NONLOCALIZED MODULATION OF PERIODIC REACTION DIFFUSION WAVES: NONLINEAR STABILITY

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Abstract. Extending results of Johnson and Zumbrun showing stability under localized (L^1) perturbations, we show that spectral stability implies nonlinear modulational stability of periodic traveling-wave solutions of reaction diffusion systems under small perturbations consisting of a nonlocalized modulation plus a localized perturbation. The main new ingredient is a detailed analysis of linear behavior under modulational data $\bar{u}'(x)h_0(x)$, where $\bar{u}$ is the background profile and h_0 is the initial modulation.

1. Introduction

Stability and behavior of modulated periodic wave trains have received considerable recent attention, both in the reaction diffusion and conservation law contexts; see, for example [S1, S2, DSSS, JZ1, JZ2, JZN, BJNRZ1, BJNRZ2, NR1, NR2] and references therein. The initial mathematical challenge of this problem is that the linearized equations, being periodic-coefficient, have purely essential spectrum when considered as problems on the whole line, so that there is no spectral gap between neutral and other modes, making difficult either the treatment of linearized behavior or the passage from linear to nonlinear estimates.

This issue was overcome in the reaction diffusion context in the late 1990s by Schneider [S1, S2], resolving an at the time 30-year open problem. Using a method of "diffusive stability," Schneider combined diffusive-type linear estimates with renormalization techniques to show that, assuming "diffusive" spectral stability in the sense to be specified later of the background periodic wave, long-time behavior under localized perturbations is essentially described by a scalar linear convection-diffusion equation in the phase variable, with the amplitude decaying more rapidly.

More recently, Oh–Zumbrun and Johnson–Zumbrun [OZ2, JZ2] in 2010 and 2011 using rather different tools coming from viscous shock theory have resolved the corresponding problem in the conservation law case, showing that, again assuming diffusive spectral stability of the background periodic wave, behavior under localized perturbations is described, roughly, by a system of viscous conservation laws in the derivative of the phase and other modulation parameters, so that the phase decays at slower, error function rate than in the reaction diffusion case. Further developments include the treatment of nonlinear stability of roll waves of the St. Venant equations [JZN], a set of partially parabolic balance laws of quasilinear type, and the resolution in [BJNRZ3] of the 35-year open problem of proving nonlinear stability of spectrally stable periodic Kuramoto–Sivashinsky waves.

Applied to the reaction diffusion context [JZ1], the approach of [JZ2] recovers and slightly extends the results of Schneider, yielding the same heat kernel rate of decay with respect to localized perturbations while removing the assumption of [S1, S2] that nearby periodic waves have constant speed. However, one lesson of the successful analysis in [JZ2], the conservation law case, is that one

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can close a nonlinear iteration for perturbations decaying at slower, error function rate, yielding hope that one could do better also for the reaction-diffusion case.

As described clearly in [DSSS], formal (WKB) asymptotics suggest the same thought, yielding the description to lowest order of a scalar viscous conservation law

$$k_t - \omega_0(k)_x = (d(k)k_x)_x \tag{1.1}$$

for the wave number $k = k_*(1+\psi_x)$, where $\omega = \psi_t$ is the time frequency and $k_*\psi$ is the phase shift. Here $\omega_0(k) = -c(k)k$ and wave speed $c(k)$ are determined through the nonlinear dispersion relation obtained from the manifold of periodic traveling-wave solutions $\bar u^k(k(x-ct))$, and $d(k)$ through higher-order asymptotics. See [OZ1, Se, OZ3, NR1, NR2] for corresponding developments in the conservation law case. Note that an integrated version of (1.1) tells that $\omega_0(k)$ is a first approximation of the time frequency $\omega = \psi_t$ in the low frequency regime. From (1.1), one might hope to treat initial shifts $k_*\psi|_{t=0} = k_*h_0$ for which, not the phase shift k_*h_0, but its derivative $k_*\partial_x h_0$ is localized (L^1), obtaining the same heat kernel rate of decay for k as found in the conservation law case [JZ2].

In this note, we show that this is indeed the case, proving by an improved version of the nonlinear iteration scheme of [JZ1, JZ2] that *diffusively spectrally stable periodic reaction diffusion waves are nonlinearly stable under initial perturbations consisting of a localized perturbation plus a nonlocalized modulation h_0, $\partial_x h_0 \in L^1(\mathbb{R})$, in phase*, with heat kernel rate of decay in the wave number and error function decay in the phase. In a companion paper [JNRZ1], we build on these basic estimates to establish, further, time-asymptotic behavior, validating description (1.1) in the long-time limit. Recall that (1.1) has been validated in [DSSS] in the related, asymptotically-large-bounded-time small-wave number limit and for data in uniformly-local Sobolev spaces[1], in the somewhat different sense of showing that there exists a δ-family of solutions of the full system δ^M-close to a formal expansion[2] in δ on intervals $[0, T/\delta^2]$, where M and T are arbitrarily large and δ is the characteristic size of the wave number of the modulation: that is, they build "prepared" data for bounded intervals rather than work with general ones globally in time.

Consider a periodic traveling-wave solution of a reaction diffusion system

$$u_t = u_{xx} + f(u), \qquad u \in \mathbb{R}^n,$$

or, equivalently, a standing-wave solution $u(x,t) = \bar u(x)$ of

$$k_*u_t = k_*^2 u_{xx} + f(u) + k_*cu_x, \tag{1.2}$$

where c is the speed of the original traveling wave, and wave number k_* is chosen so that $\bar u(x+1) = \bar u(x)$. For simplicity of notation, we will follow this convention throughout the paper; that is, all periodic functions are assumed to be periodic of period one.

We make the following standard genericity assumptions:

(H1) $f \in C^K(\mathbb{R}^n)$, $(K \geq 3)$.

(H2) Up to translation, the set of 1-periodic solutions of (1.2) (with k replacing k_*) in the vicinity of $\bar u$, $k = k_*$, forms a smooth 1-dimensional manifold $\{\bar u(k,\cdot)\} = \{\bar u^k(\cdot)\}$, $c = c(k)$.

Linearizing (1.2) about $\bar u$ yields the periodic coefficient equation

$$k_*v_t = k_*Lv := (k_*^2\partial_x^2 + k_*c\partial_x + b)v, \qquad b(x) := df(\bar u(x)). \tag{1.3}$$

Introducing the one-parameter family of closed Floquet operators

$$k_*L_\xi := e^{-i\xi x}k_*Le^{i\xi x} = k_*^2(\partial_x + i\xi)^2 + k_*c(\partial_x + i\xi) + b \tag{1.4}$$

[1] Specifically, $\psi_x \in H^s_{ul}$, $s \geq 4$, with $\|\psi_x\|_{H^s_{ul}}$ sufficiently small, where $\|w\|_{H^s_{ul}} := \sup_{x\in\mathbb{R}} \|w\|_{H^s([x,x+1])}$.

[2] In particular, δ^2-close to the expansion involving the solution of the second-order equation (1.1).

acting on $L^2_{\rm per}([0,1])$ with densely defined domains $H^2_{\rm per}([0,1])$, determined by the defining relation

$$L(e^{i\xi x}g)=e^{i\xi x}(L_\xi g) \text{ for } g \text{ periodic,} \tag{1.5}$$

we define following [S1, S2] the standard *diffusive spectral stability* conditions:

(D1) $\sigma(L)\subset\{\lambda\mid\Re\lambda<0\}\cup\{0\}$.

(D2) There exists a constant $\theta>0$ such that $\sigma(L_\xi)\subset\{\lambda\mid\Re\lambda\leq-\theta|\xi|^2\}$ for each $\xi\in[-\pi,\pi)$.

(D3) $\lambda=0$ is a simple eigenvalue of L_0.[3]

Notice in (D1) above, we consider L as a closed operator on $L^2(\mathbb{R})$ with densely defined domain $H^2(\mathbb{R})$.

Then, we have the following Main Theorem, extending the results of [JZ1] to nonlocalized perturbations. Here, and throughout the paper, given two real valued functions A and B, we say that $A\lesssim B$ or that for every $x\in\mathrm{dom}(A)\cap\mathrm{dom}(B)$, $A(x)\lesssim B(x)$ if there exists a constant $C>0$ such that $A(x)\leq CB(x)$ for each $x\in\mathrm{dom}(A)\cap\mathrm{dom}(B)$. Even in a chain of inequalities, we will also feel free to denote by C harmless constants with different values.

Theorem 1.1. *Let $K\geq3$. Assuming (H1)-(H2) and (D1)-(D3), let*

$$E_0:=\|\tilde u_0(\cdot-h_0(\cdot))-\bar u(\cdot)\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}+\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}$$

be sufficiently small, for some choice of modulation h_0. Then, there exists a global solution $\tilde u(x,t)$ of (1.2) with initial data $\tilde u_0$ and a phase function $\psi(x,t)$ with initial data h_0 such that for $t>0$ and $2\leq p\leq\infty$,

$$\begin{aligned}\|\tilde u(\cdot-\psi(\cdot,t),t)-\bar u(\cdot)\|_{L^p(\mathbb{R})},\quad\|\nabla_{x,t}\psi(\cdot,t)\|_{W^{K+1,p}(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac12(1-1/p)},\\ \|\tilde u(\cdot-\psi(\cdot,t),t)-\bar u(\cdot)\|_{H^K(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac14},\end{aligned} \tag{1.6}$$

and

$$\|\tilde u(\cdot,t)-\bar u(\cdot)\|_{L^\infty(\mathbb{R})},\quad\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R})}\lesssim E_0. \tag{1.7}$$

In particular, $\bar u$ is nonlinearly (boundedly) stable in $L^\infty(\mathbb{R})$ with respect to initial perturbations $v_0=\tilde u_0-\bar u$ for which $\|v_0\|_E:=\inf_{\partial_x h_0\in L^1(\mathbb{R})\cap H^K(\mathbb{R})}E_0$ is sufficiently small.

Remark 1.2. It may seem more natural, and indeed is so, to introduce h_0 and ψ via

$$v(x,0)=\tilde u_0(x)-\bar u(x+h_0(x)),\quad v(x,t)=\tilde u(x,t)-\bar u(x+\psi(x,t)).$$

However, in doing so one introduces in the equation for v terms involving only ψ and thus not decaying in time; see Lemma 5.1 below. For this reason we work instead with $v(x,t)=\tilde u(x-\psi(x,t),t)-\bar u(x)$, that is, $\tilde u(x,t)=\bar u(Y(x,t))+v(Y(x,t),t)$ where Y is such that

$$Y(x,t)-\psi(Y(x,t),t)=x,\quad Y(y-\psi(y,t),t)=y\ .$$

Notice that we insure the existence of such a map Y by keeping, for any t, $\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R})}$ bounded and $\|\psi_x(\cdot,t)\|_{L^\infty(\mathbb{R})}$ small.[4] It should be stressed, however, that[5]

$$Y(x,t)=x+\psi(x,t)+O(\|\psi(\cdot,t)\|_{L^\infty}\|\psi_x(\cdot,t)\|_{L^\infty})$$

so that we are not so far from the natural (but inappropriate) approach. Furthermore, notice also that introducing the map Y above enables one to go back to the original unknown $\tilde u(x,t)$ when desired.

[3] L_0 has always at least the translational zero-eigenfunction $\bar u'$.

[4] This latter comment repairs an omission in [JZ1, JZ2], where conditions for invertibility were not discussed. This issue is well-discussed in [DSSS], to which we refer the reader for further remarks. See also the more detailed derivation/estimates in [JNRZ2].

[5] This follows from $Y(X(y,t),t)-[X(y,t)+\psi(X(y,t),t)]=\psi(y,t)-\psi(y-\psi(y,t),t)$.

Remark 1.3. As noted in [JZ1], by Sturm-Liouville considerations, (D1) cannot hold in the scalar case $n = 1$; hence, our analysis *is relevant in the system case only,* that is, in the case $n > 1$.

1.1. **Discussion and open problems.** The main new observation here is that, by suitable adaptations/improvements, we can incorporate in the nonlinear iteration scheme of [JZ1, JZ2] the prescription of an initial phase shift h_0. Indeed, introducing a supplementary unknown ψ representing phase shift and performing the same nonlinear implicit change of variables (described in Remark 1.2) used in [JZ1] converts the initial nonlinear perturbation $\bar{u}(\cdot + h_0) - \bar{u}(\cdot)$ coming from such a phase shift, to its "linearization" $h_0\bar{u}'$, a term suitable for estimation by our iteration scheme based on linearized estimates. However, there are two important obstacles, both technical and conceptual, that must be overcome in implementing this simple observation to obtain a result.

The first concerns a mismatch between the prescribed initial phase shift h_0 and the phase shift predicted by our linearized estimates based on large-time asymptotics. In [JZ2], this change of variables was used to capture and factor out critical linear contributions in the direction of $\bar{u}'$. Yet, since only localized contributions were considered in [JZ1, JZ2], there was no need to prescribe $\psi(\cdot, 0)$, and this issue did not arise. More precisely, there was imposed simply $\psi(\cdot, 0) \equiv 0$, and the linear response was truncated by a smooth cutoff so as to return zero phase shift at initial time, consistent with the imposed data. Clearly, however, that will not work here, since we require that the initial shift be *nonzero*.

To this deceptively simple problem, we find a deceptively simple solution, namely to impose $\psi(\cdot, 0) = h_0$, as we must, but impose a smooth cutoff not on the linearized solution operator, but at the level of the nonlinear iteration scheme, prescribing ψ for $0 \leq t \leq 1$ as a convex average of the initial shift h_0 and the time-asymptotic value predicted by our linearized estimates; see (7.1)-(7.2) below. This effectively reproduces in our iteration scheme the same initial layer that was the cause of this technical issue in the first place, while shifting the difficulty from a conceptual to a technical one, namely, to show that the resulting perturbation variable is localized, despite nonlocalized initial data, an issue that we must in any case already face for other reasons.

This leads us to the second and, from a technical point of view, main issue, namely, to estimate the linear contribution of data $h_0\bar{u}'$, knowing only that $\partial_x h_0$ is localized. This essentially reduces to proving that $\partial_x S(t)(\bar{u}'h_0) \sim \bar{u}'s^{\mathrm{p}}(t)(\partial_x h_0)$, where $S(t)$ is the solution operator of the linear evolution about our wave $\bar{u}$ and $s^{\mathrm{p}}(t)$ is some averaged solution operator, well-approximated by the solution operator of equation (1.1) linearized about the constant wave number k_*. Note that this translates into rigorous estimates the expected modulation-type behavior that dynamics about a periodic wave reduces at first order to dynamics about a constant state in the space of parameters describing the family of nearby periodic waves (here just wave number). In proving so, one has to be able to disregard on one hand the contribution of far-from-coperiodic modes, non-critical close-to-coperiodic modes, and higher-order correction to the critical mode starting from $\bar{u}'$, and on another hand, the contribution of high frequencies of h_0 to close-to-coperiodic modes of $h_0\bar{u}'$. We perform this benefiting from the fact that the Bloch transform is both well-behaved with respect to spectral decompositions and expansions of our operator, and designed to deal with resonance-like contributions. Such details are not easily seen from the Green function description of [JZ1, JZ2, JZN]. Indeed, this seems to be an instance where frequency domain techniques detect cancelation not readily apparent by spatial-domain techniques.

In the process, we obtain an alternative proof of the basic estimates of localized terms in [JZ1], carried out entirely within the Bloch transform formulation without passing to a Green function description as in [JZ1]. This elucidates somewhat the relation between the Bloch transform based estimates of [S1, S2] and the Green function based estimates of [JZ1, JZ2], in particular the role of the Green function (integral kernel) in obtaining the estimates of [JZ1]; see Remark 3.2. We also obtain in passing, somewhat miraculously, the estimates needed to show localization of the terms introduced by our nonlinear truncation, thus overcoming the remaining technical obstacle and completing the circle of estimates needed to close our analysis; see (4.3) and Remark 4.2.3.

Our analysis suggests that one might by an elaboration of the approach used here treat still more general perturbations converging asymptotically to *any* fixed periodic perturbations at $\pm\infty$, not necessarily phase shifts of a single wave, which would then time-exponentially relax to modulations following the dynamics of the corresponding periodic problem; see Remark 4.2.2. This would be an interesting direction for further investigation.

We mention, finally, that the techniques introduced here are not limited to the reaction-diffusion case, but apply equally to the conservative case treated in [JZ2], yielding a comparable result of stability with respect to nonlocalized modulations [JNRZ2]. Indeed, this clarifies somewhat the relation between the reaction diffusion and conservation law case, revealing a continuum of models with common behavior lying between these two extremes. These issues will be reported on elsewhere.

Remark 1.4. Without change, we may treat reaction-diffusion systems with more general diagonal (rather than simple Laplacian) diffusion as considered in [DSSS]. More generally, all proofs go through whenever the linearized operator generates a C^0 semigroup and the nonlinear flow satisfies a damping estimate like (6.1); in particular, our results apply to the (sectorial) Swift–Hohenberg equations treated for localized perturbations in [S2][6] and to the general quasilinear $2r$-order parabolic case. It would be interesting to extend to the multi-dimensional case as in [U].

Note: Similar results have been obtained by different means by Sandstede and collaborators [SSSU] using a nonlinear decomposition of phase and amplitude variables as in [DSSS] together with renormalization techniques. We emphasize that, though the results are similar,[7] at a technical level these analyses are quite different. In particular, the methods of [SSSU] do not appear to apply to the conservative case treated in [JZ2, JNRZ2], for which asymptotic behavior is described by a *system* of viscous conservation laws, whereas our methods, having their origins in shock wave theory, are essentially designed for this; see Remark 5.2 for further discussion.

2. Preliminaries

Recall the Bloch solution formula for periodic-coefficient operators,

$$(S(t)g)(x) := (e^{tL}g)(x) = \int_{-\pi}^{\pi} e^{i\xi x}(e^{tL_\xi}\check{g}(\xi,\cdot))(x)d\xi \tag{2.1}$$

(i.e., $(e^{t\check{L}}g)(\xi,x) = (e^{tL_\xi}\check{g}(\xi,\cdot))(x)$, a consequence of (1.5)), where L_ξ is as in (1.4),

$$\check{g}(\xi,x) := \sum_{j\in\mathbb{Z}} \hat{g}(\xi+2j\pi)e^{i2\pi jx} \quad \text{(periodic in } x\text{)} \tag{2.2}$$

denotes the Bloch transform of g, $\hat{g}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}g(x)dx$ the Fourier transform, and

$$g(x) = \int_{-\pi}^{\pi} e^{i\xi x}\check{g}(\xi,x)d\xi \tag{2.3}$$

the inverse Bloch transform, or Bloch representation of $g \in L^2(\mathbb{R})$.

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[6]See [BJNRZ1, BJNRZ2, BJNRZ3, JZN] for related analyses in such general settings.

[7]For $\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{u}(\cdot)$ and $\partial_x h_0$ sufficiently small in $H^2_2(\mathbb{R})$, where $\|g\|_{H^2_2(\mathbb{R})} := \|\rho g\|_{H^2(\mathbb{R})}$ with $\rho(x); = (1+x^2)^{1/2}$, the results of [SSSU] yield the decay rates obtained here in (1.6) assuming smallness in $L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ (reducible to smallness in $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ as described in Remark 4.2, [JZ1]). That is, they give the same conclusions assuming the stronger localization of initial perturbations of order $\sim |x|^{-5/2}$ as compared to the order $\sim |x|^{-1}$ assumed here.

Note that, in view of the inverse Bloch transform formula (2.3), the generalized Hausdorff–Young inequality $\|u\|_{L^p(\mathbb{R})} \le \|\check{u}\|_{L^q([-\pi,\pi],L^p([0,1]))}$ for $q \le 2 \le p$ and $\frac{1}{p}+\frac{1}{q}=1$ [JZ1], yields for any 1-periodic functions $g(\xi,\cdot)$, $\xi \in [-\pi,\pi]$,

$$\| \int_{-\pi}^{\pi} e^{i\xi\cdot} g(\xi,\cdot)d\xi\|_{L^p(\mathbb{R})} \le (2\pi)^{1/p}\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} \text{ for } q\le 2\le p \text{ and } \frac{1}{p}+\frac{1}{q}=1, \tag{2.4}$$

where, here and elsewhere, we are denoting

$$\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} := \Big(\int_{-\pi}^{\pi} \|g(\xi,\cdot)\|^q_{L^p([0,1])}d\xi\Big)^{1/q}.$$

This convenient formulation is the one by which we will obtain all of our linear estimates.

Furthermore, notice by (D3) that there exists a simple zero eigenfunction $\bar{u}'$ of L_0, which by standard perturbation results [K] thus bifurcates to an eigenfunction $\phi(\xi,\cdot)$, with associated left eigenfunction $\tilde{\phi}(\xi,\cdot)$ and eigenvalue

$$\lambda(\xi) = ai\xi - d\xi^2 + O(|\xi|^3), \quad |\xi|\le \xi_0, \tag{2.5}$$

where a and d are real with $d>0$ by assumption (D2) and the complex symmetry, $\lambda(\xi)=\bar{\lambda}(-\xi)$. Again by standard perturbation results [K], each of the functions ϕ, $\tilde{\phi}$, and λ are analytic in ξ and well-defined for $|\xi|\le\xi_0$ sufficiently small.

3. Basic linear estimates

Loosely following [JZ1] decompose the solution operator as

$$S(t) = S^{\mathrm{p}}(t) + \tilde{S}(t), \qquad S^{\mathrm{p}}(t) = \bar{u}'\, s^{\mathrm{p}}(t), \tag{3.1}$$

with

$$(s^{\mathrm{p}}(t)g)(x) = \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi,\cdot),\check{g}(\xi,\cdot)\rangle_{L^2([0,1])}d\xi, \tag{3.2}$$

and

$$\begin{aligned}(\tilde{S}(t)g)(x) &:= \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check{g}(\xi))(x)d\xi + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde{\Pi}(\xi)\check{g}(\xi))(x)d\xi \\ &\quad + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}d\xi,\end{aligned} \tag{3.3}$$

where α is a smooth cutoff function such that $\alpha(\xi)=0$ for $|\xi|\ge\xi_0$ and $\alpha(\xi)=1$ for $|\xi|\le\frac{1}{2}\xi_0$,

$$\Pi^{\mathrm{p}}(\xi) := \phi(\xi)\langle\tilde{\phi}(\xi),\cdot\rangle_{L^2([0,1])}$$

denotes the eigenprojection onto the eigenspace Range$\{\phi(\xi)\}$ bifurcating from Range$\{\bar{u}'(x)\}$ at $\xi=0$, $\tilde{\phi}$ the associated left eigenfunction, and $\tilde{\Pi} := \mathrm{Id}-\Pi^{\mathrm{p}}$, each well-defined on suppt $\alpha \subset [-\xi_0,\xi_0]$.

We begin by reproving the following estimates established (in a slightly different form) in [JZ1], describing linear behavior under a localized perturbation $g\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$.

Proposition 3.1 ([JZ1])**.** *Under assumptions (H1)-(H2) and (D1)-(D3), for all $t>0$, $2\le p\le\infty$,*

$$\Big\|\partial_x^l\partial_t^m s^{\mathrm{p}}(t)\partial_x^r g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}}\|g\|_{L^1(\mathbb{R})},\\ (1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{l+m}{2}}\|g\|_{L^2(\mathbb{R})},\end{cases} \tag{3.4}$$

for $0\le r\le K+1$, and for some $\eta>0$ and $0\le l+2m, r\le K+1$,

$$\Big\|\partial_x^l\partial_t^m \tilde{S}(t)\partial_x^r g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|g\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})},\\ e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})} + (1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}\|g\|_{L^2(\mathbb{R})},\end{cases} \tag{3.5}$$

Estimates (3.4)–(3.5) were established in [JZ1] by first passing to a Green function formulation. Here, we give an alternative proof within the Bloch formulation that will be useful for what follows.

Proof. (i) (Proof of (3.4)*(i)).* First, expand

$$
\begin{aligned}
(s^{\mathrm{p}}(t)g)(x) &= \int_{-\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde{\phi}, \check{g}\rangle_{L^2([0,1])}(\xi) d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde{\phi}(\xi,\cdot), e^{i2\pi j \cdot}\rangle_{L^2([0,1])} \hat{g}(\xi + 2j\pi) d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \hat{\tilde{\phi}}_j(\xi)^* \hat{g}(\xi + 2j\pi) d\xi,
\end{aligned}
\tag{3.6}
$$

where $\hat{\tilde{\phi}}_j(\xi)$ denotes the jth Fourier coefficient in the Fourier expansion of periodic function $\tilde{\phi}(\xi,\cdot)$, and $z^* = \bar{z}$ denotes complex conjugate.

By Hausdorff-Young's inequality, $\|\hat{g}\|_{L^\infty(\mathbb{R})} \le \|g\|_{L^1(\mathbb{R})}$, and by (D2), $|e^{\lambda(\xi)t}\alpha^{1/2}(\xi)| \le e^{-\eta\xi^2 t}$, $\eta > 0$, while by Cauchy–Schwarz' inequality,

$$
\alpha^{1/2}(\xi)\sum_j |\hat{\tilde{\phi}}_j(\xi)| \le \alpha^{1/2}(\xi)\sqrt{\sum_j (1+|j|^2)|\hat{\tilde{\phi}}_j(\xi)|^2 \sum_j (1+|j|^{-2})} \le C\alpha^{1/2}(\xi)\|\tilde{\phi}(\xi)\|_{H^1([0,1])}.
$$

Combining these facts, and applying (2.4), we obtain for $1/q + 1/p = 1$

$$
\begin{aligned}
\|s^{\mathrm{p}}(t)g\|_{L^p(\mathbb{R})} &\lesssim \|\xi \mapsto e^{-\eta\xi^2 t}\|_{L^q([-\pi,\pi])} \sup_{|\xi|\le\xi_0} \|\tilde{\phi}(\xi,\cdot)\|_{H^1([0,1])}\|g\|_{L^1(\mathbb{R})} \\
&\lesssim (1+t)^{-\frac{1}{2}(1-1/p)}\|g\|_{L^1(\mathbb{R})},
\end{aligned}
\tag{3.7}
$$

yielding the result for $l = m = r = 0$. Estimates for general $l, m, r \ge 0$ go similarly, passing ∂_x^r derivatives onto $\tilde{\phi}$ in the inner product using integration by parts and noting that ∂_x^l and ∂_t^m derivatives bring down harmless bounded factors $(i\xi)^l$ and $\lambda(\xi)^m$ enhancing decay.

(ii) (Proof of (3.4)*(ii)).* The second estimate on s^{p} follows similarly, but substituting for the estimate of term $\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi), \check{g}(\xi)\rangle_{L^2([0,1])}d\xi$ the slightly simpler estimate

$$
\begin{aligned}
&\|\ x \mapsto \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi), \check{g}(\xi)\rangle_{L^2([0,1])}d\xi\ \|_{L^p(\mathbb{R})} \\
&\lesssim \|\ (\xi, x) \mapsto \alpha(\xi)e^{-\eta\xi^2 t}|\langle\tilde{\phi}(\xi), \check{g}(\xi)\rangle_{L^2([0,1])}|\ \|_{L^q([-\pi,\pi], L^p([0,1]))} \\
&\lesssim \|\ \xi \mapsto e^{-\eta\xi^2 t}\|\check{g}(\xi)\|_{L^2([0,1])}\ \|_{L^q([-\pi,\pi])} \\
&\lesssim \|\xi \mapsto e^{-\eta\xi^2 t}\|_{L^{r_1 q}([-\pi,\pi])}\ \|g\|_{L^2(\mathbb{R})} \\
&\lesssim (1+t)^{-\frac{1}{2}(1/2-1/p)}\|g\|_{L^2(\mathbb{R})},
\end{aligned}
$$

where $1/r_1 + 1/r_2 = 1$ and $qr_2 = 2$, so that $r_1 q = \frac{2q}{2-q}$ equals ∞ for $p = q = 2$, while it equals 2 for $q = 1$, $p = \infty$. This verifies the result for $l = m = r = 0$; estimates for general $l, m, r \ge 0$ go similarly, as described in (i) above.

(iii) (Proof of (3.5)*(1)).* By (D1)–(D2) and Prüss' Theorem [Pr],[8] we have

$$
|e^{L_\xi t}(1-\alpha(\xi))|_{H^{l+1}([0,1])\to H^{l+1}([0,1])},\quad |\alpha(\xi)e^{L_\xi t}\tilde{\Pi}(\xi)|_{H^{l+1}([0,1])\to H^{l+1}([0,1])} \lesssim e^{-\eta t}, \quad \eta > 0,
$$

[8] Here, along with standard parabolic resolvent estimates [He], we are using the fact (by smoothness of coefficients and basic ODE regularity theory) that $H^{l+1}([0,1])$ and $L^2([0,1])$ spectra coincide for the operators considered here.

whence, by Sobolev embedding,

$$|\partial_x^l e^{L_\xi t}(1-\alpha(\xi))|_{H^{l+1}([0,1])\to L^p([0,1])},\quad |\partial_x^j \alpha(\xi)e^{L_\xi t}\tilde\Pi(\xi)|_{H^{l+1}([0,1])\to L^p([0,1])}\lesssim e^{-\eta t}$$

for $2\le p\le\infty$. The $W^{l,p}(\mathbb{R})$ norms of the first two terms of (3.3), by (2.4) and Parseval's identity, $\|\check g\|_{L^2([-\pi,\pi],H^{l+1}([0,1]))}\sim\|g\|_{H^{l+1}(\mathbb{R})}$, more precisely

$$\begin{aligned}\frac{1}{2\pi}\|g\|^2_{H^{l+1}(\mathbb{R})} &= \|\check g\|^2_{L^2([-\pi,\pi],L^2([0,1]))}+\|\check g\|^2_{L^2([-\pi,\pi],\dot H_\xi^{l+1}([0,1]))}\\ &:= \|\check g\|^2_{L^2([-\pi,\pi],L^2([0,1]))}+\|(\partial_x+i\xi)^{l+1}\check g(\xi)\|^2_{L^2([-\pi,\pi],L^2([0,1]))},\end{aligned}$$

are thus bounded by $Ce^{-\eta t}\|g\|_{H^{l+1}(\mathbb{R})}$. The $W^{l,p}(\mathbb{R})$ norm of the third term may be bounded similarly as in the estimation of s^{p} above, noting that the factor $(\phi(\xi)-\phi(0))\sim|\xi|$ introduces an additional factor of $(1+t)^{-1/2}$ decay. This establishes the result for $m=r=0$; other cases go similarly, noting that $\partial_t e^{L_\xi t}\tilde\Pi(\xi)=L_\xi e^{L_\xi t}\tilde\Pi(\xi)$, with L_ξ a second-order operator, so that we may essentially trade one t-derivative for two x-derivatives.

(iv) (Proof of (3.5)*(ii)).* The second estimate on $\tilde S$ follows similarly, but substituting for the estimate of term $\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde\phi(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi$ the estimate

$$\begin{aligned}&\|x\mapsto\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde\phi(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi\ \|_{L^p(\mathbb{R})}\\ &\lesssim\|\ (\xi,x)\mapsto\alpha(\xi)e^{-\eta\xi^2 t}|\xi||\langle\tilde\phi(\xi),\check g(\xi)\rangle_{L^2([0,1])}|\ \|_{L^q(\xi,L^p([0,1]))}\\ &\lesssim\|\ \xi\mapsto|\xi|e^{-\eta\xi^2 t}\|\check g(\xi)\|_{L^2([0,1])}\ \|_{L^q([-\pi,\pi])}\\ &\lesssim\|\ \xi\mapsto|\xi|e^{-\eta\xi^2 t}\ \|_{L^{r_q}([-\pi,\pi])}\ \|g\|_{L^2(\mathbb{R})}\\ &\lesssim(1+t)^{-\frac12(1/2-1/p)-\frac12}\|g\|_{L^2(\mathbb{R})},\end{aligned}$$

where $1/p+1/q=1$ and $1/2+1/r_q=1/q$, so that $r_q=\frac{2q}{2-q}$ is ∞ for $p=q=2$ and 2 for $q=1$, $p=\infty$. $\square$

Remark 3.2. In [JZ1], by contrast, $s^{\mathrm{p}}(t)$ is estimated through its integral kernel

$$e(x,t;y):=(s^{\mathrm{p}}(t)\delta_y)(x)=\int_{-\pi}^{\pi}\alpha(\xi)e^{\lambda(\xi)t}e^{i\xi x}\tilde\phi(\xi,y)d\xi,$$

yielding the slightly sharper estimate

$$\frac{\|s^{\mathrm{p}}(t)g\|_{L^p(\mathbb{R})}}{\|g\|_{L^1(\mathbb{R})}}\le\sup_y\|e(\cdot,t;y)\|_{L^p(\mathbb{R})}\le\sup_{|\xi|\le\xi_0}\|\tilde\phi(\xi)\|_{L^\infty([0,1])}(1+t)^{-1/2(1-1/p)}$$

in place of the bound $C\sup_{|\xi|\le\xi_0}\|\tilde\phi(\xi)\|_{H^1([0,1])}(1+t)^{-1/2(1-1/p)}$ computed in (3.7). For our purposes, this makes no difference, and, as we shall see in Section 4, there can be an advantage in maintaining the separation into distinct frequencies afforded by the Bloch representation.

4. Linear behavior for modulational data

Next, we consider behavior of (2.1) when applied to modulational data $g=h_0\bar u'$, the linearized version of a nonlinear modulational perturbation $\bar u(x+h_0(x))-\bar u(x)\sim h_0(x)\bar u'(x)$. The following estimates, obtained by frequency-domain rather than spatial-domain (Green function) techniques as in [JZ1, JZ2, JZN], together with the associated modified nonlinear iteration scheme of Section 7, below, represent the main new technical contributions of this paper.

Proposition 4.1. *Under assumptions (H1)–(H2) and (D1)–(D3), for all $t > 0$, $2 \le p \le \infty$,*

$$\|\partial_x^l \partial_t^m s^{\mathrm{p}}(t)(h_0 \bar u')\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}, \tag{4.1}$$

for $l+m \ge 1$ or else $l = m = 0$ and $p = \infty$, and, for $0 \le l + 2m \le K+1$,

$$\|\partial_x^l \partial_t^m \tilde S(t)(h_0 \bar u')\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)} \|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})}, \tag{4.2}$$

and when $t \le 1$

$$\begin{array}{rcl} \|\partial_x^l \partial_t^m (S^{\mathrm{p}}(t) - \mathrm{Id})(h_0 \bar u')\|_{L^p(\mathbb{R})} & \lesssim & \|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})}, \\ \|\partial_x^l \partial_t^m (s^{\mathrm{p}}(t)(h_0 \bar u') - h_0)\|_{L^p(\mathbb{R})} & \lesssim & \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}. \end{array} \tag{4.3}$$

Proof. (i) (Proof of (4.1)*, $l+m \ge 1$).* We treat the case $l = 1$, $m = 0$; other cases go similarly. First, re-express

$$\begin{aligned} \partial_x (s^{\mathrm{p}}(t)(\bar u' h_0))(x) &= \int_{-\pi}^{\pi} i\xi \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde\phi \bar u', \check h_0 \rangle_{L^2([0,1])}(\xi) d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} i\xi \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde\phi(\xi,y) \bar u'(y), e^{i2\pi j y} \rangle_{L^2([0,1])} \hat h_0(\xi + 2j\pi) d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} i\xi \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \widehat{\tilde\phi(\xi)\bar u'}_j^{\,*} \hat h_0(\xi + 2j\pi) d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\xi}{\xi + 2\pi j} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \widehat{\tilde\phi(\xi)\bar u'}_j^{\,*} \widehat{\partial_x h_0}(\xi + 2j\pi) d\xi, \end{aligned} \tag{4.4}$$

where $\widehat{\tilde\phi(\xi)\bar u'}_j$ denotes the jth Fourier coefficient in the Fourier expansion of periodic function $\tilde\phi(\xi)\bar u'$. By (2.4) and $|e^{\lambda(\xi)t}\alpha^{1/2}(\xi)| \le e^{-\eta\xi^2 t}$, $\eta > 0$, we thus get

$$\|\partial_x(s^{\mathrm{p}}(t)(\bar u' h_0))\|_{L^p(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}(1-1/p)} \|\partial_x h_0\|_{L^1(\mathbb{R})} \sup_{|\xi|\le\xi_0} \sum_j \Big| \frac{\widehat{\tilde\phi(\xi)\bar u'}_j}{1+|j|} \Big|,$$

yielding the result together with the Cauchy–Schwarz estimate

$$\sum_j \Big| \frac{\widehat{\tilde\phi(\xi)\bar u'}_j}{1+|j|} \Big| \le \sqrt{\sum_j (1+|j|)^{-2} \sum_j |\widehat{\tilde\phi(\xi)\bar u'}_j|^2} \lesssim \|\tilde\phi(\xi)\bar u'\|_{L^2([0,1])}.$$

(ii) (Proof of (4.1)*, $l = m = 0$).*[9] This follows by an explicit error function decomposition analogous to that used in [JZ2], p. 18, to treat the conservative case. Computing similarly as in (4.4)

$$(s^{\mathrm{p}}(t)(\bar u' h_0))(x) = \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{1}{\xi + 2\pi j} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \widehat{\tilde\phi(\xi)\bar u'}_j^{\,*} \widehat{\partial_x h_0}(\xi + 2j\pi) d\xi \tag{4.5}$$

[9] This estimate, which lies apart from the rest of the arguments in the paper, is not needed for the nonlinear iteration. Moreover, the resulting bounds (1.7) may be recovered alternatively by the modulation decomposition of [JNRZ1] without case $l = 0$. Thus, this case might be skipped by the reader if desired.

we obtain by exactly the same computation as in (i) an estimate on high-frequency terms

$$\Big\| x \mapsto \sum_{j\neq 0} \int_{-\pi}^{\pi} e^{i\xi x} \frac{1}{\xi + 2\pi j} \alpha(\xi) e^{\lambda(\xi)t} \widehat{\tilde{\phi}(\xi)\bar{u}'}_j^{*} \widehat{\partial_x h_0}(\xi + 2j\pi) d\xi \Big\|_{L^p(\mathbb{R})}$$

$$\leq C(1+t)^{-\frac{1}{2}(1-1/p)} \|\partial_x h_0\|_{L^1(\mathbb{R})} \sup_{|\xi|\leq \xi_0} \sum_j \Big| \frac{\widehat{\tilde{\phi}(\xi)\bar{u}'}_j}{1+|j|} \Big|$$

that is stronger than we need, leaving only the estimation of the zero-frequency term

$$\widehat{\tilde{\phi}(\xi)\bar{u}'}_0^{*} \int_{-\pi}^{\pi} e^{i\xi x} \frac{\alpha(\xi) e^{\lambda(\xi)t}}{\xi} \widehat{\partial_x h_0}(\xi) d\xi.$$

Recalling expansion (2.5) of $\lambda(\xi)$ and using smoothness of α, we have

$$\begin{aligned} \int_{-\pi}^{\pi} e^{i\xi x} \frac{\alpha(\xi) e^{\lambda(\xi)t}}{\xi} \widehat{\partial_x h_0}(\xi) d\xi &= \int_{-\infty}^{+\infty} e^{i\xi x} \frac{e^{(-ia\xi - b\xi^2)t}}{\xi} \widehat{\partial_x h_0}(\xi) d\xi \\ &\quad - \int_{|\xi|\geq \pi} e^{i\xi x} \frac{e^{(-ia\xi - b\xi^2)t}}{\xi} \widehat{\partial_x h_0}(\xi) d\xi \\ &\quad + \int_{-\pi}^{\pi} e^{i\xi x} O(e^{-\eta\xi^2 t} |\widehat{\partial_x h_0}(\xi)|) d\xi \\ &=: I + II + III. \end{aligned} \tag{4.6}$$

Terms II and III are readily seen by Hausdorff–Young's inequality to be bounded by a stronger estimate $C\|\partial_x h_0\|_{L^1(\mathbb{R})}(1+t)^{-\frac{1}{2}(1-1/p)}$ than we need. Summarizing, the only non-negligible term in this computation is term I, which may be explicitly evaluated as the convolution of $\partial_x h_0$ with

$$e(x,t) := \text{P.V.} \int_{-\infty}^{+\infty} e^{i\xi x} \frac{e^{(-ai\xi - d\xi^2)t}}{i\xi} d\xi = \text{errfn}((x-at)^2/\sqrt{t}).$$

Estimating $\|e * \partial_x h_0\|_{L^\infty(\mathbb{R})} \leq \|e\|_{L^\infty(\mathbb{R})} \|\partial_x h_0\|_{L^1(\mathbb{R})} \leq \|\partial_x h_0\|_{L^1(\mathbb{R})}$, we are done.

(iii) (Proof of (4.2)*).* Likewise, this follows by re-expressing

$$\begin{aligned} \tilde{S}(t)(h_0\bar{u}')(x) &= \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check{h}_0(\xi)\bar{u}')(x)d\xi + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde{\Pi}(\xi)\check{h}_0(\xi)\bar{u}')(x)d\xi \\ &\quad + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde{\phi}(\xi), \check{h}_0(\xi)\bar{u}'\rangle_{L^2([0,1])}d\xi \\ &= \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi)) \frac{(e^{L_\xi t}(\bar{u}' e^{2i\pi j\cdot}))(x)}{i(\xi+2\pi j)} \widehat{\partial_x h_0}(\xi+2j\pi)d\xi \\ &\quad + \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi) \frac{(e^{L_\xi t}\tilde{\Pi}(\xi)(\bar{u}' e^{2ij\pi\cdot}))(x)}{i(\xi+2\pi j)} \widehat{\partial_x h_0}(\xi+2j\pi)d\xi \\ &\quad + \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t} \frac{(\phi(\xi,x)-\phi(0,x))\widehat{\tilde{\phi}\bar{u}'}_j(\xi)^*}{i(\xi+2\pi j)} \widehat{\partial_x h_0}(\xi+2j\pi)d\xi \end{aligned} \tag{4.7}$$

then estimating as before, where we are using $(1-\alpha(\xi)) \lesssim \xi$,

$$\tilde{\Pi}(\xi)\bar{u}' = \tilde{\Pi}(\xi)\Big[(\tilde{\Pi}(\xi)-\tilde{\Pi}(0))\bar{u}'\Big] \text{ with } |\tilde{\Pi}(\xi)-\tilde{\Pi}(0)|_{H^{l+1}([0,1])\to H^{l+1}([0,1])} \lesssim |\xi|,$$

and $\phi(\xi)-\phi(0)=O(\xi)$, respectively, to bound the key terms $\frac{(1-\alpha(\xi))}{i(\xi+2\pi j)}$, $\frac{\tilde\Pi(\xi)(\bar u' e^{2ij\pi\cdot})}{i(\xi+2\pi j)}$, and $\frac{\phi(\xi)-\phi(0)}{i(\xi+2\pi j)}$ appearing for $j=0$, and are using the Cauchy–Schwarz inequality

$$\sum_j \left| \frac{(2\pi ij)^{l+1}\widehat{\partial_x^{l+2}h_0}(\xi+2j\pi)}{(\xi+2j\pi j)^{l+2}} \right| \le C\sqrt{\sum_j \frac{1}{(1+|j|)^2}\sum_{j'}|\widehat{\partial_x^{l+2}h_0}(\xi+2j'\pi)|^2} \le C\|\partial_x h_0\|_{H^{l+1}(\mathbb{R})}$$

to bound the sum over $j\neq 0$ arising for the first and second terms on the righthand side of (4.7) upon estimating as in part (iii) of the proof of Proposition 3.1

$$|\partial_x^l e^{L_\xi t}(1-\alpha(\xi))|_{H^{l+1}([0,1])\to L^p([0,1])},\ \ |\partial_x^j \alpha(\xi)e^{L_\xi t}\tilde\Pi(\xi)|_{H^{l+1}([0,1])\to L^p([0,1])} \lesssim e^{-\eta t}$$

and

$$\Big\| \frac{(\bar u' e^{2i\pi j\cdot}))(x)}{i(\xi+2\pi j)}\widehat{\partial_x h_0}(\xi+2j\pi)\Big\|_{H^{l+1}([0,1])} \lesssim \Big| \frac{(2\pi ij)^{l+1}\widehat{\partial_x^{l+2}h_0}(\xi+2j\pi)}{(\xi+2\pi j)^{l+2}}\Big|.$$

This establishes the result for $m=0$; other cases go similarly, noting again that we may trade t-derivatives for x-derivatives using $\partial_t e^{L_\xi t}\tilde\Pi(\xi)=L_\xi e^{L_\xi t}\tilde\Pi(\xi)$,

(iv) (Proof of (4.3)*).* Expanding $S^{\mathrm{p}}(t)-\mathrm{Id}=(S^{\mathrm{p}}(t)-S^{\mathrm{p}}(0))-\tilde S(0)=t\partial_t S^{\mathrm{p}}(s(t))-\tilde S(0)$ for some $0<s(t)<t$, we obtain the first inequality by combining (4.1) and (4.2). Likewise, the second inequality follows by expanding $s^{\mathrm{p}}(t)(h_0\bar u')-h_0=(s^{\mathrm{p}}(t)-s^{\mathrm{p}}(0))(h_0\bar u')+(s^{\mathrm{p}}(0)(h_0\bar u')-h_0)$ and applying (4.1) together with

$$\begin{aligned}&\|s^{\mathrm{p}}(0)(h_0\bar u')-h_0\|_{L^p(\mathbb{R})} =\\ &\Big\| \textstyle\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))\sum_j \hat h_0(\xi+2j\pi)\widehat{\tilde\phi(\xi)\bar u'}^*_j d\xi \ +\int_{-\pi}^{\pi} e^{i\xi x}\sum_{j\neq 0}\hat h_0(\xi+2j\pi)\Big(\widehat{\tilde\phi(\xi)\bar u'}^*_j-1\Big)d\xi\Big\|_{L^p(\mathbb{R})}\\ &\le C\sup_{\xi\le\xi_0}\textstyle\sum_j \frac{|\xi||\widehat{\tilde\phi(\xi)\bar u'}_j|}{|\xi+2j\pi|}\|\partial_x h_0\|_{L^1(\mathbb{R})}+C\Big\|\xi\mapsto\sum_j\frac{|\widehat{\partial_x h_0}(\xi+2j\pi)|}{|\xi+2j\pi|}\Big\|_{L^2(\mathbb{R})}\le C\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}\end{aligned}$$

This establishes the case $l=m=0$; other cases go similarly. □

Remark 4.2. 1. If we split h_0 into high-frequency and low-frequency parts, then the contribution of the high-frequency part decays faster by factor $(1+t)^{-1/2}$ in all estimates (4.1)–(4.3).

2. In the estimate of s^{p}, it is easy to see that the same bounds hold if $\bar u'$ is replaced by any periodic $g\in H^1_{\mathrm{per}}([0,1])$. However, in the estimate of $\tilde S$, replacing $\bar u'$ by a periodic $g\in H^1_{\mathrm{per}}([0,1])$ that is not in $\ker\tilde\Pi(0)=\mathrm{Span}\{\bar u'\}$ introduces a time-exponentially decaying but spatially nonlocalized error $\int_{-\pi}^{\pi} e^{i\xi x}\ \widehat{h_0}(\xi)\ e^{tL_\xi}\tilde\Pi(0)(g)\,d\xi$, lying at best in $L^\infty(\mathbb{R})$ but unbounded in every $L^p(\mathbb{R})$, $p<\infty$, reflecting the dynamics of the problem on a periodic domain. That is, in this case, $\tilde S(t)(gh_0)\notin L^p(\mathbb{R})$, $p<\infty$, violating (4.2) for $l=m=0$. To extend our results to the sum of a localized perturbation and a perturbation asymptotic as $x\to\pm\infty$ to any two fixed periodic waves, not necessarily two shifted copies of the same wave, appears to require the estimation of this latter error term as a time-exponentially decaying function from $L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})$, a semigroup/Fourier multiplier problem of concrete technical nature. This would be an interesting direction for further investigation.

3. The estimate (4.3) is the key to handling the "initial layer problem" in our later nonlinear iteration, allowing us to prescribe initial data for ψ as is convenient for the analysis; see Remark 7.2.

5. Nonlinear perturbation equations

Essentially following [JZ1], for $\tilde u(x,t)$ satisfying $k_*\tilde u_t=k_*^2\tilde u_{xx}+f(\tilde u)+k_*c\tilde u_x$ and $\psi(x,t)$ to be determined, set

$$u(x,t)=\tilde u(x-\psi(x,t),t)\quad\text{and}\quad v(x,t)=u(x,t)-\bar u(x). \tag{5.1}$$

Lemma 5.1 ([JZ1]). *The nonlinear residual v defined in* (5.1) *satisfies*

$$k_*\left(\partial_t - L\right)\left(v + \psi \bar{u}_x\right) = k_* \mathcal{N}, \qquad k_* \mathcal{N} = \mathcal{Q} + \mathcal{R}_x + (k_* \partial_t + k_*^2 \partial_x^2) \mathcal{S} + \mathcal{T}, \tag{5.2}$$

where

$$\begin{aligned} \mathcal{Q} &:= f(v+\bar{u}) - f(\bar{u}) - df(\bar{u})v, \qquad \mathcal{R} := -k_* v\psi_t - k_*^2 v\psi_{xx} + k_*^2(\bar{u}_x + v_x)\frac{\psi_x^2}{1-\psi_x}, \\ \mathcal{S} &:= v\psi_x, \quad \text{and} \qquad \mathcal{T} := -\left(f(v+\bar{u}) - f(\bar{u})\right)\psi_x. \end{aligned} \tag{5.3}$$

Proof. (Following [JZ1].[10]) From definition (5.1) and the fact that $\tilde{u}$ satisfies (1.2), we obtain

$$k_*(1-\psi_x)u_t + k_*(-c+\psi_t)u_x = k_*^2\left(\frac{1}{1-\psi_x}u_x\right)_x + (1-\psi_x)f(u) \tag{5.4}$$

hence, subtracting the profile equation $-k_* c\bar{u}_x = k_*^2 \bar{u}_{xx} + f(\bar{u})$ for $\bar{u}$,

$$k_* v_t - k_* Lv + k_*\psi_t v_x - k_*\psi_x v_t + k_*\psi_t \bar{u}_x = \mathcal{Q} - \psi_x f(u) + k_*^2\left(\frac{\psi_x}{1-\psi_x}u_x\right)_x \tag{5.5}$$

or

$$k_*(\partial_t - L)v + k_*(\psi\bar{u}_x)_t = \mathcal{Q} + \mathcal{R}_x + (k_*\partial_t + k_*^2\partial_x^2)\mathcal{S} - \psi_x f(u) + k_*^2\left(\psi_x \bar{u}_x\right)_x, \tag{5.6}$$

and we finish the proof using $L\bar{u}_x = 0$ and the profile equation for $\bar{u}$ to obtain

$$k_* L(\psi\bar{u}_x) = k_* c\psi_x \bar{u}_x + k_*^2 \psi_x \bar{u}_{xx} + k_*^2\left(\psi_x\bar{u}_x\right)_x = -\psi_x f(\bar{u}) + k_*^2\left(\psi_x\bar{u}_x\right)_x \;. \tag{5.7}$$

□

Remark 5.2. The decomposition (5.2) has the effect of grouping linear order source terms $\psi_t\bar{u}_x$ and $\psi_x\bar{u}_{xx}$ that are individually too large to handle in our later nonlinear iteration into a single term $k_*\left(\partial_t - L\right)\left(\psi\bar{u}_x\right)$ with explicitly evaluable contribution $\psi\bar{u}_x$ that may be canceled with nondecaying linear translational modes s^{p} by a judicious choice of ψ (see (7.1) below). For the origins of this approach in the "instantaneous tracking method" of viscous shock theory, see [Z1] (Eq. (2.30)) and especially [HoZ] (in particular, the nearly identical Eq. (5.23) of Cor. 5.4, p. 453); for a general discussion of the method in the context of viscous shock waves, see [Z2]. For related analyses in different contexts featuring "pullback" coordinatizations analogous to (5.1), see [TZ1, Section 3] and [TZ2, Section 2.2.1] (group invariance and uniqueness), [Z3, Section 3] (translation-invariant center–stable manifold), and [TZ3, Theorem 2.2.0] (Nash–Moser uniqueness theorem), and references therein.

Similar, "pullback"-type, coordinatizations are used in an important way in [DSSS, SSSU]; however, to avoid any danger of confusion, we emphasize that our approach, here and in [JZ1, JZ2], (i) *originates from a different branch of the stability literature* [Z1, HoZ], substantially predating [DSSS],[11] and (ii) *is fundamentally different from those of* [DSSS, SSSU] in the central aspect of the way of detecting cancelation.[12] That is, the common use of coordinatization (5.1), though important, is only a necessary first step eliminating grossly infeasible nondecaying source terms and not the essential point of these various approaches. Our approach here has the advantage that it extends to the more difficult conservation law case [JZ2, JNRZ2], whose more elaborate asymptotic behavior would appear to greatly complicate an approach via normal forms.

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[10]Note that we have here followed a different convention than in [JZ1], reversing the sign of ψ in (5.1) in agreement with formal asymptotics of [Se, DSSS]. The change $\psi \to -\psi$ recovers the formulas of [JZ1].

[11]The "pullback" coordinatization was introduced in the periodic reaction-diffusion-context in [DSSS], and, separately, in the at-the-time seemingly technically-unrelated periodic conservation law context in [JZ2].

[12]Done here by subtracting out the expected principle linear response $k_*\psi\bar{u}_x$ from the solution and observing that the resulting source terms are small; done in [DSSS, SSSU] by the use of "mode filters," or approximate spectral projectors, and what might be described as the method of normal forms, i.e., successive nonlinear approximations.

6. Nonlinear damping estimate

Proposition 6.1 ([JZ1])**.** *Assuming (H1)–(H2), let $v(\cdot,0)\in H^K(\mathbb{R})$ (for v as in (5.1) and $K\ge 3$ as in (H1)) and suppose that for some $T>0$, the $H^K(\mathbb{R})$ norm of $v(t)$ and $\psi_t(t)$ and the $H^{K+1}(\mathbb{R})$ norm of $\psi_x(t)$ remain bounded by a sufficiently small constant for all $0\le t\le T$. Then there are positive constants θ and C, independent of T, such that, for all $0\le t\le T$,*

$$\|v(t)\|^2_{H^K(\mathbb{R})}\le C\,e^{-\theta t}\|v(0)\|^2_{H^K(\mathbb{R})}+C\int_0^t e^{-\theta(t-s)}\left(\|v(s)\|^2_{L^2(\mathbb{R})}+\|(\psi_t,\psi_x)(s)\|^2_{H^K(\mathbb{R})}\right)ds. \tag{6.1}$$

Proof. (Following [JZ1].) Take for writing simplicity $k_*=1$. Rewriting (5.6) as

$$\begin{aligned}(1-\psi_x)v_t-v_{xx}-cv_x&=df(\bar u)v+\mathcal{Q}-(\bar u_x+v_x)\psi_t\\&\quad+((\bar u_x+v_x)\psi_x)_x+\left((\bar u_x+v_x)\frac{\psi_x^2}{1-\psi_x}\right)_x-f(\bar u+v)\psi_x,\end{aligned}\tag{6.2}$$

taking the $L^2(\mathbb{R})$ inner product against $\sum_{j=0}^K\frac{(-1)^j\partial_x^{2j}v}{1-\psi_x}$, integrating by parts, and rearranging, we obtain

$$\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t)\le-\tilde\theta\|\partial_x^{K+1}v(t)\|^2_{L^2(\mathbb{R})}+C\left(\|v(t)\|^2_{H^K(\mathbb{R})}+\|(\psi_t,\psi_x)(t)\|^2_{H^K(\mathbb{R})}\right),$$

for some $\tilde\theta>0$, so long as $\|(v,\psi_t,\psi_x,\psi_{xx})(t)\|_{H^K(\mathbb{R})}$ remains sufficiently small. Sobolev interpolation $\|g\|^2_{H^K(\mathbb{R})}\le\tilde C^{-1}\|\partial_x^{K+1}g\|^2_{L^2(\mathbb{R})}+\tilde C\|g\|^2_{L^2(\mathbb{R})}$ gives, then, for $\tilde C>0$ sufficiently large,

$$\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t)\le-\theta\|v(t)\|^2_{H^K(\mathbb{R})}+C\left(\|v(t)\|^2_{L^2(\mathbb{R})}+\|(\psi_t,\psi_x)(t)\|^2_{H^K(\mathbb{R})}\right),$$

from which (6.1) follows by Gronwall's inequality. □

7. Nonlinear iteration scheme

The key idea is, similarly as in the localized case treated in [JZ1], starting with

$$(\partial_t-L)(v+\psi\bar u')=\mathcal{N},\qquad v|_{t=0}=d_0,\ \psi|_{t=0}=h_0,$$

where $d_0:=\tilde u_0(\cdot-h_0(\cdot))-\bar u\in L^1(\mathbb{R})\cap H^K(\mathbb{R})$, $\partial_x h_0\in L^1(\mathbb{R})\cap H^K(\mathbb{R})$, to choose ψ to cancel s^{p} contributions, as

$$\begin{aligned}\psi(t)&=s^{\mathrm{p}}(t)(h_0\bar u'+d_0)+\int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\\&\quad-(1-\chi(t))\left(s^{\mathrm{p}}(t)(d_0+h_0\bar u')-h_0+\int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\right),\end{aligned}\tag{7.1}$$

where $\chi(t)$ is a smooth cutoff that is zero for $t\le 1/2$ and one for $t\ge 1$, leaving the system

$$\begin{aligned}v(t)&=\tilde S(t)(d_0+h_0\bar u')+\int_0^t\tilde S(t-s)\mathcal{N}(s)ds\\&\quad+(1-\chi(t))\left(S^{\mathrm{p}}(t)d_0+(S^{\mathrm{p}}(t)-\mathrm{Id})h_0\bar u'+\int_0^t S^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\right).\end{aligned}\tag{7.2}$$

We may extract from (7.1)-(7.2) a closed system in (v,ψ_x,ψ_t) (and some of their derivatives), and then recover ψ through the slaved equation (7.1).

Remark 7.1. At first sight, we have accomplished nothing by introducing a ψ-dependent change of variable and choosing $\psi(0)=h_0$, since we have replaced the nonlocalized perturbation $\tilde u_0(x)-\bar u(x)$ used in the previous $h_0\equiv 0$ setup of [JZ1], by a different nonlocalized perturbation $d_0+h_0\bar u'$. However, what we really did was replace the asymptotic states $\bar u(x+c_\pm)-\bar u(x)$ by their linear approximants $c_\pm \bar u'(x)$, which removes the key difficulty of higher order remainders in the Taylor expansion of nonlinear modulations.

Remark 7.2. Notice that modulational data $\bar u' h_0$ enters in (7.2) only through operators $\tilde S(t)$ and $(1-\chi(t))(S^{\mathrm{p}}(t)-\mathrm{Id})$ for which we have Gaussian decay in $L^p(\mathbb{R})$ with respect to $\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^1(\mathbb{R})}$, hence the error incurred by defining ψ by (7.1) instead of the value $\tilde\psi(t) = s^{\mathrm{p}}(t)(h_0\bar u' + d_0) + \int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds$ exactly canceling s^{p} terms is harmless to our analysis. The choice of (7.1) reflects our need to accommodate the incompatibility between the initial value $\psi|_{t=0} = h_0$ prescribed by the spatially-asymptotic behavior of the initial perturbation and the function $\tilde\psi$ encoding time-asymptotic behavior of the perturbed solution; that is, it is a device to avoid having to resolve an initial layer near $t = 0$.[13] Whether this initial layer is an artifact of our analysis or reflects some aspect of short-time behavior is unclear; as the estimates show, this is below our level of resolution.

8. Nonlinear stability

With these preparations, the proof of stability now goes essentially as in the localized conservative case treated in [JZ2], using the new linear modulation bounds to estimate the new linear term coming from data $h_0\bar u'$ in (7.1) and (7.2). As noted in [JZ1], from differential equation (5.2) together with integral equation (7.1)-(7.2), we readily obtain short-time existence and continuity with respect to t of solution $(v,\psi_t,\psi_x)\in H^K(\mathbb{R})$ by a standard contraction-mapping argument treating the linear $df(\bar u)v$ term of the left hand side along with $\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{T},\psi\bar u'$ terms of the righthand side as sources in the heat equation. Associated with this solution define so long as it is finite,

$$\zeta(t) := \sup_{0\le s\le t} \|(v,\psi_t,\psi_x)(s)\|_{H^K(\mathbb{R})}(1+s)^{1/4}. \tag{8.1}$$

Lemma 8.1. *There exist positive constants C and ε such that if $E_0 := \|(d_0,\partial_x h_0)\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}$ is less than ϵ, then for $t\ge 0$ such that $\zeta(t)$ is finite and less than $\varepsilon^{1/2}$,*

$$\zeta(t)\le C(E_0+\zeta(t)^2). \tag{8.2}$$

Proof. [14] By (5.3) and corresponding bounds on the derivatives together with definition (8.1), and using (6.2) to bound v_t,

$$\|\mathcal{N}(t)\|_{L^1(\mathbb{R})\cap H^1(\mathbb{R})} \lesssim \|(v,\psi_t,\psi_x)(t)\|^2_{H^3(\mathbb{R})} \le C\zeta(t)^2(1+t)^{-\frac12}, \tag{8.3}$$

so long as $\zeta(t)$ remains small. Applying the bounds (3.4)(1)–(3.5)(1) and (4.1)–(4.3) of Propositions 3.1 and 4.1 to system (7.1)- (7.2), we obtain for any $2\le p<\infty$

$$\begin{aligned}\|v(t)\|_{L^p(\mathbb{R})} &\le C(1+t)^{-\frac12(1-1/p)}E_0 + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac12(1-1/p)-\frac12}(1+s)^{-\frac12}ds\\ &\le C_p(E_0+\zeta(t)^2)(1+t)^{-\frac12(1-1/p)}\end{aligned} \tag{8.4}$$

and

$$\begin{aligned}\|(\psi_t,\psi_x)(t)\|_{W^{K+1,p}(\mathbb{R})} &\le C(1+t)^{-\frac12}E_0 + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac12(1-1/p)-1/2}(1+s)^{-\frac12}ds\\ &\le C_p(E_0+\zeta(t)^2)(1+t)^{-\frac12(1-1/p)}.\end{aligned} \tag{8.5}$$

Estimate (8.5) yields in particular that $\|(\psi_t,\psi_x)(t)\|_{H^{K+1}(\mathbb{R})}$ is small, verifying the hypotheses of Proposition 6.1. From (6.1) and (8.4)–(8.5), we thus obtain

[13] In the case $h_0\equiv 0$, this essentially reduces to the simpler device used in [JZ1] to treat the localized case, of substituting $\chi(t)s^{\mathrm{p}}(t)$ for $s^{\mathrm{p}}(t)$. However, the latter is clearly too crude to treat the present case.

[14] Compare to the argument of [JZ2, Lemma 4.2] regarding localized perturbations in the conservative case.

$$\|v(t)\|_{H^K(\mathbb{R})} \le C(E_0+\zeta(t)^2)(1+t)^{-\frac14}. \tag{8.6}$$

Combining this with (8.5), $p=2$, rearranging, and recalling definition (8.1), we obtain the result. □

Proof of Theorem 1.1. By short-time H^K existence theory, $\|(v,\psi_t,\psi_x)(t)\|_{H^K(\mathbb{R})}$ is continuous so long as it remains small, hence ζ remains continuous so long as it remains small. By (8.1), therefore, it follows by continuous induction that $\zeta(t)\le 2CE_0$ for $t\ge 0$, if $E_0<1/4C$, yielding by (8.1) the result (1.6) for $p=2$. Applying (8.4)–(8.6), we obtain (1.6) for $2\le p\le p_*$ for any $p_*<\infty$, with uniform constant C. Taking $p_*>4$ and estimating

$$\|(\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{T})(t)\|_{L^2(\mathbb{R})} \lesssim \|(v,\psi_t,\psi_x,\psi_{xx})(t)\|^2_{L^4(\mathbb{R})} \le CE_0(1+t)^{-\frac34} \tag{8.7}$$

in place of the weaker (8.3), then applying (3.4)(ii) in place of (3.4)(i), we obtain

$$\begin{aligned}\|(\psi_t,\psi_x)(t)\|_{W^{K+1,p}(\mathbb{R})} &\le C(1+t)^{-\frac12}E_0 + CE_0\int_0^t (1+t-s)^{-\frac12(1/2-1/p)-1/2}(1+s)^{-\frac34}ds\\ &\le CE_0(1+t)^{-\frac12(1-1/p)},\end{aligned} \tag{8.8}$$

for $2\le p\le\infty$.[15] Likewise, using (8.7) together with bound

$$\|(\mathcal{Q},\mathcal{T})(t)\|_{H^1(\mathbb{R})} + \|\mathcal{R}(t)\|_{H^2(\mathbb{R})} + \|\mathcal{S}(t)\|_{H^3(\mathbb{R})} \lesssim \zeta(t)^2(1+t)^{-\frac12}$$

obtained in the course of proving (8.3), we may use (3.5)(ii) rather than (3.5)(i) to get

$$\begin{aligned}\|v(t)\|_{L^p(\mathbb{R})} &\le C(1+t)^{-\frac12(1-1/p)}E_0 + CE_0\int_0^t e^{-\eta(t-s)}(1+s)^{-\frac12}ds\\ &\quad + CE_0\int_0^t (1+t-s)^{-\frac12(1/2-1/p)-\frac12}(1+s)^{-\frac34}ds\\ &\le CE_0(1+t)^{-\frac12(1-1/p)}\end{aligned} \tag{8.9}$$

and achieve the proof of (1.6) for $2\le p\le\infty$.

Estimate (1.7) then follows through (7.1) using (3.4)(i), by

$$\|\psi(t)\|_{L^\infty(\mathbb{R})} \le CE_0 + CE_0\int_0^t (1+t-s)^{-\frac12}(1+s)^{-\frac12}ds \le CE_0, \tag{8.10}$$

yielding nonlinear stability by the fact that

$$\tilde u(x-\psi(x,t),t) - \bar u(x-\psi(x,t)) = v(x,t) + \bar u(x) - \bar u(x-\psi(x,t)), \tag{8.11}$$

so that

$$\|\tilde u(t)-\bar u\|_{L^\infty(\mathbb{R})} \le \|v(t)\|_{L^\infty(\mathbb{R})} + \|\bar u'\|_{L^\infty([0,1])}\|\psi(t)\|_{L^\infty(\mathbb{R})}.$$

□

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[15] We bound the $\partial_t\mathcal{S}$ contribution using $\int_0^t s^{\mathrm{p}}(t-s)\partial_t\mathcal{S}(s)ds = -\int_0^t \partial_t[s^{\mathrm{p}}](t-s)\mathcal{S}(s)ds + s^{\mathrm{p}}(0)\mathcal{S}(t) - s^{\mathrm{p}}(t)\mathcal{S}(0)$.

References

[BJNRZ1] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Witham averaged equations and modulational stability of periodic solutions of hyperbolic-parabolic balance laws,* to appear, Proceedings and seminars, Centre de Mathématiques de l'École polytechnique; Conference proceedings, "Journées equations aux derivées partielles", 2010, Port d'Albret, France.

[BJNRZ2] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Stability of periodic Kuramoto–Sivashinsky waves*, Appl. Math. Letters, 25 (2012) no. 5: 824-829.

[BJNRZ3] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinsky equation,* Preprint (2012).

[DSSS] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider, *The dynamics of modulated wave trains,* Mem. Amer. Math. Soc. 199 (2009), no. 934, viii+105 pp. ISBN: 978-0-8218-4293-5.

[He] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981).

[HoZ] *Asymptotic behavior of multi-dimensional scalar viscous shock fronts,* Indiana Univ. Math. J. 49 (2000) 427–474.

[JNRZ1] M. Johnson, P. Noble, L.M. Rodrigues, and K. Zumbrun, *Nonlocalized modulation of periodic reaction diffusion waves: The Whitham equation,* Preprint (2012).

[JNRZ2] M. Johnson, P. Noble, L.M. Rodrigues, and K. Zumbrun, *Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations,* in preparation.

[JZ1] M. Johnson and K. Zumbrun, *Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction diffusion equations,* Annales de l'Institut Henri Poincaré - Analyse non linéaire, 28 (2011), issue 4, 471-483.

[JZ2] M. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling waves of viscous conservation laws in the generic case*, J. of Diff. Eqn., 249 (2010), no. 5, 1213-1240.

[JZN] M. Johnson, K. Zumbrun, and P. Noble, *Nonlinear stability of viscous roll waves,* SIAM J. Math. Anal. 43 (2011), no. 2, 557-611.

[K] T. Kato, *Perturbation theory for linear operators*, Springer–Verlag, Berlin Heidelberg (1985).

[NR1] P. Noble, and M. Rodrigues, *Whitham's equations for modulated roll-waves in shallow flows,* unpublished manuscript (2010), arXiv:1011.2296.

[NR2] P. Noble, and M. Rodrigues, *Whitham's modulation equations and stability of periodic wave solutions of the generalized Kuramoto-Sivashinsky equations,* Preprint (2010).

[OZ1] M. Oh and K. Zumbrun, *Stability of periodic solutions of viscous conservation laws with viscosity- 1. Analysis of the Evans function*, Arch. Ration. Mech. Anal. 166 (2003), no. 2, 99–166.

[OZ2] M. Oh, and K. Zumbrun, *Stability and asymptotic behavior of traveling-wave solutions of viscous conservation laws in several dimensions*, Arch. Ration. Mech. Anal. 196 (2010), no. 1, 1-20. Erratum: Arch. Ration. Mech. Anal., 196 (2010), no. 1, 21-23.

[OZ3] M. Oh, and K. Zumbrun, *Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions*,

[Pr] J. Prüss, *On the spectrum of C^0-semigroups.* Trans. Amer. Math. Soc. 284 (1984), no. 2, 847–857.

[SSSU] B. Sandstede, A. Scheel, G. Schneider, and H. Uecker, *Diffusive mixing of periodic wave trains in reaction-diffusion systems with different phases at infinity*, to appear, J. Diff. Eq.

[S1] G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions– abstract theorem and higher space dimensions*, Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.

[S2] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation,* (English. English summary) Comm. Math. Phys. 178 (1996), no. 3, 679–702.

[Se] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis*, Comm. Partial Differential Equations 30 (2005), no. 1-3, 259–282.

[TZ1] B. Texier and K. Zumbrun, *Relative Poincaré-Hopf bifurcation and galloping instability of traveling waves,* Methods Appl. Anal. 12 (2005), no. 4, 349–380.

[TZ2] B. Texier and K. Zumbrun, *Galloping instability of viscous shock waves,* Physica D. 237 (2008) 1553-1601.

[TZ3] B. Texier and K. Zumbrun, *Nash–Moser iterates and singular perturbations,* to appear, Annales de l'Institut Henri Poincare - Analyse non lineaire; Preprint (2009).

[U] H. Uecker, *Diffusive stability of rolls in the two-dimensional real and complex Swift-Hohenberg equation,* Comm. Partial Differential Equations 24 (1999), no. 11-12, 2109–2146.

[Z1] K. Zumbrun, *Refined Wave–tracking and Nonlinear Stability of Viscous Lax Shocks.* Methods Appl. Anal. 7 (2000) 747–768.

[Z2] K. Zumbrun, Instantaneous shock location and one-dimensional nonlinear stability of viscous shock waves. Quart. Appl. Math. 69 (2011), no. 1, 177–202.

[Z3] K. Zumbrun, *Conditional stability of unstable viscous shocks,* J. Differential Equations 247 (2009), no. 2, 648–671.

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