CSP TD n°4

<u>Solutions</u>

1) On a $\delta(s) = f(\gamma(s))$ avec $f(X) = AX + b$, $A \in SO(3)$, $b \in \mathbb{R}^3$

Alors $f(\gamma(s)) = A\gamma(s) + b$ d'où

$$\delta'(s) = A\gamma'(s), \quad \delta''(s) = A\gamma''(s), \quad \delta'''(s) = A\gamma'''(s)$$

car $A : \mathbb{R}^3 \to \mathbb{R}^3$ est <u>linéaire</u> (voir CDF, Cours 3 §4)

Donc :

i) $$\|\delta'(s)\| = \|A\gamma'(s)\| = (A\gamma'(s) \cdot A\gamma'(s))^{1/2}$$
$$= (\gamma'(s) \cdot \gamma'(s))^{1/2} \quad \text{car } A \in SO(3)$$
$$= \|\gamma'(s)\|$$

ii) $$\kappa_\delta(s) := \|\delta''(s)\| = \|A\gamma''(s)\| \overset{*}{=} \|\gamma''(s)\| = \kappa_\gamma(s)$$

(* $A \in SO(3)$, voir i) ci-dessus)

iii) $$-\tau_\delta(s) = B'(s) \cdot N(s) \qquad (\text{car } B' = -\tau N)$$
$$= (T(s) \wedge N'(s)) \cdot N(s) \qquad \left(\begin{array}{l}\text{car } B = T \wedge N \\ \Rightarrow B' = T' \wedge N + T \wedge N'\end{array}\right)$$
$$\overset{*}{=} \frac{(\delta'(s) \wedge \delta'''(s)) \cdot \delta''(s)}{(\kappa(s))^2}$$

$$^{*}\left(\begin{array}{l}\text{car } T(s) = \delta'(s), \quad N(s) = \dfrac{\delta''(s)}{\kappa(s)}, \\[2mm] N'(s) = \dfrac{\kappa(s)\,\delta'''(s) - \delta''(s)\,\kappa'(s)}{(\kappa(s))^2} \\[2mm] \text{et } (T(s) \wedge \delta''(s)) \cdot N(s) = 0\end{array}\right)$$

N.B.
$(\vec{u}\wedge\vec{v})\cdot\vec{w}$ s'appelle le produit mixte. On a $(\vec{u}\wedge\vec{v})\cdot\vec{w} = \det(\vec{u}|\vec{v}|\vec{w})$

$$= \frac{\left(A\gamma'(s)\wedge A\gamma'''(s)\right)\cdot A\gamma''(s)}{(\kappa(s))^2}$$

$$= (\det A)\,\frac{\left(\gamma'(s)\wedge\gamma'''(s)\right)\cdot\gamma''(s)}{(\kappa(s))^2}$$

$$\left(\text{car } (A\vec{u}\wedge A\vec{v})\cdot A\vec{w} = \det(A)\,(\vec{u}\wedge\vec{v})\cdot\vec{w}\right)$$

$$= (\det A)\left(-\tau_\gamma(s)\right)$$

Alors $A\in SO(3) \Rightarrow \det A = 1 \Rightarrow \tau_\delta(s) = \tau_\gamma(s)$

2) $\gamma(t) = \left(a\cos\frac{t}{c},\ a\sin\frac{t}{c},\ \frac{bt}{c}\right)$ $\quad a>0,\ b\neq 0,\ c=\sqrt{a^2+b^2}$

i) $\gamma'(t) = \left(-\frac{a}{c}\sin\frac{t}{c},\ \frac{a}{c}\cos\frac{t}{c},\ \frac{b}{c}\right)$

$\Rightarrow \|\gamma'(t)\| = \left(\frac{a^2+b^2}{c^2}\right)^{1/2} = 1$ $\quad (\Rightarrow \gamma(t)$ unitaire$)$

$$\gamma''(t) = \left(-\frac{a}{c^2}\cos\frac{t}{c},\ -\frac{a}{c^2}\sin\frac{t}{c},\ 0\right) = \frac{a}{c^2}\left(-\cos\frac{t}{c},\ -\sin\frac{t}{c},0\right)$$

$$\underline{\kappa = \frac{a}{c^2} = \frac{a}{a^2+b^2}}$$

$$T(s) = \left(-\frac{a}{c}\sin\frac{t}{c},\ \frac{a}{c}\cos\frac{t}{c},\ \frac{b}{c}\right)$$

$$N(s) = \left(-\cos\frac{t}{c},\ -\sin\frac{t}{c},\ 0\right)$$

$$B(s) = T(s)\wedge N(s) = \left(\frac{b}{c}\sin\frac{t}{c},\ -\frac{b}{c}\cos\frac{t}{c},\ \frac{a}{c}\right)$$

$$\Rightarrow B'(s) = \left(\frac{b}{c^2}\cos\frac{t}{c},\ \frac{b}{c^2}\sin\frac{t}{c},\ 0\right)$$

$$= -\frac{b}{c^2}\left(-\cos\frac{t}{c},\ -\sin\frac{t}{c},\ 0\right) = -\tau(s)\,N(s)$$

d'où $\tau = \dfrac{b}{c^2} = \dfrac{b}{a^2+b^2}$

ii) Ecrivant a, b en termes de κ et τ on obtient :

$$a = \frac{\kappa}{\kappa^2+\tau^2} \quad \text{et} \quad b = \frac{\tau}{\kappa^2+\tau^2} \qquad (*)$$

Donc toute courbe $\gamma(s)$ unitaire avec courbure $\kappa > 0$ et torsion $\tau \neq 0$ constantes est une hélice circulaire à isométrie de $\mathbb{R}^3$ près (voir Théorème fondamental, Cours 4).

(*)

En effet, $\kappa = \dfrac{a}{a^2+b^2}$, $\tau = \dfrac{b}{a^2+b^2}$

$$\Rightarrow \quad \kappa^2+\tau^2 = \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}$$

Donc $a = \kappa(a^2+b^2) \Rightarrow a = \dfrac{\kappa}{\kappa^2+\tau^2}$

et

$$b = \tau(a^2+b^2) \Rightarrow b = \frac{\tau}{\kappa^2+\tau^2}$$

3) $\gamma(t) = (t, t^2, t^3)$; $\gamma'(t) = (1, 2t, 3t^2)$; $\gamma''(t) = (0, 2, 6t)$; et $\gamma'''(t) = (0, 0, 6)$

Donc $\tau(t) := \dfrac{(\gamma'(t) \wedge \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \wedge \gamma''(t)\|^2} = \dfrac{3}{1+9t^2+9t^4}$

4) Trouvez l'isométrie $f(X) = AX + \vec{b}$ qui envoie $(\gamma_1(0), T_1, N_1, B_1)$ à $(\gamma_2(0), T_2, N_2, B_2)$

Rép

i) Comme $T_1 = (0,0,1)$; $N_1 = (1,0,0)$; $B_1 = (0,1,0)$

et $T_2 = (\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2})$; $N_2 = (0,1,0)$; $B_2 = (-\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2})$

$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ envoie le repère (e_1, e_2, e_3) à (T_1, N_1, B_1)

$A_2 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix}$ envoie le repère (e_1, e_2, e_3) à (T_2, N_2, B_2)

Donc $A := A_2 A_1^{-1} = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$

envoie (T_1, N_1, B_1) à (T_2, N_2, B_2)

ii) $\gamma_1(0) = (1, 3, 1)$ d'où $A(\gamma_1(0)) = (-\sqrt2, 1, 2\sqrt2)$

$\gamma_2(0) = (2, 5, -1)$

Alors la translation par $\vec{b} = (2, 5, -1) - (-\sqrt2, 1, 2\sqrt2)$

envoie $A(\gamma_1(0))$ à $\gamma_2(0)$

iii) Finalement, l'isométrie $f(X) = AX + \vec{b}$ avec

$A = \begin{pmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2+\sqrt2 \\ 4 \\ -1-2\sqrt2 \end{pmatrix}$

envoie $(\gamma(0), T_1, N_1, B_1)$ à $(\gamma_2(0), T_2, N_2, B_2)$

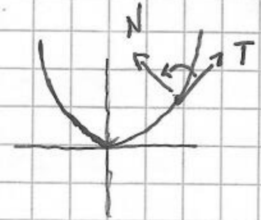
5) La développée de $\gamma(t) = (t, t^2)$

i) $\gamma'(t) = (1, 2t)$, $\gamma''(t) = (0, 2)$ d'où

$$\tilde{k}(t) = \frac{\det\left(\gamma'(t), \gamma''(t)\right)}{\|\gamma'(t)\|^3} = \frac{\det\begin{pmatrix} 1 & 0 \\ 2t & 2 \end{pmatrix}}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$$

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{(1+4t^2)^{1/2}}\,(1, 2t)$$ d'où

$$N(t) = \frac{1}{(1+4t^2)^{1/2}}\,(-2t, 1)$$



$$\left(\det\left(T(t), N(t)\right) = \frac{1}{1+4t^2} \det\begin{pmatrix} 1 & -2t \\ 2t & 1 \end{pmatrix} = 1 \right)$$

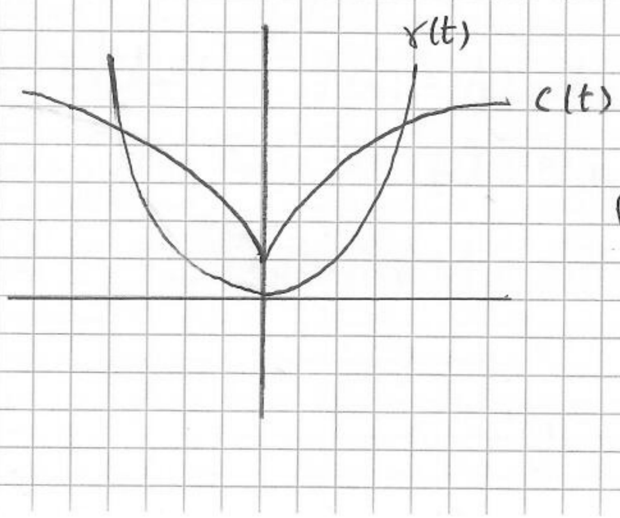
Alors

$$c(t) = (t, t^2) + \frac{(1+4t^2)^{3/2}}{2}\,\frac{(-2t, 1)}{(1+4t^2)^{1/2}}$$

$$= (t, t^2) + \frac{1+4t^2}{2}\,(-2t, 1)$$

$$= \left(-4t^3, \tfrac{1}{2} + 3t^2\right)$$

ii)



(tracées approximatives)