Stability of periodic Kuramoto–Sivashinsky waves

Blake Barker* Mathew A. Johnson† Pascal Noble‡ L.Miguel Rodrigues§ Kevin Zumbrun¶

April 10, 2011

Keywords: periodic waves; Kuramoto–Sivashinsky equations; modulational stability.

2000 MR Subject Classification: 35B35.

Abstract

In this note, we announce a general result resolving the long-standing question of nonlinear modulational stability, or stability with respect to localized perturbations, of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinski equation, establishing that spectral modulational stability, defined in the standard way, implies nonlinear modulational stability with sharp rates of decay. The approach extends readily to other second- and higher-order parabolic equations, for example, the Cahn Hilliard equation or more general thin film models.

1 Introduction

In this note, we describe recent results obtained using techniques developed in [JZ, JZN, BJNRZ2] on linear and nonlinear stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky (gKS) equation

$$u_t + \gamma\partial_x^4 u + \epsilon\partial_x^3 u + \delta\partial_x^2 u + \partial_x(u^2/2) = 0, \quad \delta > 0, \tag{1.1}$$

a canonical model for pattern formation in one spatial dimension that has been used to describe, variously, plasma instabilities, flame front propagation, turbulence in reaction-diffusion systems, and thin film flow down an incline [S1, KT, CD, DSS, PSU].

More generally, we consider (taking without loss of generality $\gamma = 1$) an equation of the form

$$u_t + \partial_x^4 u + \epsilon\partial_x^3 u + \delta\partial_x^2 u + \partial_x f(u) = 0, \tag{1.2}$$

$f \in C^2$, δ not necessarily positive. Our methods apply also, with slight modifications to accomodate quasilinear form (see [JZN]) to the Cahn Hilliard equation and other fourth-order models for thin film flow as discussed for example in [BMSZ]. Indeed, the argument, and results, extend to arbitrary

*Indiana University, Bloomington, IN 47405; bhbarker@indiana.edu: Research of B.B. was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.

†Indiana University, Bloomington, IN 47405; matjohn@indiana.edu: Research of M.J. was partially supported by an NSF Postdoctoral Fellowship under NSF grant DMS-0902192.

‡Université Lyon I, Villeurbanne, France; noble@math.univ-lyon1.fr: Research of P.N. was partially supported by the French ANR Project no. ANR-09-JCJC-0103-01.

§Université de Lyon, Université Lyon 1, Institut Camille Jordan, UMR CNRS 5208, 43 bd du 11 novembre 1918, F - 69622 Villeurbanne Cedex, France; rodrigues@math.univ-lyon1.fr: Stay of M.R. in Bloomington was supported by Frency ANR project no. ANR-09-JCJC-0103-01.

¶Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grant no. DMS-0300487.

$2r$-order parabolic systems, so is essentially completely general for the diffusive case. As shown in [JZ], they can apply also to mixed-order and relaxation type systems in some cases as well.

It has been known since 1976, almost since the introduction of the model (1.1) in 1975 [KT, S1], that there exist spectrally stable bands of solutions in parameter space; see for example the numerical studies in [CKTR, FST]. Moreover, numerical time-evolution experiments described for example in [CD] suggest that these waves are nonlinearly stable as well, serving as attractors in the chaotic dynamics of (gKS). However, up to now this conjecture had not been rigorously verified.

Here, we announce the result, resolving this open question, that *spectral modulational stability*, defined in the standard sense of the modulational stability literature,[1] *implies linear and nonlinear modulational stability*. Our analysis gives at the same time new understanding even at the formal level of Whitham averaged equations. Further details, along with numerical investigations of existence and spectral stability, will be given in [BJNRZ3].

2 The traveling-wave equation

Substituting $u = \bar{u}(x - ct)$ into (1.2), we find that the traveling-wave equation is $-cu' + u'''' + \epsilon u''' + \delta u'' + f(u)' = 0$, or, integrating once in x,

$$-cu + u''' + \epsilon u'' + \delta u' + f(u) = q, \tag{2.1}$$

where $q \in \mathbb{R}$ is a constant of integration. Written as a first-order system in (u, u', u''), this is

$$\begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}' = \begin{pmatrix} u' \\ u'' \\ c - \epsilon u'' - \delta u' - f(u) + q \end{pmatrix}. \tag{2.2}$$

It follows that periodic solutions of (1.2) correspond to values $(X, c, q, b) \in \mathbb{R}^6$, where X, c, and q denote period, speed, and constant of integration, and $b = (b_1, b_2, b_3)$ denotes the values of (u, u', u'') at $x = 0$, such that the values of (u, u', u'') at $x = X$ of the solution of (2.2) are equal to the initial values (b_1, b_2, b_3).

Following [JZ], we assume:

(H1) $f \in C^{K+1}$, $K \geq 4$.

(H2) The map $H : \mathbb{R}^6 \to \mathbb{R}^3$ taking $(X, c, q, b) \mapsto (u, u', u'')(X, c, q, b; X) - b$ is full rank at $(\bar{X}, \bar{c}, \bar{q}, \bar{b})$, where $(u, u', u'')(\cdot; \cdot)$ is the solution operator of (2.2).

By the Implicit Function Theorem, conditions (H1)–(H2) imply that the set of periodic solutions in the vicinity of $\bar{U}$ form a smooth 3-dimensional manifold (counting translations)[2]

$$\{\bar{U}^\beta(x - \alpha - c(\beta)t)\}, \text{ with } \alpha \in \mathbb{R},\ \beta \in \mathcal{B} \subset \mathbb{R}^2. \tag{2.3}$$

3 Bloch decomposition and spectral stability conditions

In co-moving coordinates, the linearized equation about $\bar{u}$ reads

$$v_t = Lv := \big((c - a)v\big)_x - v_{xxxx} - \epsilon v_{xxx} - \delta v_{xx}, \qquad a := df(\bar{u}), \tag{3.1}$$

[1] In particular, the sense verified in [CKTR, FST]

[2] That α, β may be chosen to enter in this specific way follows through a second application of the Implicit Function Theorem from the fact that translation in x generates one-dimensional fibers foliating the three-dimensional manifold of periodic solutions, along which wave form $\bar{U}(\cdot)$ and wave speed c are constant.

and the eigenvalue equation as $Lv := -v_{xxxx} - \epsilon v_{xxx} - \delta v_{xx} + \big((c - av\big)_x = \lambda v$. Following [G], we define the one-parameter family of Bloch operators

$$L_\xi := e^{-i\xi x} L e^{i\xi x}, \quad \xi \in [-\pi, \pi) \tag{3.2}$$

operating on the class of L^2 periodic functions on $[0, X]$; the (L^2) spectrum of L is equal to the union of the spectra of all L_ξ with ξ real with associated eigenfunctions $w(x, \xi, \lambda) := e^{i\xi x} q(x, \xi, \lambda)$, where q, periodic, is an eigenfunction of L_ξ. By standard considerations, the spectra of L_ξ consist of the union of countably many continuous surfaces $\lambda_j(\xi)$; see, e.g., [G].

Without loss of generality taking $X = 1$, recall now the *Bloch representation*

$$u(x) = \Big(\frac{1}{2\pi}\Big) \int_{-\pi}^{\pi} e^{i\xi \cdot x} \hat{u}(\xi, x) d\xi \tag{3.3}$$

of an L^2 function u, where $\hat{u}(\xi, x) := \sum_k e^{2\pi i k x} \hat{u}(\xi + 2\pi k)$ are periodic functions of period $X = 1$, $\hat{u}(\cdot)$ denoting with slight abuse of notation the Fourier transform of u in x. By Parseval's identity, the Bloch transform $u(x) \to \hat{u}(\xi, x)$ is an isometry in L^2: $\|u\|_{L^2(x)} = \int_{-\pi}^{\pi} \int_0^1 |\hat{u}(\xi, x)|^2 \, dx \; d\xi = \|\hat{u}\|_{L^2(\xi; L^2(x))}$, where $L^2(x)$ is taken on $[0, 1]$ and $L^2(\xi)$ on $[-\pi, \pi]$. More generally, for $q \le 2 \le p$, $\frac{1}{p} + \frac{1}{q} = 1$, there holds the generalized Hausdorff–Young's inequality [JZ]

$$\|u\|_{L^p(x)} \le \|\hat{u}\|_{L^q(\xi; L^p(x))}. \tag{3.4}$$

The Bloch transform diagonalizes the periodic-coefficient operator L, yielding the *inverse Bloch transform representation*

$$e^{Lt} u_0 = \Big(\frac{1}{2\pi}\Big) \int_{-\pi}^{\pi} e^{i\xi \cdot x} e^{L_\xi t} \hat{u}_0(\xi, x) d\xi\,. \tag{3.5}$$

Following [JZ], we assume along with (H1)–(H2) the *strong spectral stability* conditions:

(D1) $\sigma(L_\xi) \subset \{\mathrm{Re}\lambda < 0\}$ for $\xi \neq 0$.

(D2) $\mathrm{Re}\sigma(L_\xi) \le -\theta |\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}$ and $|\xi|$ sufficiently small.

(D3) $\lambda = 0$ is an eigenvalue of L_0 of multiplicity 2.[3]

Assumptions (H1)-(H2) and (D1)–(D3) imply [JZ, BJNRZ3] that there exist 2 smooth eigenvalues

$$\lambda_j(\xi) = -i a_j \xi + o(|\xi|), \quad j = 1, 2 \tag{3.6}$$

of L_ξ bifurcating from $\lambda = 0$ at $\xi = 0$. Following [JZ], we make the further nondegeneracy hypotheses:

(H3) The coefficients a_j in (3.6) are distinct.[4]

(H4) The eigenvalue 0 of L_0 is nonsemisimple, i.e., $\dim\ker L_0 = 1$.

4 Spectral stability and the Whitham averaged equations

As noted in [Se, JZ], coefficients a_j in (3.6) are characteristics of the 2×2 Whitham averaged system

$$\begin{aligned} M_t + F_x &= 0, \\ \omega_t + (c\omega)_x &= 0 \end{aligned} \tag{4.1}$$

[3]The zero eigenspace of L_0, corresponding to variations along the 3-dimensional manifold of periodic solutions in directions for which period does not change [JZ], is at least 2-dimensional by (H2).

[4]Equivalent to strict hyperbolicity of the formal Whitham averaged system (4.1).

formally governing large-time ($\sim$ small frequency) behavior, evaluated at the values c, ω of the background wave $\bar u$, where $M(c,\omega)$ is the mean of u over one period of the periodic wave with speed c and frequency $\omega = 1/X$, and $F(c,\omega)$ is the mean of $f(u)$. Here, $\omega \sim \psi_x$, $c \sim -\psi_t/\psi_x$, where ψ denotes phase in the modulation approximation

$$u(x,t) \approx \bar u(\psi(x,t)). \tag{4.2}$$

In the context of (1.1), thanks to the Galillean invariance $x \to x - ct$, $u \to u + c$, (4.1) reduces to

$$\begin{aligned} c_t + (H(\omega) - m(\omega)c)_x &= 0,\\ \omega_t + (c\omega)_x &= 0, \end{aligned} \tag{4.3}$$

where $m(\omega)$ denotes the mean over one period of u for a zero-speed wave of frequency ω, and $H(\omega)$ the mean of $u^2/2$, and in the classical situation $\varepsilon = 0$ considered in [FST], to $c_t + (H(\omega))_x = 0$, $\omega_t + (c\omega)_x = 0$, which linearized about background values $c = 0$, $\omega = \omega_0$, yields a *wave equation*

$$\psi_{tt} + \omega_0 dH(\omega_0)\psi_{xx} = 0 \tag{4.4}$$

so long as $dH(\omega_0) < 0$. Indeed, by odd symmetry, we may conclude in this case that the second-order corrections b_j in the further expansion

$$\lambda_j(\xi) = ia_j\xi - b_j\xi^2 \cdots \tag{4.5}$$

of (3.6) are equal, hence $\lambda_j(\xi)$ agree to second order with the dispersion relations of the *viscoelastic wave equation*

$$\psi_{tt} + \omega_0 dH(\omega_0)\psi_{xx} = d(\omega_0)\psi_{txx}, \quad d = 2b_1 = 2b_2. \tag{4.6}$$

This recovers the formal prediction of "viscoelastic behavior" of modulated waves carried out in [FST] and elsewhere, or "bouncing" behavior of individual periodic cells. Put more concretely, (4.6) predicts that the maxima of a perturbed periodic solution should behave approximately like point masses connected by viscoelastic springs. However, we emphasize that *such qualitative behavior*- in particular, the fact that the modulation equation is of second order- *does not derive only from Gallilean or other invariance of the underlying equations*, as might be suggested by early literature on the subject, *but rather from the more general structure of conservative* (i.e., divergence) *form* [Se, JZ].[5] Indeed, for any choice of f, $\lambda_j(\xi)$ may be seen to agree to second order with the dispersion relation for an appropriate diffusive correction of (4.1), a generalized viscoelastic wave equation. See [NR1, NR2] for further discussion of Whitham averaged equations and their derivation.

5 Linear estimates

The main difficulty in obtaining linear estimates is that, by (D3) and (H4), the zero eigenspace of L_0 has an associated 2×2 Jordan block. This means that $e^{L_0 t}$ is not only neutral but grows as $O(t)$. Viewed from the Bloch perspective, it means that the eigenprojections of L_ξ blow up as $\xi \to 0$. Performing a careful spectral perturbation analysis, separating out the singular part of the eigendecomposition of $e^{L_\xi t}$ in (3.5), and applying (3.4), as in Lemma 2.1, Prop. 3.3, and Prop. 3.4 of [JZ], we obtain the following detailed description of linearized behavior.

[5] As discussed further in [Z], conservation of mass lies outside the usual Noetherian formulation.

Proposition 5.1. *Under assumptions (H1)–(H4) and (D1)–(D3), the Green function $G(x,t;y)$ of* (3.1) *decomposes as*

$$G(x,t;y) = \bar{u}'(x)e(x,t;y) + \widetilde{G}(x,t;y), \tag{5.1}$$

where, for some $C>0$ and all $t>0$, $1 \le q \le 2 \le p \le \infty$ and $1 \le r \le 4$,

$$\left\| \int_{-\infty}^{\infty} \widetilde{G}(\cdot,t;y)f(y)dy \right\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{4}\left(\frac{1}{2}-\frac{1}{p}\right)} (1+t)^{-\frac{1}{4}\left(\frac{2}{q}-\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^q \cap L^2} \tag{5.2}$$

$$\left\| \int_{-\infty}^{\infty} \partial_y^r \widetilde{G}(\cdot,t;y)f(y)dy \right\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{4}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{r}{4}} (1+t)^{-\frac{1}{4}\left(\frac{2}{q}-\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}+\frac{r}{4}} \|f\|_{L^q \cap L^2} \tag{5.3}$$

$$\left\| \int_{-\infty}^{\infty} \partial_t \widetilde{G}(\cdot,t;y)f(y)dy \right\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{4}\left(\frac{1}{2}-\frac{1}{p}\right)-1} (1+t)^{-\frac{1}{4}\left(\frac{2}{q}-\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{2}} \|f\|_{L^q \cap L^2}, \tag{5.4}$$

$e(x,t;y) \equiv 0$ for $0 \le t \le 1$, and for all $t>0$, $1 \le q \le 2 \le p \le \infty$, $0 \le j, l, j+l \le K$, and $1 \le r \le 4$,

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} \partial_x^j \partial_t^l e(\cdot,t;y)f(y)dy \right\|_{L^p(\mathbb{R})} &\le C(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{(j+k)}{2}} \|f\|_{L^q(\mathbb{R})} \\ \left\| \int_{-\infty}^{\infty} \partial_x^j \partial_t^l \partial_y^r e(\cdot,t;y)f(y)dy \right\|_{L^p(\mathbb{R})} &\le C(1+t)^{\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{(j+k)}{2}} \|f\|_{L^q(\mathbb{R})}. \end{aligned} \tag{5.5}$$

Moreover, for some constants p_j, and a_j and b_j as in (4.5),

$$\left\| e(\cdot,t;y) - \sum_{j=1}^{2} p_j \, errfn\left(\frac{\cdot - y - a_j t}{\sqrt{4b_j t}}, t \right) \right\|_{L^p} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}, \quad t \ge 1. \tag{5.6}$$

Defining $\psi := -e$ and noting that $\bar{u}(x)+\psi(x,t)\bar{u}'(x) \sim \bar{u}(x+\psi(x,t))$, we see from (5.1)–(5.6) that linearized behavior indeed agrees to lowest order with modulation by a phase function ψ satisfying a generalized viscoelastic wave equation obtained by diffusive correction of (4.1), consisting of a *first-order hyperbolic–parabolic system* in ψ_x, ψ_t.

This observation not only generalizes the second-order scalar description (4.6) obtained in the special case $\varepsilon = 0$ [FST], but gives also new information even in that case. For, note that the formal modulation description can be a bit misleading as regards the assumption of initial data. In particular, from the description (4.6), one might be tempted to conclude, tacitly assuming $\psi_t|_{t=0} \equiv 0$, that the linear response to a compactly supported perturbation would consist of the D'Alembertian picture of two approximately compactly supported wave forms in ψ moving in opposite directions, diffusing slowly at Gaussian rate. Yet, the explicit bound (5.6) shows that this is rather a description of the derivative ψ_x![6]

Indeed, as described further in Section 1.2, [JZN], it is the variables ψ_x, ψ_t that are primary, rather than ψ, ψ_t as suggested by (4.6), and it is these variables that are most closely related to the initial perturbation $(u-\bar{u})|_{t=0}$. *Thus, our analysis gives not only technical verification of existing observations, but also new intuition regarding the nature of modulated behavior.*

[6]More generally, the explicit description (5.6) shows that the principle, noncompactly supported, response $\bar{u}'e$, corresponding to the part of the second-order wave solution associated with initial data $\psi_t|_{t=0}$, may be neglected roughly speaking when the integral of the initial perturbation is much smaller than its L^1 norm; in the case of purely modulational initial perturbations $h(x)\bar{u}'(x)$, this corresponds to the slowly-varying wave form case $|h'\bar{u}| << |h\bar{u}'|$.

6 Nonlinear stability

Using the linear bounds of Prop. 5.1 together with nonlinear cancellation estimates as in [JZ, JZN], we obtain, finally, our main result describing nonlinear behavior under localized perturbations.

Theorem 6.1. *Assuming (H1)–(H4) and (D1)–(D3), let $\bar u = (\bar\tau, \bar u)$ be a traveling-wave solution of* (1.2)*. Then, for some $C > 0$ and $\psi \in W^{2,\infty}(x,t)$,*

$$\begin{aligned} \|\tilde u - \bar u(\cdot - \psi - ct)\|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)}\|\tilde u - \bar u\|_{L^1\cap H^K}|_{t=0}, \\ \|\tilde u - \bar u(\cdot - \psi - ct)\|_{H^K}(t) &\le C(1+t)^{-\frac{1}{4}}\|\tilde u - \bar u\|_{L^1\cap H^K}|_{t=0}, \\ \|(\psi_t, \psi_x)\|_{W^{K+1,p}} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}\|\tilde u - \bar u\|_{L^1\cap H^K}|_{t=0}, \end{aligned} \tag{6.1}$$

and

$$\|\tilde u - \bar u(\cdot - ct)\|_{L^\infty}(t), \ \|\psi(t)\|_{L^\infty} \le C\|\tilde u - \bar u\|_{L^1\cap H^K}|_{t=0} \tag{6.2}$$

for all $t \ge 0$, $p \ge 2$, for solutions $\tilde u$ of (1.2) *with $\|\tilde u - \bar u\|_{L^1\cap H^K}|_{t=0}$ sufficiently small. In particular, $\bar u$ is nonlinearly bounded $L^1 \cap H^K \to L^\infty$ stable.*

Similarly as in the discussion of linear behavior, we note that Theorem 6.1 asserts asymptotic $L^1 \cap H^K \to L^p$ convergence of $\tilde u$ toward the modulated wave $\bar u(x - \psi(x,t))$, but only bounded $L^1 \cap H^K \to L^\infty$ stability about $\bar u(x)$, a quite different picture from that suggested at first sight by (4.6).

7 Application to Kuramoto–Sivashinsky

We conclude our discussion by displaying some representative traveling wave orbits and their associated spectrum, for the case $\varepsilon = 0.2$, computed respectively using MATLAB and the spectral Galerkin package SpectrUW. These indicate, similarly as in the $\varepsilon = 0$ case studied in [CKTR, FST], the existence of a band of spectrally stable periodic traveling waves. For related studies, and an animation of the spectral evolution, see http://www.math.indiana.edu/gallery/TravelingWave.phtml. For more detailed numerical verification using the periodic Evans function [G], see [BJNRZ3].

References

[BJNRZ2] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Witham averaged equations and modulational stability of periodic solutions of hyperbolic-parabolic balance laws,* to appear, Proceedings and seminars, Centre de Mathmatiques de lÉcole polytechnique; Conference proceedings, "Journees equations aux derivées partielles", 2010, Port d'Albret, France.

[BJNRZ3] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinsky equation,* in preparation.

[BMSZ] A. Bertozzi, A. Münch, M. Shearer, and K. Zumbrun, *Stability of compressive and undercompressive thin film travelling waves*, The dynamics of thin fluid films. European J. Appl. Math. 12 (2001) 253–291.

[CDKK] J. D. Carter, B. Deconick, F. Kiyak, and J. Nathan Kutz, *SpectrUW: a laboratory for the numerical exploration of spectra of linear operators*, Mathematics and Computers in Simulation 74, 370–379, 2007.

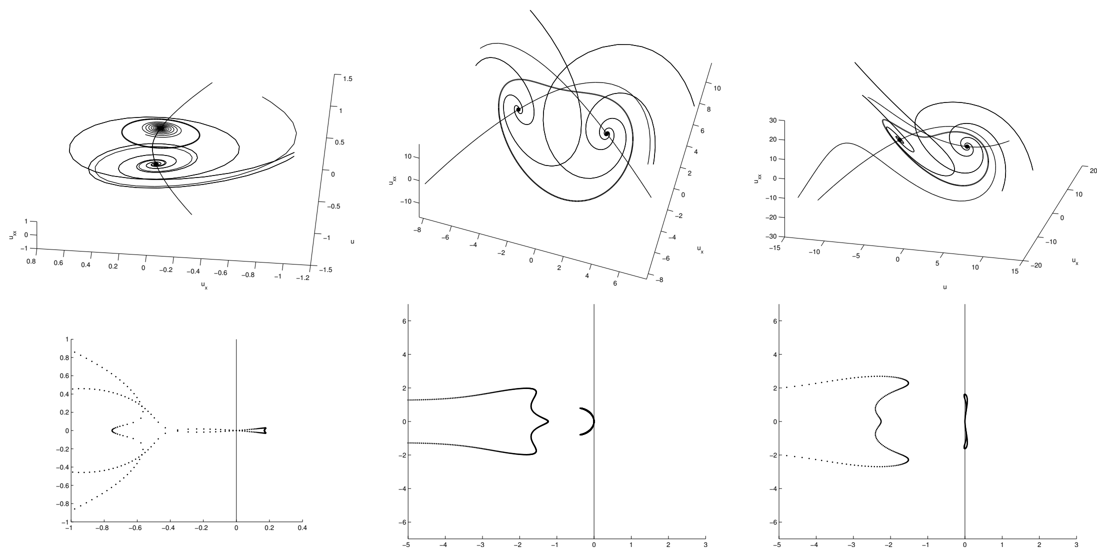

Figure 1: Evolution of spectra as period X increases with $c = 0$ held constant. Here $\varepsilon = 0.2$, $\delta = 1$, $\gamma = 1$. Running left to right in the top row, we display the evolution of the periodic orbits in the three-dimensional phase portrait as X is increased, with frames directly below depicting the spectrum of the corresponding linearized operator about the wave. Similarly as in the well-known computations [CKTR, FST] in the $\varepsilon = 0$ case, we see the familiar picture of initial instability (first frame) on an interval $[X_{Hopf}, X_*)$ from the Hopf bifurcation value $X = X_{Hopf}$ up to a special value X_*, transitioning to spectral stability (second frame) for X within a stable band (X_*, X^*), then back to instability (third and final frame) for $X > X^*$. Orbits were computed with MATLAB; spectra were computed with the SpectrUW package developed at University of Washington [CDKK].

[CD] H-C. Chang and E.A. Demekhin, *Complex wave dynamics on thin films,* Studies in Interface Science, 14. Elsevier Science B.V., Amsterdam, 2002. x+402 pp. ISBN: 0-444-50970-4,

[CKTR] B.I. Cohen, J.A. Krommes, W.M. Tang, and M.N. Rosenbluth, Nucl. Fusion 16,6 (1976).

[DSS] Arjen Doelman, Björn Sandstede, Arnd Scheel, and Guido Schneider. *The dynamics of modulated wave trains.* Mem. Amer. Math. Soc. 199 (2009) 934:viii+105.

[FST] U. Frisch, Z.S. She, and O. Thual, *Viscoelastic behaviour of cellular solutions to the Kuramoto–Sivashinsky model* J. Fluid Mech. 168 (198) 221–240.

[G] R. Gardner, *On the structure of the spectra of periodic traveling waves*, J. Math. Pures Appl. 72 (1993), 415-439.

[JZ] M. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling waves of viscous conservation laws in the generic case*, J. Diff. Eq. 249 (2010) no. 5, 1213-1240.

[JZN] M. Johnson, K. Zumbrun, and P. Noble, *Nonlinear stability of viscous roll waves,* to appear, SIAM J. Math. Anal.

[KT] Y. Kuramoto and T. Tsuzuki, *On the formation of dissipative structures in reaction-diffusion systems,* Progress of Theoretical Physics, 1975. 54:3.

[NR1] P. Noble, and M. Rodrigues, *Whitham's modulation equations for shallow flows,* preprint (2010).

[NR2] P. Noble, and M. Rodrigues, *Modulational stability of periodic waves of the generalized Kuramoto–Sivashinsky equation,* preprint (2010).

[PSU] R. Pego, H. Schneider, and H. Uecker, *Long-time persistence of Korteweg-de Vries solitons as transient dynamics in a model of inclined film flow,* Proc. Royal Soc. Edinburg 137A (2007) 133–146.

[Se] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis*, Comm. Partial Differential Equations 30 (2005), no. 1-3, 259–282.

[S1] G.I. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flame. I. Derivation of basic equations,* Acta Astron., 1977. 4:11-12. Pp.1177–1206.

[Z] K. Zumbrun, *A sharp stability criterion for soliton-type propagating phase boundaries in Korteweg's model*, Z. Anal. Anwend. 27 (2008), no. 1, 11–30.