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ERROR ESTIMATE IN ISOPARAMETRIC FINITE ELEMENT EIGENVALUE PROBLEM

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Abstract. The aim of this paper is to obtain an eigenvalue approximation for elliptic operators defined on some domain Ω with the help of isoparametric finite elements of degree k. We prove that : $\lambda - \lambda_h = O(h^{2k})$ provided the boundary of Ω is well-approximated, which is the same estimate as the one obtained in the case of conform finite elements.

1. Introduction

We consider a spectral approximation by the isoparametric finite element method for an elliptic operator L defined over a bounded domain Ω of $\mathbb{R}^2$. The goal is to approximate a simple real eigenvalue λ of L.

J. Osborn [9] developed a general spectral approximation theory for compact operators on a Banach space. He proved that the conform finite element method of degree k made up over a polygonal domain Ω involves the following result:

$$\| u - u_h \|_{L^2(\Omega)} = O(h^{k+1}) \quad \text{and} \quad | \lambda - \lambda_h | = O(h^{2k}). \tag{1.1}$$

where (λ, u) is an eigenpair of an elliptic operator. Banerjee-Osborn [4] took into account the effect of numerical integration and showed that it depends on the degree of precision of the quadrature rules and on the smoothness of the eigenfunctions. To be more precise, they found the same rate of convergence as indicated before if the quadrature rules are of degree $2k - 1$ and u regular enough. Banerjee [3] improved in some way this result: for quadrature rules of degree $2k - 2$, the estimate for the eigenfunction remains true but not for the eigenvalue where one degree is lost.

If we apply the general results of Osborn [9] to isoparametric finite element approximation over some bounded domains (see section 4), we obtain the same rate of convergence as in (1.1) for the eigenfunction u but for the eigenvalue, we only have: $| \lambda - \lambda_h | = O(h^{k+1})$. Our purpose in this article is to give a "good" construction of the approximate boundary that will involve the phenomenon of supraconvergence: $| \lambda - \lambda_h | = O(h^{2k})$.

In section 2, we briefly describe the exact problem and the approximate one. In section 3, we precise how we build up the mesh over the bounded domain Ω of interest and how we devise the external layer of the elements to obtain a good approximation of the boundary $\partial\Omega$. The main result is given in section 4, where we also recall some previous results we need next. This result is proved in two steps: first we write $\lambda - \lambda_h$ as an integral defined over $\partial\Omega$ (section 5); then the estimate of this integral (section 6) leads to the result. In the last section, some examples of triangulations satisfying the requirements of the theorem are given in the cases $k = 2$ and $k = 3$.

2. Setting for Problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with a C^∞-boundary $\partial\Omega$. We define an operator L on $C^2(\overline{\Omega})$ by:

$$Lu = - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i}\right) \tag{2.1}$$

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where a_{ij} belong to $C^\infty(\mathbb{R}^2, \mathbb{R})$. We assume that L is uniformly strongly elliptic, i.e. there is a constant $a_0 > 0$ such that:

$$\forall \xi \in \mathbb{R}^2, \forall x \in \mathbb{R}^2 \quad \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geqslant a_0 \sum_{i=1}^{2} \xi_i^2. \tag{2.2}$$

We associate with L the following bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$:

$$a_\Omega(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx. \tag{2.3}$$

It is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$; furthermore the boundedness of a_{ij} on $\overline{\Omega}$ implies that a_Ω is continuous on $H^1(\Omega)$. According to the Lax-Milgram theorem, the following problem:

$$\begin{cases} \text{Let } f \in L^2(\Omega), \text{ find } u \in H_0^1(\Omega) \text{ such that:} \\ a_\Omega(u,\varphi) = \displaystyle\int_\Omega f(x)\varphi(x)\,dx \quad \text{for all } \varphi \in H_0^1(\Omega) \end{cases}$$

has one and only one solution $u = Tf$. T is a compact operator according to the Rellich theorem. Let us denote by μ a non-zero, real and simple eigenvalue of T and u a unitary associated eigenfunction. We may then choose an eigenfunction u^* of T^* associated with μ, where T^* is the adjoint of T with respect to the $L^2(\Omega)$ inner product, in such a way that:

$$\int_\Omega u^* u\,dx = 1. \tag{2.4}$$

We consider the following problem:

$$(P_1)\begin{cases} u - \lambda T u = 0 \\ \ell^*(u) = 1 \end{cases}$$

where $\lambda = 1/\mu$ and ℓ^* is the linear form defined on $L^2(\mathbb{R}^2)$ by:

$$\ell^*(v) \stackrel{\text{def}}{=} \int_\Omega u^* v\,dx. \tag{2.5}$$

We suppose the space $W^{m,p}(\Omega)$ normed with:

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leqslant m} \|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}}$$

where $\|.\|_p$ is the usual norm of $L^p(\Omega)$. We use also the semi-norm :

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}}.$$

and we make the usual changes if $p = \infty$.

We consider the approximation of (P_2) by the isoparametric finite elements method of Lagrangian type and start by reviewing the construction of a triangulation associated with this method ([5],[6],[7]).

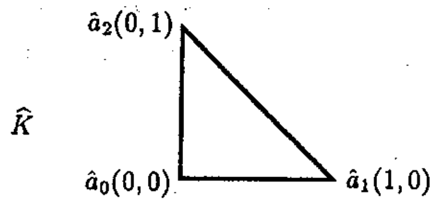

FIG. 2.1

Let k be a nonnegative integer and $(\widehat{K},\widehat{P},\widehat{\Sigma})$ the finite element of reference defined as follows:
- $\widehat{K}=\{\widehat{x}=(\widehat{x}_1,\widehat{x}_2)\ ;\ \widehat{x}_1\geqslant 0\ ;\ \widehat{x}_2\geqslant 0\ ;\ \widehat{x}_1+\widehat{x}_2\leqslant 1\}$ is a triangle whose vertices are denoted by $\widehat{a}_0,\widehat{a}_1,\widehat{a}_2$.
- $\widehat{P}=P_k$ where P_k is the space of all polynomials of degree not exceeding k defined on $\widehat{K}$
- $\widehat{\Sigma}=\{\widehat{x}=(\hat{x}_1,\hat{x}_2)\ ;\ \hat{x}_1=i/k\ ;\ \hat{x}_2=j/k\ ;\ i+j\leqslant k\ ;\ i,j\ \in \mathbb{N}\}$, the set of all Lagrangian interpolation nodes.

We consider an open set Ω_h approximation of Ω and a triangulation $\mathcal{K}_h$ of curved finite elements: an element K of $\mathcal{K}_h$ is given by $K=F_K(\widehat{K})$ where F_K is an invertible mapping each composante of which belongs to P_k. F_K is indeed determined by the data of the images $a_{i,K}$ of the nodes $\widehat{a}_i$ belonging to $\widehat{\Sigma}$. We suppose that, if an edge Γ of K is on $\partial\Omega_h$, its vertices are on $\partial\Omega$ too and that the edges which do not belong to $\partial\Omega_h$ are straight. These hypotheses are illustrated by figure (2.2).

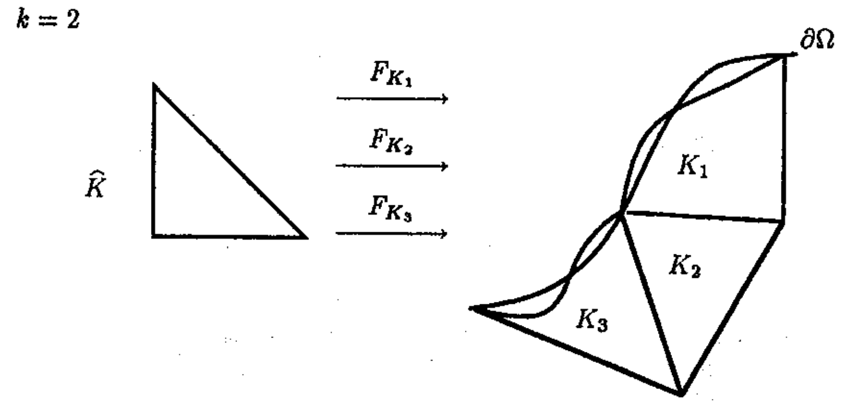

FIG. 2.2

We denote by h_K the diameter of K and suppose that all h_K are bounded by h.
We define the space of functions V_h:

$$V_h=\{v\ \in\ C^0(\mathbb{R}^2)\,;\,v(x)=0 \text{ si } x\ \notin\ \Omega_h\,;\,v_{/K}\ \in\ P_K\ \forall K\in\mathcal{K}_h\} \tag{2.6}$$

where $P_K=\{p:K\longrightarrow\mathbb{R};p\circ F_K\ \in\ P_k\}$. It is easy to check that:

$$V_h\ \subset\ H_0^1(\Omega_h). \tag{2.7}$$

We eventually suppose that this triangulation is k-regular (Ciarlet-Raviart [6]). Let us now approximate our problem. We first define an elliptic bilinear form on $V_h\times V_h$ by:

$$a_h(v_h,w_h)=\sum_{i,j=1}^{2}\int_{\Omega_h}a_{ij}\,\frac{\partial v_h}{\partial x_i}\,\frac{\partial w_h}{\partial x_j}\,dx. \tag{2.8}$$

We also define two operators T_h and T_h^* from $L^2(\mathbb{R}^2)$ to V_h by:

$$\forall f\ \in\ L^2(\mathbb{R}^2),\forall v_h\ \in\ V_h\quad \begin{cases} a_h(T_hf,v_h)=\displaystyle\int_{\mathbb{R}^2}f\,v_h\,dx\\ a_h(v_h,T_h^*f)=\displaystyle\int_{\mathbb{R}^2}f\,v_h\,dx\end{cases}$$

and u_h and λ_h are solutions of:

$$(P_2)\qquad u_h-\lambda_hT_hu_h=0.$$

We furthermore suppose that u_h is the orthogonal projection of u on the eigenspace of T_h associated with $\mu_h = 1/\lambda_h$. We now turn to estimating $\lambda - \lambda_h$.

Remark. Most of the time, Ω and Ω_h are different. We sometimes need to extend functions defined on Ω or Ω_h to $\mathbb{R}^2$ in a continuous way and use the same notation for a function or its extension. Unless explicit mention, an $H_0^1(\Omega)-$ function is extended by zero outside of Ω.

3. Curved triangles

We shall obtain the stated estimate $\lambda - \lambda_h = O(h^{2k})$ thanks to a "good approximation" of the boundary $\partial\Omega$. This needs explanations which we do in this section.

We suppose that $\partial\Omega$ is parametrized by its curvilinear abscissa: $\sigma \rightarrow x(\sigma)$ and denote by $\vec{n}\,(\sigma)$ the unitary normal vector, exterior to $\partial\Omega$ at the point $x(\sigma)$ and L the length of $\partial\Omega$.

Let us consider the application defined as follows:

$$\chi:\ (\sigma,\xi) \rightarrow \chi(\sigma,\xi) = x(\sigma) + \xi\,\vec{n}\,(\sigma). \tag{3.1}$$

If $a > 0$ is small enough, χ is a C^∞-diffeomorphism from $[0,L] \times [-a,a]$ onto a neighbourhood $\mathcal{V}$ of $\partial\Omega$ in $\mathbb{R}^2$. From now on, we suppose that h is small enough so that:

$$\partial\Omega_h \subset \mathcal{V}. \tag{3.2}$$

Remark. If $M = x(\sigma) + \xi\,\vec{n}\,(\sigma) \in \mathcal{V}$, then $x(\sigma)$ is the orthogonal projection of M on $\partial\Omega$ and $|\xi| = d(M,\partial\Omega)$ where $d(M,\partial\Omega)$ is the distance of M to $\partial\Omega$.

Now let us consider K a triangle of $\mathcal{K}_h$ with a curved edge Γ_h in $\partial\Omega_h$ and let $a_0 = x(\sigma_i)$ and $a_1 = x(\sigma_{i+1})$ be the vertices of Γ_h. We call Γ the part of $\partial\Omega$ lying between those two points and we denote by $l_i = \sigma_{i+1} - \sigma_i$ its length. We remark that:

$$l_i = O(h). \tag{3.3}$$

We suppose that $a_0 = F_K(\widehat{a}_0)$ and $a_1 = F_K(\widehat{a}_1)$ where F_K is the application of $(P_k)^2$ that defines K; thus, Γ_h is the image of the segment $[\widehat{a}_0, \widehat{a}_1]$ under F_K and letting:

$$x_h(\sigma) = F_K(\frac{\sigma - \sigma_i}{l_i}, 0), \tag{3.4}$$

we obtain a parametrized equation of Γ_h. Furthermore, x_h is a polynomial of degree k with respect of σ on $[\sigma_{i+1}, \sigma_i]$.

We supposed that for every i:

$$x_h(\sigma_i) = x(\sigma_i). \tag{3.5}$$

We furthermore assume that there is a constant $C > 0$ such that, for all i, we have:

$$|\,x_h(\sigma_i + j\,\frac{l_i}{k}) - x(\sigma_i + j\,\frac{l_i}{k})\,| \leqslant C\,l_i^{k+1} \text{ for } j = 1, \cdots, k-1. \tag{3.6}$$

Lemma 3.1. *Assumed that (3.2),(3.5) and (3.6) hold. Then, there is a constant $C > 0$ such that, for all i, we have:*

$$\|\,x_h - x\,\|_{m,\infty,[\sigma_i,\sigma_{i+1}]} \leqslant C\,h^{k+1-m} \text{ for } m = 0, \cdots, k+1.$$

Proof. Let $\sigma \rightarrow g_h x(\sigma)$ be the Lagrangian interpolation polynomial at the points $\sigma_i + j\,l_i/k$ for $j = 0, \cdots k$ of the function $\sigma \rightarrow x(\sigma)$. Thus, we have:

$$\begin{cases} g_h x(\sigma_i + j\,l_i/k) = x(\sigma_i + j\,l_i/k) \text{ for } j = 0, \cdots, k \\ g_h x \in (P_k)^2. \end{cases} \tag{3.7}$$

It is well-known that:

$$\|g_h x - x\|_{m,\infty,[\sigma_i,\sigma_{i+1}]} \leqslant C\, h^{k+1-m} \quad \text{for } m = 0,\cdots,k+1 \tag{3.8}$$

with C independent of i and of h. We define the Lagrange polynomial basis as follows:

$$\ell_j(\sigma) = \prod_{p\neq j}\Big(\frac{\sigma - \sigma_i - p\,l_i/k}{(j-p)\,l_i/k}\Big) \text{ for } j = 1,\cdots,k-1$$

Then, we can write:

$$g_h x(\sigma) - x_h(\sigma) = \sum_{m=1}^{k-1}\big(x(\sigma_i + m\,l_i/k) - x_h(\sigma_i + m\,l_i/k)\big)\,\ell_m(\sigma).$$

The result is thus a consequence of (3.7) and (3.8) and of the well-known following estimate:

$$\Big|\frac{d^m}{d\sigma^m}\ell_j(\sigma)\Big| \leqslant \frac{C}{l_i^m} \text{ for } m = 0,\cdots,k+1.$$

Remark. We deduce, from this lemma, that the function x_h and all its derivatives are bounded on $[0,L]$ independently of h.

According to (3.2), we observed, that for $\sigma \in [0,L]$, there is a unique $\xi \in]-a,a[$ such that $x(\sigma)+\xi\,\vec{n}\,(\sigma) \in \partial\Omega_h$. Let $d_h(\sigma)$ be this value of ξ and we obtain a new parametrized equation of $\partial\Omega_h$ as follows:

$$\sigma \to \tilde{x}_h(\sigma) = x(\sigma) + d_h(\sigma)\ \vec{n}\,(\sigma). \tag{3.9}$$

Lemma 3.1 then implies:

Corollary 3.1. *The application d_h is C^∞ on $[\sigma_i,\sigma_{i+1}]$ for every i and we have:*

$$d_h(\sigma) = O(h^{k+1}).$$

Proof. We have $d_h(\sigma) = \big(\tilde{x}_h(\sigma) - x(\sigma), \vec{n}\,(\sigma)\big)$, the regularity of $\tilde{x}_h$ gives the regularity of d_h □

Put $\widehat{K}_{\frac{1}{2}} = \{(x,y) \in \widehat{K}\,;\, y \geqslant 1/2\}$. There is a constant h_0 such that, for $h \leqslant h_0$, we have $\widehat{K}_{\frac{1}{2}} \subset \widehat{K\cap\Omega}$. From now on, we assume that $h \leqslant h_0$

Lemma 3.2. *There is a constant $C_1 > 0$ such that, for all $h \leqslant h_0$ and for all $\widehat{v} \in P_k$, we have, for $i = 0,1$:*

$$|\widehat{v}|_{i,\infty,\hat{K}} \leqslant C_1\,|\widehat{v}|_{i,2,\widehat{K\cap\Omega}}.$$

Proof. P_k is a space of finite dimension, thus, by equivalence of norms, we obtain:

$$|\widehat{v}|_{i,\infty,\hat{K}} \leqslant C_1\,|\widehat{v}|_{i,2,\hat{K}_{\frac{1}{2}}}.$$

We conclude by using $\widehat{K}_{\frac{1}{2}} \subset \widehat{K\cap\Omega}$ □

Lemma 3.3. *There is a constant $C > 0$ such that, for all $h > 0$ small enough and for all $v \in V_h$, we have:*

$$\| v \|_{1,2,\Delta_e} \leqslant C\, h^{\frac{k}{2}} \, \| v \|_{1,2,\Omega\cap\Omega_h}$$

where $\Delta_e = \Omega_h \setminus (\Omega \cap \Omega_h)$.

Proof. Let $\widehat{K \cap \Delta_e} = F_K^{-1}(K \cap \Delta_e)$ and $J_K(\widehat{x})$ be the Jacobian of the application F_K at the point $\widehat{x}$ of $\widehat{K}$. According to the k-regularity of the triangulation ([6]), there is a nonnegative constant C_0 such that:

$$0 < \frac{1}{C_0} \leqslant \frac{J_K(\widehat{x})}{J_K(\widehat{y})} \leqslant C_0 \text{ for all } \widehat{x}, \widehat{y} \in \widehat{K}. \tag{3.10}$$

We deduce:

$$\text{Surface}(\widehat{K \cap \Delta_e}) \leqslant \frac{\text{Surface}(K \cap \Delta_e)}{\text{Surface}(K)} \leqslant C\, h^k. \tag{3.11}$$

since:

$$\begin{cases} \text{Surface}(K \cap \Delta_e) \leqslant C\, h^{k+2}, \\ \quad \text{Surface}(K) \geqslant C\, h^2, \\ \quad \text{Surface}(\widehat{K}) = 1/2. \end{cases}$$

We then consider a function v of P_K; let $\widehat{v} = v \circ F_K$ and we have $\widehat{v} \in P_k$ thanks to the definition of P_K. Thus, we can write the following inequalities:

$$\begin{aligned} | v |_{0,2,K\cap\Delta_e} &\leqslant \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, | \widehat{v} |_{0,2,\widehat{K\cap\Delta_e}}, \\ &\leqslant \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, \big(\text{Surface}(\widehat{K \cap \Delta_e})\big)^{\frac{1}{2}} \, | \widehat{v} |_{0,\infty,\widehat{K}}, \\ &\leqslant C\, h^{\frac{k}{2}} \, \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, | \widehat{v} |_{0,\infty,\widehat{K}} \text{ according to (3.11)}, \\ &\leqslant C\, h^{\frac{k}{2}} \, \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, | \widehat{v} |_{0,2,\widehat{K\cap\Omega}} \text{ according to lemma 3.2}, \\ &\leqslant C\, h^{\frac{k}{2}} \left(\frac{(\operatorname{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x}))}{(\operatorname{Min}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x}))} \right)^{\frac{1}{2}} | v |_{0,2,K\cap\Omega}. \end{aligned}$$

The inequality (3.10) then implies:

$$| v |_{0,2,K\cap\Delta_e} \leqslant C_1\, h^{\frac{k}{2}} | v |_{0,2,K\cap\Omega}. \tag{3.12}$$

The k-regularity of the triangulation implies also that there is a constant C such that, for all $K \in \mathcal{K}_h$, we have:

$$\begin{cases} \| DF_K \|_{0,\infty,\widehat{K}} \leqslant C\, h, \\ \| DF_K^{-1} \|_{0,\infty,K} \leqslant \dfrac{C}{h}. \end{cases} \tag{3.13}$$

Furthermore:

$$\begin{aligned} | v |_{1,2,K\cap\Delta_e} &\leqslant \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, \| DF_K^{-1} \|_{0,\infty,K}^{\frac{1}{2}} \, | \widehat{v} |_{1,2,\widehat{K\cap\Delta_e}}, \\ &\leqslant C\, h^{\frac{k}{2}} \, \big(\operatorname*{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})\big)^{\frac{1}{2}} \, \| DF_K^{-1} \|_{0,\infty,K}^{\frac{1}{2}} \, | \widehat{v} |_{1,\infty,\widehat{K}} \text{ according to (3.11)}, \\ &\leqslant C\, h^{\frac{k}{2}} \left(\frac{\operatorname{Max}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})}{\operatorname{Min}_{\widehat{x}\in\widehat{K}} J_K(\widehat{x})} \right)^{\frac{1}{2}} \| DF_K^{-1} \|_{0,\infty,K}^{\frac{1}{2}} \, \| DF_K \|_{0,\infty,\widehat{K}}^{\frac{1}{2}} \, | v |_{1,2,K\cap\Omega} \end{aligned}$$

according to lemma 3.2. The inequalities (3.13) then give :

$$| v |_{1,2,K\cap\Delta_e} \leqslant C_2\, h^{\frac{k}{2}} \, | v |_{1,2,K\cap\Omega}. \tag{3.14}$$

Adding up the inequalities (3.12) and (3.14) over all the triangles that are concerned, we yield lemma 3.3. □

Remark. The inequality (3.14) is optimal, but (3.12) could be improved.

4. The main result

We use the notations defined in section 3. Recall that we assume:

(1) For all i:

$$(H_1) \qquad x(\sigma_i) = x_h(\sigma_i)$$

(2) There is a real $C > 0$ such that, for all $j \in \{1, \cdots, k-1\}$ and for all i:

$$(H_2) \qquad |\, x_h(\sigma_i + j\,\frac{l_i}{k}) - x(\sigma_i + j\,\frac{l_i}{k})\,| \leqslant C\, l_i^{k+1}.$$

Let us denote by $\theta_0 = 0, \cdots, \theta_k = 1$ the $k+1$ Gauss-Lobatto points of the interval $[0,1]$ and define:

$$\sigma_{i,j} = \sigma_i + \theta_j\, l_i \quad \text{for} \quad j = 0, \cdots, k. \tag{4.1}$$

Theorem. *If (H_1) and (H_2) hold and if the triangulation $\mathcal{K}_h$ is k-regular, then there is a constant M independent of h such that:*

$$|\, \lambda - \lambda_h\,| \leqslant M\,\big(h^{2k} + \operatorname*{Max}_{i,j} |\, x(\sigma_{i,j}) - x_h(\sigma_{i,j})\,|\big).$$

Remark. If we suppose that:

$$(H) \qquad \operatorname*{Max}_{i,j} |\, x(\sigma_{i,j}) - x_h(\sigma_{i,j})\,| = O(h^{2k}),$$

we obtain the supraconvergence phenomenon:

$$\lambda - \lambda_h = O(h^{2k}).$$

In order to prove the theorem, we establish the two following propositions.

Proposition 1. *There is a constant C_1 such that:*

$$|\, \lambda - \lambda_h - \int_{\partial\Omega} g(\sigma)\, d_h(\sigma)\, d\sigma\,| \leqslant C_1\, h^{2k}$$

where g is a regular function of σ.

We then estimate the integral with:

Proposition 2. *Let $\varphi \in W^{k-1,1}(\partial\Omega)$, then there is a constant $C_2 > 0$ such that:*

$$\begin{aligned}|\int_{\partial\Omega} \varphi(\sigma)\, d_h(\sigma)\, d\sigma\,| \leqslant\, & C_2\, h^{2k}\,(|\,\varphi\,|_{k-1,1,\partial\Omega} + L\,\|\,\varphi\,\|_{k-2,\infty,\partial\Omega}) \\ & + C_2\, L\,\|\,\varphi\,\|_{k-2,\infty,\partial\Omega}\, \operatorname*{Max}_{i,j} |\,(x - x_h)(\sigma_{i,j})\,|.\end{aligned}$$

Remark. The first proposition is valid in any dimension of space but it is not the case for the second one where the dimension two plays an important role.

These propositions clearly imply the theorem. We shall prove them in the two following sections. For later purposes, let us first recall some results.

If the triangulation is k-regular, we have:

$$\begin{aligned}&\text{For all } u \in H^{k+1}(\mathbb{R}^2), \\ &\|\,(T - T_h)u\,\|_{m,2,\Omega\cap\Omega_h} \leqslant C\, h^{k+1-m}\,\|\,Tu\,\|_{k+1,2,\Omega} \text{ for } m = 0, 1.\end{aligned} \tag{4.2}$$

One can find a proof of this statement in the articles by Zlàmal ([11],[12]) in the case of Dirichlet-type problems. It has been improved by Zenisek ([10]) for various types of nonhomogeneous boundary value problems.

We remark that the definitions of T and T_h imply that:

$$\begin{aligned}\| (T - T_h)u \|_{0,2,\mathbb{R}^2} &\leqslant \| (T - T_h)u \|_{0,2,\Omega\cap\Omega_h} + \| (T - T_h)u \|_{0,2,\mathbb{R}^2\setminus(\Omega\cap\Omega_h)} \\ &\leqslant \| (T - T_h)u \|_{0,2,\Omega\cap\Omega_h} + \| T_h u \|_{0,2,\Omega_h\setminus(\Omega\cap\Omega_h)} \\ &\quad + \| Tu \|_{0,2,\Omega\setminus(\Omega\cap\Omega_h)}\end{aligned} \tag{4.3}$$

since $T_h u = 0$ on $\mathbb{R}^2 \setminus \Omega_h$ and $Tu = 0$ on $\mathbb{R}^2 \setminus \Omega$. Then, by the Poincaré's inequality, for $u \in H^{k+1}(\mathbb{R}^2)$, such that $Tu \in H_0^1(\Omega)$:

$$\begin{aligned}\| Tu \|_{0,2,\Omega\setminus(\Omega\cap\Omega_h)} &\leqslant C\, h^{k+1} \| \nabla Tu \|_{0,2,\Omega\setminus(\Omega\cap\Omega_h)} \text{ according to lemma 3.1} \\ &\leqslant C\, h^{k+1} \| \nabla Tu \|_{1,2,\Omega}.\end{aligned}$$

For $u \in H^{k+1}(\mathbb{R}^2)$, such that $T_h u \in H_0^1(\Omega_h)$:

$$\begin{aligned}\| T_h u \|_{0,2,\Omega_h\setminus(\Omega\cap\Omega_h)} &\leqslant C\, h^{k+1} \| \nabla T_h u \|_{0,2,\Omega_h\setminus(\Omega\cap\Omega_h)} \\ &\leqslant C\, h^{k+1} \| T_h u \|_{1,2,\Omega\setminus(\Omega\cap\Omega_h)} \\ &\leqslant C\, h^{\frac{3}{2}k+1} \| T_h u \|_{1,2,\Omega\cap\Omega_h} \text{ according to lemma 3.3} \\ &\leqslant C\, h^{\frac{3}{2}k+1} \text{ according to (4.2)}\end{aligned}$$

The previous inequalities and (4.3) infer to:

$$\| (T - T_h)u \|_{0,2,\mathbb{R}^2} = O(h^{k+1}). \tag{4.4}$$

We then use two results from the general theory of the spectral approximation for compact operators by J. Osborn [9].

Let T be a compact operator of $L^2(\Omega)$ into $H_0^1(\Omega)$. We define a compact operator $\widetilde{T}$ from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ as follows:

Let $u \in L^2(\mathbb{R}^2)$, then:

$$\begin{cases} \widetilde{T}u = T(u_{/\Omega}) & \text{on} \quad \Omega \\ \widetilde{T}u = 0 & \text{on} \quad \mathbb{R}^2 \setminus \Omega \end{cases}$$

T_h is an operator of $L^2(\mathbb{R}^2)$ into V_h, thus into $L^2(\mathbb{R}^2)$. We denote by E (respectively E_h) the projection of $L^2(\mathbb{R}^2)$ onto the space of generalized eigenvectors of T (respectively T_h) corresponding to μ (respectively $\mu_h = 1/\lambda_h$). These spaces are spanned respectively by u and u_h defined by (P_1) and (P_2). We notice that $u_h = E_h u$. We let $R(E)$ be the range of the application E. Given two closed subspaces M and N of $L^2(\mathbb{R}^2)$, we set:

$$\begin{cases} \delta(M,N) = \operatorname{Sup}\{\{\operatorname{Inf}\{\| f - g \|_{0,2,\mathbb{R}^2} \,;\, g \in N\}\} f \in M \,;\, \| f \|_{0,2,\mathbb{R}^2} = 1\} \\ \widehat{\delta}(M,N) = \operatorname{Max}(\delta(M,N), \delta(N,M)). \end{cases}$$

J. Osborn proves in [9] that:

There are two constants $C_1 > 0$ and $C_2 > 0$ such that:

$$\begin{cases} \widehat{\delta}(R(E), R(E_h)) \leqslant C_1 \| (T - T_h)_{/R(E)} \| \\ \qquad | \mu - \mu_h | \leqslant C_2 \| (T - T_h)_{/R(E)} \|. \end{cases} \tag{4.5}$$

Moreover:

$$\begin{aligned}\|(T-T_h)_{/R(E)}\| &= \operatorname{Sup}\{|((T-T_h)f,\varphi)|\,;\, f\in R(E),\varphi\in L^2(\mathbb{R}^2)\,;\\ &\qquad \|f\|_{0,2,\mathbb{R}^2}=\|\varphi\|_{0,2,\mathbb{R}^2}=1\}\\ &\leqslant \operatorname{Sup}\{\|(T-T_h)f\|_{0,2\mathbb{R}^2}\,;\,\|f\|_{0,2,\mathbb{R}^2}=1\}\\ &\leqslant C\,h^{k+1} \text{ according to (4.4).}\end{aligned}$$

We then have the following results, for $u\in R(E)$ with $\|u\|_{0,2,\Omega}=1$ and $u_h=E_h u_h$:

$$\|u-u_h\|_{0,2,\mathbb{R}^2}=O(h^{k+1}), \tag{4.6}$$

$$|\lambda-\lambda_h|=O(h^{k+1}). \tag{4.7}$$

We now turn to the proof of the two propositions stated above.

5. Proof of Proposition 1

We first give some notations. We decompose $\Omega\cup\Omega_h$ in three domains:

$$\begin{cases}\Theta=\Omega\cap\Omega_h\\ \Delta_i=\Omega\setminus\Theta\\ \Delta_e=\Omega_h\setminus\Theta\\ \Gamma_i=\partial\Omega\cap\overline{\Delta}_i\\ \Gamma_e=\partial\Omega\cap\overline{\Delta}_e.\end{cases} \tag{5.1}$$

We consider $\vec{n}=\nu=(\nu_1,\nu_2)$ the unitary normal vector, exterior to $\partial\Omega$ and set:

$$\begin{cases}\partial_\nu=\dfrac{\partial}{\partial\nu}\\ \partial_{\nu_L}=\dfrac{\partial}{\partial\nu_L}=\displaystyle\sum_{i,j=1}^{2}\nu_i\,a_{ij}\frac{\partial}{\partial x_j}\\ \partial_{\nu_{L^*}}=\dfrac{\partial}{\partial\nu_{L^*}}=\displaystyle\sum_{i,j=1}^{2}\nu_j\,a_{ij}\frac{\partial}{\partial x_i}\\ A(\sigma)=\displaystyle\sum_{i,j=1}^{2}a_{ij}(x(\sigma))\nu_i\nu_j.\end{cases} \tag{5.2}$$

5.1. Proof of Proposition 1.

We decompose the proof of Proposition 1 in two lemmas:

Lemma 5.1. *We have the following estimates:*

$$\lambda-\lambda_h=-\lambda^2\ell^*((T-T_h)u)+O(h^{2k+2}). \tag{1}$$

$$\begin{aligned}\ell^*((T-T_h)u) &= a_{\Delta_i}(Tu,T^*u^*)+a_{\Delta_e}(T_hu,T_h^*u^*)\\ &+a_\Theta((T-T_h)u,(T^*-T_h^*)u^*)+\int_{\Gamma_e}A(\sigma)\,[\partial_\nu(Tu)T_h^*u^*+\partial_\nu(T^*u^*)T_hu]\,d\sigma.\end{aligned} \tag{2}$$

We introduce:

$$g(\sigma)=A(\sigma)\,\partial_\nu u\,\partial_\nu u^*(x(\sigma)). \tag{5.3}$$

Lemma 5.2. *We have the following equalities:*

$$a_{\Delta_i}(u,u^*) = -\int_{\Gamma_i} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k+2}). \tag{1}$$

$$a_{\Delta_e}(T_h u, T_h^* u^*) = \frac{1}{\lambda^2}\int_{\Gamma_e} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k}). \tag{2}$$

$$\int_{\Gamma_e} A(\sigma)\,\partial_\nu(u^*)T_h u\,d\sigma = -\frac{1}{\lambda}\int_{\Gamma_e} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k+1}). \tag{3}$$

$$\int_{\Gamma_e} A(\sigma)\,\partial_\nu(u)T_h^* u^*\,d\sigma = -\frac{1}{\lambda}\int_{\Gamma_e} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k+1}). \tag{4}$$

Assume these lemmas hold. We show that they imply Proposition 1. According to the first lemma, we have:

$$\begin{aligned}\lambda - \lambda_h = &-\lambda^2\, a_{\Delta_i}(Tu, T^*u^*) - \lambda^2\, a_{\Delta_e}(T_h u, T_h^* u^*)\\ &- \lambda^2\, a_\Theta\big((T-T_h)u, (T^*-T_h^*)u^*\big)\\ &- \lambda^2 \int_{\Gamma_e} A(\sigma)\,\big[\,\partial_\nu(Tu)T_h^*u^* + \partial_\nu(T^*u^*)T_h u\,\big]\,d\sigma.\end{aligned}$$

Using $\lambda Tu = u$ and $\lambda T^*u^* = u^*$ and the bilinearity of a, we have:

$$\begin{aligned}\lambda - \lambda_h = &-a_{\Delta_i}(u,u^*) - \lambda^2\, a_{\Delta_e}(T_h u, T_h^* u^*) - \lambda^2\, a_\Theta\big((T-T_h)u, (T^*-T_h^*)u^*\big)\\ &- \lambda \int_{\Gamma_e} A(\sigma)\,\big[\,\partial_\nu(u)T_h^*u^* + \partial_\nu(u^*)T_h u\,\big]\,d\sigma.\end{aligned}$$

We then use lemma 5.2 and we obtain:

$$\begin{aligned}\lambda - \lambda_h &= \int_{\Gamma_i} g(\sigma)d_h(\sigma)\,d\sigma - \int_{\Gamma_e} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k})\\ &\quad - \lambda^2\, a_\Theta\big((T-T_h)u, (T^*-T_h^*)u^*\big) + 2\int_{\Gamma_e} g(\sigma)d_h(\sigma)\,d\sigma,\\ &= \int_{\partial\Omega} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k}) - \lambda^2\, a_\Theta\big((T-T_h)u, (T^*-T_h^*)u^*\big),\\ &= \int_{\partial\Omega} g(\sigma)d_h(\sigma)\,d\sigma + O(h^{2k}).\end{aligned}$$

To obtain the last equality, we have used the continuity of a_Θ and the following inequality:

$$\|\,(T-T_h)u\,\|_{1,2,\Theta} + \|\,(T^*-T_h^*)u^*\,\|_{1,2,\Theta} \leqslant C\,h^k,$$

which is proposition 1. Now we prove the two lemmas stated above.

5.2. Proof of Lemma 5.1.

To show the truth of (1) in lemma 5.1, we remark that, for all $w \in \operatorname{Im}(I-\lambda T)$, we have:

$$\ell^*(w) = 0. \tag{5.4}$$

This gives us:

$$\begin{aligned}0 &= \ell^*(u_h - \lambda_h T_h u_h)\\ &= \ell^*(u_h - \lambda T u_h) + \lambda_h \,\ell^*\big((T-T_h)u_h\big) + (\lambda - \lambda_h)\,\ell^*(Tu_h)\\ &= \lambda_h \,\ell^*\big((T-T_h)u_h\big) + (\lambda-\lambda_h)\,\ell^*(Tu_h) \text{ according to (5.4).}\end{aligned}$$

Thus:

$$(\lambda - \lambda_h)\,\ell^*(Tu_h) = -\lambda_h\,\ell^*\big((T-T_h)u_h\big) \tag{5.5}$$

Furthermore:

$$\begin{aligned}\ell^*(Tu_h) &= \ell^*(Tu) - \ell^*\big(T(u-u_h)\big)\\ &= \frac{1}{\lambda} + O(h^{k+1}) \quad \text{according to (4.4)}\end{aligned} \tag{5.6}$$

$$\ell^*\big((T-T_h)u_h\big) = \ell^*\big((T-T_h)u\big) - \ell^*\big((T-T_h)(u-u_h)\big)$$

We remind that the last term satisfies:

$$\begin{aligned}\ell^*\big((T-T_h)(u-u_h)\big) &= \int_\Omega \big(T^* - T_h^*\big)\, u^*\,(u-u_h)\,dx\\ &= O(h^{2k+2}) \text{ according to (4.4) et (4.6)}\end{aligned}$$

Equalities (5.5) and (5.6) infer to:

$$\lambda - \lambda_h = -\lambda\lambda_h\,\ell^*\big((T-T_h)u\big) + O\big(h^{2k+2} + |\,\lambda-\lambda_h\,|\,h^{k+1}\big)$$

and we obtain the desired result thanks to (4.4) and (4.7) □

The second item in lemma 5.1 is a decomposition of the integral $\ell^*\big((T-T_h)u\big)$ over the domains defined in (5.1). From Green's formula:

$$\begin{cases} a_\Omega(v,w) - \displaystyle\int_\Omega v\, L^* w\, dx = \int_{\partial\Omega} \partial_{\nu_{L^*}} w\, v\, d\sigma\\ a_\Omega(w,v) - \displaystyle\int_\Omega Lv\, w\, dx = \int_{\partial\Omega} \partial_{\nu_L} v\, w\, d\sigma\end{cases} \tag{5.7}$$

Choosing $v = T_h u$ and $w = T^* u^*$ in the first one and $v = Tu$ and $w = T_h^* u^*$ in the second one, we have:

$$\begin{aligned}\ell^*(T_h u) &= a_\Omega(T_h u, T^* u^*) - \int_{\partial\Omega} \partial_{\nu_{L^*}}(T^*u^*) T_h u\, d\sigma\\ a_\Omega(Tu, T_h^* u^*) &= \int_{\partial\Omega} \partial_{\nu_L}(Tu) T_h^* u^*\, d\sigma + \int_\Omega u T_h^* u^*\, dx\\ &= \int_{\partial\Omega} \partial_{\nu_L}(Tu) T_h^* u^*\, d\sigma + a_{\Omega_h}(T_h u, T_h^* u^*)\\ &\qquad \text{by definition of } T_h.\end{aligned} \tag{5.8}$$

We know that $T_h u = 0$ on Γ_i; the first equality in (5.8) and the definition of T infer to:

$$\begin{aligned}\ell^*\big((T-T_h)u\big) &= a_\Omega(Tu, T^*u^*) - \ell^*(T_h u)\\ &= a_\Omega\big((T-T_h)u, T^*u^*\big) + \int_{\Gamma_e} \partial_{\nu_{L^*}}(T^*u^*) T_h u\, d\sigma\\ &= a_\Omega\big((T-T_h)u, (T^* - T_h^*)u^*\big) + \int_{\Gamma_e} \partial_{\nu_{L^*}}(T^*u^*) T_h u\, d\sigma\\ &\quad + a_\Omega(Tu, T_h^* u^*) - a_\Omega(T_h u, T_h^* u^*)\\ &= a_\Theta\big((T-T_h)u, (T^*-T_h^*)u^*\big) + a_{\Delta_i}(Tu, T^*u^*)\\ &\quad + \int_{\Gamma_e} \big[\,\partial_{\nu_{L^*}}(T^*u^*) T_h u + \partial_{\nu_L}(Tu) T_h^* u^*\,\big]\, d\sigma\\ &\quad + a_{\Delta_e}(T_h u, T_h^* u^*) \quad \text{according to (4.8).}\end{aligned}$$

The proof of the second item is complete after we remark that:

$$\begin{cases} \partial_{\nu_{L^*}}(T^*u^*) = A(\sigma)\partial_\nu T^*u^* \\ \partial_{\nu_L}(Tu) = A(\sigma)\partial_\nu Tu \end{cases}$$

since $Tu = T^*u^* = 0$ on $\partial\Omega$ □

5.3. Proof of Lemma 5.2.

Proof of (1): we describe $\partial\Omega_h$ with the notation defined in (3.8). Every point y of Δ_i can be written in a unique way as follows:

$$y = x(\sigma) + \xi\ \vec{n}\ (\sigma) \text{ with } \xi\ \in]\, d_h(\sigma), 0\, [.$$

The Taylor-Lagrange's formula at the point $x(\sigma)$ gives:

$$\begin{aligned} \frac{\partial u}{\partial x_i}(y)\,\frac{\partial u^*}{\partial x_j}(y) &= \frac{\partial u}{\partial x_i}(x(\sigma))\,\frac{\partial u^*}{\partial x_j}(x(\sigma)) + O(h^{k+1}) \text{ according to corollary 3.1} \\ &= \nu_i\nu_j\,\frac{\partial u}{\partial \nu}(x(\sigma))\,\frac{\partial u^*}{\partial \nu}(x(\sigma)) + O(h^{k+1}) \text{ because of } u = 0 \text{ on } \Gamma_i. \end{aligned}$$

We remark that:

$$dx_1\, dx_2 = (1 - \frac{\xi}{R(\sigma)})\, d\sigma\, d\xi = \big(1 + O(h^{k+1})\big)\, d\sigma\, d\xi$$

where $R(\sigma)$ is the radius of curvature of $\partial\Omega$ at the point $x(\sigma)$. The second equality is a consequence of corollary 3.1. This gives us:

$$\begin{aligned} a_{\Delta_i}(u,u^*) &= \int_{\Gamma_i}\sum_{i,j=1}^{2}\int_{d_h(\sigma)}^{0}\Big(a_{ij}\nu_i\nu_j\frac{\partial u}{\partial \nu}\frac{\partial u^*}{\partial \nu} + O(h^{k+1})\Big)\, d\xi\, d\sigma \\ &= -\int_{\Gamma_i} A(\sigma)\frac{\partial u}{\partial \nu}\frac{\partial u^*}{\partial \nu}(x(\sigma))d_h(\sigma)\, d\sigma + O(h^{2k+2}). \end{aligned}$$

which ends the proof of the first item □

Proof of (2): for the sake of simplicity, we let:

$$\text{(5.9)}\qquad \begin{cases} v_h = T_h u \\ v_h^* = T_h^* u^* \end{cases} \text{ and } \begin{cases} v\ = Tu \text{ on } \Omega \\ v^* = T^*u^* \text{ on } \Omega. \end{cases}$$

The boundary of Ω being regular, we can extend v and v^* to $\mathbb{R}^2 \setminus \Omega$ as C^{k+1}-functions. The bilinearity of a_{Δ_e} gives us:

$$\text{(5.10)}\qquad \begin{aligned} a_{\Delta_e}(v_h, v_h^*) = a_{\Delta_e}(v, v^*) + a_{\Delta_e}(v_h - v, v^*) + a_{\Delta_e}(v, v_h^* - v^*) \\ + a_{\Delta_e}(v_h - v, v_h^* - v^*). \end{aligned}$$

Analogously to the previous argument, we have:

$$\text{(5.11)}\qquad a_{\Delta_e}(v, v^*) = \int_{\Gamma_e} \tilde{g}(\sigma) d_h(\sigma)\, d\sigma + O(h^{2k+2}).$$

with $\tilde{g}(\sigma) = A(\sigma)\,\partial_\nu v \partial_\nu v^*\big(x(\sigma)\big)$, hence $\lambda^2\tilde{g}(\sigma) = g(\sigma)$ and we obtain the first term given in the equality we are trying to prove. Let us show that the remaining terms in the equality (5.10) are bounded by h^{2k}. We first use the continuity of a_{Δ_e}:

$$\text{(5.12)}\qquad \begin{cases} |\, a_{\Delta_e}(v - v_h, v^*)\,| \leqslant C\,\|\, v - v_h \,\|_{1,2,\Delta_e}\, \|\, v^* \,\|_{1,2,\Delta_e} \\ |\, a_{\Delta_e}(v, v^* - v_h^*)\,| \leqslant C\,\|\, v^* - v_h^* \,\|_{1,2,\Delta_e}\, \|\, v \,\|_{1,2,\Delta_e} \\ |\, a_{\Delta_e}(v - v_h, v^* - v_h^*)\,| \leqslant C\,\|\, v - v_h \,\|_{1,2,\Delta_e}\, \|\, v^* - v_h^* \,\|_{1,2,\Delta_e} \end{cases}$$

Let $r_h v$ be the Lagrangian polynomial interpolation of degree k of v. According to Ciarlet-Raviart [7], we have:

$$\| v - r_h v \|_{1,2,\Omega_h} \leqslant C h^k. \tag{5.13}$$

According to (4.2), we then obtain:

$$\| v_h - r_h v \|_{1,2,\Theta} \leqslant C h^k. \tag{5.14}$$

According to the fact that $v_h - r_h v$ belongs to V_h and lemma 3.3, we furthermore have:

$$\begin{aligned} \| v_h - r_h v \|_{1,2,\Delta_e} &\leqslant C h^{\frac{k}{2}} \| v_h - r_h v \|_{1,2,\Theta} \\ &\leqslant C h^{\frac{3k}{2}} \text{ according to (5.14)} \end{aligned} \tag{5.15}$$

We also have according to Ciarlet-Raviart [7]:

$$\| v - r_h v \|_{1,\infty,\Omega\cup\Omega_h} \leqslant C h^k.$$

It is then clear that we obtain:

$$\begin{aligned} \| v - r_h v \|_{1,2,\Delta_e} &\leqslant (\text{Surface}(\Delta_e))^{\frac{1}{2}} \| v - r_h v \|_{1,\infty,\Delta_e} \\ &\leqslant C h^{\frac{3}{2}k+\frac{1}{2}}. \end{aligned} \tag{5.16}$$

Hence, we obtain:

$$\| v - v_h \|_{1,2,\Delta_e} \leqslant C h^{\frac{3}{2}k}. \tag{5.17}$$

The same kind of estimate holds for $v^* - v_h^*$. We furthermore have the following inequality:

$$\| v^* \|_{1,2,\Delta_e} \leqslant C\,(\text{Surface}(\Delta_e))^{\frac{1}{2}} \leqslant C h^{\frac{k+1}{2}},$$

which is also true for $\| v \|_{1,2,\Delta_e}$.

Putting those two last results in (5.12), we have:

$$\begin{cases} | a_{\Delta_e}(v - v_h, v^*) | \leqslant C h^{2k+\frac{1}{2}} \\ | a_{\Delta_e}(v, v^* - v_h^*) | \leqslant C h^{2k+\frac{1}{2}} \\ | a_{\Delta_e}(v - v_h, v^* - v_h^*) | \leqslant C h^{3k} \end{cases}$$

Using these previous inequalities in the equality (5.10), we obtain the second item of lemma 5.2 thanks to (5.11) □

Proof of (3): the proof of (4) being similar, it will be omitted. We use the notations (5.9). The function v_h vanishes on $\partial\Omega_h$, hence, according to (3.8), we can write:

$$\begin{aligned} v_h(x(\sigma)) &= -d_h(\sigma)\ \vec{n}\ (\sigma) \nabla v_h(y) \quad \text{with } y \in]\, x(\sigma), x_h(\sigma)\, [, \\ &= -d_h(\sigma)\ \vec{n}\ (\sigma) \nabla v_h(x(\sigma)) + d_h(\sigma)\ \vec{n}\ (\sigma)\Big(\nabla v_h(x(\sigma)) - \nabla v_h(y)\Big), \\ &= -d_h(\sigma)\, \partial_\nu v_h(x(\sigma)) + O\Big(h^{k+1}\, |\, \vec{n}\ (\sigma)\Big(\nabla v_h(x(\sigma)) - \nabla v_h(y)\Big)\, |\Big) \\ &\qquad \text{according to the estimate on } d_h \text{ obtained in corollary 3.1.} \end{aligned}$$

We furthermore have:

$$\begin{aligned} |\, \vec{n}\ (\sigma)\Big(\nabla v_h(x(\sigma)) - \nabla v_h(y)\Big)\, | &\leqslant |\, \vec{n}\ (\sigma)\Big(\nabla v_h(x(\sigma)) - \nabla v(x(\sigma))\Big)\, | \\ &\quad + |\, \vec{n}\ (\sigma)\Big(\nabla v(x(\sigma)) - \nabla v(y)\Big)\, | \\ &\quad + |\, \vec{n}\ (\sigma)\Big(\nabla v_h(y) - \nabla v(y)\Big)\, |, \\ &\leqslant C\,(\| v - v_h \|_{1,2,\Delta_e} + | x(\sigma) - y |), \\ &\leqslant C h^{k+1} \text{ according to (5.17) and corollary 3.1,} \end{aligned}$$

which gives us:

$$v_h(x(\sigma)) = -d_h(\sigma)\partial_\nu v_h(x(\sigma)) + O(h^{2k+2}).$$

We finally obtain:

$$\int_{\Gamma_e} A(\sigma)\,\partial_\nu(u^*)T_h u\,d\sigma = -\int_{\Gamma_e} A(\sigma)\,\partial_\nu(u^*)d_h(\sigma)\partial_\nu v_h(x(\sigma))\,d\sigma + O(h^{2k+2}). \tag{5.18}$$

Since $\|v - v_h\|_{1,2,\Theta} = O(h^{k+1})$:

$$\partial_\nu v_h(x(\sigma)) = \partial_\nu v(x(\sigma)) + O(h^{k+1}).$$

Furthermore:

$$\begin{cases} v = Tu = \dfrac{1}{\lambda}u \\ d_h(\sigma) = O(h^{k+1}) \end{cases}$$

Therefore:

$$d_h(\sigma)\,\partial_\nu v_h(x(\sigma)) = \frac{1}{\lambda}\,d_h(\sigma)\,\partial_\nu u(x(\sigma)) + O(h^{2k+2}).$$

The previous equality and (5.18) complete then the proof of (3) and the lemma is completely proved □

6. Proof of Proposition 2

We use the Gauss-Lobatto formula to prove the proposition. Therefore we introduce $\theta_0 = 0, \theta_1, \cdots, \theta_k = 1$ the $k+1$ Gauss-Lobatto quadrature points of $[0,1]$ and:

$$G_i(f) = \sum_{j=0}^{k} l_j\,\lambda_j\,f(\sigma_i + \theta_j\,l_i) \tag{6.1}$$

where the coefficients λ_i are uniquely determined by:

$$G_i(p) = \int_0^1 p(x)\,dx \text{ for all } p \in P_{2k-1} \tag{6.2}$$

We recall that $\lambda_j > 0$ and $\sum_{j=0}^{k}\lambda_j = 1$.

According to the Péano theorem (see, for example, [8]), we have:

$$\text{For all } f \in C^{2k}([\sigma_i, \sigma_{i+1}]) \quad |E_i(f)| \leqslant C\,l_i^{2k+1}\,|f|_{2k,\infty,\Gamma} \tag{6.3}$$

where

$$E_i(f) = \int_{\sigma_i}^{\sigma_{i+1}} f(\sigma)\,d\sigma - G_i(f). \tag{6.4}$$

Let us begin the proof of proposition 2. Let us denote by $\gamma_i = [\sigma_i, \sigma_{i+1}]$ and consider a $W^{k-1,1}(\partial\Omega)$-function φ. Then we have for all $\sigma \in \gamma_i$:

$$\varphi(\sigma) = p_i(\sigma) + \int_{\sigma_i}^{\sigma} \varphi^{(k-1)}(s)\frac{(\sigma - s)^{k-2}}{(k-2)!}\,ds \tag{6.5}$$

where p_i is the Taylor's polynomial of degree $k-2$ of φ at the point σ_i. This equality implies that:

$$\|\,\varphi - p_i\,\|_{0,\infty,\gamma_i} \leqslant C\,h^{k-2}\,|\,\varphi\,|_{k-1,1,\gamma_i}. \tag{6.6}$$

Furthermore we can write:

$$\int_{\gamma_i} \varphi(\sigma)\, d_h(\sigma)\, d\sigma = \int_{\gamma_i} (\varphi - p_i)(\sigma)\, d_h(\sigma)\, d\sigma + E_i(p_i\, d_h) + l_i \sum_{j=0}^{k} \lambda_j\, (p_i\, d_h)(\sigma_{i,j}).$$

According to (6.4), (6.6) and lemma 3.1, we deduce:

$$\begin{aligned} &\Big| \int_{\gamma_i} \varphi(\sigma)\, d_h(\sigma)\, d\sigma \Big| \\ &\quad\leqslant C \Big(h^{2k} \,|\, \varphi \,|_{k-1,1,\gamma_i} + l_i\, h^{2k} \,|\, p_i\, d_h \,|_{2k,\infty,\gamma_i} + l_i \operatorname*{Max}_{j=0,\cdots,k} |\, (p_i\, d_h)(\sigma_{i,j}) \,| \Big) \\ &\quad\leqslant C \Big(h^{2k} \,|\, \varphi \,|_{k-1,1,\gamma_i} + l_i \,\|\, \varphi \,\|_{k-2,\infty,\gamma_i} \big(h^{2k} \,|\, d_h \,|_{2k,\infty,\gamma_i} + \operatorname*{Max}_{j=0,\cdots,k} |\, d_h(\sigma_{i,j}) \,| \big) \Big), \end{aligned}$$

where we have used the inequalities:

$$\begin{cases} \operatorname*{Max}_{j=0,\cdots,k} |\, p_i(\sigma_{i,j}) \,| \leqslant \|\, p_i \,\|_{0,\infty,\gamma_i} \leqslant C \,\|\, \varphi \,\|_{k-2,\infty,\gamma_i} \\ \quad |\, p_i\, d_h \,|_{2k,\infty,\gamma_i} \leqslant C \,\|\, \varphi \,\|_{k-2,\infty,\gamma_i} \,\|\, d_h \,\|_{2k,\infty,\gamma_i}. \end{cases}$$

Carrying out the summation over all intervals γ_i, we obtain:

$$\begin{aligned} \Big| \int_{\partial\Omega} \varphi(\sigma)\, d_h(\sigma)\, d\sigma \Big| \leqslant C \Big(& h^{2k} \,|\, \varphi \,|_{k-1,1,\partial\Omega} \\ & + L \,\|\, \varphi \,\|_{k-2,\infty,\partial\Omega} \big(\operatorname*{Max}_{j,i} |\, d_h(\sigma_{i,j}) \,| + h^{2k} \operatorname*{Max}_{i} \|\, d_h \,\|_{2k,\infty,\gamma_i} \big) \Big). \end{aligned}$$

To complete the proof of proposition 2, we need the:

Lemma 6.1. *There is a nonnegative constant C such that, for all i, we have:*

$$\begin{cases} \|\, d_h \,\|_{2k,\infty,\gamma_i} \leqslant C \\ \quad |\, d_h(\sigma_{i,j}) \,| \leqslant C \big(|\, (x - x_h)(\sigma_{i,j}) \,| + h^{2k+1} \big) \end{cases}$$

Proof. We built two parametric representations of $\partial\Omega_h$:

$$\begin{aligned} s &\in [\, \sigma_i, \sigma_{i+1} \,] \rightarrow x_h(s) = F_K\Big(\frac{s - \sigma_i}{l_i}, 0\Big) \\ \sigma &\in [\, \sigma_i, \sigma_{i+1} \,] \rightarrow \widetilde{x}_h(\sigma) = x(\sigma) + d_h(\sigma)\, \vec{n}\,(\sigma) \end{aligned}$$

This defines an homeomorphism:

$$f : \sigma \in [\, 0, L \,] \rightarrow s = f(\sigma) \in [\, 0, L \,] \text{ with:}$$

$$x_h(s) = \widetilde{x}_h(\sigma) = x(\sigma) + d_h(\sigma)\, \vec{n}\,(\sigma) \tag{6.7}$$

which gives us:

$$d_h(\sigma) = \big((x_h(s) - x(\sigma)) \,.\, \vec{n}\,(\sigma) \big). \tag{6.8}$$

We already observed that x_h and all its derivatives are bounded independently of h and i, then we obtain the first inequality provided f is C^∞ on $[\, 0, L \,]$ and has all its derivatives bounded independently of h and i. We first prove with a Taylor's expansion, that there is a constant C independent of i and h such that:

$$|\, \sigma - s \,| \leqslant C\, h^{k+1}. \tag{6.9}$$

According to (6.7), we have:

$$
(6.10)\qquad \begin{aligned} x_h(s) - x(s) &= x(\sigma) - x(s) + d_h(\sigma)\ \vec{n}\ (\sigma) \\ &= (s-\sigma)\ \vec{t}\ (\sigma) + \Big(\frac{(s-\sigma)^2}{2R(\sigma)} + d_h(\sigma)\Big)\ \vec{n}\ (\sigma) + O(|\, s-\sigma\,|^3) \end{aligned}
$$

where $R(\sigma)$ is the radius of curvature of $\partial\Omega$ at the point $x(\sigma)$.

Lemma 3.1 and corollary 3.1 say that we have $\| x - x_h \|_{0,\infty,\partial\Omega} \leqslant Ch^{k+1}$ and $d_h(\sigma) \leqslant Ch^{k+1}$, thus (6.10) yields to (6.9).

Let us study the regularity of f. According to (6.7), we have:

$$
(6.11)\qquad f(\sigma) = l_i F_K^{-1}\big(x(\sigma) + d_h(\sigma)\ \vec{n}\ (\sigma)\big) + \sigma_i.
$$

We suppose the triangulation to be k-regular, it implies that F_K is a C^kdiffeomorphism; furthermore it belongs to $(P_k)^2$. It is then clearly a C^∞-diffeomorphism. Since d_h is regular and according to (6.11), we obtain that f is regular.

In order to prove that all derivatives of f are bounded independently of h on $[0, L]$, we multiply (5.7) by $x'(\sigma) = \vec{t}\ (\sigma)$ and we have:

$$
(6.12)\qquad \phi(\sigma) \stackrel{\text{def}}{=} \big(x(\sigma) \,.\, x'(\sigma)\big) = \big(x_h(s) \,.\, x'(\sigma)\big)
$$

where ϕ is a C^∞function on $[0, L]$ which do not hinge on h. Carrying out the differentiation with respect to σ, we obtain:

$$
(6.13)\qquad \phi'(\sigma) = f'(\sigma)\big(x_h'(s) \,.\, x'(\sigma)\big) + \big(x_h(s) \,.\, x''(\sigma)\big).
$$

According to (6.9), we can write:

$$
\begin{aligned} \big(x_h'(s) \,.\, x'(\sigma)\big) &= \big(x'(\sigma) \,.\, x'(\sigma)\big) + \big(x'(\sigma) \,.\, x'(s) - x'(\sigma)\big) + \big(x'(\sigma) \,.\, x_h'(s) - x'(s)\big) \\ &= 1 + O(h^k) \end{aligned}
$$

because of $\| x' - x_h' \|_{0,\infty,\partial\Omega} = O(h^k)$ on account of lemma 3.1. We deduce from these calculations that, if h is small enough,

$$
\big(x_h'(s) \,.\, x'(\sigma)\big) \neq 0,
$$

thus f' is independent of h since x_h and all its derivatives also are. We then obtain that all derivatives of f are bounded independently of h thanks to the previous remark and with the help of an induction by carrying out the differentiation of the equation (6.13) with respect to σ. Thus all the derivatives of f are bounded independently of h on $[0, L]$ and inequality (6.8) proves the first part of the lemma and we come to the second part.

Let us define:

$$
(6.14)\qquad s_{i,j} = f(\sigma_{i,j})
$$

and write (5.8) at the points $s_{i,j}$ and $\sigma_{i,j}$:

$$
(6.15)\qquad \begin{aligned} d_h(\sigma_{i,j}) &= \big((x_h(s_{i,j}) - x(\sigma_{i,j})) \,.\, \vec{n}\ (\sigma_{i,j})\big) \\ &= \big((x_h(\sigma_{i,j}) - x(\sigma_{i,j})) \,.\, \vec{n}\ (\sigma_{i,j})\big) + \big((x_h(s_{i,j}) - x_h(\sigma_{i,j})) \,.\, \vec{n}\ (\sigma_{i,j})\big) \\ &\quad + \big((x(\sigma_{i,j}) - x(s_{i,j})) \,.\, \vec{n}\ (\sigma_{i,j})\big). \end{aligned}
$$

We know that:

$$
\begin{aligned}
x_h(s_{i,j}) - x_h(\sigma_{i,j}) &= (s_{i,j} - \sigma_{i,j})x_h'(\sigma_{i,j}) + O(h^{2k+2}) \\
&\quad \text{according to the estimate of } s - \sigma \text{ stated in (6.9)} \\
&= (s_{i,j} - \sigma_{i,j})x'(\sigma_{i,j}) + O(h^{2k+2}) \\
&\quad + (s_{i,j} - \sigma_{i,j})\big(x_h'(\sigma_{i,j}) - x'(\sigma_{i,j})\big) \\
&= (s_{i,j} - \sigma_{i,j})x'(\sigma_{i,j}) + O(h^{2k+1}) \text{ since } \| x - x_h \| = O(h^k).
\end{aligned}
$$

Replacing the last equality into (6.15), we obtain:

$$d_h(\sigma_{i,j}) = \big((x_h(\sigma_{i,j}) - x(\sigma_{i,j})) \,.\, \vec{n}\,(\sigma_{i,j})\big) + O(h^{2k+1})$$

because of $x'(\sigma_{i,j}) = \vec{t}\,(\sigma_{i,j})$. □

Remark. According to (6.15), we could change (H_2) and (H) into (H_3):

$$(H_3)\begin{cases} \left| \big((x_h(\sigma_{i,j}) - x(\sigma_{i,j})) \,.\, \vec{n}\,(\sigma_{i,j})\big) \right| \leqslant C\, h^{2k} \\ \qquad\qquad \left| \, x_h(\sigma_{i,j}) - x(\sigma_{i,j}) \right| \leqslant C\, l_i^{k+1} \end{cases}$$

7. Examples

We use again the notations of section 3. Let us consider a triangle K of the triangulation $\mathcal{K}_h$ with a curved edge Γ_h in $\partial\Omega_h$ and denote by A and B the vertices of Γ_h. We call Γ the part of $\partial\Omega$ lying between those two points. Let O be the midpoint of A and B.

For $k = 2$, we give two different constructions of the arc Γ_h; for $k = 3$, we only give an indication since it is the same idea.

7.1. k=2.

The quadrature points of Gauss-Lobatto of the segment $[0,1]$ for $k = 2$ are $0, 1/2, 1$. We suppose that A and B have $-l/2$ and $l/2$ as curvilinear abscissas. Let:

$$C' = x(0). \tag{7.1}$$

If we define Γ_h by the three points A, B and C', then the hypothesis (H) clearly holds and the triangle K is k-regular, but C' is difficult to calculate if Γ is not parametrized by its curvilinear abscissa.

We can also consider the point C intersection of Γ and the median of $[A, B]$. Let us show that this point is convenient. We must have:

$$\overrightarrow{CC'} = O(h^4). \tag{7.2}$$

Let:

$$\begin{cases} \vec{t}\,(\sigma) = \dfrac{dx(\sigma)}{d\sigma} \\ C = x(\sigma_1) \end{cases} \tag{7.3}$$

Lemma 7.1. *With the previous notations, we have:*

(1) $\sigma_1 = O(h^4)$

(2) $\overrightarrow{CC'} = O(h^4)$

Proof. σ_1 is defined by $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$.

We write the expansion of the function x at the point 0 for $\sigma = \sigma_1, -l/2$ or $l/2$:

$$x(\sigma) = x(0) + \sigma x'(0) + \frac{\sigma^2}{2} x''(0) + \frac{\sigma^3}{6} x'''(0) + O(l^4) \quad \text{hence:}$$

$$\begin{aligned} \overrightarrow{OC} &= x(\sigma_1) - \frac{1}{2}\left(x(-\frac{l}{2}) + x(\frac{l}{2})\right) \\ &= \sigma_1 x'(0) + \frac{1}{2}(\sigma_1^2 - l^2)x''(0) + \frac{1}{6}\sigma_1^3 x'''(0) + O(l^4) \\ \overrightarrow{AB} &= x(\frac{l}{2}) - x(-\frac{l}{2}) = l\left(x'(0) + \frac{1}{36} x'''(0) + O(l^4)\right) \end{aligned}$$

Thus:

$$\sigma_1 + \frac{1}{6}\sigma_1^3 + \frac{1}{36}\sigma_1 l^2 (x'(0), x'''(0)) + O(l^4) = 0. \tag{7.4}$$

We deduce that $\sigma_1 = O(l^4)$. We have already remarked that $l = O(h)$, thus we have shown the first point of the lemma.

We also have:

$$\overrightarrow{CC'} = x(\sigma_1) - x(0) = \sigma_1 \vec{t}\,(0) + 0(h^8),$$

which shows the second point □

C satisfies the hypothesis (H); it also satisfies the hypotheses needed for a k-regular triangulation ([6]). We remark that any point C'' with $\overrightarrow{C'C''} = O(h^4)$ is also convenient; we then show another way of building the third point we need to have Γ_h.

Let D and E be the two exterior nodes of the triangulation which are respectively the nearest of A and B.

We consider p a polynomial of degree three, passing through A, B, D and E and we denote by C'' the intersection of p with the median of $[A, B]$; by construction, C'' satisfies that $\overrightarrow{C'C''} = O(h^4)$ and C'' is easy to calculate.

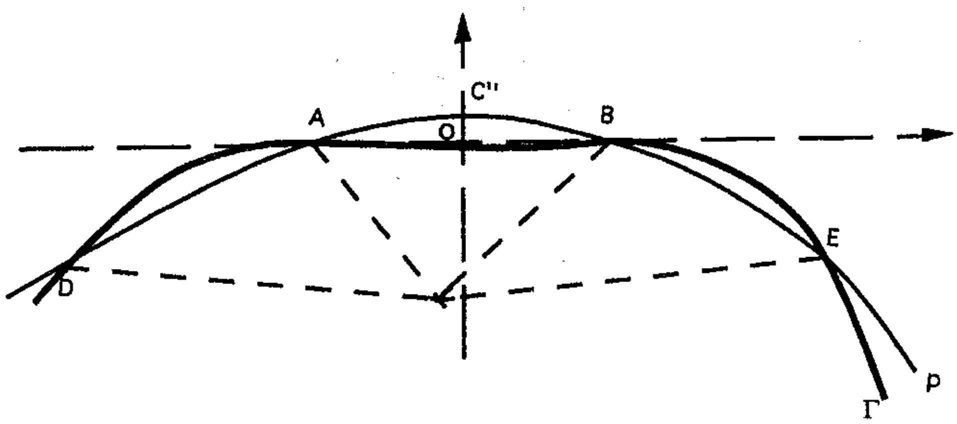

FIG. 7.1

We give an algorithm to obtain C''. We first work with the orthonormal frame of reference defined by figure (7.1) and we denote by (x_M, y_M) the coordinates of a point M in this frame of reference. We have:

$$\begin{cases} x_A = -x_B \\ y_A = y_B = 0 \\ y_{C''} = 0 \end{cases}$$

We define two polynomials p_D and p_E as follows:

$$\begin{cases} p_D(x) = \dfrac{(x - x_E)(x - x_A)(x - x_B)}{(x_D - x_E)(x_D - x_A)(x_D - x_B)} \\ p_E(x) = \dfrac{(x - x_D)(x - x_A)(x - x_B)}{(x_E - x_D)(x_E - x_A)(x_E - x_B)} \end{cases} \tag{7.5}$$

We then define C'' by:

$$x_{C''} = y_D p_D(0) + y_E p_E(0). \tag{7.6}$$

Let us now work in the original frame of reference supposed to be orthonormal and denote by (x'_M, y'_M) the coordinates of a point M in this frame of reference; we can then give an algorithm to calculate C'':

(1) **Change of frame of reference:**

$$\alpha = \frac{x'_O - x'_A}{h/2} \qquad \beta = \frac{y'_O - y'_A}{h/2}$$

$$f(x,y) = \beta(x - x'_O) + \alpha(y - y'_O)$$

$$g(x,y) = \alpha(x - x'_O) - \beta(y - y'_O)$$

$$\begin{cases} x_E = g(x'_E, y'_E) \\ y_E = f(x'_E, y'_E) \end{cases} \qquad \begin{cases} x_D = g(x'_D, y'_D) \\ y_D = f(x'_D, y'_D) \end{cases}$$

(2) **Equality (7.6):**

$$\begin{cases} p(x,y,z,t) = \dfrac{h^2}{4(x-y)}\left[\dfrac{xz}{x^2 - h^2/4} - \dfrac{yt}{y^2 - h^2/4}\right] \\ c = p(x_D, x_E, y_D, y_E) \end{cases}$$

(3) **Result:**

$$\begin{cases} x'_{C''} = \beta c + x'_O \\ y'_{C''} = \alpha c + y'_O \end{cases}$$

Remark. In the case of $k = 2$, according to Ciarlet-Raviart [6], the triangulation would be k-regular if we have $\| \overrightarrow{OC''} \| = O(h^2)$, that is the case; we build then the two other edges to obtain the other hypotheses of a k-regularity.

7.2. k=3.

The Gauss-Lobatto quadrature points $[0,1]$ in the case $k = 3$ are $0, \alpha = \frac{1}{2}(1 - \frac{1}{\sqrt{5}})$, $\beta = \frac{1}{2}(1 + \frac{1}{\sqrt{5}}), 1$. Let us call:

$$\begin{cases} B = x(\alpha l) \\ C = x(\beta l) \end{cases}$$

We then remark that all points B' et C' satisfying:

$$\begin{cases} \| \overrightarrow{BB'} \| = O(h^6) \\ \| \overrightarrow{CC'} \| = O(h^6) \end{cases}$$

are convenient to build Γ_h. We consider a polynomial p of degree five passing through six exterior and nearest nodes of the triangulation and we denote by B' (respectively C') the intersection of p with the orthogonal straight line to (A, B) passing through the point $\alpha A + (1-\alpha)B$ (respectively $\beta A + (1-\beta)B$).

Those points define a convenient arc Γ_h; we then build the two other edges of the triangle in order to have a 3-regular triangulation.

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