NONLOCALIZED MODULATION OF PERIODIC REACTION DIFFUSION WAVES: THE WHITHAM EQUATION

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Abstract. In a companion paper, we established nonlinear stability with detailed diffusive rates of decay of spectrally stable periodic traveling-wave solutions of reaction diffusion systems under small perturbations consisting of a nonlocalized modulation plus a localized (L^1) perturbation. Here, we determine time-asymptotic behavior under such perturbations, showing that solutions consist to leading order of a modulation whose parameter evolution is governed by an associated Whitham averaged equation.

1. Introduction

In this, the second part of a two-part series of papers, we determine the time-asymptotic behavior of spectrally stable periodic traveling-wave solutions of reaction diffusion systems under small perturbations consisting of a nonlocalized modulation plus a localized (L^1) perturbation, showing that solutions consist of an (in general nonlocalized) modulation governed by the formal second-order Whitham averaged equations plus a faster-decaying localized residual.

In the first part, [JNRZ1], we established nonlinear stability under such perturbations, together with detailed rates of decay, using a refined version of the argument used in [JZ1] to show stability under localized perturbations. In each of these cases, the basic approach was to introduce a phase *via* a phase-dependent change of variables, then separate out from the linearized solution operator slowly-decaying terms corresponding to linear phase shifts, and estimate separately the modulational and nonmodulational parts of the solution. While we reduced at first order the long-time dynamics of the solution to that of phase shifts, no information about the behavior of the phase was given besides decay rates.

Here, we show that, using the same basic linear estimates as in [JNRZ1], but a one-order-higher decomposition of the solution operator, we may describe at a higher precision the asymptotic behavior in terms of a full modulation instead of just a phase shift. This enables us, by a further linear estimate in the spirit of [JNRZ1], to prove that the principal part of the local wave number appearing in this modulation obeys a convected Burgers equation that is asymptotically equivalent to a solution of an associated *Whitham modulation equation* of a type derived formally in, e.g., [S1, DSSS, W].

This not only gives rigorous validation of the formal Whitham approximation in the strong sense of showing that it describes time-asymptotic behavior, but, through the explicit prescription of initial data coming from our analysis, also gives new information of predictive value not available from the formal asymptotic derivation. Indeed, as discussed in [BJNRZ], the connection between initial perturbation and initial values for the Whitham equations has remained for a long time somewhat mysterious. As a side-consequence, we show that the decay rates of [JNRZ1] are sharp.

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We recall first the nonlinear stability result of [JNRZ1]. Consider a periodic traveling-wave solution $u(x,t) = \bar u(k_*(x-ct))$ of reaction diffusion system $u_t = u_{xx} + f(u)$, or, equivalently, a standing-wave solution $u(x,t) = \bar u(x)$ of

$$k_* u_t = k_*^2 u_{xx} + f(u) - \omega_0 u_x, \tag{1.1}$$

where $\omega_0(k_*) := -k_* c(k_*)$ is the temporal frequency, c is the speed of the original traveling wave, and wave number k_* is chosen so that

$$\bar u(x+1) = \bar u(x). \tag{1.2}$$

Here and throughout the paper, all periodic functions are assumed to be periodic of period one.

We make the following standard genericity assumptions:

(H1) $f \in C^K(\mathbb{R}^n)$, $(K \geq 3)$.

(H2) Up to translation, the set of 1-periodic solutions of (1.1) (with k replacing k_*) in the vicinity of $\bar u$, $k = k_*$, forms a smooth 1-dimensional manifold $\{\bar u(\cdot;k)\} = \{\bar u^k(\cdot)\}$, $c = c(k)$.

Linearizing (1.1) about $\bar u$ yields the periodic coefficient equation

$$k_* v_t = k_* L v := (k_*^2 \partial_x^2 - \omega_0(k_*)\partial_x + b)v, \qquad b(x) := df(\bar u(x)). \tag{1.3}$$

Introducing the one-parameter family of closed Floquet operators

$$k_* L_\xi := e^{-i\xi x} k_* L e^{i\xi x} = k_*^2(\partial_x + i\xi)^2 - \omega_0(k_*)(\partial_x + i\xi) + b \tag{1.4}$$

acting on $L^2_{\mathrm{per}}([0,1])$ with densely defined domains $H^2_{\mathrm{per}}([0,1])$, determined by the defining relation

$$L(e^{i\xi x} g) = e^{i\xi x}(L_\xi g), \tag{1.5}$$

we define following [S1, S2] the standard *diffusive spectral stability* conditions:

(D1) $\sigma(L) \subset \{\lambda \mid \Re\lambda < 0\} \cup \{0\}$.

(D2) There exists a constant $\theta > 0$ such that $\sigma(L_\xi) \subset \{\lambda \mid \Re\lambda \leq -\theta|\xi|^2\}$ for each $\xi \in [-\pi, \pi)$.

(D3) $\lambda = 0$ is a simple eigenvalue of L_0.[1]

Notice in (D1) above, we consider L as a closed operator on $L^2(\mathbb{R})$ with densely defined domain $H^2(\mathbb{R})$.

The following stability result was established in [JNRZ1], generalizing results of [S1, S2, JZ1]. Here, and throughout the paper, given two real valued functions A and B, we say that $A \lesssim B$, or that for every $x \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$, $A(x) \lesssim B(x)$, if there exists a constant $C > 0$ such that $A(x) \leq C B(x)$ for each $x \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$. Even in a chain of inequalities, we will also feel free to denote by C harmless constants with different values.

Proposition 1.1 ([JNRZ1])**.** *Let $K \geq 3$. Assuming (H1)-(H2) and (D1)-(D3), let*

$$E_0 := \left\|\tilde u_0(\cdot - h_0(\cdot)) - \bar u(\cdot)\right\|_{L^1(\mathbb{R}) \cap H^K(\mathbb{R})} + \left\|\partial_x h_0\right\|_{L^1(\mathbb{R}) \cap H^K(\mathbb{R})}$$

be sufficiently small, for some choice of phase modulation h_0. Then, there exists a global solution $\tilde u(x,t)$ of (1.1) with initial data $\tilde u_0$ and a phase function $\psi(x,t)$ such that, for $t > 0$ and $2 \leq p \leq \infty$,

$$\begin{aligned} &\left\|\tilde u(\cdot - \psi(\cdot,t),t) - \bar u(\cdot)\right\|_{L^p(\mathbb{R})}, \quad \left\|\nabla_{x,t}\psi(\cdot,t)\right\|_{W^{K+1,p}(\mathbb{R})} \lesssim E_0(1+t)^{-\frac12(1-1/p)}, \\ &\qquad\qquad \left\|\tilde u(\cdot - \psi(\cdot,t),t) - \bar u(\cdot)\right\|_{H^K(\mathbb{R})} \lesssim E_0(1+t)^{-\frac14}, \end{aligned} \tag{1.6}$$

and

$$\left\|\tilde u(\cdot,t) - \bar u(\cdot)\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\psi(\cdot,t)\right\|_{L^\infty(\mathbb{R})} \lesssim E_0. \tag{1.7}$$

In particular, $\bar u$ is nonlinearly (boundedly) stable in $L^\infty(\mathbb{R})$ with respect to initial perturbations $v_0 = \tilde u_0 - \bar u$ for which $\|v_0\|_E := \inf_{\partial_x h_0 \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})} E_0$ is sufficiently small.

[1] L_0 has always at least the translational zero-eigenfunction $\bar u'$.

Recall now the formal, Whitham equation, as derived in various contexts and to varying degrees of accuracy in [W, HK, Se, DSSS, NR1, NR2]. By translation-invariance of the underling equations,

$$L_0 \bar{u}' = 0, \tag{1.8}$$

so that by (D3) the zero-eigenspace of L_0 is exactly Range$\{\bar{u}'\}$. Denote by $\bar{u}^{ad}$ the left, or adjoint, zero eigenfunction of L_0. Fixing $\bar{u}$, k_*, introduce the parametrization

$$\bar{u}^k(x-\beta) = \bar{u}(k, x-\beta) \tag{1.9}$$

of nearby periodic traveling waves, i.e., 1-periodic solutions of $\omega_0(k)\bar{u}^k_x - k^2\bar{u}^k_{xx} - f(\bar{u}^k) = 0$, $\omega_0(k) = -kc(k)$, for definiteness chosen in such a way that

$$\langle \bar{u}^{ad}(k), \partial_k \bar{u}(k)\rangle_{L^2([0,1])} = 0. \tag{1.10}$$

(This may be achieved by an appropriate translation, by the fact that $\langle \bar{u}^{ad}, \bar{u}'\rangle_{L^2([0,1])} \neq 0$.) Then, the formal approximate solution of $u_t - u_{xx} - f(u) = 0$ obtained by a nonlinear WKB expansion is

$$u(x,t) \approx \bar{u}^{\kappa(x,t)}(\Psi(x,t)), \tag{1.11}$$

where the wave number $\kappa := \Psi_x$ satisfies the Whitham equation (viscous scalar conservation law)

$$\kappa_t - (\omega_0(\kappa))_x = (d(\kappa)\kappa_x)_x \tag{1.12}$$

or equivalently, the phase Ψ satisfies its integral (viscous Hamilton–Jacobi equation)

$$\Psi_t - \omega_0(\Psi_x) = d(\Psi_x)\Psi_{xx} \tag{1.13}$$

where, taking (1.10) into account, $d(k) = 1 + 2k\langle u^{ad}(k), \partial_k \bar{u}'(k)\rangle_{L^2([0,1])}$, and $\omega_0(k) = -c(k)k$ is the nonlinear dispersion relation determined by the manifold of periodic traveling-wave solutions $\bar{u}^k(k(x-ct))$ described in (H2). See [NR1, NR2] for a detailed derivation of this kind of nonlinear Whitham's equation in the context of the Saint-Venant and Kuramoto-Sivashinsky equations when the modulation procedure yields a system rather than an equation.

In our context, we look for the evolution of a localized perturbation $k = k_* h_x$ in our co-moving frame, and so it is enough to retain the quadratic order approximants

$$k_* k_t + k_* q(k)_x = k_*^2 d(k_*) k_{xx}, \tag{1.14}$$

and

$$k_* h_t + q(k_* h_x) = k_*^2 d(k_*) h_{xx} \tag{1.15}$$

with $q(k) = -(\omega_0'(k_*) + c(k_*))k - \frac{1}{2}\omega_0''(k_*)k^2$.

For, as is well known, (1.12) and (1.14) are "asymptotically equivalent" for such data, in the sense that the difference between solutions of (1.12) and (1.14) decays faster in all $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, than does the solution itself, which, for an initial perturbation with nonzero integral, decays at the rate of a heat kernel. See [DSSS] for a direct derivation of the quadratic Whitham's equation.

The notion of asymptotic equivalence is quantified in the following result.

Lemma 1.2. *Let $\eta > 0$ be arbitrary. Let κ be a solution of (1.12) with initial datum κ_0 and k be a solution of (1.14) with initial datum $k_0 = \kappa_0(\,\cdot\,/k_*) - k_*$, $E_0 := \|k_0\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})}$ sufficiently small. Then, setting $\tilde{\kappa}(x,t) = k_* + k(k_*(x - c(k_*)t), k_* t)$,*

$$\big\|(\kappa - \tilde{\kappa})(t)\big\|_{L^p(\mathbb{R})} \lesssim E_0^2 (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \qquad 1 \leq p \leq \infty;$$

moreover, if $m_0 := \int_{\mathbb{R}} k_0 \neq 0$ and $E_1 := E_0 + \||\cdot|\, k_0\|_{L^1(\mathbb{R})}$ is sufficiently small, then

$$\big\|k(t) - \phi(\cdot, 1+t)\big\|_{L^p(\mathbb{R})}, \quad \big\|(\kappa - \tilde{\kappa})(t)\big\|_{L^p(\mathbb{R})} \lesssim E_0^2 (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \qquad 1 \leq p \leq \infty; \tag{1.16}$$

where

$$\phi(x,t) \;=\; \frac{1}{\sqrt{t}}\,\bar\phi\left(\frac{x+\omega_0'(k_*)t}{\sqrt{t}}\right) \tag{1.17}$$

is the unique self-similar[2] *solution of* (1.14) *determined by* $\int \bar\phi = \int k_0$; *in particular,* $\|\kappa(t)-k_*\|_{L^p(\mathbb{R})}$, $\|\tilde\kappa(t)-k_*\|_{L^p(\mathbb{R})} \gtrsim |m_0|\,(1+t)^{-\frac12(1-1/p)}$.

Proof. See Appendix A. □

Our main result is as follows.

Theorem 1.3. *Let* $\eta>0$ *and* $K\geq 3$. *Under the assumptions of Proposition 1.1, let* k *and* h *satisfy the quadratic approximants* (1.14) *and* (1.15) *of the second-order Whitham modulation equations* (1.12) *and* (1.13) *with initial data* $k|_{t=0}=k_*\partial_x h_0$, $h|_{t=0}=h_0$, *and let* ψ *be the phase prescribed in the proof of Proposition 1.1 in* [JNRZ1] *(see* (4.4) *below). Then, for* $t>0$, $2\leq p\leq\infty$,

$$\begin{aligned}
\|\tilde u(\cdot-\psi(\cdot,t),t)-\bar u^{k_*(1+\psi_x(\cdot,t))}(\,\cdot\,)\|_{L^p(\mathbb{R})} &\lesssim E_0\ln(2+t)\;(1+t)^{-\frac34},\\
\|k_*\partial_x\psi(t)-k(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-1/p)-\frac12+\eta},\\
\|\psi(t)-h(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-1/p)+\eta}.
\end{aligned} \tag{1.18}$$

Remark 1.4. From estimates (1.18)(i) on $\tilde u$ and (1.6)(ii) and (1.7) on ψ_x and ψ, we obtain

$$\|\tilde u(\cdot,t)-\bar u^{k_*(1+\psi_x(\cdot,t))}(\tilde\Psi(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0\ln(2+t)\;(1+t)^{-\frac34},\qquad 2\leq p\leq\infty,$$

where $\tilde\Psi(\cdot,t)$ is the inverse of $y\mapsto X(y,t):=y-\psi(y,t)$. We insure the existence of such a map by keeping, for any t, $\|\psi(t)\|_{L^\infty(\mathbb{R})}$ bounded and $\|\psi_x(t)\|_{L^\infty(\mathbb{R})}$ small. Since $\tilde\Psi(x,t)-(x+\psi(x,t))=\psi(\tilde\Psi(x,t),t)-\psi(\tilde\Psi(x,t)-\psi(\tilde\Psi(x,t),t),t)$ one may translate this into a bound

$$\|\tilde u(\cdot,t)-\bar u^{k_*\tilde\Psi_x(\cdot,t)}(\tilde\Psi(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0\ln(2+t)\;(1+t)^{-\frac34},\qquad 2\leq p\leq\infty, \tag{1.19}$$

of the form (1.11), or degrade it into[3] $\|\tilde u(\cdot,t)-\bar u^{k_*(1+\psi_x(\cdot,t))}(\cdot+\psi(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0(1+t)^{-\frac12(1-1/p)}$, obtaining thereby, for $\eta>0$ arbitrary, and $2\leq p\leq\infty$, the exact Whitham comparison

$$\|\tilde u(\cdot,t)-\bar u^{k_*+k(\cdot,t)}(\cdot+h(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0(1+t)^{-\frac12(1-1/p)+\eta}. \tag{1.20}$$

Here, we could as well write

$$\|\tilde u(\cdot,t)-\bar u(\cdot+h(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0(1+t)^{-\frac12(1-1/p)+\eta} \tag{1.21}$$

similarly as in (1.6), since k is negligible at this order of approximation.

From Theorem 1.3 and Lemma 1.2, we see immediately that the decay rates of Proposition 1.1 are sharp. At the same time, we give rigorous validation of the Whitham equation in two ways. The first, more obvious way, is to show through (1.20) that asymptotic behavior consists of modulation by a solution h of the exact Whitham equations. Here, as pointed out in Remark 1.4, modulation in wave number is asymptotically irrelevant, and only phase shift plays a role. The second, less direct, but ultimately sharper (by factor $(1+t)^{-1/4}$ to $(1+t)^{-1/2}$ in rate of decay) way, is to show through (1.19) that *a more accurate description of asymptotic behavior is modulation including both phase shift and variation in wave number* by a solution $\tilde\Psi$ of an approximate Whitham equation with rapidly decaying error term (term $\tilde r(t)$ of (5.4), together with terms of similar order coming from the difference $\tilde\Psi_x-\Psi_x=O(\psi_x^2)$; see Remark A.1). See [DSSS] for validation of the Whitham equation in the altenative sense of building a family of solutions existing on asymptotically-large

[2] More exactly self-similar in a frame moving with linear group velocity.

[3] We use here $\|\tilde\Psi(\cdot,t)-(\,\cdot\,+\psi(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R})}\|\psi_x(\cdot,t)\|_{L^p(\mathbb{R})}$; see [JNRZ2].

but bounded intervals and close to a given asymptotic expansion involving a given solution of the Whitham equation.

Remark 1.5. Similarly as in [JZ1], all of our analysis goes through in the general quasilinear $2r$-order parabolic case; in particular, our results extend to the (sectorial) Swift–Hohenberg equations treated for localized perturbations in [S2]. See [BJNRZ], Appendix B, for related analysis.

Remark 1.6. We conjecture that (1.18)(i) and thus also of (1.19) may be improved for $p > 2$ to

$$\left\|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{u}^{k_*(1+\psi_x(\cdot,t))}(\,\cdot\,)\right\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}$$

by the use of additional, $W^{k,p} \to L^p$ estimates outside the scope of this paper; see Remark 4.3.

1.1. **Discussion and open problems.** The Whitham modulation equations, and WKB approximations in general, are "magical" prescriptions of great predictive value that may be obtained by formal, "consistency-type," considerations suppressing the sometimes very complicated details of the underlying equations. As a consequence, they often hide the mechanisms leading to the behavior they predict. In particular, rigorous verification of such asymptotic expansions typically comes from techniques apart from the method of derivation of the Whitham equation. Moreover, in the course of verification, these other techniques may often give additional information not found in the formal asymptotics.

The present case is no exception. The Whitham modulation equations, based on slowly varying perturbations, can be related rather directly at the linearized, spectral level, to perturbation expansions of small Floquet-number/eigenvalue modes; see, in various contexts, [DSSS, NR1, NR2]. Moreover, the same type of critical modes expansion, followed carefully, leads to the linearized estimates of [JZ1]. However, to carry this analysis to the nonlinear level involves significant technical challenges, and appears to require more indirect methods, motivated by, but at a technical level quite different from, the formal Whitham modulation.

In [JNRZ1] and the present paper, we have broken the nonlinear analysis into two distinct pieces, focused respectively on *decay* and *asymptotic behavior*. In [JNRZ1], by a judiciously chosen nonlinear transformation involving an implicitly prescribed shift in phase, we showed how to convert rigorously the picture of behavior afforded by formal asymptotics to a system of integral equations exhibiting the expected nonlinear decay. At a technical level, this could be understood as identifying the main part of the linearized solution operator as a linearized phase shift, and separating off this part of the behavior by a counterbalancing nonlinear change in phase. Put most naively, the analysis is driven by the observation that the critical mode given by the kernel of the linearized Bloch operator L_0 about the wave at Bloch frequency $\xi = 0$ is

$$\phi(0,\cdot) = \bar{u}', \tag{1.22}$$

where $\bar{u}'$ represents instantaneous translation.

The above analysis gives a simple and self-contained argument yielding sharp rates of decay to a phase modulation of the wave. In the present analysis, we show how to extract from the system of integral equations derived in [JNRZ1] an approximate differential equation governing the phase, and to connect this equation to the formally predicted Whitham equation. The first step is to go one step further in the decomposition of the solution, Taylor expanding the critical mode $\phi(\xi,x)$ of L_ξ about $\xi = 0$, and observing that, under an appropriate normalization (see (1.10) and (2.6)), the first order corrector is

$$\partial_\xi \phi(0,\cdot) = ik_* \partial_k u(\cdot;k_*) \tag{1.23}$$

corresponding at linear level to modulation in the wave number k; see Lemma 2.1 below. As often happens for higher-order correctors, the corresponding nonlinear correction can be made in simple, *linear* fashion (see (4.6)), since this term is fast enough decaying that nonlinear effects are negligible.

This yields a new residual z decaying at rate (4.8) faster by factor $(1+t)^{-1/4}$ to $(1+t)^{-1/2}$ than that of the residual v of [JNRZ1]. Once this is done, we may, isolating explicit terms k in the $\partial_k \bar{u}$ direction, and discarding as asymptotically negligible all terms in the integral equations that are not linear or quadratic in k, obtain, evaluating the resulting quadratic interaction coefficients, an integral equation that may be recognized as the Duhamel (variation of constants) representation of a forced Burgers equation corresponding to (1.14). From this description and our previous bounds on the residual, we then readily obtain the convergence result of our main Theorem 1.3.

Taken together, these two analyses give a blueprint for connecting formal asymptotics to the integral equations natural for analysis. This should carry over to other interesting settings; indeed, we have already carried out in [JNRZ2] the extension to the much more complicated conservation law case. We note that, despite its difficulty, the analysis in the end, both for decay and for convergence to the Whithams approximant, is quite short and easily verified. In particular, there is little advance preparation by formal asymptotics, different from the approach set out in [DSSS].

We note also that our approach gives "direct" access to bounds, i.e., we "solve" rather than "impose." As noted earlier, we obtain as a result somewhat sharper bounds (1.19) than what is available from comparison to the exact Whitham equation, a result that would at least not be easily derived by starting from the exact Whitham solution to begin with. That is, the system of integral equations we derive contains *more* information than the formal Whitham approximation.

One interesting problem for future investigation is the rigorous verification of the spectral stability conditions in interesting situations either by analysis or numerical proof. Another, very interesting direction, is the derivation of pointwise estimates on nonlocalized perturbations similarly as has been done for localized solutions in [J].

A simplifying aspect of the present analysis is that we were able to separate the analyses of decay and behavior. However, there could be an advantage in combining these, in that Burgers equation is known to decay even for large perturbations, and, thanks to the maximum principle, decays in L^∞. Thus, one might hope by such a simultaneous argument structure to treat the case that ψ_x is large but bounded in L^1 but only small in L^∞. This would be a very interesting extension to carry out. Finally, as described in [DSSS], nonlinear stability of shock-type solutions, for which not only ψ but k approaches different endstates, is a challenging and very interesting open problem, which appears to require substantial new ideas beyond those introduced here; in particular, perturbation around such shock-type solutions and not a background periodic wave.

Note: Similar results have been obtained by different means by Sandstede and collaborators [SSSU] using a nonlinear decomposition of phase and amplitude variables combined with a renormalization iteration process as in [S1, S2]. Specifically, they obtain the bound (1.21) assuming the somewhat stronger localization of initial perturbations of order $\sim |x|^{-5/2}$ as compared to the order $\sim |x|^{-1}$ assumed here; see footnote 7, [JNRZ1]. Our bound (1.19), though not of the same form, gives more precise information by factor $(1+t)^{-1/4}$ to $(1+t)^{-1/2}$ in the rates of decay; note also that this takes into account modulation in wave number, whereas the approximation of [SSSU] does not. The results of [SSSU] on the other hand include also convergence (at the same rate) to a "nonlinear diffusion wave" ϕ as described in (1.17). Combining (1.20) with (1.16), we recover this bound as well, but with localization $\sim |x|^{-2}$ closer to the $|x|^{-5/2}$ assumption of [SSSU].

2. Preliminaries

Recall the Bloch solution formula for periodic-coefficient operators,

$$(S(t)g)(x) := (e^{tL}g)(x) = \int_{-\pi}^{\pi} e^{i\xi x}(e^{tL_\xi}\check{g}(\xi,\cdot))(x)d\xi, [4] \tag{2.1}$$

[4] In other words, $(e^{t\check{L}}g)(\xi,x) = (e^{tL_\xi}\check{g}(\xi,\cdot))(x)$ a consequence of (1.5).

where L_ξ is as in (1.4),

$$\check{g}(\xi,x) := \sum_{j\in\mathbb{Z}} \hat{g}(\xi+2j\pi)e^{i2\pi jx} \quad \text{(periodic in } x) \tag{2.2}$$

denotes the Bloch transform of g, $\hat{g}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}g(x)dx$ the Fourier transform, and

$$g(x) = \int_{-\pi}^{\pi} e^{i\xi x}\check{g}(\xi,x)d\xi \tag{2.3}$$

the inverse Bloch transform, or Bloch representation of $g \in L^2(\mathbb{R})$. The generalized Hausdorff–Young inequality $\|u\|_{L^p(\mathbb{R})} \leq (2\pi)^{1/p}\|\check{u}\|_{L^q([-\pi,\pi],L^p([0,1]))}$ for $q \leq 2 \leq p$ and $\frac{1}{p}+\frac{1}{q}=1$ [JZ1], together with (2.3), yields for any 1-periodic function $g(\xi,\cdot)$

$$\Big\| \int_{-\pi}^{\pi} e^{i\xi\cdot} g(\xi,\cdot)d\xi \Big\|_{L^p(\mathbb{R})} \leq (2\pi)^{1/p}\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} \text{ for } q\leq 2\leq p \text{ and } \frac{1}{p}+\frac{1}{q}=1 \tag{2.4}$$

where, here and elsewhere, we are denoting

$$\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} := \Big(\int_{-\pi}^{\pi} \|g(\xi,\cdot)\|^q_{L^p([0,1])}d\xi\Big)^{1/q}.$$

By (D3), the zero eigenfunction $\bar{u}'$ of L_0 is simple, whence by standard perturbation results bifurcates to an eigenfunction $\phi(\xi,\cdot)$, with associated left eigenfunction $\tilde{\phi}(\xi,\cdot)$ and eigenvalue

$$\lambda(\xi) = ai\xi - d\xi^2 + O(|\xi|^3), \tag{2.5}$$

where a and d are real and $d>0$ by assumption (D2) and complex symmetry $\lambda(\xi)=\bar{\lambda}(-\xi)$, each of ϕ, $\tilde{\phi}$, λ analytic in ξ and defined for $|\xi|\leq\xi_0$, ξ_0 being positive and sufficiently small.

Before refining linear estimates of [JNRZ1], we need some extra spectral preparation. For this purpose, we set $\tilde{\phi}(0)=\bar{u}^{ad}$, assume parametrization is normalized according to (1.10) and normalize eigenfunctions in such a way that

$$\langle \bar{u}^{ad},\phi(\xi)\rangle_{L^2([0,1])} \;=\; \langle\tilde{\phi}(0),\phi(\xi)\rangle_{L^2([0,1])} \;=\; 1 \tag{2.6}$$

for all $\xi\in[-\xi_0,\xi_0]$. A similar preparation was already needed in [DSSS], see [DSSS, Section 4.2].

Lemma 2.1 ([DSSS])**.** *Assume (H1)-(H2) and (D3), normalize according to* (1.10) *and* (2.6)*. Then*

$$\partial_\xi\phi(0,\cdot) \;=\; ik_*\partial_k u(\cdot;k_*) \tag{2.7}$$

and the constants a and d in (2.5) *are*

$$a = -k_*c'(k_*), \qquad d = k_*(1+2k_*\langle u^{ad},\partial_k\bar{u}'(k_*)\rangle_{L^2([0,1])}). \tag{2.8}$$

Proof. To expand the equation $L_\xi\phi(\xi)=\lambda(\xi)\phi(\xi)$, we split L_ξ

$$k_*L_\xi = L^{(0)} + ik_*\xi L^{(1)} + (ik_*\xi)^2L^{(2)}.$$

Then we find $ik_*a\bar{u}' = L^{(0)}\partial_\xi\phi(0) + ik_*L^{(1)}\bar{u}'$. Yet differentiating the profile equation with respect to k yields $-k_*c'(k_*)\bar{u}' = L^{(0)}\partial_k\bar{u}(k_*) + L^{(1)}\bar{u}'$. Thus, taking scalar product with $\bar{u}^{ad}$ leads to

$$a \;=\; \langle u^{ad}, L^{(1)}\bar{u}'\rangle_{L^2([0,1])} \;=\; -c'(k_*)$$

hence $L^{(0)}\partial_\xi\phi(0) = L^{(0)}\partial_k\bar{u}(k_*)$ and therefore $\partial_\xi\phi(0)-\partial_k\bar{u}(k_*)\in\ker L_0=\mathbb{C}\bar{u}'$. Normalizations (1.10) and (2.6) reduce it to (2.7). Afterwards, expanding further, we find

$$k_*^2c'(k_*)\partial_k\bar{u}(k_*) - k_*d\;\bar{u}' \;=\; \frac{1}{2}L^{(0)}\partial_\xi^2\phi(0) + (ik_*)^2L^{(1)}\partial_k\bar{u}(k_*) + (ik_*)^2L^{(2)}\bar{u}'.$$

Taking scalar product with $\bar u^{ad}$ again gives

$$d \;=\; k_*\langle u^{ad}, L^{(1)}\partial_k \bar u(k_*) + L^{(2)}\bar u'\rangle_{L^2([0,1])}.$$

which, thanks to the explicit form

$$L^{(1)}\partial_k \bar u' + L^{(2)}\bar u' \;=\; 2k_*\partial_k \bar u(k_*)' + c(k_*)\partial_k \bar u(k_*) + \bar u',$$

completes the proof. □

From now on, although we do not repeat them, we always assume the above normalizations.

3. Linear estimates

Now, refining slightly the decomposition of [JZ1, JNRZ1], decompose the solution operator as

$$S(t) = R^{\mathrm p}(t) + \tilde R(t), \qquad R^{\mathrm p}(t) = (\bar u' + k_*\partial_k \bar u \partial_x)s^{\mathrm p}(t), \tag{3.1}$$

with

$$(s^{\mathrm p}(t)g)(x) = \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde\phi(\xi,\cdot),\check g(\xi,\cdot)\rangle_{L^2([0,1])}d\xi, \tag{3.2}$$

and

$$\begin{aligned}(\tilde R(t)g)(x) :=& \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check g(\xi))(x)d\xi + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde\Pi(\xi)\check g(\xi))(x)d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x)-\xi\partial_\xi\phi(0,x))\langle\tilde\phi(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi,\end{aligned} \tag{3.3}$$

where α is a smooth cutoff function such that $\alpha(\xi)=0$ for $|\xi|\geq\xi_0$ and $\alpha(\xi)=1$ for $|\xi|\leq\frac12\xi_0$,

$$\Pi^{\mathrm p}(\xi) := \phi(\xi)\langle\tilde\phi(\xi),\cdot\rangle_{L^2([0,1])} \tag{3.4}$$

denotes the eigenprojection onto the eigenspace Range$\{\phi(\xi)\}$ bifurcating from Range$\{\bar u'\}$ at $\xi=0$, $\tilde\phi$ the associated left eigenfunction, and $\tilde\Pi := \mathrm{Id} - \Pi^{\mathrm p}$, each well-defined on suppt$(\alpha)\subset[-\xi_0,\xi_0]$.

Proposition 3.1 ([JZ1, JNRZ1])**.** *Under assumptions (H1)-(H2) and (D1)-(D3), for all $t>0$, $2\leq p\leq\infty$,*

$$\left\|\partial_x^l\partial_t^m s^{\mathrm p}(t)\partial_x^r g\right\|_{L^p(\mathbb R)} \lesssim \min\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac{l+m}{2}}\|g\|_{L^1(\mathbb R)},\\ (1+t)^{-\frac12(1/2-1/p)-\frac{l+m}{2}}\|g\|_{L^2(\mathbb R)},\end{cases} \tag{3.5}$$

for $0\leq r\leq K+1$, and for some $\eta>0$ and $0\leq l+2m, r\leq K+1$,

$$\left\|\partial_x^l\partial_t^m \tilde R(t)\partial_x^r g\right\|_{L^p(\mathbb R)} \lesssim \min\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac12}\|g\|_{L^1(\mathbb R)\cap H^{l+2m+1}(\mathbb R)},\\ e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb R)} + (1+t)^{-\frac12(1/2-1/p)-1}\|g\|_{L^2(\mathbb R)},\end{cases} \tag{3.6}$$

Proof. The estimates on $s^{\mathrm p}$ were proved in [JNRZ1]. Using (2.4) together with spectral projection/direct computation for low Floquet numbers and standard semigroup estimates for high Floquet numbers, estimates on $\tilde R$ are proved exactly as were the estimates on $\tilde S$ in [JNRZ1], with the observation that the substitution of factor $(\phi(\xi)-\phi(0)-\xi\partial_\xi\phi(0))$ in $\tilde R$ for factor $(\phi(\xi)-\phi(0))$ in $\tilde S$ introduces an additional factor of $|\xi|$ in the estimates, hence an additional $(1+t)^{-1/2}$ factor of decay. As we will build on the main arguments of these proofs to establish further linear estimates, we briefly recall them here.

(i) (Proof of (3.5)*).* In the case $l = m = r = 0$ estimates on s^{p} follows from, for either $s = 1$ or $s = 2$, introducing s' such that $1/s + 1/s' = 1$,

$$\begin{aligned}
&\left\| \, x \mapsto \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}d\xi \, \right\|_{L^p(\mathbb{R})} \\
&\lesssim \| \, (\xi,x) \mapsto \alpha(\xi)e^{\lambda(\xi)t}|\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}| \, \|_{L^q([-\pi,\pi],L^p([0,1]))} \\
&\lesssim \|\xi \mapsto e^{-\eta\xi^2 t}\|_{L^{r_{s,p}}([-\pi,\pi])} \, \|\xi \mapsto \alpha(\xi)^{1/2}|\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}|\|_{L^{s'}([-\pi,\pi])} \\
&\lesssim (1+t)^{-\frac{1}{2}(1/s-1/p)}\|\xi \mapsto \alpha(\xi)^{1/2}|\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}|\|_{L^{s'}([-\pi,\pi])},
\end{aligned}$$

where $1/p + 1/q = 1$ and $1/s' + 1/r_{s,p} = 1/q$, so that $1/r_{s,p} = 1/s - 1/p$. Here we have used the fact that by (D2) for some $\eta > 0$, $|e^{\lambda(\xi)t}\alpha^{1/2}(\xi)| \le e^{-\eta\xi^2 t}$.

Now for $s = 2$, we combine this with Cauchy-Schwarz inequality and Parseval identity that yield

$$\begin{aligned}
\left\|\xi \mapsto \alpha(\xi)^{1/2}|\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}\right\|_{L^2([-\pi,\pi])} &\le \sup_{|\xi|\le\xi_0} \|\tilde{\phi}(\xi,\cdot)\|_{L^2([0,1])} \, \|\check{g}\|_{L^2([-\pi,\pi],L^2([0,1])} \\
&\lesssim \sup_{|\xi|\le\xi_0} \|\tilde{\phi}(\xi,\cdot)\|_{L^2([0,1])} \, \|g\|_{L^2(\mathbb{R})}
\end{aligned}$$

while for $s = 1$ we write

$$\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])} = \sum_{j\in\mathbb{Z}} \hat{\tilde{\phi}}_j(\xi)^*\hat{g}(\xi + 2j\pi)$$

where $\hat{\tilde{\phi}}_j(\xi)$ denotes the jth Fourier coefficient in the Fourier expansion of periodic function $\tilde{\phi}(\xi,\cdot)$, and $z^* = \bar{z}$ denotes complex conjugate, and apply Hausdorff-Young's inequality, $\|\hat{g}\|_{L^\infty(\mathbb{R})} \le \|g\|_{L^1(\mathbb{R})}$ with Cauchy–Schwarz' inequality,

$$\alpha^{1/2}(\xi)\sum_j |\hat{\tilde{\phi}}_j(\xi)| \le \alpha^{1/2}(\xi)\sqrt{\sum_j (1+|j|^2)|\hat{\tilde{\phi}}_j(\xi)|^2 \sum_j (1+|j|^{-2})} \le C\alpha^{1/2}(\xi)\|\tilde{\phi}(\xi)\|_{H^1([0,1])}.$$

to get

$$\|\xi \mapsto \alpha(\xi)^{1/2}|\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}|\|_{L^\infty([-\pi,\pi])} \lesssim \sup_{|\xi|\le\xi_0} \|\tilde{\phi}(\xi,\cdot)\|_{H^1([0,1])} \, \|g\|_{L^1(\mathbb{R})}.$$

This yields the result for $l = m = r = 0$. Estimates for general $l, m, r \ge 0$ go similarly, passing ∂_x^r derivatives onto $\tilde{\phi}(\xi)$ in the inner product using integration by parts and noting that ∂_x^l and ∂_t^m derivatives bring down bounded factors $(i\xi)^l$ and $\lambda(\xi)^m$ enhancing decay through

$$\|\xi \mapsto |\xi|^{l+m}e^{-\eta\xi^2 t}\|_{L^{r_{s,p}}([-\pi,\pi])} \lesssim (1+t)^{-\frac{1}{2}(1/s-1/p)-\frac{l+m}{2}}$$

(ii) (Proof of (3.6)*).* By (D1)–(D2), benefiting from standard parabolic resolvent estimates [He] together with the fact that (by basic ODE regularity theory) $H^{l+1}([0,1])$ and $L^2([0,1])$ spectra coincide, we may apply Prüss' Theorem [Pr] and get

$$|e^{L_\xi t}(1-\alpha(\xi))|_{H^{l+1}([0,1])\to H^{l+1}([0,1])}, \quad |\alpha(\xi)e^{L_\xi t}\tilde{\Pi}(\xi)|_{H^{l+1}([0,1])\to H^{l+1}([0,1])} \lesssim e^{-\eta t}, \quad \eta > 0,$$

whence, by Sobolev embedding,

$$|\partial_x^l e^{L_\xi t}(1-\alpha(\xi))|_{H^{l+1}([0,1])\to L^p([0,1])}, \quad |\partial_x^j \alpha(\xi)e^{L_\xi t}\tilde{\Pi}(\xi)|_{H^{l+1}([0,1])\to L^p([0,1])} \lesssim e^{-\eta t}$$

for $2 \le p \le \infty$. The $W^{l,p}(\mathbb{R})$ norms of the first two terms of (3.3), by (2.4) and Parseval's identity,

$$\frac{1}{2\pi}\|g\|^2_{H^{l+1}(\mathbb{R})} = \|\check{g}\|^2_{L^2([-\pi,\pi],L^2([0,1]))} + \left\|(\partial_x + i\xi)^{l+1}\check{g}(\xi)\right\|^2_{L^2([-\pi,\pi],L^2([0,1]))},$$

are thus bounded by $Ce^{-\eta t}\|g\|_{H^{l+1}(\mathbb{R})}$. The $W^{l,p}(\mathbb{R})$ norm of the third term may be bounded similarly as in the estimation of s^{p} above, noting that the factor $(\phi(\xi)-\phi(0)-\xi\partial_\xi\phi(0))=O(\xi^2)$ introduces an additional factor of $(1+t)^{-1}$ decay. This establishes the result for $m=r=0$; other cases go similarly, noting that $\partial_t e^{L_\xi t}\tilde\Pi(\xi)=L_\xi e^{L_\xi t}\tilde\Pi(\xi)$, with L_ξ a second-order operator, so that we may essentially trade one t-derivative for two x-derivatives. □

Proposition 3.2 ([JNRZ1]). *Under assumptions (H1)–(H2) and (D1)–(D3), for all $t>0$ and $2\le p\le\infty$,*

$$\|\partial_x^l\partial_t^m s^{\mathrm{p}}(t)(h_0\bar u')\|_{L^p(\mathbb{R})}\lesssim(1+t)^{-\frac12(1-1/p)+\frac12-\frac{l+m}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})},\tag{3.7}$$

when $l+m\ge1$ or else $l=m=0$ and $p=\infty$, and, for $0\le l+2m\le K+1$,

$$\|\partial_x^l\partial_t^m\tilde R(t)(h_0\bar u')\|_{L^p(\mathbb{R})}\lesssim(1+t)^{-\frac12(1-1/p)-1}\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})},\tag{3.8}$$

and when $t\le1$

$$\begin{aligned}\|\partial_x^l\partial_t^m(R^{\mathrm{p}}(t)-\mathrm{Id})(h_0\bar u')\|_{L^p(\mathbb{R})}&\lesssim\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})},\\ \|\partial_x^l\partial_t^m(s^{\mathrm{p}}(t)(h_0\bar u')-h_0)\|_{L^p(\mathbb{R})}&\lesssim\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}.\end{aligned}\tag{3.9}$$

Proof. Again, estimates on s^{p} were proved in [JNRZ1] whereas the proof of those on R^{p} and $\tilde R$ goes exactly as the proof of the corresponding estimates for S^{p} and $\tilde S$ in [JNRZ1], noting that the substitution of factor $(\phi(\xi)-\phi(0)-\xi\partial_\xi\phi(0))$ in $\tilde R$ for factor $(\phi(\xi)-\phi(0))$ in $\tilde S$ introduces an additional factor of $(1+t)^{-1/2}$ in the decay rate for $\tilde R$. We briefly outline the new points of the arguments.

(i) (Proof of (3.7)*).* The inequality (3.7) was established in [JNRZ1], Proposition 4.1.

(ii) (Proof of (3.8)*).* Contribution of the last term in (3.3) is bounded as in the proof of (3.7). For the remaining terms, we first split

$$(h_0\bar u')\check{}\,(\xi,x)=\bar u'(x)\check{h_0}(\xi,x)=\sum_{j\in\mathbb{Z}}\bar u'(x)e^{2\pi jx}\widehat{h_0}(2\pi j+\xi)$$

then apply to each term of the sum the semigroup bounds as above and, after noting that $(1-\alpha(\xi))\lesssim\xi$ and that

$$\tilde\Pi(\xi)\bar u'=\tilde\Pi(\xi)\Big[(\tilde\Pi(\xi)-\tilde\Pi(0))\bar u'\Big]\quad\text{with}\quad|\tilde\Pi(\xi)-\tilde\Pi(0)|_{H^{l+2m+1}([0,1])\to H^{l+2m+1}([0,1])}\lesssim|\xi|,$$

achieve the proof with the Cauchy–Schwarz inequality

$$\begin{aligned}\Big\|x\mapsto\bar u'(x)\widehat{\partial_x h_0}(\xi)\Big\|_{H^{l+2m+1}([0,1])}&+\sum_{j\neq0}\Big\|x\mapsto\frac{\bar u'(x)e^{2i\pi jx}}{i(\xi+2\pi j)}\widehat{\partial_x h_0}(\xi+2\pi j)\Big\|_{H^{l+2m+1}([0,1])}\\ &\lesssim|\widehat{\partial_x h_0}(\xi)|+\sum_{j\neq0}\left|\frac{(2\pi j)^{l+2m+1}(\widehat{\partial_x^{l+2m+2}h_0})(\xi+2\pi j)}{(\xi+2\pi j)^{l+2m+2}}\right|\\ &\lesssim|\widehat{\partial_x h_0}(\xi)|+\sqrt{\sum_j\frac{1}{(1+|j|)^2}\sum_{j'}|(\widehat{\partial_x^{l+2m+2}h_0})(\xi+2\pi j')|^2}\\ &\lesssim|\widehat{\partial_x h_0}(\xi)|+\sqrt{\sum_{j\in\mathbb{Z}}|(\widehat{\partial_x^{l+2m+2}h_0})(\xi+2\pi j)|^2},\end{aligned}$$

the $L^2([-\pi,\pi])$ norm of the last quantity being bounded by $\|\partial_x h_0\|_{H^{l+2m+1}(\mathbb{R})}$.

(iii) (Proof of (3.9)*).* Expanding $R^{\mathrm{p}}(t) - \mathrm{Id} = (R^{\mathrm{p}}(t) - R^{\mathrm{p}}(0)) - \tilde{R}(0) = t\partial_t R^{\mathrm{p}}(s(t)) - \tilde{R}(0)$ for some $0 < s(t) < t$, we obtain the first inequality by combining (3.7) and (3.8). The second inequality was established in [JNRZ1], Proposition 4.1. □

We require also the following key new estimates, proved by similar techniques. The first statement reflects the fact that modulations approximately travel at reference group speed a. The second one quantifies the fact that the (parabolic) second-order linearized modulation equation is given by (3.13).

Lemma 3.3. *Under assumptions (H1)-(H2) and (D1)-(D3), for all $t > 0$, $2 \le p \le \infty$,*

$$\|(\partial_t - a\partial_x)(s^{\mathrm{p}}(t)g)\|_{L^p(\mathbb{R})} \lesssim \min \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-1}\|g\|_{L^1(\mathbb{R})}, \\ (1+t)^{-\frac{1}{2}(1/2-1/p)-1}\|g\|_{L^2(\mathbb{R})}, \end{cases} \tag{3.10}$$

$$\|(\partial_t - a\partial_x)(s^{\mathrm{p}}(t)(h_0\bar{u}'))\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}. \tag{3.11}$$

Proof. Differentiating (3.2), we have

$$(\partial_t - a\partial_x)(s^{\mathrm{p}}(t)g)(x) = \int_{-\pi}^{\pi} (\lambda(\xi) - ai\xi) e^{i\xi x} \alpha(\xi) e^{\lambda(\xi)t} \langle \tilde{\phi}(\xi), \check{g}(\xi)\rangle_{L^2([0,1])}(\xi) d\xi \tag{3.12}$$

with $\lambda(\xi) - ia\xi = O(|\xi|^2)$, whence the result follows, again by (2.4). □

Proposition 3.4. *Assuming (H1)–(H2) and (D1)–(D3), let $\sigma(t)$ be the solution operator of the convected heat equation*

$$u_t = au_x + du_{xx}, \tag{3.13}$$

where a, d are as in (2.5), and let g be a periodic function on $[0,1]$, $g \in L^2([0,1])$. Then, for all $t > 0$, $2 \le p \le \infty$, and $l \in \mathbb{N}$,

$$\|\partial_x^l s^{\mathrm{p}}(t)(h_0 g) - \langle \tilde{\phi}(0), g\rangle \ \sigma(t)(\partial_x^l h_0)\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} t^{-\frac{l-1}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}. \tag{3.14}$$

Proof. Expressing

$$\begin{aligned} \langle \tilde{\phi}(0), g\rangle_{L^2([0,1])} (\sigma(t)\partial_x^l h_0)(x) &= \int_{\mathbb{R}} (i\xi)^l e^{i\xi x} e^{(ia\xi - d\xi^2)t} \langle \tilde{\phi}(0), g\rangle_{L^2([0,1])} \widehat{h_0}(\xi) d\xi \\ \partial_x^l s^{\mathrm{p}}(t)(gh_0)(x) &= \int_{-\pi}^{\pi} (i\xi)^l e^{i\xi x} \alpha(\xi) e^{\lambda(\xi)t} \sum_j \langle \tilde{\phi}(\xi), g e^{i2\pi j\cdot}\rangle_{L^2([0,1])} \widehat{h_0}(\xi + 2j\pi) d\xi \end{aligned}$$

and subtracting, we obtain, for $l \in \mathbb{N}$,

$$\begin{aligned} \big[\langle \tilde{\phi}(0), g\rangle \sigma(t)(\partial_x^l h_0) - \partial_x^l (s^{\mathrm{p}}(t)(gh_0))\big](x) = &\int_{\mathbb{R}\setminus[-\pi,\pi]} e^{i\xi x} e^{(ia\xi - d\xi^2)t} (i\xi)^{l-1} \langle \tilde{\phi}(0), g\rangle \, \widehat{\partial_x h_0}(\xi) d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x} (1 - \alpha(\xi)) e^{(ia\xi - d\xi^2)t} (i\xi)^{l-1} \langle \tilde{\phi}(0), g\rangle \, \widehat{\partial_x h_0}(\xi) d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) (i\xi)^{l-1} (e^{(ia\xi - d\xi^2)t} - e^{\lambda(\xi)t}) \langle \tilde{\phi}(0), g\rangle \, \widehat{\partial_x h_0}(\xi) d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda(\xi)t} (i\xi)^{l-1} \langle \tilde{\phi}(0) - \tilde{\phi}(\xi), g\rangle \, \widehat{\partial_x h_0}(\xi) d\xi \\ &- \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda(\xi)t} (i\xi)^l \sum_{j \ne 0} (\widehat{\tilde{\phi}(\xi) g})_j^* \left(\frac{1}{i(\xi + 2j\pi)}\right) \widehat{\partial_x h_0}(\xi + 2j\pi) d\xi. \end{aligned}$$

where $(\widehat{\tilde{\phi}(\xi)g})_j$ denotes the jth coefficient in the Fourier expansion of periodic function $\tilde{\phi}(\xi)g$. Using Haussdorff-Young estimates twice, the first term is bounded in $L^p(\mathbb{R})$ by

$$\|\xi \mapsto |\xi|^{l-1}e^{-d\xi^2 t}\|_{L^q(\mathbb{R}\setminus[-\pi,\pi])}\ \|g\|_{L^2([0,1])}\ \|\tilde{\phi}(0)\|_{L^2([0,1])}\ \|\partial_x h_0\|_{L^1(\mathbb{R})} \lesssim e^{-\eta t}t^{-(l-1)}\ \|\partial_x h_0\|_{L^1(\mathbb{R})},$$

where q is such that $1/p + 1/q = 1$ and $\eta > 0$. Using $(1-\alpha(\xi)) \lesssim |\xi|$,

$$\|\tilde{\phi}(\xi) - \tilde{\phi}(0)\|_{L^2([0,1])} \leq |\xi| \sup_{|\xi'|\leq\xi_0} \|\partial_\xi \tilde{\phi}(\xi')\|_{L^2([0,1])},$$

$\lambda(\xi) - (ia\xi - d\xi^2) = O(|\xi|^3)$ to get

$$|e^{(ia\xi-d\xi^2)t} - e^{\lambda(\xi)t}| \ = \ e^{-d\xi^2 t}|1 - e^{(\lambda(\xi)-(ia\xi-d\xi^2))t}| \ \lesssim \ |\xi|^3 e^{-d\xi^2 t}$$

and by Cauchy-Schwarz estimate

$$\begin{aligned}
\sup_{|\xi|\leq\xi_0}\left|\sum_{j\neq 0}(\widehat{\tilde{\phi}(\xi)g})_j^*\left(\frac{1}{i(\xi+2j\pi)}\right)\widehat{\partial_x h_0}(\xi+2j\pi)\right| &\lesssim \|\partial_x h_0\|_{L^1(\mathbb{R})}\sup_{|\xi|\leq\xi_0}\sum_{j\neq 0}\frac{|(\widehat{\tilde{\phi}(\xi)g})_j|}{(1+|j|)}\\
&\lesssim \|\partial_x h_0\|_{L^1(\mathbb{R})}\sup_{|\xi|\leq\xi_0}\|\tilde{\phi}(\xi)g\|_{L^2([0,1])}\\
&\lesssim \|g\|_{L^2([0,1])}\sup_{|\xi|\leq\xi_0}\|\tilde{\phi}(\xi)\|_{L^\infty([0,1])}\|\partial_x h_0\|_{L^1(\mathbb{R})},
\end{aligned}$$

we may apply (2.4) to the other terms and bound them in $L^p(\mathbb{R})$ by

$$\|\xi \mapsto |\xi|^l e^{-\eta\xi^2 t}\|_{L^q([-\pi,\pi])}\ \|\partial_x h_0\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}}\ \|\partial_x h_0\|_{L^1(\mathbb{R})},$$

with q is such that $1/p + 1/q = 1$ and $\eta > 0$. □

4. Nonlinear stability estimates

4.1. **Perturbation equations and nonlinear decomposition.** We now refine the nonlinear perturbation equations of [JNRZ1]. First, recall the iteration scheme introduced in [JNRZ1]. For $\tilde{u}$ satisfying $k_* u_t = k_*^2 u_{xx} + f(u) + k_* c u_x$, we introduced the nonlinear perturbation

$$v(x,t) = \tilde{u}(x - \psi(x,t), t) - \bar{u}(x) \tag{4.1}$$

and showed that it satisfies

$$k_*\left(\partial_t - L\right)(v + \psi\bar{u}_x) = k_*\mathcal{N} := \mathcal{Q} + \mathcal{R}_x + (k_*\partial_t + k_*^2\partial_x^2)S + T, \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{Q} &:= f(v+\bar{u}) - f(\bar{u}) - df(\bar{u})v, \qquad \mathcal{R} := -k_* v\psi_t - k_*^2 v\psi_{xx} + k_*^2(\bar{u}_x + v_x)\frac{\psi_x^2}{1-\psi_x},\\
\mathcal{S} &:= v\psi_x, \quad \text{and} \qquad \mathcal{T} := -\left(f(v+\bar{u}) - f(\bar{u})\right)\psi_x.
\end{aligned} \tag{4.3}$$

Defining the phase ψ implicitly by

$$\begin{aligned}
\psi(t) &= s^{\mathrm{p}}(t)(d_0 + h_0\bar{u}') + \int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\\
&\quad - (1-\chi(t))\Big(s^{\mathrm{p}}(t)(d_0 + h_0\bar{u}') - h_0 + \int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big),
\end{aligned} \tag{4.4}$$

where $d_0 := \tilde{u}_0(\cdot - h_0(\cdot)) - \bar{u}$ and χ is a smooth cutoff such that $\chi(t)$ is zero for $t \le 1/2$ and one for $t \ge 1$, applying Duhamel's principle (variation of constants), and rearranging terms, we then obtained the integral representation

$$
\begin{aligned}
v(t) &= \tilde{S}(t)(d_0 + h_0\bar{u}') + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds \\
&\quad + (1-\chi(t))\Big(\bar{u}'s^{\mathrm{p}}(t)(d_0 + h_0\bar{u}') - h_0\bar{u}' + \int_0^t \bar{u}'s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big),
\end{aligned}
\tag{4.5}
$$

closing the system for (v, ψ), where $\tilde{S}(t) := S(t) - \bar{u}'s^{\mathrm{p}}(t)$.

As described in [JNRZ1], the choice (4.4) is designed to cancel, modulo an "initial layer" term $(1-\chi)(\dots)$ vanishing for $t \ge 1$, any $\bar{u}'s^{\mathrm{p}}$ terms that would otherwise occur in the description of v, that is, to extract as much as possible the expected phase shift from the solution $\tilde{u}$. Indeed, it is essentially uniquely determined by this requirement together with the requirement that $\psi(\cdot, 0) = h_0$. The crucial fact making possible the estimates of Proposition 1.1 is that the nonlinear term $\mathcal{N}$ is of quadratic order in terms of v and derivatives of ψ decaying at the rate of a heat kernel, whereas propagators $\tilde{S}$ and $\nabla_{x,t}s^{\mathrm{p}}$ decay at the rate of a differentiated heat kernel, leading to a closable iteration in variables $(\nabla_{x,t}\psi, v)$ and derivatives. This is analogous to the situation of the standard iteration scheme used to show decay with respect to localized (L^1) data of a Burgers equation $k_t - k_{xx} = (\frac{1}{2}k^2)_x$ treated as a perturbation of the heat equation, which is perhaps natural in view of the expected asymptotics (1.14). We refer the reader to [JNRZ1] for further details; see also [Z1, HoZ] for related arguments in the context of viscous shock stability.

Recall from the introduction that ψ_x approximates perturbation in wave number according to the formal Whitham approximation. Accordingly, refining the decomposition (4.1), we recall Lemma 2.1 and define the new nonlinear perturbation variable

$$
z(x,t) := \tilde{u}(x - \psi(x,t), t) - \bar{u}(x) - \partial_k\bar{u}(x)k_*\partial_x\psi(x,t) = v(x,t) - \partial_k\bar{u}(x)k_*\partial_x\psi(x,t),
\tag{4.6}
$$

approximately separating out expected modulation in the wave-number along with the phase shift.

Lemma 4.1. *For ψ defined as in* (4.4)*, the nonlinear residual z defined in* (4.6) *satisfies*

$$
\begin{aligned}
z(t) &= \tilde{R}(t)(d_0 + h_0\bar{u}') + \int_0^t \tilde{R}(t-s)\mathcal{N}(s)ds \\
&\quad + (1-\chi(t))\Big(R^{\mathrm{p}}(t)(d_0 + h_0\bar{u}') - h_0\bar{u}' - \partial_k\bar{u}\,k_*\partial_x h_0 + \int_0^t R^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big).
\end{aligned}
\tag{4.7}
$$

Proof. Using $z(t) = v(t) - \partial_k\bar{u}\,k_*\psi_x(t)$, $\tilde{R}(t) = \tilde{S}(t) - \partial_k\bar{u}\,k_*\partial_x s^{\mathrm{p}}(t)$ and $R^{\mathrm{p}}(t) = (\bar{u}' + \partial_k\bar{u}\,k_*\partial_x)s^{\mathrm{p}}(t)$, equation (4.7) follows immediately from (4.4) and (4.5). □

4.2. **Refined stability estimate.** Comparing (4.5) and (4.7), we see that we have by these manipulations effectively exchanged for $\tilde{S}$ and s^{p} the faster-decaying propagators $\tilde{R}$ and R^{p} in the representations of z vs. v. With these improvements, we may refine Proposition 1.1 as follows.

Proposition 4.2. *Under the assumptions of Proposition 1.1, for $t > 0$, $2 \le p \le \infty$, we have the sharpened estimate*

$$
\|z(t)\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}}.
\tag{4.8}
$$

Proof. From the basic bounds (1.6) of Proposition 1.1, we have $\|\mathcal{N}(t)\|_{H^1(\mathbb{R})} \le CE_0(1+t)^{-\frac{3}{4}}$. Applying the bounds of Propositions 3.1 and 3.2 to system (4.7), we obtain for any $2 \le p \le \infty$

$$\begin{aligned} \|z(t)\|_{L^p(\mathbb{R})} &\le CE_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} + CE_0\int_0^t (1+t-s)^{-\frac{1}{2}(1/2-1/p)-1}(1+s)^{-\frac{3}{4}}ds \\ &\le CE_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}}. \end{aligned} \tag{4.9}$$

□

Remark 4.3. We expect that it should be possible to improve (4.8) to

$$\|z(t)\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}, \qquad 2 \le p \le \infty,$$

by substituting in (4.9) $W^{k,p}$ bounds on $\mathcal{N}$ and $W^{k,p} \to L^p$ bounds on the solution operators R^{p} and $\tilde{R}$. However, to obtain such $W^{k,p} \to L^p$ bounds for $p > 2$ would appear to require techniques outside the Hausdorff–Young-type estimates used in this paper, perhaps pointwise bounds as in [J], or (operator-valued) Fourier multiplier techniques as suggested in Remark 4.2.2, [JNRZ1].

5. The Whitham equation

We complete our investigation by connecting to the Whitham modulation equation.

Lemma 5.1. *Assuming (H1)–(H2), (D1)–(D3),* $k_*\mathcal{N}(t) = f^{\mathrm{p}}\ k_*^2\psi_x(t)^2 + r(t)$*, where*

$$f^{\mathrm{p}} = \frac{1}{2}d^2f(\bar{u})(\partial_k\bar{u}, \partial_k\bar{u}) + k_*\partial_k\bar{u}'' - \frac{1}{k_*}df(\bar{u})\partial_k\bar{u} + \bar{u}'' - a\partial_k\bar{u}' \tag{5.1}$$

is periodic and $\|r(t)\|_{L^1(\mathbb{R})} \lesssim E_0^2(1+t)^{-1}$.

Proof. Immediate from Lemma 4.1, (1.6), (4.8), and $\|\psi_t(t) - a\psi_x(t)\|_{L^2(\mathbb{R})} = O((1+t)^{-3/4})$, a consequence of Lemma 3.3. □

Lemma 5.2. *Setting* $e := \langle\tilde{\phi}(0), f^{\mathrm{p}}\rangle_{L^2([0,1])}$*, we have* $e = \frac{1}{2}\omega_0''(k_*)$.

Proof. We may rewrite

$$f^{\mathrm{p}} = -L\partial_k\bar{u} + 2k_*\partial_k\bar{u}'' - \omega_0'(k_*)\partial_k\bar{u}' + \bar{u}'' + \frac{1}{2}d^2f(\bar{u})(\partial_k\bar{u}, \partial_k\bar{u})$$

thus

$$e = \langle\tilde{\phi}(0), 2k_*\partial_k\bar{u}'' - \omega_0'(k_*)\partial_k\bar{u}' + \bar{u}'' + \frac{1}{2}d^2f(\bar{u})(\partial_k\bar{u}, \partial_k\bar{u})\rangle_{L^2([0,1])} = \frac{1}{2}\omega_0''(k_*)\ .$$

The last equality comes differentiating twice the profile equation with respect to k (see [DSSS]). □

Proof of Theorem 1.3. With estimates on $k_*\partial_x\psi(t)$ from Proposition 1.1, the estimate on $\tilde{u}$ follows (recall Remark 1.4) from the bound

$$\|\tilde{u}(\cdot - \psi(\cdot,t), t) - \bar{u}^{k_*}(\cdot) - \partial_k\bar{u}^{k_*}(\cdot)k_*\partial_x\psi(\cdot,t)\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}} \tag{5.2}$$

established in Proposition 4.2 for $2 \le p \le \infty$.

Using Duhamel's principle we may write

$$k(t) = \sigma(t)k_*\partial_x h_0 + \int_0^t \sigma(t-s)\partial_x(ek^2(s)))ds\ , \tag{5.3}$$

where σ is the constant-coefficient solution operator defined in Proposition 3.4. On the other hand, by Propositions 3.1, 3.2, and 3.4,

$$\begin{aligned} k_*\psi_x(t) &= k_*\chi(t)\left(\partial_x s^{\mathrm{p}}(t)(d_0+h_0\bar{u}') + \int_0^t \partial_x s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\right) + (1-\chi(t))k_*\partial_x h_0, \\ &= \sigma(t)k_*\partial_x h_0 + \int_0^t \sigma(t-s)\partial_x(e(k_*\psi_x)^2(s))ds + \tilde{r}(t), \end{aligned} \tag{5.4}$$

where

$$\|\tilde{r}(t)\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}, \quad 2 \le p \le \infty\ .$$

Here, we have used (3.14) and

$$\|\partial_x(\psi_x^2(s))\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} \lesssim \|\psi_x(s)\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})}\|\psi_{xx}(s)\|_{L^2(\mathbb{R})} \lesssim (1+s)^{-1}$$

to bound

$$\begin{aligned} \| \int_0^t \partial_x s^{\mathrm{p}}(t-s) f^{\mathrm{p}} k_*^2 \psi_x(t)^2 ds - & \int_0^t \sigma(t-s)\partial_x(e(k_*\psi_x)^2(s))ds\|_{L^p(\mathbb{R})} \\ &\lesssim \int_0^t (1+t-s)^{\frac{1}{2}(1-1/p)}(t-s)^{-\frac{1}{2}}\|\partial_x(\psi_x^2(s))\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}ds \\ &\lesssim \int_0^t (1+t-s)^{\frac{1}{2}(1-1/p)}(t-s)^{-\frac{1}{2}}(1+s)^{-1}ds. \end{aligned}$$

Thus, subtracting (5.3) from (5.4), and defining $\delta := k_*\psi_x - k$, we have

$$\delta(t) = \int_0^t \sigma(t-s)\partial_x(e\delta(s)(k(s)+k_*\psi_x(s)))ds + \tilde{r}(t). \tag{5.5}$$

Defining, for some $\eta > 0$,

$$\nu(t) := \sup_{p\in[2,\infty]}\ \sup_{0\le s\le t} \|\delta(s)\|_{L^p(\mathbb{R})}(1+s)^{\frac{1}{2}(1-1/p)+\frac{1}{4}-\eta},$$

we thus obtain by the standard heat bounds $\|\sigma(t)\partial_x f\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q(\mathbb{R})}$ when $1 \le q \le p \le \infty$, that, for $2 \le p \le \infty$,

$$\begin{aligned} \|\delta(t)\|_{L^p(\mathbb{R})} &\lesssim \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|e\delta(s)(k(s)+k_*\psi_x(s))\|_{L^1(\mathbb{R})}ds \\ &\quad + \int_{t/2}^t (t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}\|e\delta(s)(k(s)+k_*\psi_x(s))\|_{L^2(\mathbb{R})}ds + \|\tilde{r}(t)\|_{L^p(\mathbb{R})} \\ &\lesssim \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\nu(t)E_0(1+s)^{-1+\eta}ds \\ &\quad + \int_{t/2}^t (t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}\nu(t)E_0(1+s)^{-\frac{5}{4}+\eta}ds \\ &\quad + E_0\ln(2+t)\ (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} \\ &\lesssim E_0(\nu(t)+1)(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \end{aligned}$$

giving

$$\nu(t) \le C_\eta E_0\left(1+\nu(t)\right),$$

whence, if $E_0 < 1/(2C_\eta)$, we have (noting that ν by standard theory remains bounded) that $\nu(t) \le 2C_\eta E_0$. This gives $\|k(t) - k_*\partial_x\psi(t)\|_{L^p(\mathbb{R})} = \|\delta(t)\|_{L^p(\mathbb{R})} \le 2C_\eta(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}$, completing the result for $k - k_*\psi_x$. A similar computation yields the result for $h - \psi$. □

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Appendix A. Asymptotic equivalence of scalar equations

Lemma 1.2 follows by a simplified (but also somewhat modified; this does not immediately follow from the results stated in [Ka, LZ]) version of the arguments used in [Ka, LZ] to prove corresponding but weaker versions in the system case. We include a proof here, both for completeness, and to motivate the more complicated comparison arguments appearing in the main body of the paper.

Proof of Lemma 1.2. By the general, system, results of [Ka],[5] provided $E_0 := \|k_0\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})}$ is sufficiently small, we have for $\tilde{k} := \kappa - k_*$ and $1 \le p \le \infty$, the "heat-type" bounds

$$\|k(t)\|_{L^p(\mathbb{R})},\ \|\tilde{k}(t)\|_{L^p(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \|k_x(t)\|_{H^1(\mathbb{R})},\ \|\tilde{k}_x(t)\|_{H^1(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{3}{4}}. \tag{A.1}$$

Setting $\delta := \tilde{\kappa} - \kappa$, we have, subtracting and rearranging,

$$\delta_t - a\delta_x - d\delta_{xx} = \partial_x \mathcal{F}, \qquad \mathcal{F} = O((|k| + |\tilde{k}|)\delta) + O(\tilde{k}^3) + O(\tilde{k}\tilde{k}_x),$$

with $\delta|_{t=0} = 0$, where $a = -dq(k_*)$, $d = d(k_*)$ are constant. By Duhamel's formula,

$$\delta(t) = \int_0^t \sigma(t-s)\partial_x \mathcal{F}(s)ds,$$

where σ is the solution operator of the convected heat equation $u_t - au_x - du_{xx} = 0$. Applying the standard heat bounds $\|\sigma(t)\partial_x^r f\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{r}{2}}\|f\|_{L^q(\mathbb{R})}$, $1 \le q \le p \le \infty$, together with

$$\|\mathcal{F}(t)\|_{L^q(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/q)-\frac{1}{4}}(\|\delta(t)\|_{L^2(\mathbb{R})} + \|\tilde{k}_x(t)\|_{L^2(\mathbb{R})}) + E_0^2(1+t)^{-\frac{1}{2}(1-1/q)-1},$$

$1 \le q \le 2$, we find, defining $\nu(t) := \sup_{0\le s\le t} \|\delta(s)\|_{L^2(\mathbb{R})}(1+s)^{\frac{3}{4}-\eta}$, that, for all $1 \le p \le \infty$,

$$\begin{aligned}
\|\delta(t)\|_{L^p(\mathbb{R})} &\lesssim \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|\mathcal{F}(s)\|_{L^1(\mathbb{R})}ds \\
&\quad + \int_{t/2}^t (t-s)^{-\frac{1}{2}(1/(\min(2,p))-1/p)-\frac{1}{2}}\|\mathcal{F}(s)\|_{L^{\min(2,p)}(\mathbb{R})}ds \\
&\lesssim \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}(\nu(t)E_0 + E_0^2)(1+s)^{-1+\eta}ds \\
&\quad + \int_{t/2}^t (t-s)^{-\frac{1}{2}(1/(\min(2,p))-1/p)-\frac{1}{2}}(\nu(t)E_0 + E_0^2)(1+s)^{-1+\eta-\frac{1}{2}(1-1/(\min(2,p)))}ds \\
&\lesssim E_0(E_0 + \nu(t))(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta},
\end{aligned} \tag{A.2}$$

whence $\nu(t) \le C_\eta E_0\,(E_0 + \nu(t))$. This implies that $\nu(t) \le 2C_\eta E_0^2$ for $E_0 < 1/(2C_\eta)$, giving

$$\|\delta(t)\|_{L^p(\mathbb{R})} \le 2C_\eta E_0^2(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \qquad 1 \le p \le \infty.$$

Finally, let $\phi(x,t) = \frac{1}{\sqrt{1+t}}\bar{\phi}\left(\frac{x+at}{\sqrt{1+t}}\right)$ define a self-similar solution of Burgers equation (1.14) such that $\int \bar{\phi} = \int k_0$. Defining $\tilde{\delta}(t) := k(t) - \phi(t+1)$, we have by the L^1 and L^1 first moment assumptions that, for $1 \le p \le \infty$,

$$\|\sigma(t)\tilde{\delta}(0)\|_{L^p(\mathbb{R})} \lesssim E_1(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}.$$

[5] See [Ka, Remark 4.2] improving the result of [Ka, Theorem 4.2] in the strictly parabolic case. Bounds (A.1) are proved by estimates similar to those of this paper and [JNRZ1]; see for example (A.2) just below.

Expressing by Duhamel's formula $\tilde\delta(t)=\sigma(t)\tilde\delta(0)+\int_0^t\sigma(t-s)\partial_x\tilde{\mathcal F}(s)ds$, where $\tilde{\mathcal F}(t)=O((|k(t)|+|\phi(t+1)|)\tilde\delta(t))$, and, for any $\eta'>0$, estimating as before, we obtain by a contraction argument similar to the above that $\|\tilde\delta(t)\|_{L^p(\mathbb R)}\ \lesssim\ E_1\ (1+t)^{-\frac12(1-1/p)-\frac12+\eta'}, \qquad 1\le p\le\infty$, provided E_1 is small enough (depending on η'). Indeed one may prove for

$$\tilde\nu(t):=\sup_{p\in[1,\infty]}\sup_{0\le s\le t}\|\tilde\delta(s)\|_{L^p(\mathbb R)}(1+s)^{\frac12(1-1/p)+\frac12-\eta'}$$

that $\tilde\nu(t)\le C'_{\eta'}E_1(1+\tilde\nu(t))$ and conclude that $\tilde\nu(t)\le 2C'_{\eta'}E_1$ if $C'_{\eta'}E_1<1/2$. □

Remark A.1. The order $O(\partial_x(k\delta))\sim O(\partial_x(kk_x))\sim O(\partial_x(k^3))$ of neglected terms in the argument above is consistent with the order neglected throughout the paper. Indeed, as emphasized by Remark 1.4, the true local wave number is $k_*\tilde\Psi_x$ (where $\tilde\Psi(\cdot,t)$ is the inverse of $y\mapsto X(y,t):=y-\psi(y,t)$) which differs from $k_*(1+\psi_x)$ by an $O(\|\psi_x\|^2)$ term since

$$\tilde\Psi_x(x,t)-[1+\psi_x(x,t)]=\psi_x(\tilde\Psi(x,t),t)-\psi_x(\tilde\Psi(x,t)-\psi(\tilde\Psi(x,t),t),t)\tilde\Psi_x(x,t)(1-\psi_x(\tilde\Psi(x,t),t)).$$

With a slight bit of additional effort, one can check that $k_*\tilde\Psi_x$ satisfies the Whitham equation or its quadratic approximant only up to a truncation of the order $\partial_x^2(\psi_x^2)\sim\partial_x(kk_x)$. Once integrated, this means that the formal Whitham equation (1.12) derived for $k_*(1+\psi_x)$ can be expected to hold for $k_*\tilde\Psi_x$ only to the (asymptotically equivalent) quadratic order considered here. In particular, if we were to derive higher order expansions for the equation satisfied by $\tilde\Psi_x$, as we could in principle do, *these would not in general agree with the Whitham equation* (1.12). In any case, while equation (1.14), most easily obtained by switching to a comoving reference frame $x\to x-at$, is designed to match, about a given wave, exactly the order of description we attain here, equation (1.12) is just a convenient way to piece together the two first orders of a nonlinear WKB expansion; thus (1.12) could in principle be meaningful even when dealing with a modulated background wave (instead of a true wave) but should not improve in accuracy its quadratic approximant about a given wave. For further discussion of this and related issues, see [DSSS, NR1, NR2].

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