On Inequalities of Korn, Friedrichs and Babuška-Aziz

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1. Introduction

KORN'S inequalities for integrals of quadratic functionals subject to certain side conditions have played a fundamental role in the development of elasticity theory (see *e.g.* [1–14]). The Korn inequalities are essential in establishing coerciveness of the differential operators of linear elastostatics and thus form the basis for existence results in that theory [9]. Among other areas in elasticity where consideration of Korn's inequalities arise we cite, for example, fundamental studies on the mathematical foundations of finite elements [15–17], stability theory [11] and qualitative analyses of solution behavior, such as those involved in the investigation of Saint-Venant's principle [18–20]. In the latter two areas of application in particular, information on the *optimal constants* appearing in the inequalities, the *Korn constants*, is of importance. Thus several investigations have been concerned with evaluating the Korn constants for specific domains and with obtaining bounds for these constants [4–6, 8, 12–13].

Our purpose in this paper is to draw attention to an equivalence, *for two dimensional simply-connected domains*, between Korn's inequality (in the second case; see § 2 here) and two other inequalities for quadratic functionals which have received comparatively little attention in the literature. The first of these is an inequality due apparently to K. O. FRIEDRICHS [2], between the real and imaginary parts of an analytic function (or equivalently, between a harmonic function and its conjugate) defined on a two-dimensional domain R. Thus if $w(z) = f + ig$ $(z = x_1 + ix_2)$ is analytic on R, Friedrichs' inequality reads

$$\int_R f^2\, dA \leqq \Gamma \int_R g^2\, dA, \tag{1.1}$$

provided

$$\int_R f\, dA = 0. \tag{1.2}$$

Here $\Gamma > 0$ is a constant which depends only on the shape of the domain R. The best possible constant in (1.1) will be denoted by Γ_R.

The third inequality that we consider is of recent origin, due to BABUŠKA & AZIZ [15] in 1972. This result, which may be interpreted as a representation theorem, involves not only an inequality between certain quadratic functionals but also the existence of a vector potential field associated with functions of zero mean value over a plane domain. We state the result of BABUŠKA-AZIZ as follows:

Theorem 1 (BABUŠKA-AZIZ [15]). *Let R be a Lipschitz domain in the plane and let q be a function bounded on R such that*

$$\int_R q\, dA = 0. \tag{1.3}$$

Then there exist a vector function ω_α (continuous on the closure $\bar{R} \equiv R + \partial R$ of R and with square-integrable first derivatives) and a constant C depending only on R such that

$$\omega_{\alpha,\alpha} = q \text{ in } R, \; \omega_\alpha = 0 \text{ on } \partial R, \tag{1.4}$$

$$\int_R \omega_{\alpha,\beta}\omega_{\alpha,\beta}\, dA \leqq C \int_R \omega^2_{\alpha,\alpha}\, dA. \tag{1.5}$$

Here the usual Cartesian tensor notation is used, with summation over repeated subscripts implied. Theorem 1 was established by BABUŠKA & AZIZ in [15, pp. 172–174] under less stringent smoothness hypotheses. (See also ODEN & REDDY [16, pp. 230–232].) In these references, this result is used in establishing mathematical foundations for the finite element method for incompressible elastic media. A version of Theorem 1, valid for both two and three dimensions, was stated and proved by LADYZHENSKAYA & SOLONNIKOV [21, pp. 265–266] in the course of investigations concerning existence of solutions of the Stokes and Navier-Stokes equations with finite energies.

We observe that the inequality (1.5) of Theorem 1 is of a different nature from Korn's or Friedrichs' inequality. Thus, while the latter two inequalities must hold for *all* functions of a certain class, Theorem 1 merely guarantees the existence of *a* vector field ω_α and *a* constant C for which (1.4), (1.5) hold. There may be other functions satisfying (1.4) for which (1.5) does not hold.

For a given value of q satisfying (1.3), the best possible constant C for which (1.5) holds will be denoted by C_R. The constant C_R depends only on the shape of R. In a recent study of the spatial evolution to fully developed flow of solutions of the stationary Stokes and Navier-Stokes equations in cylindrical pipes, HORGAN & WHEELER [22] have obtained an estimate for the *rate* of exponential flow development which depends on C_R, where R is taken to be the cross-section of the pipe.

In this paper, following a brief discussion of the foregoing inequalities of Korn, Friedrichs and Babuška-Aziz in § 2–4, we show in § 5 that these inequalities are equivalent for the case of two-dimensional simply-connected domains. (For such domains, the equivalence between Korn's and Friedrichs' inequality was established by HORGAN [13]). Moreover a simple relation (see equation (5.5) below) between the optimal constants occurring in each of the three inequalities is provided. Thus, for a given domain, knowledge of the value of any one of these constants (or an upper bound for this quantity) may be used to obtain similar information for the remaining two constants. It appears that upper bounds for the optimal constant Γ_R appearing in (1.1) are easiest to obtain. Such results for star-shaped domains are established in § 6. The paper concludes with a brief discussion of applications to problems in linear elasticity and viscous flows.

For simplicity of presentation, we assume henceforth that the domain R is simply-connected, with C^1 boundary ∂R. It will be clear from our arguments that the results hold for simply-connected Lipschitz domains.

2. Korn's Inequality

We refer to [1–14] and the references cited therein for a discussion of the various forms of Korn inequalities which have been investigated in the literature. We are concerned here only with Korn's inequality in the *second case* for two-dimensional simply-connected domains. Thus, for vector fields $\boldsymbol{u}$, with components u_α ($\alpha = 1, 2$), which have square-integrable first derivatives on a bounded plane domain R, we define the quadratic functionals

$$D(\boldsymbol{u}) = \int_R u_{\alpha,\beta} u_{\alpha,\beta}\, dA, \tag{2.1}$$

$$S(\boldsymbol{u}) = \frac{1}{4} \int_R (u_{\alpha,\beta} + u_{\beta,\alpha})(u_{\alpha,\beta} + u_{\beta,\alpha})\, dA. \tag{2.2}$$

Korn's inequality, as established by FRIEDRICHS [2, 3][1] states that there exists a constant $K > 0$ such that

$$D(\boldsymbol{u}) \leqq K S(\boldsymbol{u}) \tag{2.3}$$

for all vector fields $\boldsymbol{u}$ satisfying the constraint

$$\int_R (u_{\alpha,\beta} - u_{\beta,\alpha})\, dA = 0. \tag{2.4}$$

The best possible constant in (2.3), which depends on the shape of the domain R, is called *Korn's constant* in the second case and is denoted by K_R. As mentioned in the Introduction, several studies have been concerned with examining the domain dependence of K_R [4–6, 8, 12–13].

There are several other forms in which the inequality (2.3), (2.4) may be expressed. Introducing the functional

$$R(\boldsymbol{u}) = \frac{1}{4} \int_R (u_{\alpha,\beta} - u_{\beta,\alpha})(u_{\alpha,\beta} - u_{\beta,\alpha})\, dA \equiv \frac{1}{2} \int_R (u_{1,2} - u_{2,1})^2\, dA \tag{2.5}$$

and noting that

$$D(\boldsymbol{u}) = S(\boldsymbol{u}) + R(\boldsymbol{u}), \tag{2.6}$$

[1] For more recent proofs, see [9], [23].

we may write (2.3) as

$$R(\boldsymbol{u}) \leqq (K-1)\, S(\boldsymbol{u}), \tag{2.7}$$

or as

$$R(\boldsymbol{u}) \leqq [(K-1)/K]\, D(\boldsymbol{u}), \tag{2.8}$$

subject to the constraint

$$\int_R (u_{1,2} - u_{2,1})\, dA = 0. \tag{2.9}$$

Of course, from (2.7) or (2.8) it follows that $K \geqq 1$. Three other forms of Korn's inequality (in two dimensions) may also be obtained. Under the transformation

$$u_1 = v_2,\, u_2 = -v_1, \tag{2.10}$$

the Dirichlet integral $D(\boldsymbol{u})$ is invariant, but

$$S(\boldsymbol{u}) = \int_R \left[v_{1,2}^2 + v_{2,1}^2 + \frac{1}{2}(v_{1,1} - v_{2,2})^2 \right] dA \equiv \mathcal{S}(\boldsymbol{v}) \tag{2.11}$$

and

$$R(\boldsymbol{u}) = \frac{1}{2} \int_R v_{\alpha,\alpha}^2\, dA \equiv \mathcal{R}(\boldsymbol{v}). \tag{2.12}$$

Thus (2.3), (2.7), (2.8) may be written as

$$D(\boldsymbol{v}) \leqq K\, \mathcal{S}(\boldsymbol{v}), \tag{2.13}$$

$$\mathcal{R}(\boldsymbol{v}) \leqq (K-1)\, \mathcal{S}(\boldsymbol{v}), \tag{2.14}$$

$$\mathcal{R}(\boldsymbol{v}) \leqq (K-1)/K\, D(\boldsymbol{v}), \tag{2.15}$$

respectively, for all vector fields $\boldsymbol{v}$ satisfying the constraint

$$\int_R v_{\alpha,\alpha}\, dA = 0. \tag{2.16}$$

3. Friedrichs' Inequality for Harmonic Functions

The inequality (1.1), (1.2) relating the real and imaginary parts of an analytic function was established by FRIEDRICHS [2] in 1937. An obviously equivalent form is the inequality

$$\int_R h_{,1}^2\, dA \leqq \Gamma \int_R h_{,2}^2\, dA \tag{3.1}$$

for all *harmonic* functions $h(x_1, x_2)$ such that

$$\int_R h_{,1}\, dA = 0. \tag{3.2}$$

Other equivalent versions of this inequality may be found in [2, 13].

It is of interest to note that if f and g are conjugate harmonic functions, then

$$F = f^2 - g^2, \quad G = 2fg \tag{3.3}$$

are also conjugate harmonic. Thus the inequality (1.1), (1.2) yields

$$\int_R (f^2 - g^2)^2 \, dA \leqq \Gamma \int_R (2fg)^2 \, dA, \tag{3.4}$$

provided

$$\int_R f^2 \, dA = \int_R g^2 \, dA. \tag{3.5}$$

Hence, for any two conjugate harmonic functions f, g satisfying (3.5), one has

$$\int_R (f^4 + g^4) \, dA \leqq 2(2\Gamma + 1) \int_R f^2 g^2 \, dA. \tag{3.6}$$

4. The Babuška-Aziz Inequality

The inequality (1.5) may also be written in other forms. Since ω_α vanishes on the boundary ∂R of R, the divergence theorem may be used to write the identity

$$\int_R \omega_{\alpha,\beta}\omega_{\alpha,\beta} \, dA = \int_R (\omega_{\alpha,\alpha})^2 \, dA + \int_R (\omega_{1,2} - \omega_{2,1})^2 dA. \tag{4.1}$$

Thus Theorem 1 holds with (1.5) replaced by the inequality

$$\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA \leqq (C - 1) \int_R \omega_{\alpha,\alpha}^2 \, dA. \tag{4.2}$$

Thus, in particular, $C \geqq 1$.

A corollary to Theorem 1 also follows on making the transformation

$$\omega_1 = v_2, \omega_2 = -v_1. \tag{4.3}$$

The statement of Theorem 1 now is seen to hold with ω_α replaced by v_α and (1.4), (1.5) replaced by

$$v_{1,2} - v_{2,1} = q \quad \text{in } R, \quad v_\alpha = 0 \text{ on } \partial R, \tag{4.4}$$

$$\int_R v_{\alpha,\beta} v_{\alpha,\beta} \, dA \leqq C \int_R (v_{1,2} - v_{2,1})^2 \, dA. \tag{4.5}$$

Again by use of (4.1) for v_α, (4.5) may also be written as

$$\int_R v_{\alpha,\alpha}^2 \, dA \leqq (C - 1) \int_R (v_{1,2} - v_{2,1})^2 \, dA. \tag{4.6}$$

Recall that by virtue of (1.3) and $(4.4)_1$, both inequalities (4.5), (4.6) hold provided that the zero mean value condition,

$$\int_R (v_{1,2} - v_{2,1}) \, dA = 0, \tag{4.7}$$

is satisfied.

5. Equivalence Results

In this section, we will establish the equivalence (for simply-connected domains) of Friedrichs' inequality (1.1), (1.2), Korn's inequality (2.3), (2.4) and the inequality of Babuška-Aziz in the form (see (4.2))

$$\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA \leqq (C - 1) \int_R \omega_{\alpha,\alpha}^2 \, dA, \tag{5.1}$$

where

$$\int_R \omega_{\alpha,\alpha} \, dA = 0, \quad \omega_\alpha = 0 \text{ on } \partial R. \tag{5.2}$$

The equivalence of the first two inequalities was established in [13], where it was shown that the optimal constants in both inequalities are related by

$$\Gamma_R = \frac{K_R}{2} - 1, \quad K_R = 2(1 + \Gamma_R). \tag{5.3}$$

This result was obtained using a variational approach (*cf.* [5, 8]) relating the optimizing functions in each inequality. Here we provide a direct proof of the equivalence of (1.1), (1.2) and (5.1), (5.2) with optimal constants Γ_R, C_R, respectively, related by

$$\Gamma_R = C_R - 1, \quad C_R = 1 + \Gamma_R. \tag{5.4}$$

Thus, by virtue of (5.3), we have the simple expressions

$$K_R = 2C_R = 2(1 + \Gamma_R), \tag{5.5}$$

relating the Korn's constant and the optimal constants in the Babuška-Aziz and Friedrichs' inequalities respectively.

To prove (5.4), we begin by showing how (1.1), (1.2) with optimal constant Γ_R ensures that Theorem 1 holds with constant $C = C_R$ in the inequality (5.1) given by $(5.4)_2$. For any vector field $\omega_\alpha \in C(\bar{R})$, we write

$$\omega_1 = \phi_{,1} + \psi_{,2}, \quad \omega_2 = \phi_{,2} - \psi_{,1} \text{ on } \bar{R}, \tag{5.6}$$

so that

$$\omega_{\alpha,\alpha} = \Delta\phi, \quad \omega_{1,2} - \omega_{2,1} = \Delta\psi \text{ on } \bar{R}. \tag{5.7}$$

To ensure that (5.2) holds, the functions ϕ, ψ are chosen so that

$$\Delta\phi = q \text{ in } R, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial R, \tag{5.8}$$

$$\Delta^2\psi = 0 \text{ in } R, \quad \frac{\partial\psi}{\partial n} = \frac{\partial\phi}{\partial s}, \quad \frac{\partial\psi}{\partial s} = 0 \text{ on } \partial R. \tag{5.9}$$

Let $V(x_1, x_2)$ be conjugate to the harmonic function $\Delta\psi$. Then, on using $(5.7)_2$ it follows, on use of the divergence theorem and the boundary conditions (5.8), (5.9), (5.2), that

$$\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA = \int_R (\omega_{1,2} - \omega_{2,1}) \, \Delta\psi \, dA$$

$$= - \int_R (\omega_1 \, \Delta\psi_{,2} - \omega_2 \, \Delta\psi_{,1}) \, dA = - \int_R \omega_\alpha V_{,\alpha} \, dA \tag{5.10}$$

$$= \int_R V \omega_{\alpha,\alpha} \, dA.$$

By Schwarz's inequality, we obtain

$$\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA \leqq \left(\int_R V^2 \, dA \right)^{\frac{1}{2}} \left(\int_R \omega_{\alpha,\alpha}^2 \, dA \right)^{\frac{1}{2}}. \tag{5.11}$$

Since V is determined from $\Delta\psi$ only to within an additive constant, this constant may be chosen such that

$$\int_R V \, dA = 0. \tag{5.12}$$

Note that, in view of $(5.2)_1$, the addition of a constant to V does not affect the value of the last integral in (5.10). Thus the inequality (1.1), (1.2) (with $\Gamma = \Gamma_R$, $f \equiv V$, $g \equiv \Delta\psi = \omega_{1,2} - \omega_{2,1}$) may be used in (5.11) to yield

$$\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA \leqq \Gamma_R \int_R \omega_{\alpha,\alpha}^2 \, dA, \tag{5.13}$$

for vector fields ω_α satisfying (5.2). But this is precisely the inequality (5.1), with the constant $C = C_R = 1 + \Gamma_R$.

Conversely, we now show that Theorem 1 with the inequality (5.1) holding with optimal constant C_R, implies the validity of Friedrichs' inequality (1.1), (1.2) with $\Gamma_R = C_R - 1$. Let h be harmonic on $\bar{R}$ and satisfy

$$\int_R h \, dA = 0. \tag{5.14}$$

By virtue of Theorem 1, there exists a vector function $\omega_\alpha \in C(\bar{R})$ such that

$$\omega_{\alpha,\alpha} = h \text{ in } R, \; \omega_\alpha = 0 \text{ on } \partial R. \tag{5.15}$$

Let h^* be conjugate harmonic to h. It follows, on using the divergence theorem, that

$$\int_R h^2 \, dA = \int_R h \omega_{\alpha,\alpha} \, dA = - \int_R h_{,\alpha} \omega_\alpha \, dA$$

$$= - \int_R (h^*_{,2} \omega_1 - h^*_{,1} \omega_2) \, dA = \int_R h^* (\omega_{1,2} - \omega_{2,1}) \, dA. \tag{5.16}$$

Schwarz's inequality now yields

$$\int_R h^2 dA \leqq \left(\int_R h^{*2} \, dA \right)^{\frac{1}{2}} \left(\int_R (\omega_{1,2} - \omega_{2,1})^2 \, dA \right)^{\frac{1}{2}}. \tag{5.17}$$

The inequality (5.1), with $C = C_R$, is now used on the right hand side of (5.17), which, with $(5.15)_1$, gives

$$\int_R h^2\,dA \leqq (C_R - 1) \int_R h^{*2}\,dA, \tag{5.18}$$

for conjugate harmonic functions h, h^* satisfying (5.14). This is precisely (1.1), (1.2) with constant $\Gamma = \Gamma_R = C_R - 1$.

6. Upper Bounds for Star-Shaped Domains

In this section, we derive an upper bound for Γ_R for star-shaped domains. Thus from (5.5), we obtain upper bounds for K_R and C_R for such domains.

Before doing this, we record here the exact values of K_R, Γ_R and C_R for some simple domains. For example, if R is the interior of a disc, it has been shown by PAYNE & WEINBERGER [5] that $K_R = 4$. FRIEDRICHS [2] has demonstrated that $\Gamma_R = 1$ (see also [13]). Thus from (5.5) we find that $C_R = 2$ for a disc. For an ellipse with semi-axes a, b $(a > b)$, it is shown in [2, 13] that

$$K_R = 2\left(1 + \frac{a^2}{b^2}\right), \quad \Gamma_R = \frac{a^2}{b^2}, \tag{6.1}$$

and so from (5.5) we obtain

$$C_R = 1 + \frac{a^2}{b^2}. \tag{6.2}$$

As regards *lower bounds*, it is demonstrated in [13] that $K_R \geqq 4$, $\Gamma_R \geqq 1$ for all R, with equality for a disc. Thus (5.5) yields $C_R \geqq 2$, with equality for a disc.

To obtain an *upper bound* for the optimal constant Γ_R occurring in (1.1), it proves convenient to adopt a normalization condition different from (1.2). Thus we will be concerned with the inequality

$$\int_R h^2\,dA \leqq \tilde{\Gamma} \int_R h^{*2}\,dA, \tag{6.3}$$

for conjugate harmonic functions h, h^* such that

$$h(0, 0) = 0. \tag{6.4}$$

(*Cf.* MIKHLIN [24, pp. 508–509].) Here the origin of the Cartesian coordinate system has been chosen to lie inside R. Of course, for a disc, the mean-value theorem for harmonic functions shows that, if the origin is chosen at the center, then (6.4) and (5.14) are identical and so $\tilde{\Gamma}_R = \Gamma_R$ for a disc. Otherwise we have

$$\Gamma_R \leqq \tilde{\Gamma}_R, \tag{6.5}$$

and so to obtain an upper bound for Γ_R, it suffices to obtain an upper bound for $\tilde{\Gamma}_R$.

Suppose now that R is star-shaped with respect to a point, which we choose to be the origin. Let the boundary be represented in plane polar coordinates by

$$r = f(\theta) \quad \text{on } \partial R. \tag{6.6}$$

Let h, h^* be conjugate harmonic functions satisfying (6.4). Then $H = h^2 - h^{*2}$ and $G = 2hh^*$ are conjugate harmonic, with

$$G(0,0) = 0, \quad H(0,0) \leqq 0. \tag{6.7}$$

Thus in polar coordinates

$$H(r,\theta) - H(0,\theta) = \int_0^r \frac{\partial H}{\partial \varrho}(\varrho,\theta)\, d\varrho = \int_0^r \frac{1}{\varrho}\frac{\partial G}{\partial \theta}(\varrho,\theta)\, d\varrho, \tag{6.8}$$

and so, by $(6.7)_2$,

$$H(r,\theta) \leqq \int_0^r \frac{1}{\varrho}\frac{\partial G}{\partial \theta}(\varrho,\theta)\, d\varrho. \tag{6.9}$$

From (6.9) we obtain on integration

$$\begin{aligned}\int_R \frac{H(r,\theta)}{f^2(\theta)}\, dA &\leqq \int_0^{2\pi}\int_0^{f(\theta)} \frac{1}{f^2(\theta)}\left\{\int_0^r \frac{1}{\varrho}\frac{\partial G}{\partial \theta}(\varrho,\theta)\, d\varrho\right\} r\, dr\, d\theta \\ &= \frac{1}{2}\int_R \frac{[f^2(\theta) - \varrho^2]}{\varrho^2 f^2(\theta)}\frac{\partial G}{\partial \theta}\, dA.\end{aligned} \tag{6.10}$$

An integration by parts yields

$$\begin{aligned}\int_R \frac{H(r,\theta)}{f^2(\theta)}\, dA \leqq \frac{1}{2}&\int_{\partial R} \frac{[f^2(\theta) - r^2]}{r^2 f^2(\theta)} G(x_2 n_1 - x_1 n_2)\, ds \\ &- \int_R \frac{f'(\theta)}{f^3(\theta)} G(r,\theta)\, dA.\end{aligned} \tag{6.11}$$[2]

But, by virtue of (6.6) the first integral on the right in (6.11) vanishes and so

$$\int_R \frac{H(r,\theta)}{f^2(\theta)}\, dA \leqq -\int_R \frac{f'(\theta)}{f^3(\theta)} G(r,\theta)\, dA. \tag{6.12}$$

Thus, in view of the definition of H and G, we deduce from (6.12) that

$$\int_R \frac{h^2}{f^2(\theta)}\, dA \leqq \int_R \frac{h^{*2}}{f^2(\theta)}\, dA + 2\int_R \frac{|Q(\theta)|\,|h|\,|h^*|}{f^2(\theta)}\, dA, \tag{6.13}$$

[2] Note that there is no contribution at the origin, in view of $(6.7)_1$.

where

$$Q^2(\theta) = \left[\frac{f'(\theta)}{f(\theta)}\right]^2. \tag{6.14}$$

We may also write

$$Q^2 = \frac{1-n_r^2}{n_r^2} = \frac{r^2-p^2}{p^2} \text{ on } \partial R, \tag{6.15}$$

where

$$p^2 = r^2 n_r^2 \text{ on } \partial R, \tag{6.16}$$

and n_r denotes the radial component of the outward normal vector $\boldsymbol{n}$ on ∂R. Since $f(\theta)$ is bounded from above and below on ∂R,

$$r_m \leqq f(\theta) \leqq r_M, \tag{6.17}$$

(6.13) would provide an estimate for $\tilde{\Gamma}$ in (6.3), if an upper bound for Q were used in (6.13). A better estimate is obtained if we first use the arithmetic-geometric mean inequality

$$2\,|Q(\theta)|\,|h|\,|h^*| \leqq [1-\beta^2(\theta)]\,h^2 + [1-\beta^2(\theta)]^{-1}\,h^{*2}, \tag{6.18}$$

where $\beta(\theta)$ is an arbitrary function satisfying

$$\beta^2(\theta) < 1. \tag{6.19}$$

Using (6.18) in (6.13) leads to

$$\int_R \frac{\beta^2(\theta)\,h^2\,dA}{f^2(\theta)} \leqq \int_R \left\{1 + \frac{Q^2(\theta)}{1-\beta^2(\theta)}\right\} \frac{h^{*2}}{f^2(\theta)}\,dA. \tag{6.20}$$

We now choose

$$\beta^2(\theta) = \alpha^2 \frac{f^2(\theta)}{r_M^2}, \tag{6.21}$$

where α is an, as yet, undetermined constant satisfying

$$0 < \alpha < 1. \tag{6.22}$$

This choice of β satisfies (6.19), by virtue of (6.17) and (6.22). On substitution from (6.21) in (6.20) one finds that

$$\int_R h^2\,dA \leqq \max_{\partial R} \left\{\frac{[r_M^2 p^{-2} - \alpha]}{\alpha[1-\alpha r^2 r_M^{-2}]}\right\} \int_R h^{*2}\,dA. \tag{6.23}$$

Choosing the constant α to minimize the multiplicative factor on the right in (6.23), we obtain the upper bound

$$\tilde{\Gamma}_R \leqq \min_{\alpha\in(0,1)} \max_{\partial R} P(\alpha, \theta), \tag{6.24}$$

for the optimal constant $\tilde{\Gamma}_R$ occurring in (6.3), (6.4). Here

$$P(\alpha, \theta) = \frac{r_M^2 p^{-2} - \alpha}{\alpha(1 - \alpha r^2 r_M^{-2})}. \tag{6.25}$$

Suppose that $\min_{\alpha \in (0,1)} \max_{\partial R} P(\alpha, \theta)$ is achieved at (α_0, θ_0). Thus

$$P(\alpha_0, \theta_0) = \min_{\alpha} P(\alpha, \theta_0). \tag{6.26}$$

This gives

$$\alpha_0 = r_M^2 p_0^{-2} - [r_M^4 p_0^{-2}(p_0^{-2} + r_0^{-2})]^{\frac{1}{2}}, \tag{6.27}$$

where $p_0 = p(\theta_0)$, $r_0 = r(\theta_0)$. Note that, in view of (6.16), $r_0 \geqq p_0$ and so α_0 does indeed satisfy (6.22). With α_0 given by (6.27), (6.25) yields

$$P(\alpha_0, \theta_0) = \{r_0 p_0^{-1} + [r_0^2 p_0^{-2} - 1]^{\frac{1}{2}}\}^2. \tag{6.28}$$

Thus (6.24), together with (6.5), yields the desired upper bound

$$\Gamma_R \leqq \max_{\partial R} \{r p^{-1} + [r^2 p^{-2} - 1]^{\frac{1}{2}}\}^2 \tag{6.29}$$

$$= \max_{\theta} \left\{\left[1 + \left(\frac{f'(\theta)}{f(\theta)}\right)^2\right]^{\frac{1}{2}} + \frac{|f'(\theta)|}{f(\theta)}\right\}^2 \equiv \max_{\theta} \mathscr{F}. \tag{6.30}$$

The upper bound (6.29) is expressed in a form amenable to geometric calculation, while (6.30) is written explicitly in terms of the equation (6.6) of the boundary curve of the star-shaped domain R. By virtue of (5.5), we have thus established the upper bounds

$$K_R \leqq 2 + 2 \max_{\theta} \mathscr{F}, \tag{6.31}$$

$$C_R \leqq 1 + \max_{\theta} \mathscr{F}, \tag{6.32}$$

for star-shaped domains R.

It is easy to verify that the upper bounds (6.30), (6.31), (6.32) hold with *equality* when R is the interior of a disc or of an ellipse. (Recall our discussion at the beginning of this section concerning these cases.) For the star-shaped domain bounded by a limaçon (Pascal's limaçon), (6.6) reads

$$r = a(1 + \varepsilon \cos \theta), \; a, \; \varepsilon \text{ constants}, \; 0 \leqq \varepsilon \leqq 1. \tag{6.33}$$

When $\varepsilon = 0$, (6.33) describes a circle, while for $\varepsilon = 1$ the curve becomes a cardioid. For the limaçon, (6.30)–(6.32) yield

$$\Gamma_R \leqq \frac{1 + \varepsilon}{1 - \varepsilon}, \quad K_R \leqq \frac{4}{1 - \varepsilon}, \quad C_R \leqq \frac{2}{1 - \varepsilon}. \tag{6.34}$$

The exact values of Γ_R, K_R for the limaçon (6.33) have been obtained in [13]. From [13] and (5.5) of the present paper, we thus have

$$\Gamma_R = \frac{2 + \varepsilon^2}{2 - \varepsilon^2}, \quad K_R = \frac{8}{2 - \varepsilon^2}, \quad C_R = \frac{4}{2 - \varepsilon^2}. \tag{6.35}$$

The bounds (6.34) are seen to be sharp only for small values of ε. For a regular n-sided polygon, one obtains[3]

$$\Gamma_R \leqq \frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}, \quad K_R \leqq \frac{4}{1-\sin(\pi/n)}, \quad C_R \leqq \frac{2}{1-\sin(\pi/n)}. \tag{6.36}$$

For the special case of a square ($n = 4$), (6.36) yields

$$\Gamma_R \leqq 3 + 2\sqrt{2}, \quad K_R \leqq 8 + 4\sqrt{2}, \quad C_R \leqq 4 + 2\sqrt{2}. \tag{6.37}$$

For a square, we *conjecture* that the optimal constant Γ_R in (1.1), (1.2) is given by the conjugate harmonic pair

$$f = 2x_1x_2, \qquad g = x_2^2 - x_1^2. \tag{6.38}$$

Thus for a square we conjecture that

$$\Gamma_R = 5/2, \quad K_R = 7, \quad C_R = 7/2. \tag{6.39}$$

7. Some Applications

As was mentioned in the Introduction, information concerning the domain dependence of K_R, Γ_R and C_R is of importance in many problems in elasticity and fluid mechanics. For example, in a recent investigation of Saint-Venant's principle for the three-dimensional linearly elastic semi-infinite cylinder with traction free lateral surface and subject to self-equilibrated end loads, BERDICHEVSKII [19] has obtained an estimate for the rate of exponential decay of stresses which differs from the classical result of TOUPIN [18]. In particular, Berdichevskii's decay rate is expressed solely in terms of material parameters and *cross-sectional* properties of the cylinder. The *two-dimensional* Korn's inequality (2.3), (2.4) plays a critical role in the arguments of [19], where R is taken to be the cross-section of the cylinder. The decay estimate obtained in [19] involves the Korn constant K_R in such a way that exact values or *upper bounds* for this quantity are required to render the result fully explicit. We refer to BERDICHEVSKII's paper [19] for complete details; the results are summarized and simplified in § III, D, 2 of the review article [20].

The Babuška-Aziz theorem (Theorem 1) has been employed in a similar fashion by HORGAN & WHEELER [22] in their analysis of entry flow of a viscous incompressible fluid into a cylindrical pipe. The axial decay to fully developed flow is shown to be exponential for sufficiently small Reynolds number, and an estimate for the *rate* of exponential flow development is obtained which involves C_R, where R is the cross-section of the pipe. The constant C_R appears in such a way that exact values or *upper bounds* for this quantity are required to analyze the dependence of the decay rate on cross-sectional geometry. It should be noted that, in some instances, certain symmetries in the flow field (and hence in the function q defined in Theorem 1) may lead to sharper estimates for C_R than are indicated in § 6 of the present paper. For example, when R is a disc, if q is specified

[3] In this case, the form (6.29) is convenient to use.

a priori to be purely radial $q = q(r)$ (corresponding to axisymmetric flow [22]), then HORGAN & WHEELER [22] have shown that $C_R = 1$.

In conclusion we remark on an application of Friedrichs' inequality (1.1), (1.2) in linear elasticity theory. The displacement equations of equilibrium (with zero body force) of linear isotropic elasticity for plane strain are given by (see *e.g.* [10])

$$u_{\alpha,\beta\beta} + \frac{1}{1-2\sigma} u_{\beta,\beta\alpha} = 0 \text{ on } R, \tag{7.1}$$

where u_α $(\alpha = 1, 2)$ denote the components of displacement and the constant σ, $-1 < \sigma < 1/2$, is Poisson's ratio. It is readily verified that if u_α satisfies (7.1) on a plane domain R, then

$$u_{1,2} - u_{2,1} \text{ and } \frac{2(1-\sigma)}{1-2\sigma} u_{\alpha,\alpha} \tag{7.2}$$

are conjugate harmonic functions on R. Thus (1.1), (1.2) yield the pair of inequalities

$$\int_R (u_{1,2} - u_{2,1})^2 \, dA \leqq 4\Gamma_R \left(\frac{1-\sigma}{1-2\sigma}\right)^2 \int_R u_{\alpha,\alpha}^2 \, dA, \tag{7.3}$$

if

$$\int_R (u_{1,2} - u_{2,1}) \, dA = 0, \tag{7.4}$$

or

$$\int_R u_{\alpha,\alpha}^2 \, dA \leqq 4\Gamma_R \left(\frac{1-\sigma}{1-2\sigma}\right)^2 \int_R (u_{1,2} - u_{2,1})^2 \, dA, \tag{7.5}$$

if

$$\int_R u_{\alpha,\alpha} \, dA = 0. \tag{7.6}$$

The inequalities (7.3), (7.5) thus provide square-integral relations between the dilatation $u_{\alpha,\alpha}$ and the rotation component $u_{1,2} - u_{2,1}$ in two-dimensional elasticity.

The strain energy in an elastic solid occupying the domain R is proportional to

$$E(\boldsymbol{u}) = S(\boldsymbol{u}) + \frac{\sigma}{1-2\sigma} \int_R u_{\alpha,\alpha}^2 \, dA, \tag{7.7}$$

where the functional $S(\boldsymbol{u})$ has been defined in (2.2). We now show how the inequality (7.3) and Korn's inequality in the form (2.7) may be used to obtain a *lower bound* for $E(\boldsymbol{u})$ in terms of $R(\boldsymbol{u})$, defined in (2.5). Such lower bounds for the strain energy play an important role in applications of elasticity theory. By virtue of (2.5), the inequality (7.3) may be written as

$$R(\boldsymbol{u}) \leqq 2\Gamma_R \left(\frac{1-\sigma}{1-2\sigma}\right)^2 \int_R u_{\alpha,\alpha}^2 \, dA. \tag{7.8}$$

Since the functions u_α satisfy not only the constraint (7.4) but also the displacement equations of equilibrium (7.1), we may use a stronger version of Korn's inequality than (2.7), (2.9) alone. This *extended main case* of Korn's inequality was treated in [8]. Thus there exists a Korn's constant $K_R(\sigma)$, depending not only on the shape of R but also on the value of Poisson's ratio σ appearing in the constraint (7.1), such that

$$R(\boldsymbol{u}) \leqq [K_R(\sigma) - 1]\, S(\boldsymbol{u}), \tag{7.9}$$

for vector fields $\boldsymbol{u}$ satisfying (7.1), (7.4). This Korn's constant is such that $1 < K_R(\sigma) \leqq K_R$. Methods for obtaining explicit values and bounds for $K_R(\sigma)$ are given in [8]. In particular, for a circular domain

$$K_R(\sigma) = \frac{16\sigma^2 - 24\sigma + 11}{8\sigma^2 - 8\sigma + 3}. \tag{7.10}$$

On combining (7.8), (7.9) it is easily shown that

$$E(\boldsymbol{u}) \geqq \left[\frac{1}{K_R(\sigma) - 1} + \frac{\sigma(1 - 2\sigma)}{2\Gamma_R(1 - \sigma)^2}\right] R(\boldsymbol{u}), \tag{7.11}$$

for displacement fields $\boldsymbol{u}$ satisfying (7.1), (7.4). For a circular domain, on using (7.10) and $\Gamma_R = 1$ in (7.11), one obtains

$$E(\boldsymbol{u}) \geqq \frac{3 - 4\sigma}{8(1 - \sigma)^2} R(\boldsymbol{u}). \tag{7.12}$$

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