Principal Congruence Links: Class number greater than 1

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Abstract

In a previous paper, we started an enumeration of the finitely many levels for which a principal congruence manifold can be a link complement in S^3. In this paper we give a complete enumeration of all the principal congruence link complements in S^3, together with their levels in the case when the class number of $\mathbf{Q}(\sqrt{-d})$ is greater than 1.

1 Introduction

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbf{Q}(\sqrt{-d})$, and let Q_d denote the Bianchi orbifold $\mathbf{H}^3/\mathrm{PSL}(2,\mathrm{O_d})$. As is well-known Q_d is a finite volume hyperbolic orbifold with h_d cusps, where h_d is the class number of $\mathbf{Q}(\sqrt{-d})$ (see [12] Chapters 8 and 9 for example). A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [12] Chapter 8 for more on this).

An important class of arithmetic 3-manifolds consists of the *congruence* manifolds. Recall that a subgroup $\Gamma < \mathrm{PSL}(2,\mathrm{O_d})$ is called a *congruence subgroup* if there exists an ideal $I \subset O_d$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2,\mathrm{O_d}) \to \mathrm{PSL}(2,\mathrm{O_d/I})\},$$

where $\mathrm{PSL}(2,\mathrm{O_d/I}) = \mathrm{SL}(2,\mathrm{O_d/I})/\{\pm\mathrm{Id}\}$. The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ. For convenience, if $n \in \mathbf{Z}$, we will denote the principal O_d-ideal $< n >$ simply by n. A manifold $M = \mathbf{H}^3/\Gamma$ is called *congruence* (resp. *principal congruence*) if $\Gamma > \Gamma(I)$ (resp. $\Gamma = \Gamma(I)$) for some ideal I.

In a previous paper [6], motivated by a question of Thurston ([16], Question 19), we started an enumeration of the finitely many levels for which a principal congruence manifold can be a link complement in S^3 (see [6] or §4.1 for the proof of finiteness). In particular, for $h_d = 1$, the main result of that paper gave 9 new examples of such principal congruence link groups bringing the known total to 18.

In this paper, we give a complete enumeration of all the principal congruence link complements in S^3, together with their levels in the case when the class number of $\mathbf{Q}(\sqrt{-d})$ is greater than 1. Our main result is the following:

Theorem 1.1. *Suppose that $h_d > 1$. Then the following list of 16 pairs (d, I) describes all principal congruence subgroups $\Gamma(I) < \mathrm{PSL}(2,\mathrm{O_d})$ such that $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3.*

1. *$d = 5$: $I =< 3, (1 \pm \sqrt{-5}) >$.*

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2. $d = 15$: $I =< 2, (1 \pm \sqrt{-15})/2 >$, $I =< 3, (3 \pm \sqrt{-15})/2 >$, $I =< 4, (1 \pm \sqrt{-15})/2 >$, $I =< 5, (5 \pm \sqrt{-15})/2 >$, $I =< 6, (-3 \pm \sqrt{-15})/2 >$.
3. $d = 23$: $I =< 2, (1 \pm \sqrt{-23})/2 >$, $I =< 3, (1 \pm \sqrt{-23})/2 >$, $I =< 4, (-3 \pm \sqrt{-23})/2 >$.
4. $d = 31$: $I =< 2, (1 \pm \sqrt{-31})/2 >$, $I =< 4, (1 \pm \sqrt{-31})/2 >$, $I =< 5, (3 \pm \sqrt{-31})/2 >$.
5. $d = 47$: $I =< 2, (1 \pm \sqrt{-47})/2 >$, $I =< 3, (1 \pm \sqrt{-47})/2 >$, $I =< 4, (1 \pm \sqrt{-47})/2 >$.
6. $d = 71$: $I =< 2, (1 \pm \sqrt{-71})/2 >$.

Note that the cases $d = 15, I =< 2, (1 \pm \sqrt{-15})/2 >$, and $d = 23, I =< 2, (1 \pm \sqrt{-23})/2 >$ were treated in [5] and explicit link diagrams were given. To our knowledge, the other 14 examples are new (although we do not have link diagrams for any of these 14 examples).

Recall that the solution of the Cuspidal Cohomology Problem for $\mathrm{PSL}(2, \mathrm{O_d})$ (see [17]) showed that Q_d can have a cover homeomorphic to a link complement in S^3 only if

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$$

In [4] it was shown that for every d in this list, Q_d is indeed covered by such a link complement. Furthermore, when $h_d = 1$ ($d = 1, 2, 3, 7, 11, 19$), Q_d is covered by a principal congruence link complement (see [6]).

Now, comparing the above with Theorem 1.1 gives the following

Corollary 1.2. *When $d = 6$ or 39 there are no principal congruence link complements.*

On the other hand, we are able to prove the existence of congruence link complements when $d = 6$ or 39 (see §5), and so this, together with Theorem 1.1 and [6] proves

Theorem 1.3. *For all $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$, there exists a congruence link complement.*

As pointed out in [6], it remains an open question as to whether there are infinitely many congruence link complements.

We close the Introduction by outlining the plan of the paper. In §2 we recall material from [6] as well as some other preliminary set-up. In §3 we show that the principal congruence groups given in Theorem 1.1 are indeed link groups, and in §4, we eliminate the other (finitely many) possibilities. Finally, §5 contains the proof that there exist (non-principal) congruence link complements when $d = 6$ or 39.

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2 Preliminaries and outline of proofs

In §2.1-2.4 we gather facts and background to be used; in §2.5 we outline the proof of Theorem 1.1.

2.1

We begin by recalling the orders of the groups $\mathrm{PSL}(2,\mathcal{R})$ where $\mathcal{R}$ is a finite ring of the form O_d/I, with $I \subset O_d$ an ideal (see [8]). For such an ideal I we have a decomposition into powers of prime ideals. Assuming that $I = \mathcal{P}_1^{a_1}\dots\mathcal{P}_r^{a_r}$, we have:

$$|\mathrm{PSL}(2,\mathrm{O_d/I})| = \begin{cases} 6, \quad \text{when} \quad N(I)=2 \\ \frac{N(I)^3}{2}\prod_{\mathcal{P}|I}(1-\frac{1}{N(\mathcal{P})^2}), \ \text{otherwise} \end{cases}$$

where $N(I) = |O_d/I|$ denotes the norm of the ideal I. Since $\mathrm{PSL}(2,\mathrm{O_d})/\Gamma(\mathrm{I}) \cong \mathrm{PSL}(2,\mathrm{O_d/I})$, this gives a formula for the index of $\Gamma(I)$ in $\mathrm{PSL}(2,\mathrm{O_d})$.

2.2

The proof of Theorem 1.1 makes use of presentations for the Bianchi groups in the cases $d \in \{5,6,15,23,31,39,47,71\}$. Note that the class numbers for the relevant quadratic imaginary number fields are 2 (when $d=5,6,15$), 3 (when $d=23,31$), 4 (when $d=39$), 5 (when $d=47$) and 7 (when $d=71$).

The presentations for $d=5,6,15$ are from [15] while the rest were done by A. Page using a suite of computer packages he recently developed (see [13]) to study arithmetic Kleinian groups. We maintain the notation of [15] for $d=5,6,15$ and for the others we use the notation of the presentations provided to us by A. Page.

$\mathrm{PSL}(2,\mathrm{O_5}) = < \mathrm{a,t,u,b,c} \mid \mathrm{a^2 = b^2 = (ta)^3 = (ab)^2 = (aubu^{-1})^2 = acatc^{-1}t^{-1} = ubu^{-1}cbtc^{-1}t^{-1}} = 1, [\mathrm{t,u}] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_6}) = < \mathrm{a,t,u,b,c} \mid \mathrm{a^2 = b^2 = (ta)^3 = (atb)^3 = (atubu^{-1})^3 = t^{-1}ctubu^{-1}c^{-1}b^{-1}} = 1, [\mathrm{t,u}] = [\mathrm{a,c}] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{15}}) = < \mathrm{a,t,u,c} \mid \mathrm{a^2 = (ta)^3 = ucuatu^{-1}c^{-1}u^{-1}a^{-1}t^{-1}} = 1, [\mathrm{t,u}] = [\mathrm{a,c}] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{23}}) = < g_1,g_2,g_3,g_4,g_5 \mid g_3^3 = (g_3g_2)^2 = g_5g_2^{-1}g_3^{-1}g_5^{-1}g_1^{-1}g_2^{-1}g_3^{-1}g_1 = g_4^{-1}g_5g_3g_2g_5^{-1}g_2g_4g_3 = 1, [g_1,g_2] = [g_4,g_5] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{31}}) = < g_1,g_2,g_3,g_4,g_5 \mid g_2^3 = (g_2g_1^{-1})^2 = g_4g_1^{-1}g_3^{-1}g_2g_3g_4^{-1}g_2g_4g_3^{-1}g_1^{-1}g_2g_3g_4^{-1}g_2 = g_5g_3^{-1}g_2g_3g_4^{-1}g_2g_1^{-1}g_5^{-1}g_2^{-1}g_4g_3^{-1}g_2^{-1}g_3g_1 = g_2g_3g_4^{-1}g_2g_1^{-1}g_4g_3^{-1}g_2g_3g_4^{-1}g_1g_2^{-1}g_4g_3^{-1} = 1, [g_1,g_3] = [g_4,g_5] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{39}}) = < g_1,g_2,g_3,g_4,g_5,g_6,g_7 \mid g_3^3 = (g_3g_5)^2 = (g_1^{-1}g_3^{-1})^2 = (g_5^{-1}g_1)^3 = (g_7g_5^{-1}g_7^{-1}g_1)^3 = g_5^{-1}g_1g_6^{-1}g_4^{-1}g_5g_4g_1^{-1}g_6 = g_4^{-1}g_5g_4g_2^{-1}g_7g_5^{-1}g_7^{-1}g_2 = g_6g_1^{-1}g_5g_6^{-1}g_4^{-1}g_5g_4g_1^{-1}g_4^{-1}g_5g_4g_1^{-1} = 1, [g_2,g_1] = [g_3^{-1},g_7^{-1}] = [g_4,g_6] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{47}}) = < g_1,g_2,g_3,g_4,g_5,g_6,g_7 \mid g_1^3 = (g_2^{-1}g_1)^2 = g_2^{-1}g_1g_6g_1^{-1}g_2g_6^{-1} = g_6g_2^{-1}g_4^{-1}g_5g_3^{-1}g_6^{-1}g_4g_2g_3g_5^{-1} = g_7^{-1}g_2^{-1}g_5^{-1}g_4g_1g_4^{-1}g_2g_7g_4g_1^{-1}g_4^{-1}g_5 = g_3g_5^{-1}g_4g_1g_4^{-1}g_2g_5g_3^{-1}g_2^{-1}g_4^{-1}g_1^{-1}g_4 = g_5^{-1}g_4g_1g_4^{-1}g_7^{-1}g_2^{-1}g_4g_1^{-1}g_4^{-1}g_5g_3^{-1}g_2g_3g_7 = 1, [g_5,g_7] = [g_3,g_2] = 1 >$,

$\mathrm{PSL}(2,\mathrm{O_{71}}) = < g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8,g_9 \mid g_8^3 = (g_8g_7^{-1})^2 = g_1^{-1}g_3g_7g_3^{-1}g_1g_7^{-1} = g_6g_3g_6^{-1}g_7g_9^{-1}g_3^{-1}g_9g_7^{-1} = g_7^{-1}g_6g_3g_6^{-1}g_5^{-1}g_2g_7g_5g_6g_3^{-1}g_6^{-1}g_2^{-1} = g_8g_7^{-1}g_1g_5g_6g_3^{-1}g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_5^{-1}g_1^{-1} = g_4^{-1}g_7^{-1}g_5^{-1}g_2g_1^{-1}g_3g_7g_9g_4g_1g_7^{-1}g_2^{-1}g_5g_7g_9^{-1}g_3^{-1} = g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_7g_9g_6g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_9^{-1}g_7^{-1}g_3^{-1}g_1 = g_2g_6g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_7g_8^{-1}g_5^{-1}g_2^{-1}g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_7g_8^{-1}g_1g_5g_6g_3^{-1}g_6^{-1} = 1, [g_8^{-1},g_4] = 1 >$.

Throughout the paper we use ω_d as follows:

$$\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1+\sqrt{-15}}{2}, \frac{1+\sqrt{-23}}{2}, \frac{1+\sqrt{-31}}{2}, \frac{1+\sqrt{-39}}{2}, \frac{1+\sqrt{-47}}{2}, \frac{1+\sqrt{-71}}{2}.$$

When $d = 5, 6, 15$, $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & \omega_d \\ 0 & 1 \end{pmatrix}$ (with the obvious abuse of notation between SL and PSL). Matrix representatives for the other generators used are given in the Appendix.

2.3

Let $\Gamma \leq \mathrm{PSL}(2, \mathrm{O_d})$ be a finite index subgroup. For convenience in what follows, we assume that $d \neq 1, 3$. Then

- A *cusp*, $[c]$, of Γ is a Γ-orbit of points in $\mathbf{P}^1(\mathbf{Q}(\sqrt{-d}))$
- A *peripheral subgroup* of Γ for $[c]$ is a maximal parabolic subgroup, $P_x < \Gamma$, fixing $x \in [c]$. Note that if $y \in [c]$, then P_x and P_y are conjugate; hence a peripheral subgroup for $[c]$ is determined up to conjugacy.
- A *set of peripheral subgroups for* Γ is the choice of one peripheral subgroup for each cusp of Γ.

We will use the term *cusp* to mean $[c]$, a choice of point x in $[c]$, as well as the end of $\mathbf{H}^3/\Gamma$ corresponding to $[c]$. Which one is meant should be clear from the context. Note that since $d \neq 1, 3$, each peripheral subgroup is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

2.4

As noted in the Introduction, Q_d has h_d cusps (or equivalently $\mathrm{PSL}(2, \mathrm{O_d})$ has h_d conjugacy classes of peripheral subgroups). In the setting of this paper, $h_d > 1$ and so the orbifold Q_d will have more than one cusp.

In the cases of $d = 5, 6$, and 15, the peripheral subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ that fixes ∞ is given by $< t, u >$ (in terms of the generators given in §2.2). Below we give a set of peripheral subgroups for each of the Bianchi groups in §2.2. We describe a choice of peripheral subgroup and cusp by (x, P_x) (following the notation described in §2.3).

d=5: $(\infty, < t, u >)$, $(\frac{1-\sqrt{-5}}{2}, < tb, tu^{-1}ct^{-1} >)$.

d=6: $(\infty, < t, u >)$, $(\frac{-\sqrt{-6}}{2}, < tb, cu >)$.

d=15: $(\infty, < t, u >)$, $(\frac{1+\sqrt{-15}}{4}, < uca, c^{-1}au^{-1}c^{-1}u^{-1}ta >)$.

d=23: $(\infty, < g_1, g_2 >)$, $(\frac{1-\sqrt{-23}}{4}, < g_4, g_5)$, $(\frac{-1-\sqrt{-23}}{4}, < g_4g_3g_2, g_2^{-1}g_5g_3g_2 >)$.

d=31: $(\infty, < g_1, g_3 >)$, $(\frac{1-\sqrt{-31}}{4}, < g_4, g_5)$, $(\frac{-1-\sqrt{-31}}{4}, < g_1g_5, g_3^{-1}g_2g_3g_4^{-1}g_2g_5 >)$.

d=39: $(\infty, < g_1, g_2 >)$, $(\frac{1-\sqrt{-39}}{4}, < g_4, g_6 >)$, $(\frac{1-\sqrt{-39}}{5}, < g_5^{-1}g_6, g_4g_1^{-1}g_6 >)$, $(\frac{3-\sqrt{-39}}{6}, < g_5, g_4g_2^{-1}g_7 >)$.

d=47: $(\infty, < g_2, g_3 >)$, $(\frac{1-\sqrt{-47}}{4}, < g_5, g_7 >)$,$(\frac{3-\sqrt{-47}}{4}, < g_2g_7, g_4g_1^{-1}g_4^{-1}g_5 >)$, $(\frac{1-\sqrt{-47}}{6}, < g_6g_2^{-1}g_4^{-1}, g_5g_3^{-1}g_2^{-1}g_4^{-1} >)$, $(\frac{1+\sqrt{-47}}{6}, < g_6^{-1}g_1^{-1}g_4, g_3g_5^{-1}g_4g_1 >)$.

d=71: $(\infty, < g_7, g_1^{-1}g_3 >)$, $(\frac{1-\sqrt{-71}}{4}, < g_2, g_6g_1g_5g_7g_8^{-1}g_5^{-1}g_1^{-1}g_3g_6^{-1}g_7g_8^{-1}g_5^{-1} >)$,
$(\frac{1+\sqrt{-71}}{6}, < g_3, g_6^{-1}g_7g_9^{-1} >)$, $(\frac{-1-\sqrt{-71}}{4}, < g_7g_2, g_6g_3g_6^{-1}g_5^{-1}g_7^{-1} >)$,
$(\frac{-1+\sqrt{-71}}{6}, < g_7g_9g_6, g_3^{-1}g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1} >)$, $(\frac{3+\sqrt{-71}}{8}, < g_3g_9g_4, g_4^{-1}g_7^{-1}g_5^{-1}g_2g_7g_1^{-1} >)$,
$(\frac{3-\sqrt{-71}}{8}, < p_1, p_2 >)$
where $p_1 = g_4g_1g_7^{-1}g_2^{-1}g_5g_7g_9^{-1}g_3^{-1}g_8^{-1}g_4^{-1}$ and $p_2 = g_6g_3^{-1}g_1g_5g_8g_7^{-1}g_5^{-1}g_1^{-1}g_6^{-1}g_9^{-1}g_8^{-1}g_4^{-1}$.

We remark that finding these peripheral subgroups and expressing them in terms of the given generators was a highly non-trivial exercise. One can check that the h_d cusps correspond to different elements of the ideal class group of $\mathbf{Q}(\sqrt{-d})$, hence they are inequivalent. Also, the generators of each of the above peripheral subgroups commute (using the relations in the corresponding Bianchi group, or by direct matrix calculation using the generators given in the Appendix), and these generators correspond to primitive parabolic matrices with the correct fixed point.

2.5

We conclude this section with an outline of the proof of Theorem 1.1. The methods used are those in [6] adapted to the case $h_d > 1$.

We first note that there are only finitely many groups $\Gamma(I)$ that can be link groups. Indeed, as mentioned above, the cuspidal cohomology of $\mathrm{PSL}(2, \mathrm{O_d})$ reduces consideration to the 8 Bianchi groups given in §2.2. Furthermore, as explained in §4.1 below, if $\Gamma(I)$ is a link group then $N(I) < 39$.

To establish the 16 principal congruence link groups in Theorem 1.1, and eliminate the remaining $\Gamma(I)$, we use the following properties of hyperbolic link complements, which we state for the case of principal congruence groups.

1. If $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$, then $\Gamma(I)$ is generated by parabolic elements.
2. $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$ if and only if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1.

Indeed, if $\mathbf{H}^3/\Gamma(I) \cong S^3 \setminus L$, then for each component L_i of L, there is a meridian curve x_i so that Dehn filling $S^3 \setminus L$ along the totality of these curves gives S^3. Thus, trivializing the corresponding parabolic elements $[x_i]$ in $\Gamma(I)$ gives the trivial group. Conversely, given Perelman's resolution of the Geometrization Conjecture, if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1, then $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3.

Given this, our method is:

Step 1: *Check whether* $\Gamma(I)$ *is generated by parabolic elements.*

Let $\Gamma(I) < \mathrm{PSL}(2, \mathrm{O_d})$, and let P_i be the peripheral subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ fixing the cusp c_i for $i = 1, \ldots, h_d$ as given in §2.3. Now, comparing with the discussion in [6], $P_i(I) = \mathrm{P}_i \cap \Gamma(I)$ is the peripheral subgroup of $\Gamma(I)$ fixing c_i.

Let $N_d(I)$ denote the normal closure in $\mathrm{PSL}(2, \mathrm{O_d})$ of $\{P_1(I), \ldots, P_{h_d}(I)\}$. Note that $N_d(I) < \Gamma(I)$ since $\Gamma(I)$ is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$. It is clear that $\Gamma(I)$ is generated by parabolic elements if and only if $N_d(I) = \Gamma(I)$.

We use Magma [7] to test whether $\Gamma(I) = N_d(I)$. In §3 we show that the groups in Theorem 1.1 are generated by parabolics, while §4 is devoted to showing that the remaining $\Gamma(I)$ are not and hence can't be link groups.

Step 2: *Find parabolic elements in $\Gamma(I)$, one for each cusp, so that trivializing these elements trivializes the group.*

First, we obtain a set of peripheral subgroups for $\Gamma(I)$ as follows. Let $\{P_1, \ldots, P_{h_d}\}$ be a set of peripheral subgroups for $\mathrm{PSL}(2, \mathrm{O_d})$. Then, as in Step 1, set $P_i(I) = P_i \cap \Gamma(I)$. Note that $S = \{P_1(I), \ldots, P_{h_d}(I)\}$ is a partial set of peripheral subgroups for $\Gamma(I)$. One obtains a full set of peripheral subgroups for $\Gamma(I)$ by adding certain conjugates of the $P_i(I)$ to the partial set S as explained in §3.

Next, given a (full) set of peripheral subgroups for $\Gamma(I)$, we choose one parabolic from each of these peripheral subgroups and use Magma to check that trivializing these elements trivializes $\Gamma(I)$. This choice of parabolics involves trial and error. However, in all cases these parabolics are linear combinations of small powers of the generators of the $P_i(I)$ and their conjugates.

3 Proof of Theorem 1.1: Determining the principal congruence link groups

We now prove, as described in §2.5, that the groups given in Theorem 1.1 do correspond to link groups. We use the notation (d, I) to indicate the Bianchi group and level given in Theorem 1.1. The ideals involved are of norm 2 (5 groups), 3 (4 groups), 4 (4 groups), 5 (2 groups), and 6 (1 group). Note that if $\Gamma(I)$ is determined to be a link group in S^3, then $\Gamma(\overline{I})$ also is. Therefore, in what follows we simply refer to one of the complex conjugate pair.

3.1

Doing Step 1 above, we check that $\Gamma(I)$ is generated by parabolic elements. Since $N_d(I) < \Gamma(I)$, it suffices to show that $N_d(I)$ and $\Gamma(I)$ have the same index in $\mathrm{PSL}(2, \mathrm{O_d})$. The index of $\Gamma(I)$ is given by the formula in §2.1 while the index of $N_d(I)$ is calculated using Magma.

We now do this explicitly for the group $(d, I) = (15, < 2, \omega_{15} >)$. Recall from §2.4 that the peripheral subgroups for $\mathrm{PSL}(2, \mathrm{O}_{15})$ are given by $P_1 =< t, u >$, $P_2 =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. In terms of the matrices for the presentation of $\mathrm{PSL}(2, \mathrm{O}_{15})$, we have $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & \omega_{15} \\ 0 & 1 \end{pmatrix}$, $uca = \begin{pmatrix} 1+2\omega_{15} & 4-\omega_{15} \\ 4 & 1-2\omega_{15} \end{pmatrix}$, and $(c^{-1}au^{-1}c^{-1}u^{-1}ta) = \begin{pmatrix} -3+4\omega_{15} & 8-\omega_{15} \\ 7+\omega_{15} & 1-4\omega_{15} \end{pmatrix}$. Thus we obtain the peripheral subgroups $P_i(I) = P_i \cap \Gamma(I)$ by reducing these matrices modulo I which gives $P_1(I) =< t^2, u >$, $P_2(I) =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2 >$.

Now from §2.1 we have $[\mathrm{PSL}(2, \mathrm{O}_{15}) : \Gamma(\mathrm{I})] = 6$ and Magma gives $[\mathrm{PSL}(2, \mathrm{O}_{15}) : \mathrm{N_d(I)}] = 6$, hence $\Gamma(I) = N_{15}(I)$ so that $\Gamma(I)$ is generated by parabolics. In our Magma routines below, $G = \mathrm{PSL}(2, \mathrm{O_d})$, $H =< P_1(I), \ldots, P_{h_d}(I) >$, and $N = N_d(I) =<< H >>$ (the normal closure of H).

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G<a,c,t,u>:=Group<a,c,t,u|a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1, (t,u),(a,c)>;
H:=sub<G|t^2,u,(c^-1*a*u^-1*c^-1*u^-1*t*a)^2,u*c*a>;
N:=NormalClosure(G,H);

print Index(G,N);
6
\\
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The remaining 15 groups are done in exactly the same way. The set of peripheral subgroups for each $\Gamma(I)$ can be read off from H in its Magma routine.

3.2

Next, we implement Step 2: find parabolic elements from $\Gamma(I)$, one for each cusp of $\Gamma(I)$, and use Magma to show that trivializing these elements trivializes $\Gamma(I)$. We start by obtaining a set of peripheral subgroups for $\Gamma(I)$ and then choose one parabolic from each subgroup. The method depends on the norm of the ideal I.

3.2.1 Cases (d,I), N(I) = 2

Here the Bianchi groups involved are $d = 15, 23, 31, 47, 71$. Since $N(I) = 2$, we have $\mathrm{PSL}(2, \mathrm{O_d/I}) \cong \mathrm{S_3}$. In order to find a set of peripheral subgroups for $\Gamma(I)$, we use the following sequence of regular covers:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{3} \mathbf{H}^3/\Gamma_1 \xrightarrow{2} Q_d$$

where $\Gamma_1 =< \Gamma(I), \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} >$.

Since each peripheral subgroup P_i of $\mathrm{PSL}(2, \mathrm{O_d})$ maps to a subgroup of order 2 in $\mathrm{PSL}(2, \mathrm{O_d/I})$ it follows that each cusp of Q_d is covered by 3 cusps of $\mathbf{H}^3/\Gamma(I)$ each with covering degree 2. Now $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/3\mathbf{Z} < \mathrm{PSL}(2, \mathrm{O_d/I})$, hence each cusp of Q_d is covered by 1 cusp of $\mathbf{H}^3/\Gamma_1$ with covering degree 2. Thus a set of peripheral subgroups for Γ_1 is given by $\{P_1(I), \ldots, P_{h_d}(I)\}$. Conjugating this set by the elements $\{Id, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\}$ gives a set of peripheral subgroups for $\Gamma(I)$.

The case $(15, < 2, \omega_{15} >)$**:**

Let $I =< 2, \omega_{15} >$. We analyze this $\Gamma(I)$; the other four cases involving N(I) = 2 are dealt with in exactly the same way.

In §3.1 above, we computed the peripheral subgroups $P_1(I)$ and $P_2(I)$ and showed that $\Gamma(I)$ was generated by parabolics. This is given again in the following Magma routine.

Since Q_{15} has two cusps, the manifold $\mathbf{H}^3/\Gamma(I)$ has 6 cusps. Recall that $P_1(I) =< t^2, u >$, $P_2(I) =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2 >$. Since $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = ta$, conjugating $P_1(I)$ and $P_2(I)$ by the elements $\{Id, ta, (ta)^2\}$ gives a set of 6 peripheral subgroups for $\Gamma(I)$. Now, we choose one element from each of these 6 peripheral subgroups:

$$\{t^2, (ta)u(ta)^{-1}, (ta)^2u(ta)^{-2}, uca, (ta)uca(ta)^{-1}(ta)^2(c^{-1}au^{-1}c^{-1}u^{-1}ta)^2(ta)^{-2}\}$$

In the Magma routine, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these 6 parabolics, and Magma calculates that $Q =< 1 >$ which shows that $\Gamma(I)$ is trivialized by setting these 6 elements equal to 1. Thus $\Gamma(< 2, \omega_{15} >)$ is indeed a 6 component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1, (t,u),(a,c)>;
H:=sub<G|t^2,u,(c^-1*a*u^-1*c^-1*u^-1*t*a)^2,u*c*a>;
N:=NormalClosure(G,H);

print Index(G,N);
6
\\
```

```
Q:=quo<N|t^2,(t*a)*u*(t*a)^-1,(t*a)^2*u*(t*a)^-2,
u*c*a,(t*a)*u*c*a*(t*a)^-1,(t*a)^2*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(t*a)^-2>;

print Order(Q);
1
\\
```

The case $(71, < 2, \omega_{71} >)$**:**

Let $I =< 2, \omega_{71} >$. The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is a 21 component link complement in S^3. Note that Q_{71} has 7 cusps and that $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = {g_8}^{-1}$. The 7 peripheral subgroups $P_i(I)$ can be read off from H and Q is $\Gamma(I)$ modulo the 21 parabolic elements to be trivialized.

```
G<g1,g2,g3,g4,g5,g6,g7,g8,g9>:=Group<g1,g2,g3,g4,g5,g6,g7,g8,g9|g8^3,(g8^-1,g4),
(g8*g7^-1)^2,g1^-1*g3*g7*g3^-1*g1*g7^-1,g6*g3*g6^-1*g7*g9^-1*g3^-1*g9*g7^-1,
g7^-1*g6*g3*g6^-1*g5^-1*g2*g7*g5*g6*g3^-1*g6^-1*g2^-1,g8*g7^-1*g1*g5*g6*g3^-1*g1*g5*
g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*g5^-1*g1^-1,g4^-1*g7^-1*g5^-1*g2*g1^-1*g3*g7*g9*g4
*g1*g7^-1*g2^-1*g5*g7*g9^-1*g3^-1,g5*g8*g7^-1*g5^-1*g1^-1*g7*g9*g6*g1*g5*g8*g7^-
1*g5^-1*g1^-1*g3*g6^-1*g9^-1*g7^-1*g3^-1*g1,g2*g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*
g6^-1*g7*g8^-1*g5^-1*g2^-1*g5*g7*g8^-1*g5^-1*g1^-1*g7*g8^-1*g1*g5*g6*g3^-1*g6^-1>;

H:=sub<G|g7^2, g7*g1^-1*g3, g2^2, g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*g7*g8^-1*g5^-1,
g3^2, g3*g6^-1*g7*g9^-1, g7*g2, (g6*g3*g6^-1*g5^-1*g7^-1)^2, (g7*g9*g6)^2,
g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1, (g3*g9*g4)^2, g4^-1*g7^-1*g5^-1*g2*g7*g1^-1,
(g4*g1*g7^-1*g2^-1*g5*g7*g9^-1*g3^-1*g8^-1*g4^-1)^2,
g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*g6^-1*g9^-1*g8^-1*g4^-1>;

N:=NormalClosure(G,H);
print Index(G,N);
6
\\
Q:=quo<N|g7*g1^-1*g3, g8*g7^2*g8^-1, g8^-1*g7^2*g8,
g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*g7*g8^-1*g5^-1,
g8*(g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*g7*g8^-1*g5^-1)*g8^-1,
g8^2*(g6*g1*g5*g7*g8^-1*g5^-1*g1^-1*g3*g6^-1*g7*g8^-1*g5^-1)*g8^-2,
g3*g6^-1*g7*g9^-1, g8*(g3*g6^-1*g7*g9^-1)*g8^-1,
g8^2*(g3*g6^-1*g7*g9^-1)*g8^-2,g7*g2,g8*(g7*g2)*g8^-1,g8^2*(g7*g2)*g8^-2,
g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1,g8*(g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1)*g8^-1,
g8^2*(g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1)*g8^-2,g4^-1*g7^-1*g5^-1*g2*g7*g1^-1,
g8*(g4^-1*g7^-1*g5^-1*g2*g7*g1^-1)*g8^-1,
g8^2*(g4^-1*g7^-1*g5^-1*g2*g7*g1^-1)*g8^-2,
g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*g6^-1*g9^-1*g8^-1*g4^-1,
g8*(g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*g6^-1*g9^-1*g8^-1*g4^-1)*g8^-1,
g8^2*(g6*g3^-1*g1*g5*g8*g7^-1*g5^-1*g1^-1*g6^-1*g9^-1*g8^-1*g4^-1)*g8^-2>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

3.2.2 Cases (d,I), N(I) = 3

Here the Bianchi groups involved are $d = 5, 15, 23, 47$. Since $N(I) = 3$, we have $\mathrm{PSL}(2, \mathrm{O_d/I}) \cong \mathrm{A}_4$.

In order to find a set of peripheral subgroups for $\Gamma(I)$, we use the following sequence of regular covers:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{4} \mathbf{H}^3/\Gamma_1 \xrightarrow{3} Q_d$$

where $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} < A_4$.

Since each peripheral subgroup P_i of $\mathrm{PSL}(2, \mathrm{O_d})$ maps to a subgroup of order 3 in $\mathrm{PSL}(2, \mathrm{O_d/I})$, it follows that each cusp of Q_d is covered by 4 cusps of $\mathbf{H}^3/\Gamma(I)$ each with covering degree 3. Also, each cusp of Q_d is covered by 1 cusp of $\mathbf{H}^3/\Gamma_1$ with covering degree 3. Thus a set of peripheral subgroups for $\mathbf{H}^3/\Gamma_1$ is given by $\{P_1(I), \ldots, P_{h_d}(I)\}$. Conjugating this set by four elements of Γ_1 that correspond to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ gives a set of peripheral subgroups for $\Gamma(I)$.

The case $(15, < 3, 1 + \omega_{15} >)$**:**

Let $I =< 3, 1 + \omega_{15} >$. We do this case in detail; the remaining 3 cases are done in the same way. Since Q_{15} has 2 cusps, $\mathbf{H}^3/\Gamma(I)$ has 8 cusps.

Reducing the matrix presentations of the P_i for Q_{15} given in §3.1 by the ideal $< 3, 1 + \omega_{15} >$, we obtain $P_1(I) =< t^3, tu >$, $P_2(I) =< (uca)^3, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. Setting $\Gamma_1 =< \Gamma(I), a, h >$, for $h = ata(at)^{-1}$, Magma checks that $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ so that conjugating $P_1(I)$ and $P_2(I)$ by $\{Id, a, h, ah\}$ gives a set of 8 peripheral subgroups for $\Gamma(I)$. The following set contains one element from each of these 8 peripheral subgroups:

$\{t^{-2}u, atua^{-1}, ht^{-2}uh^{-1}, (ha)tu(ha)^{-1}, (c^{-1}au^{-1}c^{-1}u^{-1}ta)(uca)^3,$
$a(c^{-1}au^{-1}c^{-1}u^{-1}ta)a^{-1}, h(c^{-1}au^{-1}c^{-1}u^{-1}ta)h^{-1}, ha(c^{-1}au^{-1}c^{-1}u^{-1}ta)(ha)^{-1}\}$

As above, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these 8 parabolics, and Magma calculates that $Q =< 1 >$; hence $\Gamma(I)$ is trivialized by setting these 8 elements equal to 1. Thus $\Gamma(< 3, 1 + \omega_{15} >)$ is an 8 component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1>;
H:=sub<G|t^3,t*u,(c^-1*a*u^-1*c^-1*u^-1*t*a),(u*c*a)^3>;
N:=NormalClosure(G,H);

print Index(G,N);
12
\\

h:=a*t*a*t^-1*a;
A:=sub<G|N,a,h>;
print Index(A,N);
4
\\

Q:=quo<N| t^-2*u, a*t*u*a,  h*t^-2*u*h^-1,  h*a*t*u*a*h^-1,
(c^-1*a*u^-1*c^-1*u^-1*t*a)*(u*c*a)^3,a*(c^-1*a*u^-1*c^-1*u^-1*t*a)*a,
h*(c^-1*a*u^-1*c^-1*u^-1*t*a)*h^-1,h*a*(c^-1*a*u^-1*c^-1*u^-1*t*a)*a*h^-1>;

print Order(Q);
```

```
1
\\
```

The case $(5, < 3, 1+\omega_5 >)$**:**

Let $I =< 3, 1+\omega_5 >$. The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is an 8 component link complement in S^3. Note that Q_5 has 2 cusps, the 2 peripheral subgroups $P_i(I)$ can be read off from H, and Q is $\Gamma(I)$ modulo the 8 parabolic elements to be trivialized.

```
G<a,b,c,t,u>:=Group<a,b,c,t,u|a^2,b^2,(t*a)^3,(a*b)^2,(a*u*b*u^-1)^2,
a*c*a*t*c^-1*t^-1,u*b*u^-1*c*b*t*c^-1*t^-1, (t,u)>;
H:=sub<G|t^3,t*u,(t*b)^3,(t*b)*(t*u^-1*c*t^-1)>;
N:=NormalClosure(G,H);

print Index(G,N);
12
\\
A:=sub<G|N,a,b>;
print Index(A,N);
4
\\

Q:=quo<N|t*u,a*t*u*a^-1,b*t*u*b^-1,a*b*t^3*b^-1*a^-1,
(t*b)*(t*u^-1*c*t^-1),a*(t*b)*(t*u^-1*c*t^-1)*a^-1,
b*(t*b)*(t*u^-1*c*t^-1)*b^-1,a*b*(t*b)^-2*(t*u^-1*c*t^-1)*b^-1*a^-1>;

print Order(Q);
1
\\
```

3.2.3 Cases (d,I), N(I) = 4

Here the Bianchi groups involved are $d = 15, 23, 31, 47$. In each case, $I = \mathcal{P}^2$, where $\mathrm{N}(\mathcal{P}) = 2$. Hence $[\mathrm{PSL}(2, \mathrm{O_d}) : \Gamma(\mathrm{I})] = 24$. Each cusp of Q_d is covered by 6 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 4. Since $\Gamma(\mathcal{P})/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $\Gamma(\mathcal{P}) =< \Gamma(I), \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} >$, we exploit the intermediate covers

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{2} \mathbf{H}^3/\Gamma_1 \xrightarrow{2} \mathbf{H}^3/\Gamma(\mathcal{P}),$$

where each covering has degree 2 and $\Gamma_1 =< \Gamma(I), \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} >$.

In the cases $d = 15, 23$, we show that $\mathbf{H}^3/\Gamma_1$ is a link complement in $\mathbf{RP}^3$, from which it follows that $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3. For the cases $d = 31, 47$ we show that $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3.

The case $(15, < 2, \omega_{15} >^2)$**:**

Here $\mathcal{P} = < 2, \omega_{15} >$ so that $I = \mathcal{P}^2 =< 4, \omega_{15} >$. Note also that $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = at^2a^{-1}$.

As shown in §3.2.1, $\mathbf{H}^3/\Gamma(\mathcal{P})$ has 6 cusps, each cusp of Q_{15} being covered by 3 cusps. Note that $\mathbf{H}^3/\Gamma(I)$ has 12 cusps and the intermediate cover $\mathbf{H}^3/\Gamma_1$ has 8 cusps.

In the following Magma routine, $P_1(I) =< t^4, u >$ and $P_2(I) =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^4 >$ are read off from H, $A = \Gamma_1$, and Q equals Γ_1 modulo 8 parabolics, one from each peripheral subgroup of Γ_1. Since Magma gives $Q \cong \mathbf{Z}/2\mathbf{Z}$, it follows that $\mathbf{H}^3/\Gamma_1$ is an 8 component link in $\mathbf{RP^3}$. Finally, since the 8 peripheral elements trivialized are also in $\Gamma(I)$, we have that $\mathbf{H}^3/\Gamma(I)$ is a 12 component link complement in S^3.

```
G<a,t,u,c>:=Group<a,t,u,c|a^2,(t*a)^3,
u*c*u*a*t*u^-1*c^-1*u^-1*a^-1*t^-1, (t,u),(a,c)>;

H:=sub<g|t^4,u,(c^-1*a*u^-1*c^-1*u^-1*t*a)^4,u*c*a>;
N:=NormalClosure(G,H);
print Index(G,N);
24
\\
A:=sub<G|N,a*t^2*a>;
print Index(G,K);
12
\\
Q:=quo<K|t^4,t*a*u*(t*a)^-1,(t*a)^-1*t^4*u*t*a,u*c*a,
(t*a)*u*c*a*(t*a)^-1,(t*a)^-1*u*c*a*(t*a),t^2*(t*a)*t^4*u*(t*a)^-1*t^-2,
t^2*u*c*a*t^-2>;

print Order(Q);
2
\\
```

The case $(47, < 4, 1 + \omega_{47} >)$**:**

Let $I =< 4, 1 + \omega_{47} >$. The following Magma routine that shows that $\mathbf{H}^3/\Gamma(I)$ is a 30 component link complement in S^3. Note that Q_{47} has 5 cusps, $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = g_1$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = g_2^2$, $\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} = h := g_1^{-1} g_2^2 g_1$. The peripheral subgroups $P_i(I)$ can be read off from H, and one obtains a set of peripheral subgroups for $\Gamma(I)$ by conjugating the $P_i(I)$ first by the elements $\{Id, g_1, g_1^2\}$ and then by those of $\{Id, g_2^2, h, g_2^2 h\}$. Here some care must be taken to remove 30 redundant groups, leaving a set of 30 peripheral subgroups for $\Gamma(I)$.

Q is the quotient of $\Gamma(I)$ by the normal closure of 30 parabolic elements, 1 from each of the peripheral subgroups. Magma calculates that $Q =< 1 >$, hence $\mathbf{H}^3/\Gamma(I)$ is a 30 component link complement in S^3.

```
G<g1,g2,g3,g4,g5,g6,g7>:=Group<g1,g2,g3,g4,g5,g6,g7|g1^3,(g3,g2),(g2^-1*g1)^2,
(g5,g7),g2^-1*g1*g6*g1^-1*g2*g6^-1,g6*g2^-1*g4^-1*g5*g3^-1*g6^-1*g4*g2*g3*g5^-1,
g7^-1*g2^-1*g5^-1*g4*g1*g4^-1*g2*g7*g4*g1^-1*g4^-1*g5,
g3*g5^-1*g4*g1*g4^-1*g2*g5*g3^-1*g2^-1*g4^-1*g1^-1*g4>;

H:=sub<G|g2^4,g2*g3,g5^4,g5*g7,g2*g7,(g4*g1^-1*g4^-1*g5)^4,(g6*g2^-1*g4^-1)^2,
(g6*g2^-1*g4^-1)*(g5*g3^-1*g2^-1*g4^-1)^2,(g3*g5^-1*g4*g1)^2,
(g6^-1*g1^-1*g4)^2*(g3*g5^-1*g4*g1)>;
N:=NormalClosure(G,H);
```

```
print Index(G,N);
24
\\
h:=g1^-1*g2^2*g1;

Q:=quo<N|g2^-3*g3, h*g2^-3*g3*h^-1,g1*(g2*g3)*g1^-1, h*g1*(g2^-3*g3)*g1^-1*h^-1,
g1^-1*(g2^-3*g3)*g1, g2^2*g1^-1*(g2*g3)*g1*g2^-2,g5*g7,h*g5*g7*h^-1,
g1*(g5*g7)*g1^-1,h*g1*(g5*g7)*g1^-1*h^-1, g1^-1*(g5*g7)*g1,
g2^2*g1^-1*(g5*g7)*g1*g2^-2, g2*g7,h*g2*g7*h^-1,g1*(g2*g7)*g1^-1,
g2^2*g1*(g2*g7)*g1^-1*g2^-2, g1^-1*(g2*g7)*g1, h*g1^-1*(g2*g7)*g1*h^-1,
(g6*g2^-1*g4^-1)*(g5*g3^-1*g2^-1*g4^-1)^2,h*(g6*g2^-1*g4^-1)^2*h^-1,
g1*(g6*g2^-1*g4^-1)^-1*(g5*g3^-1*g2^-1*g4^-1)^2*g1^-1,
g2^2*g1*(g6*g2^-1*g4^-1)*(g5*g3^-1*g2^-1*g4^-1)^2*g1^-1*g2^-2,
g1^-1*(g6*g2^-1*g4^-1)^-1*(g5*g3^-1*g2^-1*g4^-1)^2*g1,
h*g1^-1*(g6*g2^-1*g4^-1)^-1*(g5*g3^-1*g2^-1*g4^-1)^2*g1*h^-1,
(g6^-1*g1^-1*g4)^2*(g3*g5^-1*g4*g1),g2^2*(g3*g5^-1*g4*g1)^2*g2^-2,
g1*(g3*g5^-1*g4*g1)^2*g1^-1, h*g1*(g3*g5^-1*g4*g1)^2*g1^-1*h^-1,
g1^-1*(g6^-1*g1^-1*g4)^2*(g3*g5^-1*g4*g1)^-1*g1,h*g1^-1*(g3*g5^-1*g4*g1)^2*g1*h^-1>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

3.2.4 Cases (d,I), N(I) = 5

Here the Bianchi groups involved are $d = 15, 31$. Since $N(I) = 5$, we have $\mathrm{PSL}(2, \mathrm{O_d/I}) \cong \mathrm{A}_5$. Since each P_i maps to a subgroup of order 5 in $\mathrm{PSL}(2, \mathrm{O_d/I})$, it follows that each cusp of Q_d is covered by 12 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 5. We use the following sequence of covers of Q_d to find a set of peripheral subgroups for $\Gamma(I)$:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{4} \mathbf{H}^3/\Gamma_1 \xrightarrow{3} \mathbf{H}^3/\Gamma_2 \xrightarrow{5} Q_d$$

where $\Gamma_1/\Gamma(I) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $\Gamma_2/\Gamma(I) \cong A_4$. It follows that $\mathbf{H}^3/\Gamma_2$ has h_d cusps, $\mathbf{H}^3/\Gamma_1$ has $3h_d$ cusps and $\mathbf{H}^3/\Gamma(I)$ has $12h_d$ cusps.

If $\Gamma_1 =< \Gamma(I), a, b >$ and $\Gamma_2 =< \Gamma_1, r >$, we obtain a set of peripheral subgroups for $\mathbf{H}^3/\Gamma(I)$ by conjugating $\{P_1(I), \ldots, P_{h_d}(I)\}$ first by the elements $\{Id, r, r^2\}$, and then further conjugating by $\{Id, a, b, ab\}$.

The case $(15, < 5, 2 + \omega_{15} >)$**:**

The following Magma routine shows that $\mathbf{H}^3/\Gamma(I)$ is a 24 component link complement in S^3. Note that $\mathbf{H}^3/\Gamma(I)$ has 24 cusps. Reducing the matrix presentations for P_1 and P_2 modulo I gives $P_1(I) =< t^5, t^2u >$ and $P_2 =< (uca)^5, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. The group Γ_1 (resp. Γ_2) is given by A (resp. B). A set of peripheral subgroups for $\Gamma(I)$ is calculated as described above. Magma then calculates that quotient of $\Gamma(I)$ by the normal closure of the 24 peripheral elements given below trivializes the group.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1>;

H:=sub<G|t^5,t^2*u,(c^-1*a*u^-1*c^-1*u^-1*t*a),(u*c*a)^5>;
```

```
N:=NormalClosure(G,H);
print Index(G,N);
60
\\
b:=(t*a*u)*a*(t*a*u)^-1;
r:=(t*c)*(t*a)*(t*c)^-1;

A:=sub<G|N,a,b>;
B:=sub<G|N,a,b,r>;
print Index(A,N);
4
\\
print Index(B,N);
12
\\
Q:=quo<N|t^-3*u, a*t^2*u*a, b*t^2*u*b, a*b*t^2*u*b*a,
r*t^2*u*r^-1, a*r*t^2*u*r^-1*a, b*r*t^2*u*r^-1*b, a*b*r*t^2*u*r^-1*b*a,
r^-1*t^-3*u*r, a*r^-1*t^-3*u*r*a, b*r^-1*t^-3*u*r*b, a*b*r^-1*t^-3*u*r*b*a,
(u*c*a)^-5, a*(c^-1*a*u^-1*c^-1*u^-1*t*a)*a*a*(u*c*a)^5*a,
b*(c^-1*a*u^-1*c^-1*u^-1*t*a)*(u*c*a)^5*b,
a*b*(c^-1*a*u^-1*c^-1*u^-1*t*a)*(u*c*a)^5*b*a,
r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r^-1, a*r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r^-1*a,
b*r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r^-1*b, a*b*r*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r^-1*b*a,
r^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r, a*r^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r*a,
b*r^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r*b, a*b*r^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)*r*b*a>;

Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

The case $(31, <5, 1+\omega_{31}>)$**:**

Let $I=<5,1+\omega_{31}>$. Q_{31} has 3 cusps, hence $\mathbf{H}^3/\Gamma(I)$ has 36 cusps, and the following Magma routine shows that it is a 36 component link complement in S^3. Note that $g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $g_2 g_1^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\Gamma_1 = A$, and $\Gamma_2 = B$.

```
G<g1,g2,g3,g4,g5>:=Group<g1,g2,g3,g4,g5|(g1,g3),(g2)^3,(g2*g1^-1)^2,
(g5,g4),g4*g1^-1*g3^-1*g2*g3*g4^-1*g2*g4*g3^-1*g1^-1*g2*g3*g4^-1*g2,
g5*g3^-1*g2*g3*g4^-1*g2*g1^-1*g5^-1*g2^-1*g4*g3^-1*g2^-1*g3*g1,
g2*g3*g4^-1*g2*g1^-1*g4*g3^-1*g2*g3*g4^-1*g1*g2^-1*g4*g3^-1>;

H:=sub<G|g1^5,g1^-1*g3, g4^5,g4*g5, (g1*g5)^5,(g1*g5)*(g3^-1*g2*g3*g4^-1*g2*g5)>;
N:=NormalClosure(G,H);
print Index(G,N);
60
\\
a:=g1*(g2*g1^-1)*g1^-1;
```

```
b:=g2*g5*(g2*g1^-1)*g5^-1*g2^-1;
A:=sub<G|N,a,b>;
B:=sub<G|N,a,b,g2>;
print Index(A,N);
4
\\
print Index(B,N);
12
\\

Q:=quo<N|g1^3*g3^2, b*g1^4*g3*b, a*(g1^-1*g3)*a, b*a*(g1^-1*g3)*a*b,
g2*(g1^-1*g3)*g2^-1,b*g2*(g1^-1*g3)*g2^-1*b,
 a*g2*(g1^-1*g3)*g2^-1*a,b*a*g2*(g1^4*g3)*g2^-1*a*b, g2^-1*(g1^-1*g3)*g2,
b*g2^-1*(g1^-1*g3)*g2*b, a*g2^-1*(g1^-1*g3)*g2*a,b*a*g2^-1*(g1^-1*g3)*g2*a*b,g4*g5,
b*g4*g5*b, a*g4*g5*a,b*a*(g4*g5)*a*b,g2*(g4*g5)*g2^-1,b*g2*(g4*g5)*g2^-1*b,
a*g2*(g4*g5)*g2^-1*a,b*a*g2*(g4*g5)*g2^-1*a*b, g2^-1*(g4*g5)*g2,
b*g2^-1*(g4*g5)*g2*b, a*g2^-1*(g4*g5)*g2*a,b*a*g2^-1*(g4*g5)*g2*a*b,
(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2,b*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*b,
a*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*a,b*a*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*a*b,
g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1,
b*g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1*b,
a*g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1*a,
b*a*g2*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2^-1*a*b,
g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2,
b*g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2*b,
a*g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2*a,
b*a*g2^-1*(g1*g5)^-3*(g3^-1*g2*g3*g4^-1*g2*g5)^2*g2*a*b>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

3.2.5 Case (d,I), N(I) = 6

Here the Bianchi group is $d = 15$ and $I =< 6, -2 + \omega_{15} >=< 2, \omega_{15} >< 3, 1 + \omega_{15} >$. Note that $N(I) = 6$ and so using §2.1 we deduce that $[\mathrm{PSL}(2, \mathrm{O}_{15}) : \Gamma(\mathrm{I})] = 72$. Indeed, in this case we have $\mathrm{PSL}(2, \mathrm{O}_{15}/\mathrm{I}) \cong \mathrm{PSL}(2, \mathbf{Z}/2\mathbf{Z}) \times \mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z}) \cong \mathrm{S}_3 \times \mathrm{A}_4$. Now P_1 and P_2 both have order 6 in $\mathrm{PSL}(2, \mathrm{O}_{15}/\mathrm{I})$, so that each cusp of Q_{15} is covered by 12 cusps of $\mathbf{H}^3/\Gamma(I)$, each with covering degree 6. We will show that $\mathbf{H}^3/\Gamma(I)$ is a 24 component link group.

We use the following sequence of covers of Q_{15}:

$$\mathbf{H}^3/\Gamma(I) \xrightarrow{12} \mathbf{H}^3/\Gamma(\mathcal{P}) \xrightarrow{6} Q_{15}$$

where $\mathcal{P} =< 2, \omega_{15} >$. Since $\Gamma(\mathcal{P})/\Gamma(I) \cong A_4$, and $\Gamma(\mathcal{P}) =< \Gamma(I), (ta), r, s >$, we obtain a set of peripheral subgroups for $\Gamma(I)$ by first conjugating $\{P_1(I), P_2(I)\}$ by the elements $\{Id, ta, (ta)^2\}$ and then further conjugating by the elements $\{Id, r, s, rs\}$. In the following Magma routine, $A = \Gamma(\mathcal{P})$, $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = ta$, $r = t^2at^{-2}a$, and $s = at^{-2}at^2$.

```
G<a,c,t,u>:=Group<a,c,t,u|(t,u),(a,c),a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1>;
```

```
H:=sub<G|t^6,t^-2*u,(c^-1*a*u^-1*c^-1*u^-1*t*a)^2, (u*c*a)^3>;
N:=NormalClosure(G,H);
print Index(G,N);
72
\\
r:=(t^2*a*t^-2*a);
s:=(a*t^-2*a*t^2);
A:=sub<G|N,at,r,s);
print Index(A,N);
12
\\
Q:=quo<N|t^-2*u,
r*t^-2*u*r^-1,
s*t^-2*u*s^-1,
r*s*t^-2*u*s^-1*r^-1,

(t*a)*t^4*u*(t*a)^-1,
r*(t*a)*t^-2*u*(t*a)^-1*r^-1,
s*(t*a)*t^-2*u*(t*a)^-1*s^-1,
r*s*(t*a)*t^-2*u*(t*a)^-1*s^-1*r^-1,

(t*a)^-1*t^-2*u*(t*a),
r*(t*a)^-1*t^-2*u*(t*a)*r^-1,
s*(t*a)^-1*t^-2*u*(t*a)*s^-1,
r*s*(t*a)^-1*t^-2*u*(t*a)*s^-1*r^-1,

(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3,
r*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*r^-1,
s*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*s^-1,
r*s*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*s^-1*r^-1,

(t*a)*(u*c*a)^3*(t*a)^-1,
r*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)^-1*r^-1,
s*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)^-1*s^-1,
r*s*(t*a)*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)^-1*s^-1*r^-1,

(t*a)^-1*(u*c*a)^3*(t*a),
r*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)*r^-1,
s*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)*s^-1,
r*s*(t*a)^-1*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(u*c*a)^3*(t*a)*s^-1*r^-1>;

Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

The remaining cases of Theorem 1.1 are handled in similar fashion. Magma routines are available from the authors upon request.

4 Proof of Theorem 1.1: Excluding the remaining levels

In this section we eliminate (for the 8 values of d in §2.2) the groups $\Gamma(I)$ that don't appear in Theorem 1.1 using several techniques. In §4.1 we establish a bound on the norm of I for which $\Gamma(I)$ can be a link complement. In §4.2 we use results on cuspidal cohomology to quickly eliminate certain $\Gamma(I)$. Finally, in §4.3 we follow the procedure described in Step 1 of §2.5 using Magma to show that the remaining $\Gamma(I)$ aren't generated by parabolics hence aren't link groups.

4.1

First, we remind the reader that there are only finitely many levels that can give principal congruence link complements in S^3 (see [6] Proposition 2.3 for example). We recall a proof of that here, and exhibit an explicit bound to the norm of the ideal.

From [1] we know that if $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3, its systole (ie the length of a shortest closed geodesic) is at most $7.35534\ldots$. Note that the argument in [1] used the 2π-Theorem of Gromov and Thurston, but using the 6-Theorem of Agol [2] and Lackenby [11] the argument of [1] can be redone to improve this systole bound to $7.171646\ldots$ (see also [14]).

Lemma 4.1. *Suppose that $I \subset O_d$ is an ideal such that $\mathbf{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3. Then $N(I) < 39$.*

Proof: If $\gamma \in \Gamma(I)$ is a hyperbolic element, its complex length is $\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma)$, where $\ell_0(\gamma)$ is the translation length of γ and $\theta(\gamma)$ is the angle incurred in translating along the axis of γ by distance $\ell_0(\gamma)$. Now, as is well-known $\cosh(\ell(\gamma)/2) = \pm\mathrm{tr}(\gamma)/2$, and so we get the following inequality for $\ell_0(\gamma)$: $|\mathrm{tr}(\gamma)|/2 \le \cosh(\ell_0(\gamma)/2)$. With the systole bound given above it follows that

$$|\mathrm{tr}(\gamma)/2| \le \cosh(7.1717/2) \le 18.1 \text{ and so } |\mathrm{tr}(\gamma)| < 37.$$

From Lemma 2.5 of [6] we have that if $\gamma \in \Gamma(I)$ is a hyperbolic element, then $\mathrm{tr}\ \gamma = \pm 2 \bmod \mathrm{I}^2$. Hence $\mathrm{tr}(\gamma) \pm 2 \in I^2$, and this together with the bound on $|\mathrm{tr}(\gamma)|$ quickly gives $N(I) < 39$. $\square$

Remark: The proof of Lemma 4.1 actually shows more: if $\Gamma(I)$ is a link group, then there exists $x \in I$ such that $|x|^2 < 39$. Since $h_d > 1$, there are ideals I of norm less than 39 for which no such element exists and hence $\Gamma(I)$ is not a link group. In particular, this eliminates $\Gamma(I)$ for the levels $I =< 23, 8 + \omega_5 >$, $I =< 29, 9 + \omega_6 >$ and $I =< 13, -1 + 2\omega_{39} >$.

4.2

Given Lemma 4.1, and the above Remark, we further reduce the number of groups $\Gamma(I)$ of $N(I) < 39$ that can be link groups. The case of rational integer level can be dealt with by the following result of the first author [3].

Theorem 4.2. *If $h_d > 1$, and $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$, then $\mathbf{H}^3/\Gamma(n)$ is not homeomorphic to a link complement in S^3.*

Indeed, for certain d, it will be useful in what follows to note a stronger version of Theorem 4.2 when the primes have small norm. To state this, we recall the definition of *degree 1 cuspidal cohomology* in a form that is useful to us. Suppose that $X = \mathbf{H}^3/\Gamma$ is an orientable, non-compact, finite volume hyperbolic 3-orbifold, and U_Γ the normal subgroup of Γ generated by the parabolic elements of Γ. Then the subspace of $H_1(X, \mathbf{Q})$ which defines the degree 1 cuspidal cohomology of X (or Γ) can be identified with:

$$V_X(\text{ or } V_\Gamma) = (\Gamma/U_\Gamma)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Theorem 4.3. *If $h_d > 1$, then $\Gamma(n) < \mathrm{PSL(2, O_d)}$, has non-trivial degree 1 cuspidal cohomology in the following cases (using the notation introduced earlier to indicate the level and d):*

$$(23,3),(23,5),(31,2),(47,2),(47,3),(71,2),(71,3).$$

Proof: To prove this we use Zimmert sets as in the proof of Proposition 4.6 of [6] (following [10]). As in [6] for the cases stated in the theorem, the Zimmert sets all have at least 2 elements, and this allows one to conclude the existence of non-trivial cuspidal cohomology. □

Note that as a corollary to this we have the following.

Corollary 4.4. *Let $d \in \{23, 31, 47, 71\}$, $I \subset O_d$ an ideal and $p = 2, 3, 5$. Suppose that (d, p) is as in Theorem 4.3 and I is divisble by $< p >$. Then $\Gamma(I)$ has non-trivial degree 1 cuspidal cohomology. In particular $\mathbf{H}^3/\Gamma(I)$ is not homeomorphic to a link complement in S^3.*

4.3

We now use Magma to deal with non-integral ideals of norm less than 39 for which Corollary 4.4 does not apply.

Recall from §2.4 that $\mathbf{H}^3/\Gamma(I)$ a link complement implies that $\Gamma(I)$ is generated by parabolics. Using the presentations for the the Bianchi groups in §2.2, the peripheral subgroups P_i of $\mathrm{PSL(2, O_d)}$ in §2.4, and the matrix representatives in the Appendix, we identify the peripheral subgroups $P_i(I) = P_i \cap \Gamma(I)$ for $i = 1, \dots, h_d$. As in §2.5, $N_d(I)$ denotes the normal closure in $\mathrm{PSL(2, O_d)}$ of $< P_1(I), \dots, \mathrm{P}_{h_d}(I) >$.

Consider the quotient group $B_d(I) = \mathrm{PSL(2, O_d)/N_d(I)}$. If $\Gamma(I)$ is a link group then $B_d(I)$ is a finite group with order equal to $|\mathrm{PSL(2, O_d/I)}|$. Hence if $B_d(I)$ is infinite or has order greater than $|\mathrm{PSL(2, O_d/I)}|$, then $\Gamma(I)$ cannot be a link group. Since the groups $N_d(I)$ are often not of finite index in $\mathrm{PSL(2, O_d)}$, we calculate $B_d(I)$ in the Magma routines below as:

$$B_d(I) =< \mathrm{PSL(2, O_d)|P_1(I) = \ldots = P_{h_d}(I) = 1} >$$

that is by adding the peripheral subgroups $P_i(I)$ to the relations of $\mathrm{PSL(2, O_d)}$. We distinguish two cases:

Case 1: *$B_d(I)$ is a finite group but has order larger than $|\mathrm{PSL(O_d/I)}|$.*

Case 2: *$B_d(I)$ has a finite index subgroup with infinite abelianization.*

Remark: Note that if $B_d(I)$ is infinite or larger than $|\mathrm{PSL(2, O_d/J)}|$ for an ideal $J \subset I$, then so is $B_d(J)$ and hence $\Gamma(J)$ is not a link group.

We single out the case of $d = 6$ and discuss it in some detail in §4.3.1. The remaining values of d are discussed in the following subsections. All Magma routines not included are available on request.

4.3.1 d = 6

We begin with some comments about the behavior of rational primes of norm < 39 in the extension $\mathbf{Q}(\sqrt{-6})/\mathbf{Q}$. The primes 2 and 3 are the only ramified primes; 5, 7, 11, 29, and 31 split; and 13, 17, 19, 23 and 37 are inert and so have norm exceeding 39. Hence we exclude these and any ideal that they divide from further consideration. Recall also that $I =< 29, 9 + \omega_6 >$ is excluded by the remark following Lemma 4.1.

The following table gives the ideals in O_6 whose norms are < 39, which were ruled out using Magma. We give the order of $B_d(I)$ as ∞ if it contains a finite index subgroup with infinite abelianization, and as $\gg 1$ if its order is $\geq 10^6$. We also include peripheral subgroups $P_1(I)$ and $P_2(I)$.

The ideals of norm 8, 9, 12, 14, 15, 16, 18, 20, 21, 24, 27, 28, 30, 32, 33, 35, and 36 are sub-ideals of those in the table and are eliminated by the above Remark.

Ideal	**Norm**	**Peripheral Subgroups**	**Order($\mathbf{B_d(I)}$)**
$< 2, \omega_6 >$	2	$< t^2, u >, < (tb)^2, cu >$	24
$< 3, \omega_6 >$	3	$< t^3, u >, < (tb)^3, cu >$	∞
$< 2 >$	4	$< t^2, u^2 >, < (tb)^2, (cu)^2 >$	∞
$< 5, 2+\omega_6 >$	5	$< t^5, t^2u >, < (tb)^5, (tb)^{-2}cu >$	$\gg 1$
$< 7, 1+\omega_6 >$	7	$< t^7, tu >, < (tb)^7, tb(cu)^{-1} >$	∞
$< 11, 4+\omega_6 >$	11	$< t^{11}, t^4u >, < (tb)^{11}, (tb)^4(cu)^{-1} >$	$\gg 1$
$< 31, 5+\omega_6 >$	31	$< t^{31}, t^5u >, < (tb)^{31}, (tb)^5(cu)^{-1} >$	$\gg 1$

Magma routine for $I =< 2, \omega_6 >$:

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 6.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
t^2,u,(t*b)^2,c*u>;

print Order(B);
24
\\
```

Magma routine for $I =< 3, \omega_6 >$:

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 12.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
t^3,u,(t*b)^3,c*u>;

L:=LowIndexNormalSubgroups(G,48);
print #L;
8
\\
print AbelianQuotientInvariants(L[8]`Group);
[0,0,0]
\\
```

Magma routine for $I =< 2 >$:

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 48.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
```

```
t^2,u^2,(t*b)^2,(c*u)^2>;

L:=LowIndexNormalSubgroups(B,48);
print #L;
98
\\
print AbelianQuotientInvariants(L[20]‘Group);
[ 0 ]
\\
```

Magma routine for $I =< 5, 2 + \omega_6 >$**:**

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 60.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
t^5,t^2*u,(t*b)^5,(t*b)^-2*(c*u)>;

print Order(B);
1966080
\\
```

Magma routine for $I =< 11, 4 + \omega_6 >$**:**

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 660.

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
t^11,t^4*u,(t*b)^11,(t*b)^4*(c*u)^-1>;

L:=LowIndexNormalSubgroups(B,660);
print #L;
2
\\
print Index(B,L[2]‘Group);
660
\\
print AbelianQuotientInvariants(L[2]‘Group);
[ 4, 120, 120, 120, 120, 120, 120, 120, 120, 120, 120 ]
\\
```

Magma routine for $I =< 31, 5 + \omega_6 >$**:**

Note that from §2.1, the order of $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ is 14880.

```
G<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t,u),(t*a)^3,(a,c),t^-1*c*t*u*b*u^-1
*c^-1*b^-1,(a*t*b)^3,(a*t*u*b*u^-1)^3,
t^31,t^5*u,(t*b)^31,(t*b)^5*(c*u)^-1>;
```

```
L:=LowIndexNormalSubgroups(G,14880);
print #L;
2
\\
print Index(G,L[2]‘Group);
14880
\\
M:=sub<G|(L[2]‘Group),t>;
print Index(M,(L[2]‘Group));
31
\\
print AbelianQuotientInvariants(M);
[ 3, 3, 3, 3, 3, 3, 68433864, 1847714328 ]
```

In the tables below we summarize our Magma calculations that treat all of the remaining cases to be eliminated. We do not list those ideals $J \subset I$ that are eliminated directly by the size of $B_d(I)$ (as per the Remark in §4.3).

4.3.2 d = 5

Note that the group $\Gamma(<2, 1+\omega_5>)$ contains the element $\begin{pmatrix} \omega_5 & 2 \\ 2 & -\omega_5 \end{pmatrix}$ which has order 2.

Ideal	**Norm**	**Peripheral Subgroups**	$\mathbf{Order(B_d(I))}$
$<2>$	4	$<t^2, u^2>, <(tb)^2, (tu^{-1}ct^{-1})^2>$	∞
$<\omega_5>$	5	$<t^5, u>, <(tb)^5, tu^{-1}ct^{-1}>$	∞
$<6, 1+\omega_5>$	6	$<t^6u, tu>, <(tb)^6, tb(tu^{-1}ct^{-1})>$	144
$<7, 3+\omega_5>$	7	$<t^7, t^3u>, <(tb)^7, (tb)^{-4}(tu^{-1}ct^{-1})>$	$\geq 168 \times 3^6$
$<3, 1+\omega_5>^2$	9	$<t^9, t^{-2}u>, <(tb)^9, (tb)^{-2}(tu^{-1}ct^{-1})>$	$\geq 324 \times 2^{12}$
$<3>$	9	$<t^3, u^3>, <(tb)^3, (tu^{-1}ct^{-1})^3>$	∞
$<29, -13+\omega_5>$	29	$<t^{29}, t^{-13}u>, <(tb)^{29}, (tb)^{-13}(tu^{-1}ct^{-1})>$	∞

We make one comment on $I = <29, -13+\omega_5>$. In this case we found, as for $I = <31, 5+\omega_6>$, an intermediate subgroup with infinite abelianization as described below.

```
B<a,b,c,t,u>:=Group<a,b,c,t,u|(t,u),a^2,b^2,(t*a)^3,(a*b)^2,
(a*u*b*u^-1)^2,a*c*a*t*c^-1*t^-1,u*b*u^-1*c*b*t*c^-1*t^-1,
t^29,t^-13*u,(t*b)^29,(t*b)^-13*(t*u^-1*c*t^-1)>;

L:=LowIndexNormalSubgroups(B,12180);
print #L;
2
\\
print Index(B,L[2]‘Group);
12180
\\
M:=sub<B|L[2]‘Group,t>;
print Index(B,M);
420
\\
```

```
AbelianQuotientInvariants(M);
[ 2, 2, 58, 58, 116, 116, 0, 0 ]
\\
```

4.3.3 d = 15

For convenience set $x = c^{-1}au^{-1}c^{-1}u^{-1}ta$ be the parabolic element commuting with uca.

Ideal	**Norm**	**Peripheral Subgroups**	**Order($\mathbf{B_d(I)}$)**
$< 2, \omega_{15} >^3$	8	$< t^8, t^{-4}u >, < (uca)x^4, (uca)^2 >$	192×3^5
$< 3 >$	9	$< t^3, u^3 >, < (uca)^3, x^3 >$	∞
$< 10, 2 + \omega_{15} >$	10	$< t^{10}, t^2u >, < (uca)^5, x^2 >$	∞
$< 12, 4 + \omega_{15} >$	12	$< t^{12}, t^4u >, < (uca)^3, x^4 >$	∞
$< 15, -8 + \omega_{15} >$	15	$< t^{15}, t^7u >, < (uca)^{15}, x >$	∞
$< 17, 5 + \omega_{15} >$	17	$< t^{17}, t^5u >, < (uca)^{17}, (uca)^{-8}x >$	∞
$< 19, 10 + \omega_{15} >$	19	$< t^{19}, t^{10}u >, < (uca)^{19}, (uca)^4x >$	$\gg 1$
$< 31, -14 + \omega_{15} >$	31	$< t^{31}, t^{-14}u >, < (uca)^{31}, (uca)^{13}x >$	∞

4.3.4 d = 23

For convenience set $p_1 = g_4g_3g_2$ and $p_2 = g_2^{-1}g_5g_3g_2$ be the parabolic elements generating the peripheral subgroup of $\mathrm{PSL}(2, \mathrm{O}_{23})$ fixing $\frac{-1-\sqrt{-23}}{4}$.

Ideal	**Norm**	**Peripheral Subgroups**	**Order($\mathbf{B_d(I)}$)**
$< 2 >$	4	$< g_1^2, g_2^2 >, < g_4^2, g_5^2 >, < p_1^2, p_2^2 >$	∞
$< 6, \omega_{23} >$	6	$< g_1^6, g_1g_2 >, < g_4^3, g_4^{-1}g_5^2 >, < p_1^6, p_1^{-3}p_2 >$	∞
$< 2, \omega_{23} >^3$	8	$< g_1^8, g_1^{-1}g_2 >, < g_4^4, g_4g_5^2 >, < p_1^2p_2^{-2}, p_1^3p_2 >$	$\geq 192 \times 3^6$
$< 3, \omega_{23} >^2$	9	$< g_2^9, g_2^4g_1 >, < g_4^9, g_4^{-2}g_5 >, < p_1^9, p_1^6p_2 >$	$\geq 324 \times 2^8$
$< 13, 4 + \omega_{23} >$	13	$< g_2^{13}, g_2^5g_1 >, < g_4^{13}, g_4^{-4}g_5 >, < p_1^{13}, p_1^{-2}p_2 >$	$\gg 1$
$< 23, \omega_{23} >$	23	$< g_2^{23}, g_2^{-11}g_1 >, < g_4^{23}, g_4^{-6}g_5 >, < p_1^{23}, p_1^{-6}p_2 >$	∞

4.3.5 d = 31

For convenience set $p = g_3^{-1}g_2g_3g_4^{-1}g_2g_5$ be the parabolic element commuting with g_1g_5. These elements generate the peripheral subgroup of $\mathrm{PSL}(2, \mathrm{O}_{31})$ fixing $\frac{-1-\sqrt{-31}}{4}$.

Ideal	**Norm**	**Peripheral Subgroups**	**Order($\mathbf{B_d(I)}$)**
$< 7, 2 + \omega_{31} >$	7	$< g_1^7, g_1^{-2}g_3 >, < g_4^7, g_4^{-2}g_5 >, < (g_1g_5)^7, p >$	∞
$< 2, \omega_{31} >^3$	8	$< g_1^8, g_3 >, < g_4^4, g_4g_5^2 >, < (g_1g_5)^2, (g_1g_5)^{-1}p^4 >$	∞
$< 3 >$	9	$< g_1^3, g_3^3 >, < g_4^3, g_5^3 >, < (g_1g_5)^3, p^3 >$	∞
$< 10, 1 + \omega_{31} >$	10	$< g_1^{10}, g_1^{-1}g_3 >, < g_4^{10}, g_4^{-4}g_5 >, < (g_1g_5)^{10}, (g_1g_5)p >$	$\gg 1$
$< 19, 5 + \omega_{31} >$	19	$< g_1^{19}, g_1^5g_3 >, < g_4^{19}, g_4^3g_5 >, < (g_1g_5)^{19}, (g_1g_5)^7p >$	$\gg 1$
$< -1 + 2\omega_{31} >$	31	$< g_1^{31}, g_1^{16}g_3 >, < g_4^{31}, g_4^8g_5 >, < (g_1g_5)^{31}, (g_1g_5)^5p^{-1} >$	∞

4.3.6 d = 39

Ideal	Norm	Peripheral Subgroups	Order $(B_d(I))$
$< 2, \omega_{39} >$	2	$< g_1^2, g_2 >, < g_4^2, g_6 >, < (g_5^{-1} g_6)^2, g_4 g_1^{-1} g_6 >, < g_5^2, g_5^{-1}(g_4 g_2^{-1} g_7) >$	18
$< 3, 1+\omega_{39} >$	3	$< g_1^3, g_1 g_2 >, < g_4, g_6^3 >, < (g_5^{-1} g_6)^3, (g_5^{-1} g_6)^2 (g_4 g_1^{-1} g_6) >, < g_5^3, g_5^{-1}(g_4 g_2^{-1} g_7) >$	∞
$< 2, \omega_{39} >^2$	4	$< g_1^4, g_1^{-1} g_2 >, < g_4^4, g_6 >, < (g_5^{-1} g_6)^4, g_4 g_1 g_6 >, < g_5^4, g_5^{-1}(g_4 g_2^{-1} g_7) >$	72
$< 2 >$	4	$< g_1^2, g_2^2 >, < g_4^2, g_6^2 >, < (g_5^{-1} g_6)^2, (g_4 g_1^{-1} g_6)^2 >, < g_5^2, (g_4 g_2^{-1} g_7)^2 >$	∞
$< 5, \omega_{39} >$	5	$< g_1^5, g_2 >, < g_4^5, g_4^{-1} g_6 >, < (g_4 g_1^{-1} g_6)^5, (g_4 g_1^{-1} g_6)^{-2} (g_5^{-1} g_6 >, < g_5^5, g_5 (g_4 g_2^{-1} g_7) >$	∞
$< 2, \omega_{39} >^3$	8	$< g_1^8, g_1^2 g_2 >, < g_4^4, g_6^2 >, < (g_4 g_1^{-1} g_6)^2, (g_5^{-1} g_6)^4 (g_4 g_1^{-1} g_6) >, < g_5^4, g_5^{-2} (g_4 g_2^{-2} g_7)^2 >$	∞
$< 3 >$	9	$< g_1^3, g_2^3 >, < g_4^3, g_6^3 >, < (g_5^{-1} g_6)^3, (g_4 g_1^{-1} g_6)^3 >, < g_5^3, (g_4 g_2^{-1} g_7)^3 >$	∞
$< 11, 3+\omega_{39} >$	11	$< g_1^{11}, g_1^3 g_2 >, < g_4^{11}, g_4^{-2} g_6 >, < (g_5^{-1} g_6)^{11}, (g_5^{-1} g_6)^3 (g_4 g_1^{-1} g_6) >, < g_5^{11}, g_5^4 (g_4 g_2^{-1} g_7) >$	$\gg 1$

4.3.7 d = 47 and d = 71

For the remaining two Bianchi groups, we simply comment on a few cases which together with Theorems 4.2, 4.3 and Corollary 4.4 eliminate all other possible levels.

When $d = 47$, $B_{47}(I)$ is infinite (resp. $\gg 1$) for $I =< 6, \omega_{47} >$ (resp. for $I =< 7, 1+\omega_{47} >$, $< 8, 4+\omega_{47} >$ and $< 9, 2+\omega_{47} >$).

When $d = 71$, $B_{71}(I)$ is infinite for $I =< 3, \omega_{71} >$, (resp. $\geq 24 \times (33)^3$ for $I =< 4, -2+\omega_{71} >$), (resp. $\gg 1$ for $I =< 5, 1+\omega_{71} >$).

5 Congruence links when $d = 6$ and $d = 39$

In this section we exhibit congruence link groups for $d = 6$ and $d = 39$ which proves Theorem 1.3. We begin with some notation.

For an ideal $I \subset O_d$, let $\Gamma_0(I)$ (resp. $\Gamma_1(I)$) denote the congruence subgroup obtained as the preimage of the subgroup $\{\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} : a, x \in O_d/I\}/\{\pm Id\}$ (resp. $\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in O_d/I\}/\{\pm Id\}$) under the reduction homomorphism $\mathrm{PSL}(2, \mathrm{O_d}) \to \mathrm{PSL}(2, \mathrm{O_d/I})$.

Proposition 5.1. *$\Gamma_1(< 7, 1+\omega_6 >)$ and $\Gamma_1(< 5, \omega_{39} >)$ are congruence link subgroups of $\mathrm{PSL}(2, \mathrm{O_6})$ and $\mathrm{PSL}(2, \mathrm{O_{39}})$ respectively.*

Note that $\Gamma_1(I)$ is indeed congruence since $\Gamma(I) < \Gamma_1(I)$. Also, as long as $N(I) \geq 5$, the group $\Gamma_1(I)$ is torsion-free.

Moreover, if $I = \mathcal{P}$ is a prime ideal of O_d lying over a split prime $p \in \mathbf{Z}$ (as is the case in Proposition 5.1), then $[\mathrm{PSL}(2, \mathrm{O_d}) : \Gamma_0(\mathcal{P})] = \mathrm{p} + 1$, the quotient group $\Gamma_0(\mathcal{P})/\Gamma_1(\mathcal{P})$ is cyclic of order $(p-1)/2$, and $\Gamma_1(\mathcal{P})/\Gamma(\mathcal{P}) \cong \mathbf{Z}/p\mathbf{Z}$ with $\Gamma_1(\mathcal{P}) =< \Gamma(\mathcal{P}), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$.

Finally, note that by definiton, a peripheral subgroup of $\Gamma_0(I)$ is necessarily contained in $\Gamma_1(I)$, so that a set of peripheral subgroups for $\Gamma_0(I)$ is a partial set of peripheral subgroups for $\Gamma_1(I)$.

Proof: The proof follows the method in §3. We do $\Gamma_1(I)$ for $I =< 7, 1+\omega_6 >$ in detail, and then comment on the case $I =< 5, 1+\omega_{39} >$. Both Magma routines are given below.

Let $I =< 7, 1+\omega_6 >$. From the remarks following the statement of Proposition 5.1, we have $[\mathrm{PSL}(2, \mathrm{O_6}) : \Gamma_0(\mathrm{I})] = 8$ and $[\Gamma_0(I) : \Gamma_1(I)] = 3$. Now Q_6 has 2 cusps, and in the covering $\mathbf{H}^3/\Gamma_0(I) \longrightarrow Q_6$ each one of these 2 cusps is covered by two cusps, one of which has covering degree 1 and the other has degree 7.

Hence we deduce that $\mathbf{H}^3/\Gamma_1(I)$ has 12 cusps, and a set of peripheral subgroups for $\Gamma_1(I)$ is obtained by conjugating those of $\Gamma_0(I)$ by the elements $\{Id, x, x^{-1}\}$ where $\Gamma_0(I) =< \Gamma_1(I), x >$. A particular choice of x is $(at^{-2}a^{-1})t^{-2}(at^{-1}a^{-1})$.

Using the matrix representations for the peripheral subgroups P_1 and P_2 of $\mathrm{PSL}(2, \mathrm{O}_6)$ given in §2.4, we find the following set of peripheral subgroups for $\Gamma_0(I)$:

$$(\infty, < t, u >),\ (0, a < t^7, tu > a^{-1}),\ (\tfrac{-\sqrt{-6}}{2}, < (tb)^7, (tb)(cu)^{-1} >),$$
$$(\tfrac{-6-\sqrt{-6}}{14}, (at^2a^{-1}) < tb, cu > (at^2a^{-1})^{-1})$$

Now we obtain a set of peripheral subgroups for $\Gamma_1(I)$ by conjugating the above set for $\Gamma_0(I)$ by the elements $\{Id, x, x^{-1}\}$.

In the following Magma routine, $G = \mathrm{PSL}(2, \mathrm{O}_6)$, K is the subgroup generated by the four peripheral subgroups of $\Gamma_0(I)$, $L[1] = \Gamma_0(I)$, and $M[1] = \Gamma_1(I)$. As before, Q is the quotient of $\Gamma_1(I)$ by the normal closure the 12 parabolic elements to be trivialized. Since Magma calculates $Q =< 1 >$, it follows that $\Gamma_1(I)$ is indeed a 12 component link group.

Note that we use the `LowIndexSubgroups` routine in Magma to obtain $\Gamma_0(I)$ from $\mathrm{PSL}(2, \mathrm{O}_6)$ and $\Gamma_1(I)$ from $\Gamma_0(I)$. Indeed, since $N_d(I)$ is not of finite index in $\Gamma(I)$ (see §4.3.1), we cannot use Magma to obain a presentation for $\Gamma(I)$ as in §3, and hence we cannot present $\Gamma_1(I) =< \Gamma(I), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$ in this way.

```
G<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3,(a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c)>;

K:=sub<G|t,u,a*t^7*a,a*t*u*a,(t*b)^7,(t*b)*(c*u)^-1,a*t^2*a*t*b*a*t^-2*a,
a*t^2*a*c*u*a*t^-2*a>;

L:=LowIndexSubgroups(G,<8,8>:Subgroup:=K);
print #L;
1
\\
M:=LowIndexSubgroups(L[1],<3,3>:Subgroup:=K);
print #M;
1
\\
x:=a*t^-2*a*t^2*a*t^-1*a;
A:=sub<G|M[1],x>;
A eq L[1];
true
\\
Q:=quo<M[1]|u,x*t*x^-1,x^-1*u*x,a*t*u*a,x*a*t*u*a*x^-1,x^-1*a*t*u*a*x,
(t*b)*(c*u)^-1,x*t*b*(c*u)^-1*x^-1,x^-1*t*b*(c*u)^-1*x,a*t^2*a*t*b*a*t^-2*a,
x*a*t^2*a*t*b*a*t^-2*a*x^-1,x^-1*a*t^2*a*t*b*a*t^-2*a*x>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

Now let $I =< 5, \omega_{39} >$. Note that Q_{39} has 4 cusps and in the cover $\mathbf{H}^3/\Gamma_0(I) \longrightarrow Q_{39}$ each one of these 4 cusps is covered by two cusps, one of which has covering degree 1 and the other has degree 5. Thus we deduce that $\mathbf{H}^3/\Gamma_1(I)$ has 16 cusps, and a set of peripheral subgroups for $\Gamma_1(I)$ is obtained by conjugating one for $\Gamma_0(I)$ by the elements $\{Id, x\}$ where $\Gamma_0(I) =< \Gamma_1(I), x >$.

In the following Magma routine, $G = \mathrm{PSL}(2, \mathrm{O}_{39})$, K gives a set of peripheral subgroups for $\Gamma_0(I)$, $L[1] = \Gamma_0(I)$, and $M[1] = \Gamma_1(I)$. Magma confirms that $\Gamma_0(I) =< \Gamma_1(I), x >$ for $x =$

$(g_3g_1g_1^2g_3g_1)g_3^{-1}(g_3g_1g_1^2g_3g_1)$. Note that Magma returns two possibilities for $\Gamma_0(I)$: $L[1]$ and $L[2]$; however $L[2]$ is eliminated by homology considerations. Finally, $Q =< 1 >$, so that $\Gamma_1(I)$ is a 16 component link group.

```
G<g1,g2,g3,g4,g5,g6,g7>:=Group<g1,g2,g3,g4,g5,g6,g7|g3^3,(g4,g6),
(g3*g5)^2,(g2,g1),(g1^-1*g3^-1)^2,(g3^-1,g7^-1),(g5^-1*g1)^3,
g5^-1*g1*g6^-1*g4^-1*g5*g4*g1^-1*g6,g4^-1*g5*g4*g2^-1*g7*g5^-1*g7^-1*g2,
(g7*g5^-1*g7^-1*g1)^3, g6*g1^-1*g5*g6^-1*g4^-1*g5*g4*g1^-1*g4^-1*g5*g4*g1^-1>;

K:=sub<G|g1,g2,
(g3*g1)*g1^5*(g3*g1), (g3*g1)*g2*(g3*g1),
g5^-1*g6, g4*g1^-1*g6,
(g3*g1)*(g4*g1^-1*g6)^5*(g3*g1), (g3*g1)*(g4*g1^-1*g6)^-2*(g5^-1*g6)*(g3*g1),
g4^5, g4*g6^-1,
(g3*g1)*g1^2*g4*g1^-2*(g3*g1), (g3*g1)*g1^2*g6*g1^-2*(g3*g1),
g5^5, g5*(g4*g2^-1*g7),
(g3*g1)*g1*(g5)*g1^-1*(g3*g1), (g3*g1)*g1*(g4*g2^-1*g7)*g1^-1*(g3*g1)>;

L:=LowIndexSubgroups(G,<6,6>:Subgroup:=K);
print #L;
2
\\
print AbelianQuotientInvariants(L[1]);
[2,2,0,0,0,0,0,0,0,0]
\\
print AbelianQuotientInvariants(L[2]);
[2,0,0,0,0,0,0,0,0,0,0]
\\
M:=LowIndexSubgroups(L[1],<2,2>:Subgroup:=K);
print #M;
1
\\
x:=g3*g1*g1^2*g3*g1*g3^-1*g3*g1*g1^2*g3*g1;
A:=sub<G|M[1],x>;
A eq L[1];
true
\\
Q:=quo<M|g1,  x*g2*x^-1,  (g3*g1)*g2*g3*g1,  x*g3*g1*g2*g3*g1*x^-1,
g4*g1^-1*g6,  x*g4*g1^-1*g6*x^-1,  (g3*g1)*(g4*g1^-1*g6)^5*(g3*g1),
x*(g3*g1)*(g4*g1^-1*g6)^-2*(g5^-1*g6)*(g3*g1)*x^-1,
g4*g6^-1,  x*g4*g6^-1*x^-1,  g3*g1*g1^2*g6*g1^-2*g3*g1,
x*g3*g1*g1^2*g6*g1^-2*g3*g1*x^-1,  g5*g4*g2^-1*g7,  x*g5^5*x^-1,
g3*g1*g1*g5*g1^-1*g3*g1,  x*g3*g1*g1*g5*g1^-1*g3*g1*x^-1>;
Q:=ReduceGenerators(Q);
print Order(Q);
1
\\
```

6 Appendix:

In this appendix we gather together the matrix generators for the groups $\mathrm{PSL}(2, \mathrm{O_d})$ as given in §2.2 and used throughout. For $d = 5, 6, 15$ $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & \omega_d \\ 0 & 1 \end{pmatrix}$, and $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Also, recall that:

$$\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1+\sqrt{-15}}{2}, \frac{1+\sqrt{-23}}{2}, \frac{1+\sqrt{-31}}{2}, \frac{1+\sqrt{-39}}{2}, \frac{1+\sqrt{-47}}{2}, \frac{1+\sqrt{-71}}{2}.$$

d=5: a, $b = \begin{pmatrix} -\omega_5 & 2 \\ 2 & \omega_5 \end{pmatrix}$, $c = \begin{pmatrix} -\omega_5 - 4 & -2\omega_5 \\ 2\omega_5 & \omega_5 - 4 \end{pmatrix}$, t, u.

d=6: a, $b = \begin{pmatrix} -1-\omega_6 & 2-\omega_6 \\ 2 & 1+\omega_6 \end{pmatrix}$, $c = \begin{pmatrix} 5 & -2\omega_6 \\ 2\omega_6 & 5 \end{pmatrix}$, t, u.

d=15: a, $c = \begin{pmatrix} 4 & 1-2\omega_{15} \\ 2\omega_{15}-1 & 4 \end{pmatrix}$, t, u.

d=23: $g_1 = \begin{pmatrix} 1 & -1+\omega_{23} \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} 3+\omega_{23} & -4+\omega_{23} \\ -2+\omega_{23} & -1-\omega_{23} \end{pmatrix}$, $g_5 = \begin{pmatrix} 5-\omega_{23} & 1+2\omega_{23} \\ 2+\omega_{23} & -3+\omega_{23} \end{pmatrix}$.

d=31: $g_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 1 & \omega_{31} \\ 0 & 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} 3 & -2+2\omega_{31} \\ \omega_{31} & -5 \end{pmatrix}$, $g_5 = \begin{pmatrix} 3-2\omega_{31} & 7+\omega_{31} \\ 4 & -1+2\omega_{31} \end{pmatrix}$.

d=39: $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & \omega_{39} \\ 0 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} -3-\omega_{39} & 7-2\omega_{39} \\ 2-\omega_{39} & 5+\omega_{39} \end{pmatrix}$, $g_5 = \begin{pmatrix} 3-\omega_{39} & 2+\omega_{39} \\ 3 & -1+\omega_{39} \end{pmatrix}$, $g_6 = \begin{pmatrix} 7-\omega_{39} & 2+3\omega_{39} \\ 2+\omega_{39} & -5+\omega_{39} \end{pmatrix}$, $g_7 = \begin{pmatrix} 6-\omega_{39} & -1+2\omega_{39} \\ 1-2\omega_{39} & 5+\omega_{39} \end{pmatrix}$.

d=47: $g_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} 1 & -1+\omega_{47} \\ 0 & 1 \end{pmatrix}$, $g_4 = \begin{pmatrix} -2+\omega_{47} & 5 \\ -3 & 1+\omega_{47} \end{pmatrix}$, $g_5 = \begin{pmatrix} 5 & -6+3\omega_{47} \\ \omega_{47} & -7 \end{pmatrix}$, $g_6 = \begin{pmatrix} -4+\omega_{47} & 3+\omega_{47} \\ -3-\omega_{47} & -4+\omega_{47} \end{pmatrix}$, $g_7 = \begin{pmatrix} -1+2\omega_{47} & 11+\omega_{47} \\ 4 & -3+2\omega_{47} \end{pmatrix}$.

d=71: $g_1 = \begin{pmatrix} -5 & 5-3\omega_{71} \\ -1+\omega_{71} & -10-\omega_{71} \end{pmatrix}$, $g_2 = \begin{pmatrix} -3+2\omega_{71} & -17-\omega_{71} \\ -4 & 1-2\omega_{71} \end{pmatrix}$, $g_3 = \begin{pmatrix} 5 & -2\omega_{71} \\ 1-\omega_{71} & -7 \end{pmatrix}$, $g_4 = \begin{pmatrix} -5 & 2+\omega_{71} \\ -2-\omega_{71} & -3+\omega_{71} \end{pmatrix}$, $g_5 = \begin{pmatrix} -6-3\omega_{71} & 13-2\omega_{71} \\ 5-\omega_{71} & 4+\omega_{71} \end{pmatrix}$, $g_6 = \begin{pmatrix} -1+2\omega_{71} & 12 \\ -6 & -1+2\omega_{71} \end{pmatrix}$, $g_7 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $g_8 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $g_9 = \begin{pmatrix} 1+\omega_{71} & -7 \\ 3 & -2+\omega_{71} \end{pmatrix}$.

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