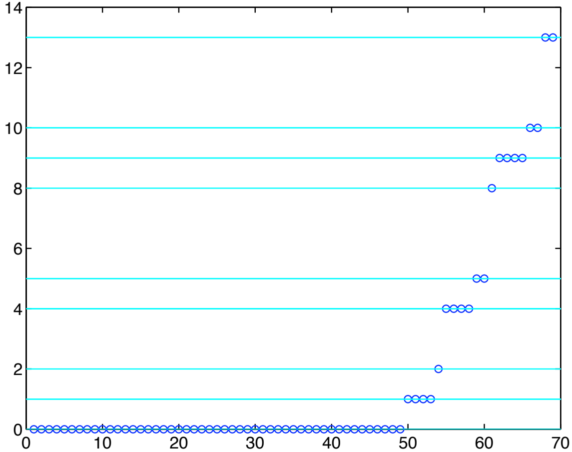

Wrong multiplicities !

Qualities of a good spectral approximation

1. Number the increasing sequence of non-zero eigenfrequencies ω , repeated according to multiplicity

$$0 < \omega^{(1)} \leq \omega^{(2)} \leq \ldots \leq \omega^{(i)} \leq \ldots$$

2. Let $\varepsilon \in (0, \omega^{(1)})$. Number the increasing sequence of discrete eigenfrequencies $\omega_N > \varepsilon$, repeated according to multiplicity

$$\varepsilon < \omega_N^{(1)} \leq \omega_N^{(2)} \leq \ldots \leq \omega_N^{(i)} \leq \ldots$$

Good spectral approximation

1. (SFA) ***Spurious-Free Approximation***

$$\exists \alpha > 0, \quad \forall N \in \mathbb{N}, \quad \omega_N \notin (0, \alpha] \quad \text{for all } \omega_N$$

2. (SCA) ***Spectrally Correct Approximation***

$$\forall i \in \mathbb{N}, \quad \lim_{N \to \infty} \omega_N^{(i)} = \omega^{(i)} \quad \text{and the eigenspaces converge.}$$

Necessary (non-sufficient) conditions for convergence

(CAS)

Completeness of the Approximating Subspaces

$$\forall \mathbf{u} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) : \qquad \lim_{N\to\infty} \inf_{\mathbf{u}_N \in \mathscr{V}_N} \|\mathbf{u} - \mathbf{u}_N\|_{H(\mathbf{curl},\Omega)} = 0$$

(CAS) $\Longrightarrow$ any eigenvector can be approximated by $\mathscr{V}_N$ as $N \to \infty$.

But $\omega = 0$ has infinite multiplicity
$\Longrightarrow$ All discrete eigenvalues will converge to 0 !

We need to handle the kernel. Two different possible directions:

1. Blow up of the kernel
Regularization [old], weighted regularization [Co.-Dauge 2002]

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) \longrightarrow (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + s(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_{L^2_w(\Omega)}$$

2. Separation of the kernel
Commuting diagrams ("cochain projections") + some conditions...

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Necessary (non-sufficient) conditions for convergence

Commuting diagram

$$
\begin{array}{ccccc}
\overset{\circ}{H}{}^1(\Omega) & \xrightarrow{\textbf{grad}} & \overset{\circ}{\boldsymbol{H}}(\textbf{curl},\Omega) & \xrightarrow{\textbf{curl}} & \\
\Big\downarrow \pi_N^0 & & \Big\downarrow \pi_N^1 & & \\
\mathscr{V}_N^0 & \xrightarrow{\textbf{grad}} & \mathscr{V}_N^1 = \mathscr{V}_N & \xrightarrow{\textbf{curl}} &
\end{array}
$$

Separation of the kernel:

Kernel $\mathscr{K} := \ker(\textbf{curl}\big|_{\overset{\circ}{\boldsymbol{H}}(\textbf{curl},\Omega)})$ *Discrete kernel* $\mathscr{K}_N := \mathscr{V}_N \cap \mathscr{K}$

(CDK)

Completeness of the Discrete Kernels

$$
\forall \mathbf{k} \in \mathscr{K} : \qquad \lim_{N\to\infty} \inf_{\mathbf{k}_N \in \mathscr{K}_N} \|\mathbf{k} - \mathbf{k}_N\|_{L^2(\Omega)} = 0.
$$

The decisive condition: Discrete Compactness

At the continuous level:

$$\mathring{\boldsymbol{H}}(\mathbf{curl},\Omega)\cap\mathscr{K}^{\perp} = \mathring{\boldsymbol{H}}(\mathbf{curl},\Omega)\cap H(\mathrm{div}\,0,\Omega)$$

is compactly embedded in $L^2(\Omega)$.

We need the corresponding property at the discrete level.

(DCP) [KIKUCHI 1989]

Discrete Compactness Property

Any sequence $\{\mathbf{u}_N\}_{N\in\mathbb{N}}$ with

$$\mathbf{u}_N \in \mathscr{V}_N \cap (\mathscr{K}_N)^{\perp} \quad \text{and} \quad \|\mathbf{u}_N\|_{H(\mathbf{curl},\Omega)} \leq 1$$

contains a subsequence that *converges in $L^2(\Omega)$*

$\mathscr{K}^{\perp}$: "divergence-free"
$(\mathscr{K}_N)^{\perp}$: "discrete divergence-free"

"The divergence of discrete divergence-free elements has to remain controlled"

Spurious-free spectrally correct approximation

The holy grail of eigenvalue approximation is to have (SFA) + (SCA): With

$$0 < \omega^{(1)} \le \omega^{(2)} \le \ldots \le \omega^{(i)} \le \ldots$$

$$0 < \omega_N^{(1)} \le \omega_N^{(2)} \le \ldots \le \omega_N^{(i)} \le \ldots$$

$\omega_N^{(i)}$ converges to $\omega^{(i)}$, together with the eigenspaces.

Theorem [Caorsi et al. 2000]

1. (CAS) + (DCP) $\Longrightarrow$ (SCA)
2. (CDK) + (DCP) $\Longrightarrow$ (SFA)

Corollary

(CAS) + (CDK) + (DCP) $\Longrightarrow$ (SFA) + (SCA)

It remains to show (DCP)...

Two versions of the Finite Element Method

1. The *h*-version :
   Refine the mesh $\mathfrak{M}_h$, keep the polynomial degree fixed,
   $N \approx [\frac{1}{h}]$
2. The *p*-version :
   Keep the mesh $\mathfrak{M}$ fixed, increase the polynomial degree p,
   $N \approx p$

Spurious Free Spectrally Correct Approximation :
Various situations

1. The *h*-version :
   - (DCP) proved for Maxwell ($d = 2, 3$) with Nedelec edge elements
   - Recent general results for differential forms by ARNOLD, FALK, WINTHER
2. The *p*-version :
   - (DCP) proved for 2d Maxwell with rectangular elements [BCDD 2006] proved modulo conjecture for triangular elements [BCD 2003]
   - General conditions ensuring (DCP) for differential forms [BCDDH 2009]

Motivation

Some puzzles from vector analysis

$\Omega \subset \mathbb{R}^n$: bounded Lipschitz domain

1. Gradient in negative Sobolev spaces

Question: $u \in H^{-1}(\Omega)$, $\mathbf{grad}\, u \in H^{-1}(\Omega) \Longrightarrow u \in L^2(\Omega)$

$$\|u\|_0 \leq C\left(\|\mathbf{grad}\, u\|_{-1} + \|u\|_{-1}\right)$$

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Application: Korn's inequality
Proof: Lions ca. 1958 (Ω smooth), Nečas 1967 (Ω Lipschitz)

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$\boldsymbol{u} \in H^{-1}(\Omega)$, $\mathbf{curl}\, \boldsymbol{u} = 0$ in $H^{-2}(\Omega) \Longrightarrow \exists \phi \in L^2(\Omega) : \boldsymbol{u} = \mathbf{grad}\, \phi$

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Application: Saint-Venant's characterization of strain tensors in L^2,
existence for strain-based variational formulation in elasticity,
Korn's inequality
Proof: Ciarlet jr. & Ciarlet 2005

In their proof, they show...

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$$\|\boldsymbol{v}\|_2 \leq C\,\|\boldsymbol{u}\|_1$$

Application: Proof of previous result

Proof: Ciarlet jr. & Ciarlet 2005

Question: What if Ω is not simply connected?

$\boldsymbol{u} \in \mathring{H}(\mathbf{curl}, \Omega)$, $\operatorname{div} \boldsymbol{u} = 0$

$$\Longrightarrow \exists \boldsymbol{v} \in \mathring{H}^2(\Omega), \alpha_1, \ldots, \alpha_b : \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} + \sum_{j=1}^{b} \alpha_j \boldsymbol{h}_j$$

Regularity of the cohomology forms $\boldsymbol{h}_j$? $\boldsymbol{h}_j \in C^\infty(\overline{\Omega})$

Application: Proof of previous result for general bounded Lipschitz domains

Proof: New

2. Curl with Dirichlet conditions

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3. Divergence with Dirichlet conditions

Question: $u \in L^2(\Omega)$, $\int_\Omega u = 0$, $\Longrightarrow \exists \boldsymbol{v} \in \mathring{H}^1(\Omega) : u = \operatorname{div} \boldsymbol{v}$

$$\|\boldsymbol{v}\|_1 \leq C\,\|u\|_0$$

Application: Inf-sup condition, Stokes, Maxwell etc.
Proof: Old

Question: $m \geq 0$, $1 < p < \infty$,
$u \in W_0^{m,p}(\Omega)$, $\int_\Omega u = 0$, $\Longrightarrow \exists \boldsymbol{v} \in W_0^{m+1,p}(\Omega) : u = \operatorname{div} \boldsymbol{v}$

$$\|\boldsymbol{v}\|_{m+1} \leq C\,\|u\|_m$$

Application: Stokes
Proof: Bogovskiĭ 1979, book by G.P. Galdi 1994,
but still conjectured in 2002...

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4. Divergence in polynomial spaces, L^2 - H^{-1} estimate

Question: K reference element, $p \in \mathbb{N}$,

$$u \in \mathbb{P}^p(K), \Longrightarrow \exists \boldsymbol{v} \in \mathbf{RT}^p(K) : u = \operatorname{div} \boldsymbol{v}$$

$$\|\boldsymbol{v}\|_0 \leq C\,\|u\|_{-1}\,, \qquad C \text{ independent of } p$$

Application: Uniform *hp*-efficiency of residual-based error estimator

Proof: Braess, Pillwein, Schöberl 2009 for rectangles K

For simplex K, general polyhedral K: New

Question: K simplex, $p \in \mathbb{N}$, $W^p(K)$ edge elements of degree p, $0 < \varepsilon < 1$

$$\boldsymbol{u} \in H^\varepsilon(K),\ \mathbf{curl}\,\boldsymbol{u} \in \mathbf{curl}\,W^p(K)$$

$$\Longrightarrow \quad \exists \boldsymbol{v} \in W^p(K), \phi \in H^{1+\varepsilon}(K):$$

$$\mathbf{curl}\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v}, \qquad \boldsymbol{u} = \boldsymbol{v} + \mathbf{grad}\,\phi$$

$$\|\phi\|_{1+\varepsilon} \le C(\|\boldsymbol{u}\|_\varepsilon + \|\mathbf{curl}\,\boldsymbol{u}\|_\varepsilon), \qquad C \text{ independent of } p$$

Application: Discrete compactness and spectrally correct convergence for the p version of FEM approximation of Maxwell eigenvalue problem

Proof: New

Boffi, Costabel, Dauge, Demkowicz, Hiptmair 2009

The Integral Operators

- M. COSTABEL, A. MCINTOSH
  On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains
  Math. Z., to appear (2009).
  DOI 10.1007/s00209-009-0517-8.

- M. E. BOGOVSKIĬ (1979)
- G. P. GALDI (1994)
- M. MITREA, D. MITREA, S. MONNIAUX (2004–2009)

The Poincaré operators: Definition

Let $D \subset \mathbb{R}^3$ be star-shaped with respect to $a \in D$

$$\mathfrak{R}_a^{\mathbf{grad}} \boldsymbol{u}(x) \quad = \quad (x-a) \cdot \int_0^1 \boldsymbol{u}\big(a + t(x-a)\big)\, dt = \int_a^x \boldsymbol{u} \cdot \mathbf{ds}$$

Known:

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\mathfrak{R}_a^{\mathbf{curl}} \boldsymbol{u}(x) &= -(x-a)\times \int_0^1 t\,\boldsymbol{u}\big(a+t(x-a)\big)\,dt \\
\mathfrak{R}_a^{\mathrm{div}} u(x) &= (x-a) \int_0^1 t^2\, u\big(a+t(x-a)\big)\,dt
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\end{aligned}$$

Known: 1. Polynomials are mapped to polynomials:

$$\mathbb{P}^p \xrightarrow{\mathfrak{R}_a^{\mathrm{div}}} \underset{\text{Raviart-Thomas}}{\mathbf{RT}^p} \xrightarrow{\mathfrak{R}_a^{\mathbf{curl}}} \underset{\text{Nedelec}}{\boldsymbol{W}^{p+1}} \xrightarrow{\mathfrak{R}_a^{\mathbf{grad}}} \mathbb{P}^{p+1}$$

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\end{aligned}$$

Known: 2. Homotopy relations:

$$\begin{aligned}
\mathfrak{R}_a^{\mathbf{grad}}\,\mathbf{grad}\,u &= u - u(a) \\
\mathfrak{R}_a^{\mathbf{curl}}\,\mathbf{curl}\,\boldsymbol{u} + \mathbf{grad}\,\mathfrak{R}_a^{\mathbf{grad}}\boldsymbol{u} &= \boldsymbol{u} \\
\mathfrak{R}_a^{\mathrm{div}}\,\mathrm{div}\,\boldsymbol{u} + \mathbf{curl}\,\mathfrak{R}_a^{\mathbf{curl}}\boldsymbol{u} &= \boldsymbol{u} \\
\mathrm{div}\,\mathfrak{R}_a^{\mathrm{div}} u &= u
\end{aligned}$$

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\end{aligned}
$$

Known: **3.** Continuity [Gopalakrishnan, Demkowicz 2004]:

$$\mathfrak{R}_a^{\mathbf{curl}}, \mathfrak{R}_a^{\mathrm{div}} : L^2(D) \to L^2(D)$$

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Known: **3.** Continuity [Gopalakrishnan, Demkowicz 2004]:

$$\mathfrak{R}_a^{\mathbf{curl}}, \mathfrak{R}_a^{\mathrm{div}} : \ L^2(D) \to L^2(D)$$

This is Not Good Enough

$$\Big(\mathfrak{R}_a^{\mathbf{grad}} : L^2(D) \to H^1(D) \Big)$$

The Poincaré operators: Definition

Regularized Poincaré operator:

$\theta \in C_0^\infty(B)$, D star-shaped with respect to B, $\int \theta(a)da = 1$

$$\mathfrak{R}^{\mathbf{curl}} u(x) = -\int_B \theta(a)(x-a) \times \int_0^1 t\, u\big(a + t(x-a)\big)\, dt\, da$$

$$\mathfrak{R}^{\mathrm{div}} u(x) = \int_B \theta(a)(x-a) \cdot \int_0^1 t^2\, u\big(a + t(x-a)\big)\, dt\, da$$

and for differential ℓ-forms u

$$\mathfrak{R}_\ell u(x) = \int_B \theta(a)(x-a) \lrcorner \int_0^1 t^{\ell-1}\, u\big(a + t(x-a)\big)\, dt\, da$$

Ω bounded domain in $\mathbb{R}^d$, $d \geq 2$.

1. $\ell \in \{0, 1, \dots, d\}$
2. $C^\infty(\Omega, \Lambda^\ell)$ space of *smooth differential ℓ-forms* on Ω.
   Fiber dimension $= \binom{d}{\ell}$
3. Scalar product on ℓ-forms $(\mathbf{u}, \mathbf{v})_\Omega$ and associated space $L^2(\Omega, \Lambda^\ell)$
4. *Exterior derivative* $\mathrm{d}_\ell : C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell+1})$.
   Co-chain complex
   $$\boxed{\mathrm{d}_\ell \circ \mathrm{d}_{\ell-1} = 0}$$
5. Domain of d_ℓ
   $$H(\mathrm{d}_\ell, \Omega) := \{\mathbf{v} \in L^2(\Omega, \Lambda^\ell) : \mathrm{d}_\ell \mathbf{v} \in L^2(\Omega, \Lambda^{\ell+1})\}$$
   Closure of C_0^∞ in $H(\mathrm{d}_\ell, \Omega)$ denoted by $\mathring{H}(\mathrm{d}_\ell, \Omega)$.

De Rham complex in dimensions $d = 2$ and $d = 3$

$d = 2$: The De Rham complex

$$\mathring{H}(\mathrm{d}_0,\Omega) \xrightarrow{\mathrm{d}_0} \mathring{H}(\mathrm{d}_1,\Omega) \xrightarrow{\mathrm{d}_1} \mathring{H}(\mathrm{d}_2,\Omega) \xrightarrow{\mathrm{d}_2} 0$$

coincides with

$$\mathring{H}^1(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\boldsymbol{H}}(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega) \xrightarrow{0} 0$$

$d = 3$: The De Rham complex

$$\mathring{H}(\mathrm{d}_0,\Omega) \xrightarrow{\mathrm{d}_0} \mathring{H}(\mathrm{d}_1,\Omega) \xrightarrow{\mathrm{d}_1} \mathring{H}(\mathrm{d}_2,\Omega) \xrightarrow{\mathrm{d}_2} \mathring{H}(\mathrm{d}_3,\Omega) \xrightarrow{\mathrm{d}_3} 0$$

coincides with

$$\mathring{H}^1(\Omega) \xrightarrow{\mathbf{grad}} \mathring{\boldsymbol{H}}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \mathring{\boldsymbol{H}}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} 0$$

The Poincaré operators: Definition

Regularized Poincaré operator:

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$$\mathfrak{R}^{\mathbf{curl}} u(x) = -\int_B \theta(a)(x-a) \times \int_0^1 t\, u\big(a+t(x-a)\big)\, dt\, da$$

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and for differential ℓ-forms u

$$\mathfrak{R}_\ell u(x) = \int_B \theta(a)(x-a) \lrcorner \int_0^1 t^{\ell-1}\, u\big(a+t(x-a)\big)\, dt\, da$$

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Change of variables... Weakly singular kernel

The Poincaré operators: Definition

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Change of variables... Weakly singular kernel

$$\mathfrak{R}^{\mathbf{curl}} u(x) = \int \int_0^\infty \Big(r^2 \frac{x-y}{|x-y|^3} + r \frac{x-y}{|x-y|^2} \Big) \theta\Big(y - r \frac{x-y}{|x-y|} \Big)\, dr \times u(y)\, dy$$

Regularized Poincaré operator:

$$\mathfrak{R}_\ell u(x) = \int_B \theta(a)(x-a) \lrcorner \int_0^1 t^{\ell-1}\, u\big(a+t(x-a)\big)\, dt\, da$$

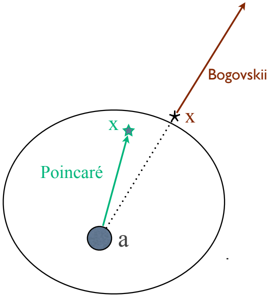
Bogovskiĭ integral operator:

$$\mathfrak{T}_\ell u(x) = -\int_B \theta(a)(x-a) \lrcorner \int_1^\infty t^{\ell-1}\, u\big(a+t(x-a)\big)\, dt\, da$$

Duality: $\mathfrak{T}_\ell = \star\, (\mathfrak{R}_{n-\ell+1})' \star$

Support properties:

- For $x \in D$, $\mathfrak{R}_\ell u(x)$ depends only on $u\big|_D$
- If $u = 0$ on $\mathbb{R}^n \setminus D$, then $\mathfrak{T}_\ell u = 0$ on $\mathbb{R}^n \setminus D$.



Theorem

* $\mathfrak{R}_\ell, \mathfrak{T}_\ell$ are pseudodifferential operators of order -1 on $\mathbb{R}^n$
* $\mathfrak{R}_\ell$ maps polynomials to polynomials
* $d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$
* $d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u$
* $\mathfrak{R}_\ell : H^s(D, \Lambda^\ell) \to H^{s+1}(D, \Lambda^{\ell-1}) \quad \forall s \in \mathbb{R}$
* $\mathfrak{T}_\ell : \widetilde{H}^s(D, \Lambda^\ell) \to \widetilde{H}^{s+1}(D, \Lambda^{\ell-1}) \quad \forall s \in \mathbb{R}$

$$\widetilde{H}^s(D) = H^s_{\overline{D}}(\mathbb{R}^n)$$

Star-shaped domain D vs general bounded Lipschitz domain Ω

On a star-shaped domain D:

$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$$
$$d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u$$

and $\mathfrak{R}_\ell$, $\mathfrak{T}_\ell$ have support properties with respect to D.

Consequence 1

$$d_\ell u = 0 \quad \Longrightarrow \quad u = d_{\ell-1}\mathfrak{R}_\ell u = d_{\ell-1}\mathfrak{T}_\ell u$$

For any $s \in \mathbb{R}$:

$$u \in H^s(D,\Lambda^\ell) \text{ and } d_\ell u = 0 \quad \Longrightarrow \quad \exists v \in H^{s+1}(D,\Lambda^{\ell-1}) : \quad u = d_{\ell-1} v$$
$$u \in \widetilde{H}^s(D,\Lambda^\ell) \text{ and } d_\ell u = 0 \quad \Longrightarrow \quad \exists v \in \widetilde{H}^{s+1}(D,\Lambda^{\ell-1}) : \quad u = d_{\ell-1} v$$

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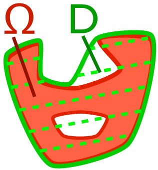
Consequence 1

$$d_\ell u = 0 \quad \Longrightarrow \quad u = d_{\ell-1}\mathfrak{R}_\ell u = d_{\ell-1}\mathfrak{T}_\ell u$$

Consequence 2. For any $s \in \mathbb{R}$:

$$u \in H^s(D,\Lambda^\ell) \text{ and } d_\ell u = 0 \quad \Longrightarrow \quad \exists\, v \in H^{s+1}(D,\Lambda^{\ell-1}) : \quad u = d_{\ell-1} v$$
$$u \in \widetilde{H}^s(D,\Lambda^\ell) \text{ and } d_\ell u = 0 \quad \Longrightarrow \quad \exists\, v \in \widetilde{H}^{s+1}(D,\Lambda^{\ell-1}) : \quad u = d_{\ell-1} v$$

On a bounded Lipschitz domain Ω with star-shaped hull D:



$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$$
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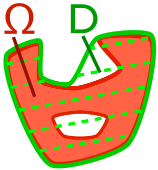
Star-shaped domain D vs general bounded Lipschitz domain Ω

On a bounded Lipschitz domain Ω with star-shaped hull D:



$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$$
$$d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u$$

and $\mathfrak{R}_\ell$, $\mathfrak{T}_\ell$ have support properties with respect to D.

Consequence 1

$$d_\ell u = 0 \quad \Longrightarrow \quad u = d_{\ell-1}\mathfrak{R}_\ell u = d_{\ell-1}\mathfrak{T}_\ell u$$

Consequence 2. ???

Star-shaped domain D vs general bounded Lipschitz domain Ω

New result: On a bounded Lipschitz domain Ω:
There exist infinitely smoothing integral operators $\mathfrak{K}_\ell$, $\mathfrak{L}_\ell$

$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u + \mathfrak{K}_\ell u$$
$$d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u + \mathfrak{L}_\ell u$$

and $\mathfrak{R}_\ell$, $\mathfrak{K}_\ell$ and $\mathfrak{T}_\ell$, $\mathfrak{L}_\ell$ have support properties with respect to Ω.

Consequence 1

$$d_\ell u = 0 \quad \Longrightarrow \quad (1 + \mathfrak{K}_\ell) u = d_{\ell-1}\mathfrak{R}_\ell u$$

$$u = d_{\ell-1} v \quad \Longrightarrow \quad u = d_{\ell-1}(\mathfrak{R}_\ell u - \mathfrak{K}_{\ell-1} v)$$

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$$u = d_{\ell-1} v \quad \Longrightarrow \quad u = d_{\ell-1}(\mathfrak{R}_\ell u - \mathfrak{K}_{\ell-1} v)$$

Consequence 2. See below

Corollary 1

For any $s \in \mathbb{R}$ we have:

(a) $u \in H^s(\Omega, \Lambda^\ell)$, $u = d_{\ell-1} v$, $v \in H^t(\Omega, \Lambda^{\ell-1})$, any $t \in \mathbb{R}$

$$\Longrightarrow \quad \exists\, w \in H^{s+1}(\Omega, \Lambda^{\ell-1}) : \; u = d_{\ell-1} w$$

$$\|w\|_{H^{s+1}(\Omega)} \leq C \left(\|u\|_{H^s(\Omega)} + \|v\|_{H^t(\Omega)} \right) .$$

(b) $u \in \widetilde{H}^s(\Omega, \Lambda^\ell)$, $u = d_{\ell-1} v$, $v \in \widetilde{H}^t(\Omega, \Lambda^{\ell-1})$, any $t \in \mathbb{R}$

$$\Longrightarrow \quad \exists\, w \in \widetilde{H}^{s+1}(\Omega, \Lambda^{\ell-1}) : \; u = d_{\ell-1} w$$

$$\|w\|_{H^{s+1}(\mathbb{R}^n)} \leq C \left(\|u\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^t(\mathbb{R}^n)} \right) .$$

Corollary 2

For any $s \in \mathbb{R}$ we have:

(a) $u \in H^s(\Omega, \Lambda^\ell),\ d_\ell u = 0$ in $\Omega \implies u = d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{K}_\ell u$ in Ω

$$\mathfrak{R}_\ell u \in H^{s+1}(\Omega, \Lambda^{\ell-1}), \quad \mathfrak{K}_\ell u \in C^\infty(\overline{\Omega}, \Lambda^\ell)$$

(b) $u \in \widetilde{H}^s(\Omega, \Lambda^\ell),\ d_\ell u = 0$ in $\mathbb{R}^n \implies u = d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{L}_\ell u$ in $\mathbb{R}^n$

$$\mathfrak{T}_\ell u \in \widetilde{H}^{s+1}(\Omega, \Lambda^{\ell-1}), \quad \mathfrak{L}_\ell u \in \widetilde{C}^\infty(\Omega, \Lambda^\ell)$$

Corollary 3, Regularity of cohomology spaces

$$\ker\big(d_\ell\big|_{H^s(\Omega,\Lambda^\ell)}\big)\,/\,\mathrm{im}\big(d_{\ell-1}\big|_{H^{s+1}(\Omega,\Lambda^{\ell-1})}\big)$$

$$\ker\big(d_\ell\big|_{\widetilde{H}^s(\Omega,\Lambda^\ell)}\big)\,/\,\mathrm{im}\big(d_{\ell-1}\big|_{\widetilde{H}^{s+1}(\Omega,\Lambda^{\ell-1})}\big)$$

are of finite dimension independent of s and can be represented by C^∞ functions.

Back to the eigenvalue problem...

The conditions

(CAS)

Completeness of the Approximating Subspaces

$$\forall \mathbf{u} \in \mathring{H}(\mathrm{d}_\ell, \Omega), \qquad \lim_{N\to\infty} \inf_{\mathbf{u}_N \in \mathscr{V}_N^\ell} \|\mathbf{u} - \mathbf{u}_N\|_{H(\mathrm{d}_\ell,\Omega)} = 0$$

(CDK)

Completeness of the Discrete Kernels

$$\forall \mathbf{k} \in \mathrm{Ker}(\mathrm{d}_\ell, \Omega), \qquad \lim_{N\to\infty} \inf_{\mathbf{k}_N \in \mathscr{K}_N^\ell} \|\mathbf{k} - \mathbf{k}_N\|_{L^2(\Omega,\Lambda^\ell)} = 0.$$

(DCP)

Discrete Compactness Property Any sequence $\{\mathbf{u}_N\}_{N\in\mathbb{N}}$ with

$$\mathbf{u}_N \in \mathscr{V}_N^\ell \cap (\mathscr{K}_N^\ell)^\perp \quad \text{and} \quad \|\mathbf{u}_N\|_{H(\mathrm{d}_\ell,\Omega)} \le 1$$

contains a subsequence that *converges in $L^2(\Omega, \Lambda^\ell)$*

Sub-complexes and co-chain projections

We assume:

1. Two compatible families of approximations at levels ℓ and $\ell-1$

$$\mathscr{V}_N^{\ell-1} \xrightarrow{\mathrm{d}_{\ell-1}} \mathscr{V}_N^{\ell}$$

2. Projection operators π_N^k with domain $S(\Omega,\Lambda^k)$ & *commuting diagram*

$$\begin{array}{ccc} S(\Omega,\Lambda^{\ell-1}) & \xrightarrow{\mathrm{d}_{\ell-1}} & S(\Omega,\Lambda^{\ell}) \\ \pi_N^{\ell-1}\big\downarrow & & \big\downarrow\pi_N^{\ell} \\ \mathscr{V}_N^{\ell-1} & \xrightarrow{\mathrm{d}_{\ell-1}} & \mathscr{V}_N^{\ell} \end{array}$$

$\Longrightarrow$ $\mathrm{d}_{\ell-1}\,\mathscr{V}_N^{\ell-1}$ subspace of *discrete kernel* $\mathscr{K}_N^{\ell} := \mathscr{V}_N^{\ell} \cap \mathrm{Ker}(\mathrm{d}_{\ell},\Omega)$

The *h*-version by ARNOLD, FALK, WINTHER

Co-chain projection defined on $S(\Omega,\Lambda^\ell) = L^2(\Omega,\Lambda^\ell)$ and satisfying

(UBP)

Uniformly L^2-Bounded Projections $\quad \exists\beta > 0, \quad \forall h$

$$\left\|\pi_h^k \mathbf{u}\right\|_{L^2(\Omega,\Lambda^k)} \le \beta \left\|\mathbf{u}\right\|_{L^2(\Omega,\Lambda^k)} \qquad \mathbf{u} \in L^2(\Omega,\Lambda^k),\ k = \ell - 1, \ell$$

Theorem [AFW 2009]

(CAS) + (CDK) + (UBP) $\implies$ (SFA) + (SCA)

Proof uses a “regularized” mixed formulation and Hodge decomposition.

Regularized eigenvalue problem: $s > 0$

Find $\mathbf{u} \in \mathring{H}(\mathrm{d}_\ell,\Omega)$ with $\mathbf{u} \neq 0$, $\varphi \in \mathring{H}(\mathrm{d}_{\ell-1},\Omega)$ and $\omega \ge 0$ such that

$$(\mathfrak{R}) \quad \begin{cases} (\mathrm{d}_\ell\,\mathbf{u}, \mathrm{d}_\ell\,\mathbf{v})_\Omega + (\mathrm{d}_{\ell-1}\,\varphi, \mathbf{v})_\Omega = \omega^2 (\mathbf{u},\mathbf{v})_\Omega & \forall \mathbf{v} \in \mathring{H}(\mathrm{d}_\ell,\Omega) \\ -(\mathbf{u}, \mathrm{d}_{\ell-1}\,\psi)_\Omega + \boxed{s}\,(\varphi,\psi)_\Omega = 0 & \forall \psi \in \mathring{H}(\mathrm{d}_{\ell-1},\Omega) \end{cases}$$

Issues

1. Like in any regularized formulation, must *sort* eigenvalues:
   For $s = 1$ $(\mathfrak{R})$ gives all eigenvalues of the Hodge Laplacian

$$\delta_{\ell+1} \circ \mathsf{d}_\ell + \mathsf{d}_{\ell-1} \circ \delta_\ell$$

2. More serious: The proof of (UBP).
   In *h*-version, done in [AFW 2006, Th.5.6] for simplicial meshes by an extension-regularization process.

Now the question is

Is it possible to prove (UBP) for *p*-version?

In absence of positive answer, we prove (DCP) in a quite general framework.

Discrete compactness

Proof of discrete compactness under general hypotheses

We assume that all objects in the co-chain projection exist elementwise: $K \in \mathfrak{M}$

$$\begin{array}{ccc} S(K,\Lambda^{\ell-1}) & \xrightarrow{\mathrm{d}_{\ell-1}} & S(K,\Lambda^{\ell}) \\ \pi^{\ell-1}_{p,K}\downarrow & & \downarrow\pi^{\ell}_{p,K} \\ \mathscr{V}^{\ell-1}_p(K) & \xrightarrow{\mathrm{d}_{\ell-1}} & \mathscr{V}^{\ell}_p(K) \end{array}$$

and

$$S(\Omega,\Lambda^{\ell-1}) = \{\psi \in \mathring{H}(\mathrm{d}_{\ell-1},\Omega) : \psi\big|_K \in S(K,\Lambda^{\ell-1}) \ \ \forall K \in \mathfrak{M}\}$$

H1 — Convergence at level $\ell-1$

$\exists$ function $\varepsilon : \mathbb{N} \mapsto \mathbb{R}^+$ with $\lim_{p\to\infty} \varepsilon(p) = 0$ so that $\forall K \in \mathfrak{M}$

$$\left\| \mathrm{d}_{\ell-1}(\phi - \pi^{\ell-1}_{p,K}\phi) \right\|_{L^2(K,\Lambda^{\ell})} \le \varepsilon(p)\, \|\phi\|_{S(K,\Lambda^{\ell-1})} \quad \forall \phi \in S(K,\Lambda^{\ell-1})$$

Hypothesis H2

For each $K \in \mathfrak{M}$, there exists an intermediate space $X(K,\Lambda^\ell)$

$$S(K,\Lambda^\ell) \subset X(K,\Lambda^\ell) \subset H(\mathrm{d}_\ell, K)$$

and *lifting operators* $\mathrm{R}_{\ell,K}$ and $\mathrm{R}_{\ell+1,K}$ satisfying H2

H2 Lifting operators

They are bounded

$$L^2(K,\Lambda^{\ell+1}) \underset{\mathrm{d}_\ell}{\overset{\mathrm{R}_{\ell+1,K}}{\rightleftarrows}} X(K,\Lambda^\ell) \underset{\mathrm{d}_{\ell-1}}{\overset{\mathrm{R}_{\ell,K}}{\rightleftarrows}} S(K,\Lambda^{\ell-1})$$

such that

$$(*) \qquad \forall \mathbf{x} \in X(K,\Lambda^\ell), \qquad \mathrm{d}_{\ell-1} \circ \mathrm{R}_{\ell,K}\,\mathbf{x} + \mathrm{R}_{\ell+1,K} \circ \mathrm{d}_\ell\,\mathbf{x} = \mathbf{x}$$

and

$$\forall \mathbf{u}_p \in \mathscr{V}_p^\ell(K), \qquad \mathrm{R}_{\ell+1,K} \circ \mathrm{d}_\ell\,\mathbf{u}_p \in \mathscr{V}_p^\ell(K)$$

We set

$$X(\Omega,\Lambda^\ell) = \left\{\mathbf{v} \in \mathring{H}(\mathrm{d}_\ell,\Omega) : \mathbf{v}\big|_K \in X(K,\Lambda^\ell)\ \ \forall K \in \mathfrak{M}\right\},$$

with norm

$$\|\mathbf{u}\|^2_{X(\Omega,\Lambda^\ell)} = \|\mathbf{u}\|^2_{H(\mathrm{d}_\ell,\Omega)} + \sum_{K\in\mathfrak{M}} \left\|\mathbf{u}\big|_K\right\|^2_{X(K,\Lambda^\ell)} .$$

Lemma 1

Under hypotheses H1 and H2

$$\forall \mathbf{u} \in X(\Omega,\Lambda^\ell) \quad \text{such that} \quad \mathrm{d}_\ell\,\mathbf{u} \in \mathrm{d}_\ell\,\mathscr{V}^\ell_p$$

we have the estimate

$$\left\|\mathbf{u} - \pi^\ell_p \mathbf{u}\right\|_{L^2(\Omega,\Lambda^\ell)} \le C\,\varepsilon(p)\,\|\mathbf{u}\|_{X(\Omega,\Lambda^\ell)}$$

with C independent of p and **u**.

From formula $(*)$, we deduce that

$$\mathbf{x} \in X(K,\Lambda^\ell) \cap \mathrm{Im}(\mathrm{d}_{\ell-1}, K) \quad \Longrightarrow \quad \exists \sigma \in S(\Omega,\Lambda^{\ell-1}), \;\; \mathbf{x} = \mathrm{d}_{\ell-1}\,\sigma$$

(we simply take $\sigma = \mathrm{R}_{\ell,K}\,\mathbf{x}$)

We need this property globally

H3 — **Maximal image**

$$X(\Omega,\Lambda^\ell) \cap \mathrm{Im}(\mathrm{d}_{\ell-1}, \Omega) = \mathrm{d}_{\ell-1}\, S(\Omega,\Lambda^{\ell-1})$$

Hypotheses H4 and H5

We need further properties for the intermediate global space $X(\Omega, \Lambda^\ell)$

H4 Compact embedding

$$X(\Omega, \Lambda^\ell) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega, \Lambda^\ell)$$

H5 Regularity

$$X(\Omega, \Lambda^\ell) \supset \mathring{H}(\mathrm{d}_\ell, \Omega) \cap \mathrm{Im}(\mathrm{d}_{\ell-1}, \Omega)^\perp$$

NB: Recall that [Picard 1984]

$$\mathring{H}(\mathrm{d}_\ell, \Omega) \cap \mathrm{Im}(\mathrm{d}_{\ell-1}, \Omega)^\perp \overset{\text{comp}}{\hookrightarrow} L^2(\Omega, \Lambda^\ell)$$

The main theorem

Theorem of Discrete Compactness [BCDDH 2009]

Under hypotheses H1 H2 H3 H4 and H5, the (DCP) holds.

Tools used in the proof of the hypotheses:

1. Demkowicz's projection-based interpolation operators
2. The regularized Poincaré operators

Corollary

Using the p-version of Nédélec's elements on

- triangles or tetrahedra (first or second family) or on
- quadrilaterals or affine hexahedra (first family)

we obtain a spurious-free spectrally correct approximation of Maxwell eigenpairs.

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Thank you for your attention!