A Boundary Integral Treatment of Domain Integral Term for Initial Condition in Time-Domain Boundary Element Method for Diffusion Equation
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Abstract
In the time-domain boundary element method for diffusion equation, the domain integral term originated from initial condition is involved. Hence, we have to discretize the domain into internal cells to evaluate the domain integral for non-zero initial conditions. In the present paper, a method to convert the domain integral to boundary integrals by applying the multiple-reciprocity method and approximating the initial condition in terms of thin-plate spline utilized in the dual-reciprocity method. By using particular solutions for the Laplace and bi-harmonic operator corresponding to the time-domain fundamental solution, the domain integral term for the initial condition is converted to boundary integrals and a domain integral in which the bi-harmonic operator is now moved to the initial condition. By approximating the initial condition in terms of thin-plate splines in the same way as in the dual-reciprocity method, this domain integral term results in boundary integrals. The effectiveness of the present approach is demonstrated through some numerical results for several initial temperature variations.

1 Introduction
In the time-domain boundary element formulation for transient problems, a domain integral originated from the initial condition remains in the derived integral representation. Therefore, the domain must also be discretized into cells in addition to the boundary when the initial condition is not zero. Although there is no unknown quantity in the domain, its discretization into cells is very cumbersome especially in analyzing complicated three-dimensional body like injection molds. Also, numerical evaluations of the domain integrals are time-consuming jobs not only for three-dimensional volume cells but also for two-dimensional cells.

In this paper, we present a new time-domain boundary element method[1] for the 2-d diffusion equation without a domain integral for the initial condition. Multiple reciprocity method (MRM)[2], is applied to transform the original domain integral term into boundary integrals. In MRM, integration by parts of the domain integral is performed repeatedly until it can be neglected. In this study, we approximate the initial condition in terms of thin-plate splines[3] which are particular solutions of the bi-harmonic equation. This approximation is actually the same as that used in the dual reciprocity method[4]. By applying the reciprocity theorem twice, the bi-harmonic operator moves to the initial temperature, which becomes a linear combination of Dirac’s delta functions. Then, the domain integral simply results in a series of functions without domain integrals. A computer code is developed based on the present formulation and the effectiveness of the present approach is demonstrated through some numerical results comparing with those obtained with internal cells.
2 Theory

We start with the following diffusion equation:

$$k \nabla^2 u(x,t) = \frac{\partial u(x,t)}{\partial t}$$  (1)

where $k$ is the thermal diffusivity, $u$ is temperature. By using a fundamental solution of the adjoint form of Eq.(1), the boundary integral equation is derived as

$$c(y) u(y,t_F) + k \int_{t_0}^{t_F} \int_{\Gamma} q^*(x,t; y,t_F) u(x,t) \, d\Gamma \, dt$$

$$- k \int_{t_0}^{t_F} \int_{\Gamma} u^*(x,t; y,t_F) q(x,t) \, d\Gamma \, dt = \int_{\Omega} u^*(x,t_0; y,t_F) u_0(x) \, d\Omega$$  (2)

where $q$ is the derivative of the temperature in the direction of outward normal vector, $\Omega$ is the domain, $\Gamma$ is the boundary, $u^*$ is the fundamental solution, and $q^*$ is the normal derivative of $u^*$. Notice that we find a domain integral term originating from the initial temperature over the domain.

The fundamental solution for two-dimensional problems is given by

$$u^*(x,t_0; y,t_F) = \frac{1}{4\pi k(t_F-t_0)} \exp\left[-\frac{r^2}{4k(t_F-t_0)}\right]$$  (3)

In order to apply MRM, we employ particular solutions $\hat{u}^*$ and $\hat{\hat{u}}^*$ of the following equations:

$$\nabla^2 \hat{u}^* = u^*$$  (4)

$$\nabla^2 \hat{\hat{u}}^* = \hat{u}^*$$  (5)

The explicit form of $\hat{u}^*$ and $\hat{\hat{u}}^*$ can be easily obtained as

$$\hat{u}^* = \frac{1}{4\pi} \left[ E_1\left(\frac{r^2}{4k(t_F-t_0)}\right) + \ln\left(\frac{r^2}{4k(t_F-t_0)}\right) + C \right]$$  (6)

$$\hat{\hat{u}}^* = \frac{r^2}{16\pi} \left[ E_1\left(\frac{r^2}{4k(t_F-t_0)}\right) + \ln\left(\frac{r^2}{4k(t_F-t_0)}\right) + C \right.$$  

$$\left. + \frac{4k(t_F-t_0)}{r^2} \left(1 - \exp\left(-\frac{r^2}{4k(t_F-t_0)}\right)\right)\right]$$

$$+ \left. \frac{4k(t_F-t_0) E_1\left(\frac{r^2}{4k(t_F-t_0)}\right) + \ln\left(\frac{r^2}{4k(t_F-t_0)}\right) + C}{r^2} \right]$$  (7)

where $C = 0.57721566490153\cdots$ is Euler's constant and $E_1$ is the exponential integral. By using these particular solutions and integrating by parts four times, the domain integral in Eq.(2) is transformed into the following form:
To eliminate the domain integral in Eq.(8), we approximate $u_0$ in terms of thin-plate splines in the form:

$$u_0(x) = \sum_{l=1}^{N+L} \left[ \alpha^l (r^l)^2 \ln r^l \right] + ax_1^2 + bx_2^2 + cx_1x_2 + dx_1^2 + ex_2^2 + fx_1 + gx_2 + hx_1x_2 + j$$

(9)

where $N$ denotes the number of the boundary nodes, $L$ the number of the internal nodes, $r^l$ is the distance between a boundary point $x$ and an internal collocation point $z^l$ as shown in Figure 1. The unknown coefficients $\alpha^l$ ($l = 1, 2, \cdots, N + L$) must satisfy the
The approximation given by Eq.(9) satisfies
\[
\nabla^4 u_0(x) = \sum_{l=1}^{N+L} \alpha^l (z^l_1)^3 = 0, \quad \sum_{l=1}^{N+L} \alpha^l (z^l_2)^3 = 0
\]
\[
\nabla^4 u_0(x) = \sum_{l=1}^{N+L} \alpha^l (z^l_1)^2 z^l_2 = 0, \quad \sum_{l=1}^{N+L} \alpha^l (z^l_2)^2 = 0
\]
\[
\nabla^4 u_0(x) = \sum_{l=1}^{N+L} \alpha^l z^l_1 z^l_2 = 0, \quad \sum_{l=1}^{N+L} \alpha^l = 0
\]
(10)

Substituting Eq.(11) into (8) gives
\[
\int_{\Omega} u^* (x, t_0; y, t_F) u_0(x) \, d\Omega = \int_{\Gamma} \frac{\partial u^*}{\partial n} u_0(x) \, d\Gamma - \int_{\Gamma} \hat{u}^* \frac{\partial u_0(x)}{\partial n} \, d\Gamma
\]
\[
+ \int_{\Gamma} \frac{\partial \hat{\nabla}^2 u_0(x)}{\partial n} \, d\Gamma - \int_{\Gamma} \hat{u}^* \frac{\partial \nabla^2 u_0(x)}{\partial n} \, d\Gamma + 8\pi \sum_{l=1}^{N+L} \left\{ \frac{\theta}{2\pi} \alpha^l \hat{\nabla}^2 u^* (z^l_1, t_0; y, t_F) \right\}
\]
(12)

Since the normal derivative \( \frac{\partial \nabla^2 u_0(x)}{\partial n} \) in the fourth term of the right-hand side of Eq.(12)
has a singularity of Cauchy principle value, we regularize it in advance.

First, we modify it by subtracting and adding back \( \hat{u}^* (z^l_1, t_0; y, t_F) \) as
\[
\int_{\Gamma} \hat{u}^* \frac{\partial \nabla^2 u_0(x)}{\partial n} \, d\Gamma
\]
\[
= \sum_{l=1}^{N+L} \int_{\Gamma} \left[ \hat{u}^* (x, t_0; y, t_F) - \hat{u}^* (z^l_1, t_0; y, t_F) \right] \alpha^l \frac{4}{r^l} \frac{\partial r^l}{\partial n} \, d\Gamma
\]
\[
+ \sum_{l=1}^{N+L} \int_{\Gamma} \hat{u}^* (z^l_1, t_0; y, t_F) \alpha^l \frac{4}{r^l} \frac{\partial r^l}{\partial n} \, d\Gamma
\]
(13)
The second term of the above equation results in the following simple form:

$$\sum_{l=1}^{N+L} \int_{\Gamma} \hat{u}^*(z^l, t_0; y, t_F) d\Gamma \left( \frac{4}{r} \frac{\partial r^l}{\partial n} \right) d\Gamma$$

$$= 8\pi \sum_{l=1}^{N+L} \int_{\Gamma} \alpha^l \frac{1}{2\pi r^2} \frac{\partial r^l}{\partial n} d\Gamma \hat{u}^*(z^l, t_0; y, t_F)$$

$$= 8\pi \sum_{l=1}^{N+L} \left[ \int_{\Omega} \delta(x - z^l) d\Omega \right] \alpha^l \hat{u}^*(z^l, t_0; y, t_F)$$

$$= 8\pi \sum_{l=1}^{N+L} \alpha^l \hat{u}^*(z^l, t_0; y, t_F)$$

(14)

Finally, by using Eqs. (12) and (14), we obtain a boundary integral expression of the domain integral for the initial temperature, as follows:

$$\int_{\Omega} u^*(x, t_0; y, t_F) u_0(x) d\Omega$$

$$= \int_{\Gamma} \frac{\partial \hat{u}^*(x, t_0; y, t_F)}{\partial n} u_0(x) d\Gamma - \int_{\Gamma} \hat{u}^*(x, t_0; y, t_F) \frac{\partial u_0(x)}{\partial n(x)} d\Gamma$$

$$+ \int_{\Gamma} \frac{\partial \hat{u}^*(x, t_0; y, t_F)}{\partial n(x)} \nabla^2 u_0(x) d\Gamma$$

$$- \sum_{l=1}^{N+L} \int_{\Gamma} \left[ \frac{\hat{u}^*(z^l, t_0; y, t_F) - \hat{u}^*(z^l, t_0; y, t_F)}{r^l} \right] \alpha^l \frac{4}{r^l} \frac{\partial r^l}{\partial n} d\Gamma$$

(15)

### 3 Numerical results

One-dimensional heat flow in \(x_1\)-direction in a rectangular region shown in Figure 2 (a) is computed with several initial conditions. The boundary is discretized into quadratic elements. Piecewise constant interpolation is employed for discretizing the time axis. The boundary elements and the internal collocation points are also shown in Figure 2 (b). The thermal diffusivity and the time-step size are assumed as \(k = 1.39 \times 10^{-5} [m^2/s]\) and \(\Delta t = 2 [s],\) respectively.

First, we consider the initial temperature variation only in \(x_1\)-direction given by

$$u_0(x_1, x_2) = 30 \sin (20\pi x_1) + 30$$

(16)

Figure 3 shows the results for the temperature variation along the edge AB. The results obtained by the present approach are compared with those obtained by evaluating the domain integral for the initial temperature with internal cells. The present results show good agreement with those by the approach with internal cells. The reason why the temperature at the middle point of the plate at \(t = 10[s]\) is lower than 30°C which is the initial
temperature at the same point is that the initial temperature is not uniform but given by a function of Eq.(16) which becomes lower at a point near the boundary. Although the boundary temperature is raised up to 100°C suddenly, there is a heat flow from the middle part of the plate to the region near the boundary for some initial period and the temperature at the middle point becomes lower than the initial temperature.

Next, the initial temperature given by

\[ u_0(x_1, x_2) = 30 \sin(10\pi x_1) \] (17)

is considered. Figure 4 show the obtained results for the temperature variation along the edge AB in comparison with those obtained by evaluating the domain integral with internal cells. Although the initial condition is more complicated than in the first example, the results show good agreement with those with internal cells.
Figure 3  Results for temperature along AB

Figure 4  Results for temperature along AB
Finally, we show in Figure 5 the results for an initial condition with two-dimensional variation given by

$$u_0(x_1, x_2) = 30 (\sin(10\pi x_1) + 1) (\sin(5\pi x_2) + 1)$$

(18)

Again, the results by the present MRM-based approach has shown good agreements with those by the calculation based on the internal cell discretization.

4 Conclusions

A boundary only treatment of the domain integral for the initial condition has newly been proposed for the time-domain BEM for two-dimensional diffusion equation. Thin-plate splines were used for the approximation of the initial temperature to eliminate the domain integral occurring in the process of MRM. Present approach has been demonstrated to give accurate results without discretizing the domain into cells through some numerical examples. Extension of the present approach to three-dimensional problems is easily be achieved by using the particular solution $\hat{u}^*$ and $\hat{\hat{u}}^*$ corresponding to the fundamental solution of the three-dimensional diffusion equation.

References

