CSP TD n°8

Solutions

1) $\varphi(u,v) = (u, v, u^2 - v^2)$. Donc

$$\left.\begin{array}{l}\varphi_u = (1, 0, 2u)\\ \varphi_v = (0, 1, -2v)\end{array}\right\} \qquad \underline{\mathcal{F}_I(\varphi) = \begin{pmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{pmatrix}}$$

$$N_\varphi = \frac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|} = \frac{1}{(1+4u^2+4v^2)^{1/2}} \left(-2u, 2v, 1\right)$$

$$\left.\begin{array}{l}\varphi_{uu} = (0, 0, 2)\\ \varphi_{uv} = (0, 0, 0)\\ \varphi_{vv} = (0, 0, -2)\end{array}\right\} \qquad \underline{\mathcal{F}_{II}(\varphi) = \frac{1}{(1+4u^2+4v^2)^{1/2}} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}}$$

<u>Donc</u> $\underline{K(u,v)} = \det\left(\mathcal{F}_I^{-1} \mathcal{F}_{II}\right)$

$$= \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)}$$

$$= \frac{-4}{1+4u^2+4v^2} \cdot \frac{1}{1+4u^2+4v^2} = \underline{\frac{-4}{(1+4u^2+4v^2)^2}}$$

2) $\varphi(u,v) = (R\cos u \cos v, R\cos u \sin v, R\sin u)$. Donc (voir Cours 6, p. 8) :

$$\left.\begin{array}{l}\varphi_u = (-R\sin u\cos v, -R\sin u \sin v, R\cos u)\\ \varphi_v = (-R\cos u \sin v, R\cos u \cos v, 0)\end{array}\right\} \qquad \underline{\mathcal{F}_I(\varphi) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\cos^2 u \end{pmatrix}}$$

Maintenant,

$$N_\varphi = \frac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|} = \left(-\cos u \cos v, -\cos u \sin v, -\sin u\right)$$

$\varphi_{uu} = (-R\cos u\cos v, -R\cos u\sin v, -R\sin u)$

$\varphi_{uu}\cdot N_\varphi = R$

$\varphi_{uv} = \varphi_{vu} = (R\sin u\sin v, -R\sin u\cos v, 0)$

$\varphi_{uv}\cdot N_\varphi = \varphi_{vu}\cdot N_\varphi = 0$

$\varphi_{vv} = (-R\cos u\cos v, -R\cos u\sin v, 0)$

$\varphi_{vv}\cdot N_\varphi = R\cos^2 u$

$\Bigg\}\quad \mathcal{F}_{II}(\varphi) = \begin{pmatrix} R & 0 \\ 0 & R\cos^2 u \end{pmatrix}$

<u>Donc</u> $\quad K(u,v) = \det\left(\mathcal{F}_I^{-1}\mathcal{F}_{II}\right) = \dfrac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)} = \dfrac{R^2\cos^2 u}{R^4\cos^2 u} = \underline{\dfrac{1}{R^2}}$

$H(u,v) = \frac{1}{2}\,\mathrm{trace}\left(\mathcal{F}_I^{-1}\mathcal{F}_{II}\right) = \underline{\dfrac{1}{R}}$

3) Si $S\subset\mathbb{R}^3$ est une surface minimale, alors S ne peut pas avoir de point elliptique. En effet,

S minimale $\Rightarrow H(p) = \frac{1}{2}(k_1(p)+k_2(p)) = 0 \quad \forall p\in S$

$\Rightarrow k_1(p) = -k_2(p) \Rightarrow K(p) = k_1(p)\,k_2(p) \leq 0.$

Mais p elliptique veut dire que $K(p)>0$.

4) $S\subset\mathbb{R}^3$ paramétrée par $\varphi(u,v) = (u\cos v, u\sin v, v)$

$\varphi_u = (\cos v, \sin v, 0)$

$\varphi_v = (-u\sin v, u\cos v, 1)$

$\Bigg\}\Rightarrow \mathcal{F}_I(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix}$

$N_\varphi = \dfrac{\varphi_u\wedge\varphi_v}{\|\varphi_u\wedge\varphi_v\|} = \dfrac{1}{(1+u^2)^{1/2}}\,(\sin v, -\cos v, u)$

$\varphi_{uu}\cdot N_\varphi = 0$

$\varphi_{uv}\cdot N_\varphi = \dfrac{-1}{(1+u^2)^{1/2}}$

$\varphi_{vv}\cdot N_\varphi = 0$

$\Bigg\}\Rightarrow \tilde{\mathcal{F}}_{II}(\varphi) = \dfrac{1}{(1+u^2)^{1/2}}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Donc $\mathcal{F}_I^{-1}\mathcal{F}_{II} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+u^2} \end{pmatrix} \left[\frac{1}{(1+u^2)^{1/2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \frac{-1}{(1+u^2)^{1/2}} \\ \frac{-1}{(1+u^2)^{3/2}} & 0 \end{pmatrix}$

$\Rightarrow \quad K(u,v) = \dfrac{-1}{(1+u^2)^2}$ et $H(u,v) = 0$

(L'hélicoïde est une surface minimale)

5) $\tilde{\mathcal{F}}_{II}(\varphi) \equiv 0$ implique que :

$\varphi_{uu}\cdot N = \varphi_{uv}\cdot N = \varphi_{vu}\cdot N = \varphi_{vv}\cdot N = 0 \qquad (N = N_\varphi)$

Alors :

i) $\varphi_u \cdot N = \varphi_v \cdot N = 0$ d'où

$\underbrace{\varphi_{uu}\cdot N}_{0} + \varphi_u \cdot N_u = 0 \Rightarrow \underline{\varphi_u \cdot N_u = 0}$

De même $\underline{\varphi_u \cdot N_v = \varphi_v \cdot N_u = \varphi_v \cdot N_v = 0}$

ii) $N \cdot N = 1$ d'où

$N_u \cdot N = N_v \cdot N = 0$

Donc i) + ii) impliquent

$N_u = N_v = 0$ car N_u et N_v sont orthogonaux aux trois vecteurs φ_u, φ_v, N qui forment un repère.

iii) $N_u = N_v = 0$ implique N est constante

Donc $(\varphi\cdot N)_u = (\varphi \cdot N)_v = 0 \Rightarrow \varphi\cdot N = c$, constante

$\Rightarrow \quad S = \operatorname{Im}\varphi$ fait partie du plan $\varphi \cdot N = c$