Asymptotic stability of Oseen vortices for a density-dependent incompressible viscous fluid

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Abstract

In the analysis of the long-time behavior of two-dimensional incompressible viscous fluids, Oseen vortices play a major role as attractors of *any* homogeneous solution with integrable initial vorticity [8]. As a first step in the study of the *density-dependent* case, the present paper establishes the asymptotic stability of Oseen vortices for slightly inhomogeneous fluids with respect to localized perturbations.

Résumé

Les tourbillons d'Oseen occupent une place majeure dans la description du comportement asymptotique en temps des fluides bidimensionnelles incompressibles et visqueux, en tant qu'attracteurs de *toute* solution homogène de vorticité initiale intégrable [8]. Première étape dans l'analyse du cas *inhomogène*, cet article établit la stabilité asymptotique des tourbillons d'Oseen, vis-à-vis de perturbations localisées, en tant que fluides à densité variable.

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Introduction

In this paper we consider the motion of a weakly inhomogeneous incompressible viscous fluid in the two-dimensional Euclidean space. We can describe the fluid by the pair (ρ, u), $\rho = \rho(t, x) \in \mathbf{R}^+$ being the density field and $u = u(t, x) \in \mathbf{R}^2$ the velocity field. The evolution we consider here is governed by the density-dependent incompressible Navier-Stokes equations:

$$\left\{ \begin{array}{rcl} \partial_t \, \rho + (u \cdot \nabla) \, \rho & = & 0 \\ \partial_t \, u + (u \cdot \nabla) \, u & = & \frac{1}{\rho} \, (\triangle u - \nabla p) \\ \operatorname{div} u & = & 0 \end{array} \right. \tag{1}$$

where $p = p(t,x) \in \mathbf{R}$ is the pressure field, which is determined (up to a constant) by the incompressibility condition which yields the elliptic equation:

$$\operatorname{div} \Big(\frac{1}{\rho} \, \nabla p \Big) = \operatorname{div} \Big(\frac{1}{\rho} \triangle u - (u \cdot \nabla) \, u \Big) \ . \tag{2}$$

Alternatively, we can represent the fluid motion using the vorticity field $\omega = \text{curl } u \in \mathbf{R}$ rather than the velocity. Note that, in the two-dimensional context, $\text{curl}(f_1, f_2)$ stands for $\partial_1 f_2 - \partial_2 f_1$. Therefore the evolution equations for (ρ, ω) become

$$\left\{ \begin{array}{rcl} \partial_t \, \rho + (u \cdot \nabla) \, \rho & = & 0 \\ \partial_t \, \omega + (u \cdot \nabla) \, \omega & = & \operatorname{div} \big(\frac{1}{\rho} \, (\nabla \omega + \nabla^{\perp} p) \big) \end{array} \right. \tag{3}$$

where p is again determined by (2), and u is recovered from ω *via* the Biot-Savart law:

$$u(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \, \omega(y) \, dy \tag{4}$$

for $x \in \mathbf{R}^2$, with $(z_1, z_2)^{\perp} = (-z_2, z_1)$. We also denote $u = K_{BS} \star \omega$, where K_{BS} is the Biot-Savart kernel: $K_{BS}(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2}$. Without loss of generality, we assume throughout the present paper that the viscosity of the fluid is equal to one.

We refer to the monograph [12] for a general presentation of the available mathematical results on incompressible Navier-Stokes equations. We also mention the work of B.Desjardins on the global existence of weak solutions [4, 3], and, closer to the spirit of the present paper, the work of R.Danchin on well-posedness in Besov spaces [1]. Let us emphasize that both Danchin and Desjardins work with the velocity formulation (1) and do not assume the density ρ to be bounded away from zero. In more physical terms, they allow for regions of (almost complete) vacuum, which create technical difficulties.

In contrast, not only shall we not allow the density to be close to zero but we shall only consider *weakly* inhomogeneous fluids, namely we shall assume that the density ρ is close to a positive constant which, without loss of generality, we take equal to one. Remark that if the initial density is constant in space *i.e.* if the fluid is initially homogeneous, then the density remains equal to this constant for all subsequent times. Therefore, in such a case, system (1) reduces to the usual incompressible Navier-Stokes equations. Moreover, since $\operatorname{div}(\nabla^{\perp} p) = 0$, the pressure term disappears from system (3) which thus reduces to

$$\partial_t \, \omega + (u \cdot \nabla) \, \omega = \triangle \, \omega \ . \tag{5}$$

Again, a wealth of information on the Cauchy problem for the *homogeneous* incompressible Navier-Stokes equations can be found in [12] or [10]. Concerning the long-time behavior of the solutions of the vorticity equation (5), the work of Th.Gallay and C.E.Wayne has revealed the important role played by a family of explicit self-similar solutions, *Oseen vortices*, given by $\rho \equiv 1$, $u = \alpha\, u^G$ and $\omega = \alpha\, \omega^G$, where $\alpha \in \mathbf{R}$ is a parameter and

$$\omega^G(t,x) = \frac{1}{t}\, G\big(\frac{x}{\sqrt{t}}\big) \ , \quad u^G(t,x) = \frac{1}{\sqrt{t}}\, v^G\big(\frac{x}{\sqrt{t}}\big)$$

with

$$G(\xi) = \frac{1}{4\pi}\, e^{-|\xi|^2/4} \ , \quad v^G(\xi) = \frac{1}{2\pi}\, \frac{\xi^\perp}{|\xi|^2}\, \big(1 - e^{-|\xi|^2/4}\big) \ .$$

For the Oseen vortex $(1, \alpha\, \omega^G)$, the quantity $|\alpha|$ is actually its Reynolds number. If the initial vorticity ω_0 is integrable, it is proved in [8] that the corresponding solution of (5) converges to $\alpha\, \omega^G$ in L^1-norm as $t \to \infty$, where $\alpha := \int_{\mathbf{R}^2} \omega_0$. Moreover, it was shown in [5, 6] that $\alpha\, \omega^G$ is the unique solution of the vorticity equation (5) with initial data $\alpha\, \delta_0$. Note also that u^G is not square integrable, since $|v^G(\xi)| \sim \frac{1}{|\xi|}$ as $|\xi| \to \infty$, which means that Oseen vortices are not finite energy solutions in the sense of Leray [11]. More generally, when dealing with incompressible flows of integrable vorticity and non-zero global circulation, one needs to consider infinite energy solutions.

Even though the homogeneous incompressible Navier-Stokes equations provide a good model in many situations, all real fluids are, at least slightly, inhomogeneous and it is therefore important and relevant, both from a practical and theoretical point of view, to investigate whether the predictions of the homogeneous model are meaningful for the *density-dependent* model, especially in the weakly inhomogeneous regime. The goal of this paper is to address this question in the particular case of Oseen vortices. These explicit solutions persist in the density-dependent case if we assume $\rho \equiv 1$, and it is therefore natural to ask whether they play there the same role as in the homogeneous case. While the general answer to this question is unknown, we treat here one important aspect: are Oseen vortices stable solutions for the density-dependent incompressible Navier-Stokes equations ? In other words, does the theory predicts that these self-similar solutions may be observed ?

Before stating what we mean exactly by stability, let us recall an important property of the Navier-Stokes equations: *scaling invariance.* For any $\lambda > 0$, if $(\rho(t,x), \omega(t,x))$ is a solution of (3), so is

$$D_\lambda(\rho, \omega) = (\rho\,(\lambda^2 t, \lambda x), \lambda^2\, \omega\,(\lambda^2 t, \lambda x)) \ .$$

Correspondingly, the velocity field $u(t,x)$ and the pressure $p(t,x)$ are rescaled into $\lambda\, u(\lambda^2 t, \lambda x)$ and $\lambda^2\, p(\lambda^2 t, \lambda x)$. As is easily verified, Oseen vortices are *self-similar*, in the sense that $D_\lambda(1, \alpha\, \omega^G) = (1, \alpha\, \omega^G)$, for any $\alpha \in \mathbf{R}$ and any $\lambda > 0$. To study these solutions, it is therefore more convenient to rewrite (3) in self-similar variables. Following [7] , we set

$$(\tau, \xi) := (\ln t, \frac{x}{\sqrt{t}}) \ . \tag{6}$$

Motivated by scaling invariance, we will work with new quantities (r, w, v, Π) related to the former by

$$\begin{array}{rclcrcl} r(\tau,\xi) & = & \rho\,(e^{\tau}, e^{\frac{\tau}{2}}\xi) & , & v(\tau,\xi) & = & e^{\frac{\tau}{2}}\,u\,(e^{\tau}, e^{\frac{\tau}{2}}\xi) \quad , \\ w(\tau,\xi) & = & e^{\tau}\,\omega\,(e^{\tau}, e^{\frac{\tau}{2}}\xi) & , & \Pi(\tau,\xi) & = & e^{\tau}\,p\,(e^{\tau}, e^{\frac{\tau}{2}}\xi) \quad . \end{array} \tag{7}$$

Then the corresponding evolution equations for (r, w) are

$$\left\{ \begin{array}{rcl} \partial_\tau\, r + \big((v - \frac{1}{2}\,\xi)\cdot\nabla\big)\, r & = & 0 \\ \partial_\tau\, w + \big((v - \frac{1}{2}\,\xi)\cdot\nabla\big)\, w - w & = & \mathrm{div}\,\big(\,\frac{1}{r}\,(\nabla\omega + \nabla^{\perp}\Pi)\,\big) \end{array} \right. \tag{8}$$

where again v is obtained from w by the Biot-Savart law and $\nabla\Pi$ by solving

$$\mathrm{div}\,\Big(\,\frac{1}{r}\,\nabla\Pi\,\Big) = \mathrm{div}\,\Big(\,\frac{1}{r}\,\triangle\, v - (v\cdot\nabla)\,v\Big)\ . \tag{9}$$

By construction, Oseen vortices correspond to stationary solutions of (8) of the form $(1, \alpha\, G)$, where α is any real number. We prescribe initial data for the original equations at $t = 1$ rather than at $t = 0$, hence at $\tau = 0$ for the new equations.

In order to state our main result, we now write down the evolution equations for a perturbation of an Oseen vortex. Given $\alpha \in \mathbf{R}$, we work with new quantities $(b, \widetilde{w})$ related to the former by $b = \frac{1}{r} - 1$ and $\widetilde{w} = w - \alpha\, G$. The evolution equations for $(b, \widetilde{w})$ are

$$\left\{ \begin{array}{rcl} \partial_\tau\, b + \big((v - \frac{1}{2}\,\xi)\cdot\nabla\big)\, b & = & 0 \\ \partial_\tau\, \widetilde{w} - (\mathcal{L} - \alpha\,\Lambda)\,\widetilde{w} + \big(\widetilde{v}\cdot\nabla\big)\,\widetilde{w} & = & \mathrm{div}\,\big(\,b\,(\nabla w + \nabla^{\perp}\Pi)\,\big) \end{array} \right. \tag{10}$$

where

$$\begin{array}{rcl} \mathcal{L}f & = & \triangle f + \frac{1}{2}\,\xi\cdot\nabla f + f\ , \\ \Lambda f & = & v^{G}\cdot\nabla f + (K_{BS}\star f)\cdot\nabla G\ . \end{array} \tag{11}$$

Here $\widetilde{v}$ is obtained from $\widetilde{w}$ by the Biot-Savart law, w and v are recovered by

$$w = \alpha\, G + \widetilde{w}\ , \quad v = \alpha\, v^{G} + \widetilde{v}\ , \tag{12}$$

and $\nabla\Pi$ is obtained by solving

$$\mathrm{div}\,\big(\,(1+b)\,\nabla\Pi\,\big) = \mathrm{div}\,\big(\,(1+b)\,\triangle v - (v\cdot\nabla)\,v\,\big)\ . \tag{13}$$

Let us point out that, thanks to the linearity of the Biot-Savart law, we both have $v = K_{BS}\star w$ and $\widetilde{v} = K_{BS}\star\widetilde{w}$.

Note that $\mathcal{L}$ is the usual Fokker-Planck type operator which generates the evolution corresponding to the heat equation in self-similar variables. On the other hand, Λ is the non-local first-order operator resulting from the linearization around $w = G$ of the transport term of (8). More precisely, if $\widetilde{v} = K_{BS}\star\widetilde{w}$ and (v, w) satisfies (12), then $v\cdot\nabla w = \alpha\,\Lambda\,\widetilde{w} + \widetilde{v}\cdot\nabla\widetilde{w}$, since $v^{G}\perp\nabla G$.

Before stating our result of stability, we introduce the function spaces and the norms that we will encounter throughout the present paper. For $1 \le p \le \infty$, we denote by $L^{p}(\mathbf{R}^{2})$ the usual L^{p}-space and by $|\,f\,|_{p}$ the usual L^{p}-norm of a function $f : \mathbf{R}^2 \to \mathbf{R}$ or $f : \mathbf{R}^2 \to \mathbf{R}^2$. Similarly, for $s \in \mathbf{R}$, we denote by $H^s(\mathbf{R}^2)$ the usual Sobolev space and by $|\, f \,|_{H^s}$ the corresponding norm. Sobolev norms will be convenient to specify the regularity of our perturbations, but we also need weighted norms to describe the localization properties. Indeed, even in the homogeneous case, we know that it is impossible to get a convergence rate in time if we do not assume that the perturbations decay sufficiently fast at infinity in space (see [8]). Instead of using polynomial weights as in [8], we choose here the Gaussian weight $G^{-\frac{1}{2}}$, which is more restrictive but naturally related to the Oseen vortices and has several technical advantages. For instance, on the Hilbert space $L^2_w(\mathbf{R}^2) := \{f \mid G^{-\frac{1}{2}} f \in L^2(\mathbf{R}^2)\}$, the linear operators $\mathcal{L}$ and Λ become respectively symmetric and skew-symmetric. More generally, we shall use the weighted L^p-space defined as follows, for any $1 \le p \le +\infty$,

$$L^p_w(\mathbf{R}^2) := \{\ f \mid G^{-\frac{1}{2}} f \in L^p(\mathbf{R}^2)\ \} \tag{14}$$

with corresponding norms $|\, f \,|_{w,p} := |\, G^{-\frac{1}{2}} f \,|_p$.

We are now able to state the main result of this paper:

Theorem 1 *Let* $\alpha \in \mathbf{R}$, $0 < s < 1$ *and* $0 < \gamma < \frac{1}{2}$. *There exist* $\varepsilon_0 > 0$ *and* $K > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$, *if*

1. b_0 *belongs to* $L^2_w(\mathbf{R}^2) \cap L^\infty_w(\mathbf{R}^2) \cap H^{s+2}(\mathbf{R}^2)$ *with* $|\, b_0 \,|_{w,2} \le \varepsilon$, $|\, b_0 \,|_{w,\infty} \le \varepsilon$, *and* $G^{-\frac{1}{2}} \nabla b_0$ *belongs to* $L^q(\mathbf{R}^2)$ *for some* $q > \max(4, \frac{2}{s})$,
2. $\widetilde{w}_0$ *belongs to* $L^2_w(\mathbf{R}^2) \cap H^s(\mathbf{R}^2)$ *with* $|\, \widetilde{w}_0 \,|_{w,2} \le \varepsilon$ *and* $\int_{\mathbf{R}^2} \widetilde{w}_0 = 0$,

then there exists a unique solution $(b, \widetilde{w})$ *of (10) with initial data* $(b_0, \widetilde{w}_0)$ *such that*

1. $b \in L^\infty_{loc}(\mathbf{R}^+; H^{s+2}(\mathbf{R}^2))$,
2. $G^{-\frac{1}{2}} b \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$, $G^{-\frac{1}{2}} \nabla b \in L^\infty_{loc}(\mathbf{R}^+; L^q(\mathbf{R}^2))$
3. $\widetilde{w} \in L^\infty_{loc}(\mathbf{R}^+; H^s(\mathbf{R}^2)) \cap L^2_{loc}(\mathbf{R}^+; H^{s+1}(\mathbf{R}^2))$,
4. $G^{-\frac{1}{2}} \widetilde{w} \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}^2)) \cap L^2(\mathbf{R}^+; L^2(\mathbf{R}^2))$,
   $G^{-\frac{1}{2}} \nabla \widetilde{w} \in L^2(\mathbf{R}^+; L^2(\mathbf{R}^2))$, $G^{-\frac{1}{2}} |\xi|\, \widetilde{w} \in L^2(\mathbf{R}^+; L^2(\mathbf{R}^2))$.

Moreover this solution satisfies $|\, \widetilde{w}(\tau) \,|_{w,2} \le K\, \varepsilon\, e^{-\gamma \tau}$, *for any* $\tau > 0$.

Theorem 1 shows that Oseen vortices, which are self-similar solutions of the density-dependent Navier-Stokes equations, are stable with respect to small localized perturbations of the density and the vorticity. It is very important to note that we do not make any smallness assumption on the parameter $\alpha \in \mathbf{R}$ so that we do treat Oseen vortices with arbitrarily high Reynolds numbers $|\alpha|$. However, unlike in the homogeneous case [8], we are not able to consider large perturbations of the vorticity, due to the lack of appropriate Lyapunov functions for the density-dependent system, and unfortunately, in **Theorem 1**, the allowed size ε_0 of the perturbations is a decreasing function of the Reynolds number $|\alpha|$ tending to zero as $|\alpha|$ goes to infinity.

Without loss of generality, we assume in **Theorem 1** and throughout this paper that the vorticity perturbation $\widetilde{w}$ has zero average. Indeed, if $w = \alpha\, G + \bar{w}$ with $\bar{w}$ small and $\delta := \int_{\mathbf{R}^2} \bar{w} \neq 0$, one can always rewrite $w = (\alpha + \delta)\, G + \widetilde{w}$ with $\widetilde{w}$ small and $\int_{\mathbf{R}^2} \widetilde{w} = 0$. This zero-mean condition is preserved under the evolution defined by (10) and is necessary to show that $\widetilde{w}(\tau)$ converges to zero as $\tau \to \infty$, *i.e.* to obtain an *asymptotic* stability result for the vorticity.

To make this asymptotic stability more concrete, we express it now in the original variables. Under the assumptions of **Theorem 1**, if (ρ, ω) is the solution of (3) defined by (7) with $w = \alpha\, G + \widetilde{w}$ and $r = \frac{1}{1+b}$, then the vorticity $\omega(t, x)$ satisfies

$$t^{\frac{1}{2}}\, |\, e^{\frac{|x|^2}{8t}}\, (\omega(t) - \alpha\, \omega^G(t))\, |_2 \,\leq\, \frac{C}{t^\gamma}\ , \quad t \geq 1\ .$$

Moreover this implies

$$t^{1-\frac{1}{p}}\, |\, \omega(t) - \alpha\, \omega^G(t)\, |_p \,\leq\, \frac{C_p}{t^\gamma}, \quad t^{\frac{1}{2}-\frac{1}{q}}\, |\, u(t) - \alpha\, u^G(t)\, |_q \,\leq\, \frac{C_q}{t^\gamma}$$

for any $1 \leq p \leq 2$, $\;2 < q < +\infty$, $\;t \geq 1$. Note that self-similarity implies $|\, \omega^G(t)\, |_p = C\, t^{-(1-\frac{1}{p})}$ and $|\, u^G(t)\, |_p = C\, t^{-(\frac{1}{2}-\frac{1}{q})}$. In contrast, since the density perturbation $\frac{1}{\rho} - 1$ is just transported by the divergence-free velocity field u, formally it remains constant in law, thus in any L^p-norm.

The rest of the paper is organized as follows. In a preliminary section, we collect estimates on the Biot-Savart kernel, thus on the velocity in terms of the vorticity, and on the pressure. In the second section, we establish various estimates for the solution of a linearized density equation. Similarly, in the third one, we study a linearized vorticity equation. The final section is devoted to the proof of **Theorem 1**.

However there is another underlying structure that the reader may find useful to keep in mind. Indeed, to establish the existence part of **Theorem 1**, following [1], we build a sequence of solutions of a linearization of (10). In order to show the convergence of this sequence, we will use local-in-time estimates in Sobolev norms for solutions of this linearized system. To be more precise, in a *first step*, we establish global estimates for solutions of the linearized system, which control the density in weighted L^p spaces and the vorticity in H^1. In a *second step*, we use the previous results to prove first global estimates of the density in $H^{2-\varepsilon}$ (with a loss of regularity), then local estimates of the vorticity in H^{s+1}, and finally local estimates of the density in H^{s+2}. Then we use these results to establish the existence and uniqueness parts of the theorem *via* estimates on differences of two solutions. In both second section and third section, where we study the linearized density and vorticity equations, we shall clearly indicate which estimates are needed for the first step and the second step respectively.

Let us make clear that the global estimates show decay in time and enable us to keep some quantities small, leading to stability, whereas the local estimates are only used to show the existence for all time of a unique solution, allowing growth in time but precluding blow-up in a finite time. It should also be emphasized that since we consider infinite energy solutions even the existence for small time was unknown. Actually, concerning local existence, one should also note that the localization of the vorticity necessary to obtain a decay rate is also useful to get low regularity estimates of the velocity field as provided by inequality (21) below.

In what follows, the original (and physical) time will never appear again, so for notational convenience henceforth we use the letter t to denote the rescaled time τ.

1 Preliminaries

If f is integrable over $\mathbf{R}^2$, we define its Fourier transform to be the function $\hat{f}$ defined for any $\eta \in \mathbf{R}^2$ by

$$\hat{f}(\eta) = \int_{\mathbf{R}^2} f(\zeta)\, e^{i\eta\cdot\zeta}\, d\zeta \ .$$

Concerning function spaces, we will also need the following convention. For any $\sigma \in \mathbf{R}$, we denote by $\dot{H}^\sigma(\mathbf{R}^2)$ the usual homogeneous Sobolev space on $\mathbf{R}^2$ equipped with $|\, f\,|_{\dot{H}^\sigma} := |\, I^\sigma f\,|_2$, where $I := (-\triangle)^{\frac{1}{2}}$.

1.1 Biot-Savart kernel

The goal of this subsection is to enable us to estimate the velocity in terms of the vorticity. For this purpose, we collect here some estimates on v in terms of w when v is obtained from w by the Biot-Savart law. Recall that, in this case, for almost every $\xi \in \mathbf{R}^2$,

$$v(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{(\xi-\eta)^\perp}{|\xi-\eta|^2}\, w(\eta)\, d\eta \tag{15}$$

where $(x_1, x_2)^\perp = (-x_2, x_1)$, that is $v = K_{BS} \star w$, where K_{BS} is the Biot-Savart kernel $K_{BS}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$. Also note that in terms of Fourier transform, (15) becomes

$$\hat{v}(\eta) = \frac{i\,\eta^\perp}{|\eta|^2}\, \hat{w}(\eta)\,. \tag{16}$$

Most of these estimates are already known, but for the sake of completeness we try to give proofs rather than references whenever this is not too long.

The following proposition gathers estimates in L^p-spaces.

Proposition 1

1. *Let $1 < p < 2 < q < +\infty$ be such that $1 + \frac{1}{q} = \frac{1}{2} + \frac{1}{p}$.*
   There exists $C > 0$ such that, if w belongs to $L^p(\mathbf{R}^2)$, then (15) defines a v in $L^q(\mathbf{R}^2)$ and

$$|\, v\,|_q \leq C\, |\, w\,|_p \ . \tag{17}$$

2. Let $1 \le p < 2 < q \le +\infty$ and $0 < \theta < 1$ be such that $\frac{\theta}{p} + \frac{1-\theta}{q} = \frac{1}{2}$. There exists $C > 0$ such that, if w belongs to $L^p(\mathbf{R}^2) \cap L^q(\mathbf{R}^2)$, then (15) defines a v in $L^\infty(\mathbf{R}^2)$ and

$$|\, v\,|_\infty \le C\,|\, w\,|_p^\theta\,|\, w\,|_q^{1-\theta}\ . \tag{18}$$

3. Let $1 < p < +\infty$. There exists $C > 0$ such that, if w belongs to $L^p(\mathbf{R}^2)$ and v is defined from w by (15), then ∇v belongs to $L^p(\mathbf{R}^2)$ and

$$|\,\nabla v\,|_p \le C\,|\, w\,|_p\ . \tag{19}$$

In addition, in all cases, we have $\operatorname{div} v = 0$ *and* $\operatorname{curl} v = w$.

Let us emphasize that we do not estimate v in $L^2(\mathbf{R}^2)$. Indeed, one can easily derive from (16) that to make v square integrable one must assume that w has zero circulation, that is $\int_{\mathbf{R}^2} w = 0$. This would exclude Oseen vortices.

Proof. Part 1 follows from a Young-like inequality called Hardy-Littlewood-Sobolev inequality (see for instance **Theorem V.1** in [13]). Indeed, K_{BS} is weakly L^2 but not square integrable.

Part 2 is trivial when $w \equiv 0$. If not, we remark that from Hölder's inequalities, we obtain

$$\begin{aligned} |v(\xi)| &\le \frac{1}{2\pi}\int_{\{|\eta|\le R\}} |w(\xi-\eta)|\,\frac{1}{|\eta|}\,d\eta + \frac{1}{2\pi}\int_{\{|\eta|\ge R\}} |w(\xi-\eta)|\,\frac{1}{|\eta|}\,d\eta \\ &\le C\,|\, w\,|_q\, R^{1-\frac{2}{q}} + C\,|\, w\,|_p\, \frac{1}{R^{\frac{2}{p}-1}}\ , \end{aligned}$$

for almost every $\xi \in \mathbf{R}^2$ and any $R > 0$. Aiming to optimize this inequality, we choose $R = (\frac{|\, w\,|_p}{|\, w\,|_q})^\beta$ with $\beta = \frac{1-\theta}{\frac{2}{p}-1} = \frac{\theta}{1-\frac{2}{q}}$ and derive (18).

Part 3 holds since differentiating (15) yields that ∇v is obtained from w by a singular integral kernel of Calderón-Zygmund type (see **Theorem II.3** in [13]). ∎

The following proposition gathers estimates in Sobolev spaces.

Proposition 2

1. Let $s \in \mathbf{R}$. There exists $C > 0$ such that, if w belongs to $\dot H^{s-1}(\mathbf{R}^2)$, then, if v is defined by (15), v belongs to $\dot H^s(\mathbf{R}^2)$ and

$$|\, v\,|_{\dot H^s} \le C\,|\, w\,|_{\dot H^{s-1}}\ . \tag{20}$$

2. Let $0 < s < 1$. There exists $C > 0$ such that, if $(1+|\xi|)\,w$ belongs to $L^2(\mathbf{R}^2)$, then, if v is defined by (15), v belongs to $\dot H^s(\mathbf{R}^2)$ and

$$|\, v\,|_{\dot H^s} \le C\,|\,(1+|\xi|)\,w\,|_2\ . \tag{21}$$

Proof. Part 1 is a direct consequence of the Fourier formulation (16).

Part 2 is thus reduced to estimate $|\,w\,|_{\dot H^{s-1}}$. If $0 < s < 1$, we note that

$$|\,w\,|^2_{\dot H^{s-1}} = C \int_{\mathbf{R}^2} \frac{|\hat w(\xi)|^2}{|\xi|^{2(1-s)}}\, d\xi \;\le\; C \int_{|\,\xi\,|\le 1} \frac{|\hat w(\xi)|^2}{|\xi|^{2(1-s)}}\, d\xi + C \int_{|\,\xi\,|\ge 1} |\hat w(\xi)|^2\, d\xi \ .$$

The second term of the right member is dominated by $|\,w\,|_2^2$. Choosing p such that $p > \frac{2}{s}$ and applying first Hölder's inequalities then Sobolev's embeddings, we obtain for the first term

$$\int_{|\,\xi\,|\le 1} \frac{|\hat w(\xi)|^2}{|\xi|^{2(1-s)}}\, d\xi \;\le\; C\,|\,\hat w\,|_p^2 \;\le\; C\,|\,\hat w\,|^2_{H^1} \ .$$

Finally, gathering every piece yields

$$|\,w\,|^2_{\dot H^{s-1}} \;\le\; C\,|\,\hat w\,|^2_{H^1} + C\,|\,w\,|_2^2 \;\le\; C\,|\,(1+|\xi|)\,w\,|_2^2 \ .$$

This concludes the proof. ∎

1.2 Pressure estimates

Keeping in mind equation (13), we gather some estimates for a solution Π of the following equation:

$$\operatorname{div}\big((1+b)\nabla\Pi\big) = \operatorname{div} F \ . \tag{22}$$

We begin with estimates in L^p-spaces.

Proposition 3

1. *Let $1 < p < +\infty$.*
There exist $C > 0$ and $\kappa > 0$ such that if F belongs to $L^p(\mathbf{R}^2)$ and b to $L^\infty(\mathbf{R}^2)$ with $\kappa\,|\,b\,|_\infty < 1$, then (22) has a unique solution Π (up to a constant) such that $\nabla\Pi$ belongs to $L^p(\mathbf{R}^2)$, and

$$|\,\nabla\Pi\,|_p \le \frac{C}{1-\kappa\,|\,b\,|_\infty}\,|\,F\,|_p \ . \tag{23}$$

2. *Let $1 < p \le +\infty$ and $1 < q, r < +\infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.*
There exist $C > 0$ and $\kappa > 0$ such that if F belongs to $L^q(\mathbf{R}^2)$ and, for $i = 1, 2$, b_i belongs to $L^\infty(\mathbf{R}^2) \cap L^p(\mathbf{R}^2)$ with $\kappa\,|\,b_i\,|_\infty < 1$ and Π_i solves

$$\operatorname{div}\big((1+b_i)\nabla\Pi_i\big) = \operatorname{div} F \ ,$$

then $\nabla\,(\Pi_2 - \Pi_1)$ belongs to $L^r(\mathbf{R}^2)$ and

$$|\,\nabla\,(\Pi_2-\Pi_1)\,|_r \le \frac{C}{(1-\kappa\,|\,b\,|_\infty)^2}\,|\,b_2-b_1\,|_p\,|\,F\,|_q \ . \tag{24}$$

Proof. To prove part 1 we want to obtain $\nabla\Pi$, the solution of (22), in terms of F as a perturbation of Leray projectors. Let $\mathbf{P}$ be the Leray projector, that is the projector on divergence-free vector fields along gradients, and let $\mathbf{Q} = \mathbf{I} - \mathbf{P}$. Remark that $\mathbf{Q}\,F$ gives the solution $\nabla\Pi$ of (22) when $b \equiv 0$. Now we can rewrite (22) as

$$\operatorname{div} \nabla\Pi = \operatorname{div} (\, F - b\,\nabla\Pi\,)$$

then $\nabla\Pi = \mathbf{Q}\,(F - b\,\nabla\Pi)$ thus $(\mathbf{I} + \mathbf{Q}\,b)\,\nabla\Pi = \mathbf{Q}\,F$. Since $\mathbf{Q}$ is continuous on L^p, there exists $\kappa > 0$ such that

$$|\,\mathbf{Q}\,b\,f\,|_p \;\le\; \kappa\,|\,b\,|_\infty\,|\,f\,|_p$$

for $f \in L^p(\mathbf{R}^2)$. Thus, if $\kappa\,|\,b\,|_\infty < 1$, $\mathbf{I} + \mathbf{Q}\,b$ is invertible on L^p, and

$$\nabla\Pi = (\mathbf{I} + \mathbf{Q}\,b)^{-1}\,\mathbf{Q}\,F \tag{25}$$

gives the unique solution, with the expected bound.

To prove part 2 reminding (25) we write

$$\begin{aligned} \nabla\Pi_1 &= (\mathbf{I} + \mathbf{Q}\,b_2)^{-1}\,(\mathbf{I} + \mathbf{Q}\,b_2)\,(\mathbf{I} + \mathbf{Q}\,b_1)^{-1}\,\mathbf{Q}\,F \\ \nabla\Pi_2 &= (\mathbf{I} + \mathbf{Q}\,b_2)^{-1}\,(\mathbf{I} + \mathbf{Q}\,b_1)\,(\mathbf{I} + \mathbf{Q}\,b_1)^{-1}\,\mathbf{Q}\,F \end{aligned}$$

then subtracting and factorizing yields

$$\nabla(\Pi_2 - \Pi_1) = (\mathbf{I} + \mathbf{Q}\,b_2)^{-1}\,\mathbf{Q}\,(b_1 - b_2)\,(\mathbf{I} + \mathbf{Q}\,b_1)^{-1}\,\mathbf{Q}\,F\;.$$

Now the continuity of the operator $\mathbf{Q}$ on L^r reduces (24) to an estimate on $|\,(b_1 - b_2)\,(\mathbf{I} + \mathbf{Q}\,b_1)^{-1}\,\mathbf{Q}\,F\,|_r$. At last applying first Hölder's inequalities then the continuity on L^q concludes the proof. ∎

In order to estimate solutions of (22) in Sobolev spaces, we first state some useful commutator estimates of Kato-Ponce type (see [9]). Let us recall that $I = (-\triangle)^{\frac{1}{2}}$.

Lemma 1 *Let $0 < s \le 1$ and $\sigma > 1$.*

1. *There exists $C > 0$ such that if $I^s f$ belongs to $L^2(\mathbf{R}^2)$ and g to $H^\sigma(\mathbf{R}^2)$, then $I^s(fg) - f\,I^s g$ belongs to $L^2(\mathbf{R}^2)$ and*

$$|\,I^s(fg) - f\,I^s g\,|_2 \le C\,|\,I^s f\,|_2\,|\,g\,|_{H^\sigma}. \tag{26}$$

2. *There exists $C > 0$ such that if $I^s f$ belongs to $H^\sigma(\mathbf{R}^2)$ and g to $L^2(\mathbf{R}^2)$, then $I^s(fg) - f\,I^s g$ belongs to $L^2(\mathbf{R}^2)$ and*

$$|\,I^s(fg) - f\,I^s g\,|_2 \le C\,|\,I^s f\,|_{H^\sigma}\,|\,g\,|_2. \tag{27}$$

Proof. Let us first note that there exists $C > 0$ such that

$$| \hat{h} |_1 \leq C | h |_{H^\sigma} ,$$

for any h in $H^\sigma(\mathbf{R}^2)$. This comes applying Hölder's inequalities to

$$\int_{\mathbf{R}^2} |\hat{h}(\eta)| \, d\eta = \int_{\mathbf{R}^2} \frac{1}{(1+|\eta|^2)^{\frac{\sigma}{2}}} (1+|\eta|^2)^{\frac{\sigma}{2}} |\hat{h}(\eta)| \, d\eta .$$

Therefore in order to prove the lemma it is sufficient to establish

$$| I^s(fg) - f I^s g |_2 \leq C | I^s f |_2 | \hat{g} |_1 , \tag{28}$$

$$| I^s(fg) - f I^s g |_2 \leq C | \widehat{I^s f} |_1 | g |_2 . \tag{29}$$

Now set $h = I^s(fg) - f I^s g$. We have

$$\hat{h}(\eta) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} (|\eta|^s - |\eta - \zeta|^s) \hat{f}(\zeta) \hat{g}(\eta - \zeta) \, d\zeta$$

for almost every $\eta \in \mathbf{R}^2$. Thanks to the following basic fact:

$$||\eta|^s - |\eta'|^s| \leq |\eta - \eta'|^s , \quad 0 < s \leq 1 \tag{30}$$

for $\eta, \eta' \in \mathbf{R}^2$, we obtain

$$|\hat{h}(\eta)| \leq \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} |\zeta|^s |\hat{f}(\zeta)| \, |\hat{g}(\eta - \zeta)| \, d\zeta .$$

At last depending on how we apply Young's inequalities we obtain either (28) or (29). ∎

We now state the announced estimates in Sobolev norms.

Proposition 4 *Let* $0 < s < 1$ *and* $\sigma > 1$.
There exists $C > 0$ *and* $\kappa > 0$ *such that if* F *belongs to* $H^s(\mathbf{R}^2)$, b *to* $L^\infty(\mathbf{R}^2)$ *with* $\kappa | b |_\infty < 1$ *and* $I^s b$ *belongs to* $H^\sigma(\mathbf{R}^2)$, *then, if* Π *solves (22),* $I^s \nabla \Pi$ *belongs to* $L^2(\mathbf{R}^2)$ *and*

$$| I^s \nabla \Pi |_2 \leq \frac{C}{1 - \kappa | b |_\infty} \left(| I^s F |_2 + \frac{1}{1 - \kappa | b |_\infty} | I^s b |_{H^\sigma} | F |_2 \right) . \tag{31}$$

Proof. Applying I^s to (22) and commuting b and I^s yields

$$\operatorname{div}\big((1+b)\nabla I^s \Pi\big) = \operatorname{div}\big([b, I^s]\nabla \Pi\big) + \operatorname{div}\big(I^s F\big) .$$

Applying then (23) to this equation we obtain

$$| I^s \nabla \Pi |_2 \leq \frac{C}{1 - \kappa | b |_\infty} \left(| I^s F |_2 + | [b, I^s] \nabla \Pi |_2 \right) .$$

Now using first (27) then applying (23) once again yields

$$| [b, I^s]\nabla\Pi |_2 \le C | I^s b |_{H^\sigma} | \nabla\Pi |_2 \le \frac{C}{1-\kappa | b |_\infty} | I^s b |_{H^\sigma} | F |_2 .$$

Gathering everything leads to (31). ∎

2 Density equation

In this section, we gather information on the following linearization of the density equation:

$$\partial_t b + \big((\nu - \frac{1}{2}\xi) \cdot \nabla \big) b = 0 \tag{32}$$

where $\widetilde{\nu}$ is a divergence-free vector-field, $\alpha \in \mathbf{R}$ and $\nu = \alpha v^G + \widetilde{\nu}$. By linearization, we mean that we do not assume that $\widetilde{\nu}$ is obtained from a solution $\widetilde{w}$ of the vorticity equation in (10), which involves b.

Actually in the cases we will consider $\nu - \frac{1}{2}\xi$ generates a flow since $\nabla\widetilde{\nu}$ belongs to $L^2_{loc}(\mathbf{R}^+; L^\infty(\mathbf{R}^2))$ (and $\widetilde{\nu}$ belongs to $L^\infty_{loc}(\mathbf{R}^+; L^\infty(\mathbf{R}^2))$. Thus our point is not to prove the existence of a solution to (32) but to establish bounds for such a solution.

We begin with estimates in L^p-spaces, weighted or not. We recall that $L^p_w(\mathbf{R}^2)$ for $1 \le p \le \infty$ is the weighted space defined in (14). The next proposition is the density part of the announced *first step* of the proof.

Proposition 5 *Let $T > 0$.*
Assume that $\widetilde{\nu}$ is a divergence-free vector field belonging to $L^2(0, T; L^\infty(\mathbf{R}^2))$. Then b, the solution of (32) with initial data b_0, satisfies

1. *for any $1 \le p \le +\infty$, provided $b_0 \in L^p(\mathbf{R}^2)$,*

$$| b(t) |_p \le | b_0 |_p \, e^{-\frac{t}{p}} , \quad \textit{for } 0 < t < T ; \tag{33}$$

2. *for any $1 \le p \le +\infty$, provided $G^{-\frac{1}{2}} b_0 \in L^p(\mathbf{R}^2)$,*

$$| b(t) |_{w,p} \le | b_0 |_{w,p} \, e^{-\frac{t}{p}} \, e^{\frac{1}{8}\int_0^t | \widetilde{\nu}(s) |_\infty^2 \, ds} , \quad \textit{for } 0 < t < T. \tag{34}$$

Proof. In order to prove part 1 for $1 \le p < +\infty$, multiply (32) by $\mathrm{sgn}(b)\,|b|^{p-1}$, where sgn is the usual sign function, and integrate by parts to obtain

$$\frac{d}{dt} | b |_p^p = - \int_{\mathbf{R}^2} \big((\nu - \frac{1}{2}\xi) \cdot \nabla \big) |b|^p = -| b |_p^p$$

since $\operatorname{div} \nu = 0$ and $\operatorname{div} \xi = 2$. Integrating gives (33) in this case. The case $p = +\infty$ follows letting p go to infinity.

To prove the weighted part of the proposition for $1 \leq p < +\infty$, start as in the former and obtain

$$\frac{d}{dt} |\, b\,|_{w,p}^p = -\int_{\mathbf{R}^2} G^{-\frac{p}{2}} \big(\, (\nu - \frac{1}{2}\,\xi) \cdot \nabla\,\big)\, |b|^p = \int_{\mathbf{R}^2} G^{-\frac{p}{2}}\, \big(\, \frac{p}{4}\,\xi \cdot (\nu - \frac{1}{2}\,\xi) - 1\,\big) |b|^p$$

since $\nabla G^{-\frac{p}{2}} = G^{-\frac{p}{2}}\, \frac{p}{4}\,\xi$. Now, since $\xi \perp v^G(\xi)$, we have

$$\xi \cdot (\nu(\xi) - \frac{1}{2}\,\xi) \;=\; \xi \cdot \widetilde{\nu}(\xi) - \frac{1}{2}\,|\xi|^2 \;\leq\; \frac{1}{2}\,|\widetilde{\nu}(\xi)|^2$$

hence

$$\frac{d}{dt} |\, b\,|_{w,p}^p \leq (-1 + \frac{p}{8}\,|\,\widetilde{\nu}\,|_\infty^2)\; |\, b\,|_{w,p}^p \;.$$

Again integrating achieves the proof for finite p and the case $p = +\infty$ follows letting p go to infinity. ∎

The next proposition corresponds to the density part of the announced *second step*: local-in-time estimates in Sobolev norms. In order to prove some part of it we will need the following commutator lemma.

Lemma 2 *Let $s \geq 1$ and $\sigma > 1$.*
There exists $C > 0$ such that if $I^s f$ belongs to $L^2(\mathbf{R}^2)$ and g to $H^\sigma(\mathbf{R}^2)$, then $I^s(fg) - f\,I^s g$ belongs to $L^2(\mathbf{R}^2)$ and

$$|\, I^s(fg) - f\,I^s g\,|_2 \;\leq C\,|\, I^s f\,|_2\,|\, g\,|_{H^\sigma} + C\,|\, \nabla f\,|_{H^\sigma}\,|\, I^{s-1} g\,|_2 \;. \tag{35}$$

Proof. The proof is essentially the same as **Lemma 1**'s except that here (30) is replaced by

$$||\zeta|^s - |\zeta'|^s| \;\leq\; C\,|\zeta - \zeta'|\,(|\zeta - \zeta'|^{s-1} + |\zeta'|^{s-1}), \quad \text{for } \zeta, \zeta' \in \mathbf{R}^2 \;. \tag{36}$$

Note that we could have obtained, as in **Lemma 1**, various estimates depending on the way we apply Young's inequalities. ∎

Proposition 6 *Let $T > 0$.*

1. *Let $0 < s \leq 2$ and $0 < \varepsilon < s$.*
   Assume $\widetilde{\nu}$ is a divergence-free vector field with $\nabla\widetilde{\nu} \in L^1(0,T;H^1(\mathbf{R}^2))$.
   Then there exists $C_T > 0$ independent of $\widetilde{\nu}$ such that, for any initial data $b_0 \in H^s(\mathbf{R}^2)$, any solution $b \in L^\infty(0,T;H^{s-\epsilon}(\mathbf{R}^2))$ of (32) satisfies

$$|\, b(t)\,|_{H^{s-\varepsilon}} \leq C_T\,|\, b_0\,|_{H^s}\, \exp\big((C_T \int_0^t |\, \nabla\nu(\tau)\,|_{H^1}\, d\tau)^2\big)\;, \quad \textit{for } 0 \leq t \leq T. \tag{37}$$

2. Let $s > 2$.
Assume $\widetilde{\nu}$ is a divergence-free vector field with $\nabla\widetilde{\nu} \in L^1(0,T;H^{s-1}(\mathbf{R}^2))$. Then (32) has a unique solution $b \in L^\infty(0,T;H^s(\mathbf{R}^2))$, for any initial data $b_0 \in H^s(\mathbf{R}^2)$. Moreover there exists $C > 0$ independent of b_0 and $\widetilde{\nu}$ such that b satisfies

$$|\, b(t)\,|_{H^s} \leq C\,|\, b_0\,|_{H^s}\, e^{\frac{s-1}{2}t} \exp\big(C\int_0^t |\, \nabla\nu(\tau)\,|_{H^{s-1}}\, d\tau\big)\ , \quad \text{for } 0 \leq t \leq T. \tag{38}$$

Proof. We refer to **Theorem 0.1** in [2] for the first part of the proposition. Let us only add that here we apply the former theorem with $\sigma = s$, $\epsilon = \varepsilon$ and $p = r = p_2 = r_2 = 2$.

To prove the second part, we start computing $[I^s, \frac{\xi}{2}] \cdot f = -\frac{s}{2}\, I^{s-2}\, \mathrm{div}\, f$, for any vector field f. Thus applying I^s to equation (32) and commuting yield

$$\partial_t\, I^s b + \big(\,(\nu - \frac{1}{2}\,\xi)\cdot\nabla\,\big)\, I^s b = \frac{s}{2}\, I^s b - [I^s, \nu]\cdot\nabla b\ .$$

Then multiplying by $I^s b$ and integrating lead to

$$\frac{1}{2}\frac{d}{dt}|\, I^s b\,|_2^2 - \frac{s-1}{2}|\, I^s b\,|_2^2 = -\int_{\mathbf{R}^2} I^s b\ [I^s,\nu]\cdot\nabla b$$

since $\mathrm{div}\,\nu = 0$. Now use Cauchy-Schwarz' inequality and apply **Lemma 2** (with $\sigma = s - 1$) to get

$$\frac{1}{2}\frac{d}{dt}|\, I^s b\,|_2^2 - \frac{s-1}{2}\,|\, I^s b\,|_2^2 \leq C\,|\,\nabla\nu\,|_{H^{s-1}}\,|\, b\,|_{H^s}^2\ .$$

At last combine the former with $\frac{1}{2}\frac{d}{dt}|\, b\,|_2^2 + \frac{1}{2}|\, b\,|_2^2 \leq 0$ to obtain

$$\frac{1}{2}\frac{d}{dt}|\, b\,|_{H^s}^2 - \frac{s-1}{2}\,|\, b\,|_{H^s}^2 \leq C\,|\,\nabla\nu\,|_{H^{s-1}}\,|\, b\,|_{H^s}^2$$

which yields (38) by a mere integration. ∎

The last estimate we state for the linearized transport equation (32) is intended to be used for the proofs of the convergence of our iterative scheme and of the uniqueness of our solutions. Indeed we estimate the difference of two solutions of equations of type (32).

Proposition 7 *Let $T > 0$.*
Assume that, for $i = 1, 2$, $\widetilde{\nu}_i$ is a divergence-free vector field belonging to $L^2(0,T;W^{1,\infty}(\mathbf{R}^2))$. If, for $i = 1, 2$, b_i is a solution of

$$\partial_t\, b_i + \big(\,(\nu_i - \frac{1}{2}\,\xi)\cdot\nabla\,\big)\, b_i = 0\ ,$$

where $\nu_i = \alpha\, v^G + \widetilde{\nu}_i$, with initial data b_0, then b_1 and b_2 satisfy

1. *provided that $G^{-\frac{1}{2}}\nabla b_0$ belongs to $L^p(\mathbf{R}^2)$, for some $1 \leq p \leq +\infty$,*

$$|\,\nabla b_i(t)\,|_{w,p} \leq |\,\nabla b_0\,|_{w,p}\, e^{-t(\frac{1}{p}-\frac{1}{2})}\, e^{\frac{1}{8}\int_0^t |\,\widetilde{\nu}_i(s)\,|_\infty^2\, ds}\, e^{\int_0^t |\,\nabla \nu_i(s)\,|_\infty\, ds} \tag{39}$$

for $i = 1,2$ and $0 \leq t \leq T$;

2. *provided that $G^{-\frac{1}{2}} b_0$ belongs to $L^p(\mathbf{R}^2)$ and $G^{-\frac{1}{2}}\nabla b_0$ belongs to $L^q(\mathbf{R}^2)$, for some $1 \leq p < q \leq +\infty$,*

$$|\,(b_2 - b_1)(t)\,|_{w,p} \leq e^{\frac{1}{8}\int_0^t |\,\widetilde{\nu}_2(s)\,|_\infty^2\, ds} \sup_{0\leq s\leq t} |\,\nabla b_1(s)\,|_{w,q} \int_0^t |\,(\widetilde{\nu}_2 - \widetilde{\nu}_1)(s)\,|_r\, ds \tag{40}$$

for $0 \leq t \leq T$, where r is such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Proof. In order to prove the first part of the proposition, we start differentiating the equation for b_1 to get for $j = 1,2$,

$$\partial_t\, \partial_j b_1 + \big(\,(\nu_1 - \frac{1}{2}\,\xi) \cdot \nabla\,\big)\, \partial_j b_1 = -\partial_j \nu_1 \cdot \nabla b_1 + \frac{1}{2}\,\partial_j b_1\ .$$

From this, following the proof of (34), we obtain for $j = 1,2$,

$$\frac{d}{dt}|\,\partial_j b_1\,|_{w,p}^p \leq (-1 + \frac{p}{2} + \frac{p}{8}\,|\,\widetilde{\nu}_1\,|_\infty^2)\,|\,\partial_j b_1\,|_{w,p}^p + p\,|\,\nabla\nu_1\,|_\infty\,|\,\nabla b_1\,|_{w,p}^p\ .$$

Now combining the inequalities for $j = 1,2$ and integrating lead to (39).

Then to prove part 2 observe that $b_2 - b_1$ satisfies

$$\partial_t\,(b_2 - b_1) + \big(\,(\nu_2 - \frac{1}{2}\,\xi) \cdot \nabla\,\big)\,(b_2 - b_1) = -(\widetilde{\nu}_2 - \widetilde{\nu}_1) \cdot \nabla b_1\ .$$

Now following again the proof of (34) yields

$$\frac{d}{dt}\,|\,\delta\, b\,|_{w,p}^p\ \leq\ (-1 + \frac{p}{8}\,|\,\widetilde{\nu}_2\,|_\infty^2)\ |\,\delta\, b\,|_{w,p}^p + p\,|\,\delta\, b\,|_{w,p}^{p-1}\,|\,\nabla b_1\,|_{w,q}\,|\,\widetilde{\nu}_2 - \widetilde{\nu}_1\,|_r$$

where $\delta\, b = b_2 - b_1$. It is now straightforward to derive (40). ∎

3 Vorticity equation

As announced, we now study a linearization of the vorticity equation:

$$\partial_t\, \widetilde{w} - (\mathcal{L} - \alpha\,\Lambda)\,\widetilde{w} + \big(\,\widetilde{\nu} \cdot \nabla\,\big)\,\widetilde{w} = \mathrm{div}\,\big(b\,(\nabla w + \nabla^\perp \Pi)\big) \tag{41}$$

where $\mathcal{L}$ and Λ are as in (11), b is a real function, $\widetilde{\nu}$ is a divergence-free vector field, $\alpha \in \mathbf{R}$,

$$\begin{array}{rclcrcl} \widetilde{v} & = & K_{BS} \star \widetilde{w} & , & v & = & \alpha\, v^G + \widetilde{v}\ , \\ w & = & \alpha\, G + \widetilde{w} & , & \nu & = & \alpha\, v^G + \widetilde{\nu}\ , \end{array}$$

and $\nabla\Pi$ is obtained by solving

$$\mathrm{div}\,\big((1+b)\,\nabla\Pi\big) = \mathrm{div}\,\big((1+b)\,\triangle v - (\nu\cdot\nabla)\,v\big)\ . \tag{42}$$

Remind that we always assume $\int_{\mathbf{R}^2}\widetilde{w}_0 = 0$.

3.1 Weighted estimate

In this subsection we establish a global-in-time estimate in weighted L^2-spaces.

Proposition 8 *Let $\alpha\in\mathbf{R}$, $K_0>0$. There exist $\varepsilon_0>0$ and $C>0$ such that if b is a real function and $\widetilde{\nu}$ a divergence-free vector field such that*

1. *for $0<t<T$, for any $1\le p\le+\infty$, $2\le q\le+\infty$,*

$$|\,b(t)\,|_p \;\le\; |\,b_0\,|_p\,e^{-\frac{t}{p}} \quad , \quad |\,b(t)\,|_{w,q} \;\le\; |\,b_0\,|_{w,q}\,e^{-\frac{t}{q}}\,e^{K_0}$$

2. *for $0<t<T$,*

$$|\,\widetilde{\nu}(t)\,|_8 \;\le\; K_0 \quad , \quad \int_0^t|\,\widetilde{\nu}\,|_\infty^2 \;\le\; \tfrac{1}{24}$$

3. *and*

$$|\,b_0\,|_{w,4} \;\le\; \varepsilon_0 \quad , \quad |\,b_0\,|_{w,\infty} \;\le\; \varepsilon_0$$

then any solution $\widetilde{w}\in L^\infty(0,T\,;L^2_w(\mathbf{R}^2))$ of (41), with initial data $\widetilde{w}_0\in L^2_w(\mathbf{R}^2)$, satisfies, for any $0<t<T$,

$$\begin{aligned}|\,\widetilde{w}(t)\,|_{w,2}^2 \quad &+ \quad C\int_0^t\big(|\,\widetilde{w}\,|_{w,2}^2+|\,\nabla\widetilde{w}\,|_{w,2}^2+|\,|\xi|\widetilde{w}\,|_{w,2}^2\big)\\ &\le\quad 2\,|\,\widetilde{w}_0\,|_{w,2}^2\; + C\,|\alpha|\,|\,b_0\,|_{w,4}\,.\end{aligned} \tag{43}$$

Note that the assumptions on b corresponds to the first proposition of the previous section. Note also that once α and K_0 are fixed, since L^2_w is embedded in any L^p, $1\le p\le 2$, and H^1 is embedded in any L^q, $2\le q<\infty$, inequalities (18) and (43) enable us to make $\int_0^t|\,\widetilde{v}\,|_\infty^2$ as small as we want provided we take $\widetilde{w}_0$ and b_0 small enough. At last note that the proposition enables us to bound $\int_0^t|\,\nabla\widetilde{v}\,|_{H^1}^2$, which can be used in (37).

Proof. Our strategy is to multiply (41) by $G^{-1}\,\widetilde{w}$ and integrate to bound $\frac{d}{dt}|\,\widetilde{w}\,|_{w,2}^2$. In what follows, we examine each term arising once multiplied by $G^{-1}\,\widetilde{w}$ and integrated on $\mathbf{R}^2$.

• Let us emphasize first that (41) preserves $\int_{\mathbf{R}^2}\widetilde{w}$. Hence $\int_{\mathbf{R}^2}\widetilde{w}=0$.

Let $\mathrm{L}:=G^{-\frac12}\,(-\mathcal{L})\,G^{\frac12}$. A direct calculation shows that $\mathrm{L}=-\triangle+\frac{|\xi|^2}{16}-\frac12$ is a harmonic oscillator with spectrum $\{0,\frac12,1,\frac32,\ldots\}$. Moreover 0 is a simple eigenvalue with eigenvector $G^{\frac12}$. In particular, if f belongs to the domain of L with $\int_{\mathbf{R}^2} G^{\frac12} f = 0$, then $\int_{\mathbf{R}^2} f\,\mathrm{L} f \geq \frac12 |\, f\,|_2^2$. Coming back to $\mathcal{L}$, we obtain: if $G^{-\frac12}\widetilde{w}$ belongs to the domain of L with $\int_{\mathbf{R}^2} \widetilde{w} = 0$, then, for any $0 < \gamma < \frac12$,

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\mathcal{L}\,\widetilde{w} \leq -\frac12\,(1-\gamma)\,|\,\widetilde{w}\,|_{w,2}^2 + \gamma \int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\mathcal{L}\,\widetilde{w}$$

thus integrating by part from the formula for L

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\mathcal{L}\,\widetilde{w} \leq -\frac12\,(1-2\gamma)\,|\,\widetilde{w}\,|_{w,2}^2 - \gamma\,\big(|\,\nabla(G^{-\frac12}\,\widetilde{w})\,|_2^2 + |\,\frac{|\xi|}{4}\,\widetilde{w}\,|_{w,2}^2\big)$$

and expanding

$$\begin{aligned}\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\mathcal{L}\,\widetilde{w} \quad &\leq \quad -\frac12\,(1-2\gamma)\,|\,\widetilde{w}\,|_{w,2}^2 \\ &\quad -\gamma\,\big(\,|\,\nabla\widetilde{w}\,|_{w,2}^2 + 2\,|\,\frac{|\xi|}{4}\,\widetilde{w}\,|_{w,2}^2 + 2\int_{\mathbf{R}^2} G^{-1}\,\nabla\widetilde{w}\cdot\frac{\xi}{4}\,\widetilde{w}\,\big)\end{aligned}$$

hence

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\mathcal{L}\,\widetilde{w} \leq -\frac12\,(1-2\gamma)\,|\,\widetilde{w}\,|_{w,2}^2 - \gamma\,\big(\frac13\,|\,\nabla\widetilde{w}\,|_{w,2}^2 + \frac12\,|\,\frac{|\xi|}{4}\,\widetilde{w}\,|_{w,2}^2\big)\,. \quad (44)$$

- Recalling that $\Lambda\widetilde{w} = v^G\cdot\nabla\widetilde{w} + \widetilde{v}\cdot\nabla G$, we obtain

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\Lambda\widetilde{w} = 0\ . \quad (45)$$

Indeed , from $v^G(\xi)\perp\xi$ and $\nabla G^{-1} = -\frac{\xi}{2}G^{-1}$, we derive

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,v^G\cdot\nabla\widetilde{w} = -\frac12\int_{\mathbf{R}^2} G^{-1}\,\frac{\xi}{2}\,\cdot\,v^G\,\widetilde{w}^2 = 0\ .$$

And, on the other hand, using the identity $\eta^\perp\cdot\xi = -\xi^\perp\cdot\eta$ and the explicit formula (15) for the Biot-Savart law, we derive

$$\begin{aligned}\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\widetilde{v}\cdot\nabla G \quad &= \quad -\int_{\mathbf{R}^2} \widetilde{w}(\xi)\,\widetilde{v}(\xi)\cdot\frac{\xi}{2}\,d\xi \\ &= \quad -\frac{1}{4\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2} \widetilde{w}(\xi)\,\frac{(\xi-\eta)^\perp\cdot\xi}{|\xi-\eta|^2}\,\widetilde{w}(\eta)\,d\eta\,d\xi \\ &= \quad \frac{1}{4\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2} \widetilde{w}(\xi)\,\frac{\eta^\perp\cdot\xi}{|\xi-\eta|^2}\,\widetilde{w}(\eta)\,d\eta\,d\xi \\ &= \quad -\frac{1}{4\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2} \widetilde{w}(\xi)\,\frac{\xi^\perp\cdot\eta}{|\xi-\eta|^2}\,\widetilde{w}(\eta)\,d\eta\,d\xi \\ &= \quad -\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\widetilde{v}\cdot\nabla G\ .\end{aligned}$$

Thus $\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\widetilde{v}\cdot\nabla G = 0$.

• Using Hölder's inequalities, we also obtain

$$|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\widetilde{\nu}\cdot\nabla\widetilde{w}| \;\le\; 6\,|\,\widetilde{\nu}\,|_\infty^2\,|\,\widetilde{w}\,|_{w,2}^2 + \frac{1}{24}\,|\,\nabla\widetilde{w}\,|_{w,2}^2\ . \tag{46}$$

• Integrating by part, we obtain

$$\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\ \mathrm{div}\,(b\,\nabla\widetilde{w}) = -\int_{\mathbf{R}^2} G^{-1}\,b\,|\nabla\widetilde{w}|^2 - \frac{1}{2}\int_{\mathbf{R}^2} G^{-1}\,b\,\widetilde{w}\,\xi\cdot\nabla\widetilde{w}$$

then, using Hölder's inequalities and the fact that $|\,b(t)\,|_\infty \le |\,b_0\,|_\infty$,

$$|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\ \mathrm{div}\,(b\,\nabla\widetilde{w})| \;\le\; \frac{5}{4}\,|\,b_0\,|_\infty\,|\,\nabla\widetilde{w}\,|_{w,2}^2 + \frac{1}{4}\,|\,b_0\,|_\infty\,|\,|\xi|\,\widetilde{w}\,|_{w,2}^2\ . \tag{47}$$

• In the same way, since $|\,b(t)\,|_2 \le |\,b_0\,|_2\,e^{-\frac{t}{2}}$, we have

$$\begin{aligned}|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\ \mathrm{div}\,(b\,\nabla G)| \;&=\; |\,\frac{1}{2}\int_{\mathbf{R}^2} b\,\xi\,\cdot\,\nabla\widetilde{w} + \frac{1}{4}\int_{\mathbf{R}^2} b\,\widetilde{w}\,|\xi|^2\,| \\ &\le\; C\,|\,b_0\,|_2\,e^{-\frac{t}{2}}\,(|\,\nabla\widetilde{w}\,|_{w,2} + |\,\widetilde{w}\,|_{w,2})\end{aligned}$$

thus

$$|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\ \mathrm{div}\,(b\,\nabla G)| \;\le\; C\,|\,b_0\,|_2\,(\,e^{-t} + |\,\nabla\widetilde{w}\,|_{w,2}^2 + |\,\widetilde{w}\,|_{w,2}^2)\ . \tag{48}$$

• Finally, integrating by part, using Hölder's inequalities and applying inequality (23), we obtain, for b_0 small enough in L^∞,

$$\begin{aligned}|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\ \mathrm{div}\,(b\,\nabla^\perp\Pi)\,| \;&=\; |\,\frac{1}{2}\int_{\mathbf{R}^2} G^{-1}\,b\,\widetilde{w}\,\xi\cdot\nabla^\perp\Pi + \int_{\mathbf{R}^2} G^{-1}\,b\,\nabla\widetilde{w}\cdot\nabla^\perp\Pi\,| \\ &\le\; \frac{C}{1-\kappa\,|\,b_0\,|_\infty}\,(|\,|\xi|\widetilde{w}\,|_{w,2} + |\,\nabla\widetilde{w}\,|_{w,2}) \\ &\times\; (\;|\,b_0\,|_{w,\infty}\,e^{K_0}\,|\,(1+b)\,\triangle\widetilde{v}\,|_2 \\ &\quad + |\,b_0\,|_{w,4}\,e^{K_0}\,e^{-\frac{t}{4}}\,|\,\alpha\,(1+b)\,\triangle v^G - (\nu\cdot\nabla)\,v\,|_4\,).\end{aligned}$$

Now, on one hand, since $|\,b(t)\,|_\infty \le |\,b_0\,|_\infty$, estimate (20) yields

$$|\,(1+b)\,\triangle\widetilde{v}\,|_2 \le C\,(1 + |\,b_0\,|_\infty)\,|\,\nabla\widetilde{w}\,|_2\ .$$

On the other hand, similarly, we have

$$|\,(1+b)\,\triangle v^G\,|_4 \le C\,(1 + |\,b_0\,|_\infty)\ .$$

At last, Hölder's inequalities, estimate (19) and Sobolev's embeddings yields

$$|\,(\nu\cdot\nabla)\,v\,|_4 \le C\,(|\alpha| + |\,\widetilde{\nu}\,|_8)\,(|\alpha| + |\,\widetilde{w}\,|_8) \le C\,(|\alpha| + |\,\widetilde{\nu}\,|_8)\,(|\alpha| + |\,\widetilde{w}\,|_{H^1})\ .$$

Taking this into account yields when $\kappa\,|\,b_0\,|_\infty \leq \frac{1}{2}$

$$
\begin{array}{rll}
|\displaystyle\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w} & \times & \mathrm{div}\,(b\,\nabla^\perp \Pi)| \\
\leq & e^{-\frac{t}{2}} & C\,|\alpha|\,e^{K_0}\,|\,b_0\,|_{w,4}\,(1+|\alpha|+|\,\widetilde{\nu}\,|_8)^2 \\
+ & |\,\widetilde{w}\,|^2_{w,2} & C\,e^{K_0}\,|\,b_0\,|_{w,4}\,(1+|\alpha|+|\,\widetilde{\nu}\,|_8)^2 \\
+ & |\,\nabla\widetilde{w}\,|^2_{w,2} & C\,e^{K_0}(|\,b_0\,|_{w,\infty}+|\,b_0\,|_{w,4}(1+|\alpha|+|\,\widetilde{\nu}\,|_8)) \\
+ & |\,|\xi|\widetilde{w}\,|^2_{w,2} & C\,e^{K_0}\,(|\,b_0\,|_{w,\infty}+|\,b_0\,|_{w,4}(1+|\alpha|))\ .
\end{array} \tag{49}
$$

It only remains to us to gather everything after setting $\gamma = \frac{1}{4}$ in (44) and integrate in time in order to obtain, when $\kappa\,|\,b_0\,|_\infty \leq \frac{1}{2}$,

$$
\begin{array}{rlcl}
|\,\widetilde{w}(t)\,|^2_{w,2} & \times\quad (1-12\int_0^t |\,\widetilde{\nu}\,|^2_\infty) & & \\
+ & C\int_0^t |\,\widetilde{w}\,|^2_{w,2} & \times & (1-e^{K_0}\,|\,b_0\,|_{w,4}\,(1+|\alpha|+|\,\widetilde{\nu}\,|_8)^2) \\
+ & C\int_0^t |\,\nabla\widetilde{w}\,|^2_{w,2} & \times & (1-\ e^{K_0}\,(\,|\,b_0\,|_{w,\infty} \\
 & & & \qquad +|\,b_0\,|_{w,4}\,(1+|\alpha|+|\,\widetilde{\nu}\,|_8)) \\
+ & C\int_0^t |\,|\xi|\,\widetilde{w}\,|^2_{w,2} & \times & (1-\ e^{K_0}\,(|\,b_0\,|_{w,\infty}+|\,b_0\,|_{w,4}\,(1+|\alpha|))) \\
\leq & |\,\widetilde{w}_0\,|^2_{w,2}\ +\ C\,|\alpha| & \times & |\,b_0\,|_{w,4}\,e^{K_0}\,(1+|\alpha|+|\,\widetilde{\nu}\,|_8)^2
\end{array}
$$

which yields the proposition since $\int_0^t |\,\widetilde{\nu}\,|^2_\infty \ \leq\ \frac{1}{24}$. ■

3.2 Sobolev estimate

In this subsection we prove a local-in-time estimate in Sobolev norms for solutions of equation (41). Remind that $I = (-\triangle)^{\frac{1}{2}}$.

Proposition 9
Let $0 < s < 1$, $1+s < s' < 2$, $1 < s'' < 2-s$ and $\alpha \in \mathbf{R}$.
There exists $\varepsilon_0 > 0$ and, for $K > 0$, there exists $C > 0$ such that if b is a real function and $\widetilde{\nu}$ a divergence-free vector field such that

$$
\begin{array}{lllllll}
\sup_{[0,T]}|\,b\,|_\infty & \leq & |\,b_0\,|_\infty \leq \varepsilon_0 & , & \sup_{[0,T]}|\,b\,|_{H^{s'}} & \leq & K\ , \\
\sup_{[0,T]}|\,\widetilde{w}\,|_2 & \leq & K & , & \int_0^T |\,\widetilde{w}\,|^2_{w,2} & \leq & K\ , \\
\int_0^T |\,\nabla\widetilde{w}\,|^2_2 & \leq & K & , & \int_0^T |\,\widetilde{\nu}\,|^2_\infty & \leq & K\ , \\
\sup_{[0,T]}|\,I^s\widetilde{\nu}\,|_2 & \leq & \varepsilon_0 & , & \int_0^T |\,I^s\widetilde{\nu}\,|^2_{H^{s''}} & \leq & K\ ,
\end{array}
$$

then any solution $\widetilde{w} \in L^\infty(0,T;\dot{H}^s(\mathbf{R}^2))$ of (41), with initial data $\widetilde{w}_0 \in \dot{H}^s(\mathbf{R}^2)$, satisfies for any $0 < t < T$,

$$
|\,I^s\widetilde{w}\,(t)\,|^2_2 + C\int_0^t |\,I^s\nabla\widetilde{w}\,|^2_2\ \leq\ C\ e^{Ct}\ (\,|\,I^s\widetilde{w}_0\,|^2_2 + K\,)\ . \tag{50}
$$

Note that (33) and (37) can provide us the validity of the assumptions on b, and (43) both the validity of the assumptions on $\widetilde{w}$ and, thanks to estimates (18), (20) and (21), the validity of estimates on $\widetilde{\nu}$ when $\widetilde{\nu} = \widetilde{v}$. Conversely (50) can be used in (38), again when $\widetilde{\nu} = \widetilde{v}$.

Proof. We choose σ such that $1 < \sigma < 1 + s$, $s + \sigma < s'$ and $\sigma < s''$. The only role of σ is to make clearer our use of commutator estimates (26) and (27).

Our strategy is to apply I^s to (41), then multiply by $I^s\widetilde{w}$ and estimate each term arising, in order to bound $\frac{d}{dt}|\, I^s\widetilde{w}\,|_2^2$.

• First we compute the commutator $[I^s, \mathcal{L}] = \frac{s}{2} I^s$ and obtain

$$\int_{\mathbf{R}^2} I^s\widetilde{w} \quad I^s\mathcal{L}\widetilde{w} = -\int_{\mathbf{R}^2} |\nabla I^s\widetilde{w}|^2 + \frac{1+s}{2}\int_{\mathbf{R}^2} |I^s\widetilde{w}|^2 \tag{51}$$

since integrating by parts yields $\int_{\mathbf{R}^2} f\ \mathcal{L}f = -\int_{\mathbf{R}^2} |\nabla f|^2 + \frac{1}{2}\int_{\mathbf{R}^2} |f|^2$.

• On one hand, we have

$$\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ I^s((v^G\cdot\nabla)\,\widetilde{w}) = \int_{\mathbf{R}^2} I^s\widetilde{w}\ \ (v^G\cdot\nabla)\, I^s\widetilde{w} + \int_{\mathbf{R}^2} I^s\widetilde{w}\ \ ([I^s, v^G]\cdot\nabla)\,\widetilde{w}$$

with, integrating by parts,

$$\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ (v^G\cdot\nabla)\, I^s\widetilde{w} \;=\; \frac{1}{2}\int_{\mathbf{R}^2} (v^G\cdot\nabla)\,|I^s\widetilde{w}|^2 \;=\; 0$$

and, using Hölder's inequalities and inequality (27),

$$|\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ ([I^s, v^G]\cdot\nabla)\,\widetilde{w}| \;\le\; C\,|\, I^s\widetilde{w}\,|_2\,|\, I^s v^G\,|_{H^\sigma}\,|\,\nabla\widetilde{w}\,|_2\ .$$

On the other hand, we have

$$\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ I^s((\widetilde{v}\cdot\nabla)\, G) = \int_{\mathbf{R}^2} I^s\widetilde{w}\ \ (\widetilde{v}\cdot\nabla)\, I^s G + \int_{\mathbf{R}^2} I^s\widetilde{w}\ \ ([I^s, \widetilde{v}]\cdot\nabla)\, G$$

with, using Hölder's inequalities,

$$|\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ (\widetilde{v}\cdot\nabla)\, I^s G| \le C\,|\, I^s\widetilde{w}\,|_2|\,\widetilde{v}\,|_\infty|\,\nabla I^s G\,|_2$$

and, using both Hölder's inequalities and inequality (26),

$$|\int_{\mathbf{R}^2} I^s\widetilde{w}\ \ ([I^s, \widetilde{v}]\cdot\nabla)\, G| \le C\,|\, I^s\widetilde{w}\,|_2\,|\, I^s\widetilde{v}\,|_2\,|\,\nabla G\,|_{H^\sigma}\ .$$

Using estimate (21) and estimate (18) combined with Sobolev's embeddings, we derive

$$|\int_{\mathbf{R}^2} I^s\widetilde{w} \quad I^s\Lambda\widetilde{w}| \;\le\; C\,(|\,\widetilde{w}\,|_{w,2}^2 + |\,\nabla\widetilde{w}\,|_2^2)\ . \tag{52}$$

• In the same way, we have

$$\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; (\widetilde{\nu}\cdot\nabla)\, I^s\widetilde{w} = 0$$

and, applying Hölder's inequalities and inequality (26),

$$|\int_{\mathbf{R}^2} I^s\nabla\widetilde{w}\cdot([I^s,\widetilde{\nu}]\,\widetilde{w})| \;\leq\; C\,|\,I^s\nabla\widetilde{w}\,|_2\,|\,I^s\widetilde{\nu}\,|_2\,|\,\widetilde{w}\,|_{H^\sigma}$$

thus, since $0\leq\sigma\leq 1+s$,

$$|\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; I^s((\widetilde{\nu}\cdot\nabla)\,\widetilde{w})| \;\leq\; C\,|\,I^s\widetilde{\nu}\,|_2\,|\,I^s\nabla\widetilde{w}\,|_2^2 \;+\; C\,|\,I^s\widetilde{\nu}\,|_2\,|\,\widetilde{w}\,|_2^2\;. \tag{53}$$

• Integrating by parts yields

$$\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; I^s\mathrm{div}(\,b\nabla\widetilde{w}) = -\int_{\mathbf{R}^2} b\,|I^s\nabla\widetilde{w}|^2 - \int_{\mathbf{R}^2} I^s\nabla\widetilde{w} \;\; [I^s,b]\,\nabla\widetilde{w}\;.$$

Applying (27), we derive

$$|\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; I^s\mathrm{div}(\,b\nabla\widetilde{w})| \;\leq\; (|\,b_0\,|_\infty+\varepsilon)\,|\,I^s\nabla\widetilde{w}\,|_2^2 + \frac{C}{\varepsilon}\,|\,b\,|_{H^{s+\sigma}}^2\,|\,\nabla\widetilde{w}\,|_2^2 \tag{54}$$

where $\varepsilon>0$ is intended to be chosen small enough.

• Similarly,

$$|\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; I^s\mathrm{div}(\,b\nabla G)| \;\leq\; (|\,b_0\,|_\infty+\varepsilon)\,|\,I^s\nabla\widetilde{w}\,|_2^2 + C\,(|\,b_0\,|_\infty+\frac{1}{\varepsilon}\,|\,b\,|_{H^{s+\sigma}}^2)\;. \tag{55}$$

• First, integrating by parts yields

$$\int_{\mathbf{R}^2} I^s\widetilde{w} \;\; I^s\mathrm{div}(b\nabla^\perp\Pi) = -\int_{\mathbf{R}^2} I^s\nabla\widetilde{w}\;\cdot\;b\,I^s\nabla^\perp\Pi \;-\int_{\mathbf{R}^2} I^s\nabla\widetilde{w}\;\cdot\;([I^s,b]\nabla^\perp\Pi)$$

with, using Hölder's inequalities,

$$|\int_{\mathbf{R}^2} I^s\nabla\widetilde{w} \;\cdot\; b\,I^s\nabla^\perp\Pi| \;\leq\; |\,I^s\nabla\widetilde{w}\,|_2\,|\,b_0\,|_\infty\,|\,I^s\nabla\Pi\,|_2$$

and, using Hölder's inequalities and inequality (27),

$$|\int_{\mathbf{R}^2} I^s\nabla\widetilde{w}\cdot([I^s,b]\nabla^\perp\Pi)| \;\leq\; C\,|\,I^s\nabla\widetilde{w}\,|_2\,|\,b\,|_{H^{s+\sigma}}\,|\,\nabla\Pi\,|_2\;.$$

Besides, on one hand, estimate (23) applied to equation (42) and Hölder's inequalities imply

$$|\,\nabla\Pi\,|_2 \;\leq\; C\,(|\alpha|+|\alpha|^2+|\,\triangle\widetilde{v}\,|_2+|\alpha||\,\nabla\widetilde{v}\,|_2+|\,\widetilde{\nu}\,|_\infty|\,\nabla\widetilde{v}\,|_2+|\alpha||\,\widetilde{\nu}\,|_\infty)$$

thus with estimates on Biot-Savart kernel

$$|\,\nabla\Pi\,|_2 \;\le\; C\,(|\alpha| + |\alpha||\,\widetilde{\nu}\,|_\infty + |\alpha|^2 + (|\alpha| + |\,\widetilde{\nu}\,|_\infty)\,|\,\widetilde{w}\,|_2 + |\,\nabla\widetilde{w}\,|_2)\;.$$

On the other hand, estimate (31) applied to (42) and Hölder's inequalities imply

$$|\,I^s\nabla\Pi\,|_2 \;\le\; C\,(|\,b\,|_{H^{s+\sigma}}|\,\nabla\Pi\,|_2 + |\,I^s((1+b)\,\triangle v)\,|_2 + |\,I^s(\nu\cdot\nabla)\,v\,|_2)$$

with, commuting I^s and b thanks to (27), after some calculation,

$$|\,I^s((1+b)\,\triangle v)\,|_2 \le C\;|\,b\,|_{H^{s+\sigma}}(|\alpha| + |\,\nabla\widetilde{w}\,|_2) + C\;(1+|\,b_0\,|_\infty)\;(|\alpha| + |\,I^s\nabla\widetilde{w}\,|_2)$$

and, in the same way, commuting I^s and ν,

$$|\,I^s((\nu\cdot\nabla)\,v)\,|_2 \le C\;(|\alpha|+|\,I^s\widetilde{\nu}\,|_{H^\sigma})\;(|\alpha|+|\,\widetilde{w}\,|_2)+C\;(|\alpha|+|\,\widetilde{\nu}\,|_\infty)\;\;(|\alpha|+|\,I^s\widetilde{w}\,|_2)\;.$$

Therefore, gathering these inequalities,

$$\begin{aligned}
|\textstyle\int_{\mathbf{R}^2} I^s\widetilde{w}\;\; & \times\; I^s\,\mathrm{div}(\,b\,\nabla^\perp\Pi)\,| \\
& \le C\;|\,I^s\widetilde{w}\,|_2^2\;|\,b_0\,|_\infty\,(|\alpha|^2 + |\,\widetilde{\nu}\,|_\infty^2) \\
& + C\;|\,I^s\nabla\widetilde{w}\,|_2^2\;(|\,b_0\,|_\infty + \varepsilon) \\
& + C\;|\,b\,|_{H^{s+\sigma}}^2\;\big(\tfrac{1}{\varepsilon} + |\,b_0\,|_\infty\big) \\
& \quad\times\big(\,|\alpha|^2 + |\alpha|^2\,|\,\widetilde{\nu}\,|_\infty^2 + |\alpha|^4 + (|\alpha|^2 + |\,\widetilde{\nu}\,|_\infty^2)\,|\,\widetilde{w}\,|_2^2 + |\,\nabla\widetilde{w}\,|_2^2\,\big) \\
& + C\;|\,b_0\,|_\infty\;\big(\,|\alpha|^2\,(1 + |\,b_0\,|_\infty^2 + |\alpha|^2 + |\,\widetilde{\nu}\,|_\infty^2) \\
& \quad +(|\alpha|^2 + |\,I^s\widetilde{\nu}\,|_{H^\sigma}^2)\,(|\alpha|^2 + |\,\widetilde{w}\,|_2^2) + |\,b\,|_{H^{s+\sigma}}^2(|\alpha|^2 + |\,\nabla\widetilde{w}\,|_2^2)\,\big).
\end{aligned} \tag{56}$$

Putting all these points together and integrating yields

$$\begin{array}{lll}
 & |\,I^s\widetilde{w}\,|_2^2 & \\
+ & C\int_0^t|\,I^s\nabla\widetilde{w}\,|_2^2 & \big(1 - |\,b_0\,|_\infty\,(1+|\alpha|) - \sup_{[0,t]}|\,I^s\widetilde{\nu}\,|_2\,\big) \\
\le & |\,I^s\widetilde{w}_0\,|_2^2 & \\
+ & C\int_0^t|\,I^s\widetilde{w}\,|_2^2 & \big(1 + |\,b_0\,|_\infty|\,\widetilde{\nu}\,|_\infty^2 + |\alpha|\,(1 + |\,b_0\,|_\infty\int_0^t|\,\widetilde{\nu}\,|_\infty^2|\alpha|\,|\,b_0\,|_\infty)\big) \\
+ & C\int_0^t|\,\widetilde{\nu}\,|_\infty^2 & \big(\,|\,b_0\,|_\infty|\alpha|^2 \\
 & & \quad + \;(|\alpha|^2 + \sup_{[0,t]}|\,\widetilde{w}\,|_2^2)\,(1 + |\,b_0\,|_\infty)\;\sup_{[0,t]}|\,b\,|_{H^{s+\sigma}}^2\,\big) \\
+ & C\int_0^t|\,\nabla\widetilde{w}\,|_2^2 & \big(\,|\alpha| + (1 + |\,b_0\,|_\infty)\;\sup_{[0,t]}|\,b\,|_{H^{s+\sigma}}^2\,\big) \\
+ & C\int_0^t|\,\widetilde{w}\,|_{w,2}^2 & \big(\,|\alpha| + |\alpha|^2\,|\,b_0\,|_\infty + \sup_{[0,t]}|\,I^s\widetilde{\nu}\,|_2 \\
 & & \quad + \;|\alpha|^2\,(1 + |\,b_0\,|_\infty)\;\sup_{[0,t]}|\,b\,|_{H^{s+\sigma}}^2\,\big) \\
+ & C\int_0^t|\,I^s\widetilde{\nu}\,|_{H^\sigma}^2 & |\,b_0\,|_\infty\,(|\alpha|^2 + \sup_{[0,t]}|\,\widetilde{w}\,|_2^2) \\
+ & C\qquad t & |\alpha|\,(1 + |\alpha|^3)\,(1 + |\,b_0\,|_\infty)\,(1 + \sup_{[0,t]}|\,b\,|_{H^{s+\sigma}}^2)\;.
\end{array}$$

A Gronwall-type argument achieves the proof. ■

3.3 Estimate for convergence

We now establish an estimate on the difference of two solutions of equations of type (41), intending to prove convergence of our iterative scheme and uniqueness of solutions of (10).

For $i = 1, 2$, consider

$$\partial_t \widetilde{w}_i - (\mathcal{L} - \alpha \Lambda) \widetilde{w}_i + (\widetilde{\nu}_i \cdot \nabla) \widetilde{w}_i = \operatorname{div}\big(b_i (\nabla w_i + \nabla^\perp \Pi_i)\big) \tag{57}$$

where $\mathcal{L}$ and Λ are as in (11), b_i, $\widetilde{\Omega}_i$ are real functions, $\alpha \in \mathbf{R}$,

$$\begin{array}{rclcrcl} \widetilde{v}_i & = & K_{BS} \star \widetilde{w}_i & , & \widetilde{\nu}_i & = & K_{BS} \star \widetilde{\Omega}_i \ , \\ v_i & = & \alpha\, v^G + \widetilde{v}_i & , & \nu_i & = & \alpha\, v^G + \widetilde{\nu}_i \ , \\ w_i & = & \alpha\, G + \widetilde{w}_i & , & \Omega_i & = & \alpha\, G + \widetilde{\Omega}_i \ , \end{array}$$

and $\nabla \Pi_i$ is obtained by solving

$$\operatorname{div}\big((1 + b_i) \nabla \Pi_i\big) = \operatorname{div}\big((1 + b_i) \triangle v_i - (\nu_i \cdot \nabla) v_i\big) \ . \tag{58}$$

Note that we choose to write $\widetilde{\nu} = K_{BS} \star \widetilde{\Omega}$ to stress the symmetry of the hypotheses on $\widetilde{\Omega}$ and $\widetilde{w}$.

In what follows for concision's sake we denote $\delta f = f_2 - f_1$ for any functions f_1, f_2.

Proposition 10
Let $\alpha \in \mathbf{R}$, $K > 0$, $\sigma > 2$, $0 < \eta < s < 1$ and $\max(\frac{2}{\eta}, 4) < p < +\infty$.
There exists $\varepsilon_0 > 0$ and, for $K', T > 0$, there exists $C > 0$ such that if $\widetilde{w}_1$, $\widetilde{w}_2$ satisfy (57) with

1. $|\, b_0 \,|_{w,4} \leq \varepsilon_0$, $|\, b_0 \,|_{w,\infty} \leq \varepsilon_0$
2. *for $0 < t < T$, for $i = 1, 2$, for any $1 \leq r \leq +\infty$,*
$$|\, b_i(t) \,|_r \ \leq \ |\, b_0 \,|_r \, e^{-\frac{t}{r}} \quad , \quad |\, b_i(t) \,|_{w,r} \ \leq \ K \,|\, b_0 \,|_{w,r} \, e^{-\frac{t}{r}}$$
3. *for $0 < t < T$, for $i = 1, 2$,* $|\, b_i(t) \,|_{H^\sigma} \leq K'$
4. *for $0 < t < T$, for $i = 1, 2$,*
$$\begin{array}{rcl} |\, \widetilde{\Omega}_i(t) \,|_{w,2}^2 + \int_0^t |\, \nabla \widetilde{\Omega}_i \,|_{w,2}^2 & \leq & \varepsilon_0 \\ |\, \widetilde{\Omega}_i(t) \,|_{H^s} + \int_0^t |\, \nabla \widetilde{\Omega}_i \,|_{H^s}^2 & \leq & K' \end{array}$$
5. *for $0 < t < T$, for $i = 1, 2$,*
$$\begin{array}{rcl} |\, \widetilde{w}_i(t) \,|_{w,2}^2 + \int_0^t |\, \nabla \widetilde{w}_i \,|_{w,2}^2 & \leq & \varepsilon_0 \\ |\, \widetilde{w}_i(t) \,|_{H^s} + \int_0^t |\, \nabla \widetilde{w}_i \,|_{H^s}^2 & \leq & K' \end{array}$$

then for $0 < t < T$*,*

$$
\begin{aligned}
|\,\delta\widetilde{w}(t)\,|_{w,2}^2 \quad &+ \quad C\int_0^t |\,\delta\widetilde{w}\,|_{w,2}^2 + |\,\nabla(\delta\widetilde{w})\,|_{w,2}^2 + |\,|\xi|\,(\delta\widetilde{w})\,|_{w,2}^2 \\
&\leq \quad C\int_0^t (1+|\,\widetilde{w}_1\,|_{w,p}^2 + |\,\nabla\widetilde{w}_1\,|_{H^\eta}^2)\ (|\,\delta b\,|_{w,p}^2 + |\,\delta\widetilde{\Omega}\,|_{w,2}^2)\ . \quad (59)
\end{aligned}
$$

Proof. Combining (60) for $i = 1, 2$, we derive

$$
\begin{aligned}
\partial_t(\delta\widetilde{w}) \quad &- \quad (\mathcal{L} - \alpha\,\Lambda)\,(\delta\widetilde{w}) + \big(\widetilde{\nu}_2\cdot\nabla\big)\,(\delta\widetilde{w}) - \operatorname{div}\big(b_2\,(\nabla(\delta\widetilde{w}) + \nabla^\perp\Pi)\big) \\
&= \quad -((\delta\widetilde{\nu})\cdot\nabla)\,\widetilde{w}_1 + \operatorname{div}\big((\delta b)\nabla w_1\big) \qquad (60)\\
&\quad\ + \operatorname{div}(b_2\nabla^\perp R) + \operatorname{div}(b_2\nabla^\perp(\delta S)) + \operatorname{div}\big((\delta b)\nabla^\perp\Pi_1\big)
\end{aligned}
$$

with Π, R, S_1 and S_2 obtained by solving

$$
\begin{cases}
\operatorname{div}\big((1+b_2)\nabla\Pi\big) &= \operatorname{div}\big((1+b_2)\triangle(\delta\widetilde{v}) - (\widetilde{\nu}_2\cdot\nabla)\,(\delta\widetilde{\nu})\big) \\
\operatorname{div}\big((1+b_2)\nabla R\big) &= \operatorname{div}\big(-\alpha(v^G\cdot\nabla)\,(\delta\widetilde{v}) + (\delta b)\triangle\widetilde{v}_1 - ((\delta\widetilde{\nu})\cdot\nabla)\,\widetilde{v}_1\big) \\
\operatorname{div}\big((1+b_i)\nabla S_i\big) &= \operatorname{div}\big((1+b_1)\triangle v_1 - (\nu_1\cdot\nabla)\,v_1\big)\ , \qquad \text{for } i = 1, 2\ .
\end{cases}
$$

Note that $\Pi_2 = \Pi + R + S_2$ and $\Pi_1 = S_1$.

Our strategy is again to multiply (60) by $G^{-1}(\delta\widetilde{w})$, integrate on $\mathbf{R}^2$ and estimate each term arising to bound $\frac{d}{dt}|\,\delta\widetilde{w}\,|_{w,2}^2$. We deal with each term coming from the left member of equality (60) as we did in the linearized vorticity equation (41). Let us only show how to deal with the other terms.

First of all, let us emphasize that $|\,G^r\,\nabla(G^{r'}\,f)\,|_2^2$ is controlled by $|\,\nabla f\,|_{w,2}^2 + |\,|\xi|\,f\,|_{w,2}^2$ provided that $r + r' = -\frac{1}{2}$.

• First integrating by parts and applying Hölder's inequalities, we obtain

$$
|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \operatorname{div}(\widetilde{w}_1(\delta\widetilde{\nu}))|\ \leq\ |\,G^{\frac{1}{2}}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2\,|\,G^{-\frac{1}{2}}\widetilde{w}_1\,|_p|\,\delta\widetilde{\nu}\,|_q
$$

where $2 < q < +\infty$ is such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then using (17)

$$
\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \operatorname{div}(\widetilde{w}_1(\delta\widetilde{\nu}))|\ \leq\ \varepsilon\,|\,G^{\frac{1}{2}}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2^2 + \frac{C}{\varepsilon}\,|\,\widetilde{w}_1\,|_{w,p}^2\ |\,\delta\widetilde{\Omega}\,|_{w,2}^2 \quad (61)
$$

where ε is intended to be chosen small enough.

• Similarly, with the same q, we also have

$$
|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \operatorname{div}((\delta b)\nabla w_1)| \leq |\,G^{\frac{1}{2}}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2\,|\,G^{-\frac{1}{2}}(\delta b)\,|_p|\,\nabla w_1\,|_q
$$

thus using Sobolev's embeddings

$$
|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \operatorname{div}((\delta b)\nabla w_1)| \leq \varepsilon\,|\,G^{\frac{1}{2}}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2^2 + \frac{C}{\varepsilon}\,|\,\nabla w_1\,|_{H^\eta}^2\ |\,\delta b\,|_{w,p}^2 \quad (62)
$$

where ε is again intended to be chosen small enough.

• In quite the same way, we obtain

$$|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \mathrm{div}(b_2\nabla^{\perp}R)| \ \leq\ C\,|\,G^{\frac12}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2|\,b_0\,|_{w,\infty}|\,\nabla R\,|_2\ .$$

With pressure estimate (23), estimate on Biot-Savart law (17), Hölder's inequalities and Sobolev's embeddings we can derive

$$\begin{aligned}|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \mathrm{div}(b_2\nabla^{\perp}R)| \ &\leq\ \varepsilon\,|\,G^{\frac12}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2^2\\ &+\frac{C}{\varepsilon}\,|\,b_0\,|_{w,\infty}^2|\,\nabla\widetilde{w}_1\,|_{H^\eta}^2\ |\,\delta b\,|_p^2\\ &+\frac{C}{\varepsilon}\,|\alpha||\,b_0\,|_{w,\infty}^2\ |\,\delta\widetilde{w}\,|_2^2\\ &+\frac{C}{\varepsilon}\,|\,b_0\,|_{w,\infty}^2|\,\widetilde{w}_1\,|_{w,2}^2\ |\,\delta\widetilde{\Omega}\,|_{w,2}^2 \qquad (63)\end{aligned}$$

where ε is once again intended to be chosen small enough.

• Using estimate (24) instead of estimate (23) and inequalities (17) and (19), since $\frac{1}{p} < \frac12 - \frac1p$, we can obtain

$$\begin{aligned}|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \mathrm{div}(b_2\nabla^{\perp}(\delta S))| \ &\leq\ \varepsilon\,|\,G^{\frac12}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2^2\\ &+\ \frac{C}{\varepsilon}\,|\,b_0\,|_{w,\infty}^2\ |\,\delta b\,|_p^2\\ &\quad\times\big[\,|\,\nabla w_1\,|_{H^\eta}^2+|\,\Omega_1\,|_{w,2}^2|\,w_1\,|_{w,p}^2\,\big] \quad (64)\end{aligned}$$

where ε is still intended to be chosen small enough.

• At last, integrating by parts and applying Hölder's inequalities, we have

$$|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \mathrm{div}((\delta b)\nabla^{\perp}\Pi_1| \leq |\,G^{\frac12}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2\,|\,\delta b\,|_{w,p}|\,\nabla^{\perp}\Pi_1\,|_q\ .$$

where $2 < q < +\infty$ is again such that $\frac1p + \frac1q = \frac12$. Again with pressure estimate (23), estimates on Biot-Savart law (17) and (19), Hölder's inequalities and Sobolev's embeddings we can derive

$$\begin{aligned}|\int_{\mathbf{R}^2} G^{-1}(\delta\widetilde{w})\ \mathrm{div}((\delta b)\nabla^{\perp}\Pi_1| \ &\leq\ \varepsilon\,|\,G^{\frac12}\,\nabla(G^{-1}\,(\delta\widetilde{w}))\,|_2^2\\ &+\ \frac{C}{\varepsilon}\,|\,\delta b\,|_{w,p}^2\,\big[\,|\,\nabla w_1\,|_{H^\eta}^2+|\,\Omega\,|_{w,2}^2|\,w_1\,|_{w,p}^2\,\big]\ (65)\end{aligned}$$

where ε is once again intended to be chosen small enough.

Gathering everything and integrating yield (59). ∎

4 Main results

We now use our various estimates to derive our main results.

4.1 Existence and uniqueness

Before proving a result of existence and uniqueness of solution of equations (10), we state a lemma that will make a link between norm estimates and convergence of the iterative scheme.

Lemma 3 *Let $T > 0$ and $1 < p \le +\infty$. Let (f_k) be a sequence in $L^\infty(0,T;\mathbf{R}^+)$, and (g_k) a bounded sequence in $L^p(0,T;\mathbf{R}^+)$ be such that for $0 < t < T$ and $k \in \mathbf{N}$,*

$$f_{k+1}(t) \le \int_0^t f_k\, g_k \ .$$

Then (f_k) is uniformly summable, namely, for $0 < t < T$,

$$\sum_{k\ge 0} f_k(t) \le C_T \ .$$

Proof. Using Hölder inequalities and iterating yield $f_k(t) \le K\, C^k\, (\frac{t^k}{k!})^{1-\frac{1}{p}}$, where C is a bound for (g_k) in $L^p(0,T;\mathbf{R}^+)$. ■

We can now prove the existence and uniqueness parts of **Theorem 1**.

Proof.

- *Existence.* We build a sequence $((b_k, \widetilde{w}_k))_{k\in\mathbf{N}^*}$ such that, for any $k \in \mathbf{N}^*$,

$$\begin{cases} \partial_t\, b_{k+1} + \big((v_k - \frac{1}{2}\xi)\cdot\nabla\big)\, b_{k+1} = 0 \\ \partial_t \widetilde{w}_{k+1} - (\mathcal{L} - \alpha\,\Lambda)\,\widetilde{w}_{k+1} + \big(\widetilde{v}_k\cdot\nabla\big)\,\widetilde{w}_{k+1} = \mathrm{div}\big(b_k\,(\nabla w_{k+1} + \nabla^\perp \Pi_{k+1})\big) \end{cases}$$

where $\mathcal{L}$ and Λ are as in (11), $(\widetilde{v}_k)$ is obtained from $(\widetilde{w}_k)$ by the Biot-Savart law,

$$v_k = \alpha\, v^G + \widetilde{v}_k \ , \quad w_k = \alpha\, G + \widetilde{w}_k \ , \qquad \text{for} \quad k \in \mathbf{N}^*,$$

and $(\nabla\Pi_k)$ is obtained by solving, for any $k \in \mathbf{N}^*$,

$$\mathrm{div}\,\big((1+b_k)\,\nabla\Pi_{k+1}\big) = \mathrm{div}\,\big((1+b_k)\,\triangle v_{k+1} - (v_k\cdot\nabla)\,v_{k+1}\big)$$

with initial data $(b_0, \widetilde{w}_0)$. For $k = 0$, we solve the system with $\widetilde{v}_k(t) \equiv 0$ and $b_k(t) \equiv 0$.

Let us show how we propagate bounds on $(b_k, \widetilde{w}_k)$.

Step 1 Fix $K_0 > 0$ and choose $\varepsilon_0 > 0$ small enough. We can propagate

1. for any $1 \le p \le +\infty$, $2 \le q \le +\infty$, thanks to **Proposition 5**,

$$|\, b_k(t)\,|_p \ \le\ |\, b_0\,|_p\, e^{-\frac{t}{p}} \quad , \quad |\, b_k(t)\,|_{w,q} \ \le\ |\, b_0\,|_{w,q}\, e^{-\frac{t}{q}}\, e^{K_0}$$

and thanks to **Proposition 8**,

$$\begin{aligned} |\,\widetilde{w}_k(t)\,|^2_{w,2} \ &+\ C_{K_0}\int_0^t \big(|\,\widetilde{w}_k\,|^2_{w,2} + |\,\nabla\widetilde{w}_k\,|^2_{w,2} + |\,|\xi|\widetilde{w}_k\,|^2_{w,2}\big) \\ &\le\ C_{K_0}\,(|\,\widetilde{w}_0\,|^2_{w,2} + |\,b_0\,|_{w,4}) \end{aligned}$$

which provides us, thanks to **Proposition 1** and Sobolev's embeddings,

$$\begin{array}{c} |\,\widetilde{v}_k\,|_8 \;\leq\; C\,|\,\widetilde{w}_k\,|_{\frac{8}{5}} \;\leq\; C\,|\,\widetilde{w}_k\,|_{w,2} \;\leq\; K_0 \\ \int_0^t |\,\widetilde{v}_k\,|_\infty^2 \;\leq\; C\int_0^t \left(|\,\widetilde{w}_k\,|_{w,2}^2 + |\,\nabla\widetilde{w}_k\,|_2^2\right) \;\leq\; \min(\frac{1}{24}, K_0)\ . \end{array}$$

Step 2 Again choosing ε_0 small enough independently of t and using **Proposition 2** we can obtain, when $0 < s < 1$ and $1 < s'' < 2 - s$,

$$\begin{array}{ccccc} \int_0^t |\,\nabla v_k\,|_{H^1} & \leq & C(t + t^{\frac{1}{2}}\,(\int_0^t |\,\nabla\widetilde{v}_k\,|_{H^1}^2)^{\frac{1}{2}}) & \leq & C(t + \int_0^t |\,\widetilde{w}_k\,|_{H^1}^2) \\ |\,I^s\widetilde{v}_k(t)\,|_2 & \leq & C\,|\,\widetilde{w}_k(t)\,|_{w,2} & \leq & C_{K_0}\,\varepsilon_0 \\ \int_0^t |\,I^s\widetilde{v}_k\,|_{H^{s''}}^2 & \leq & C\,\int_0^t(|\,\widetilde{w}_k\,|_{w,2}^2 + |\,\nabla\widetilde{w}_k\,|_2^2) & \leq & K_0 \end{array}$$

and propagate, for $0 < t < T$,

1. when $1 + s < s' < 2$, thanks to **Proposition 6**,

$$|\,b_k(t)\,|_{H^{s'}} \leq C_{K_0,T}$$

2. and thanks to **Proposition 9**,

$$|\,I^s\widetilde{w}_k(t)\,|_2^2 + C\int_0^t |\,I^s\nabla\widetilde{w}_k\,|_2^2 \;\leq\; C_{K_0,T}$$

which provides us, for $0 < t < T$,

$$\int_0^t |\,\nabla v_k\,|_{H^{s+1}} \;\leq\; C\,(t + \int_0^t (|\,\widetilde{w}_k\,|_2^2 + |\,I^s\nabla\widetilde{w}_k\,|_2^2)) \;\leq\; C_{K_0,T}$$

thus, thanks to **Proposition 6**, for $0 < t < T$,

$$|\,b_k(t)\,|_{H^{s+2}} \;\leq\; C_{K_0,T}\ .$$

Step 3 We now prove the convergence of the scheme. Set $(\delta b)_k = b_{k+1} - b_k$ and $(\delta\widetilde{w})_k = \widetilde{w}_{k+1} - \widetilde{w}_k$. Choose $\max(4, \frac{2}{s}) < p < q$. **Propositions 7 & 10** give us, for $T > 0$, for any $0 < t < T$ and any $k \in \mathbf{N}^*$,

$$\begin{array}{l} |\,(\delta b)_{k+1}(t)\,|_{w,p}^2 + |\,(\delta\widetilde{w})_{k+1}(t)\,|_{w,2}^2 \\ \leq\; C_T\int_0^t\,(1 + |\,\widetilde{w}_k\,|_{w,p}^2 + |\,\nabla\widetilde{w}_k\,|_{H^\eta}^2)\;\;(|\,(\delta b)_k\,|_{w,p}^2 + |\,(\delta\widetilde{w})_k\,|_{w,2}^2) \end{array} \tag{66}$$

for some $0 < \eta < s$ such that $\frac{2}{\eta} < p < +\infty$.

Now in order to apply **Lemma 3** with $f_k = |\,(\delta b)_k\,|_{w,p}^2 + |\,(\delta\widetilde{w})_k\,|_{w,2}^2$, remark that

– since $(G^{-\frac{1}{2}}\widetilde{w}_k)$ is bounded in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}^2))$ and $(\nabla(G^{-\frac{1}{2}}\widetilde{w}_k))$ in $L^2(\mathbf{R}^+; L^2(\mathbf{R}^2))$, $(G^{-\frac{1}{2}}\widetilde{w}_k)$ is bounded in $L^r(\mathbf{R}^+; L^p(\mathbf{R}^2))$, for some $2 < r < +\infty$, by interpolation and Sobolev embeddings

– since $(\widetilde{w}_k)$ is bounded in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}^2)) \cap L^2(0,T; H^{s+1}(\mathbf{R}^2))$, $(\nabla \widetilde{w}_k)$ is bounded in $L^{r'}(0,T; H^\eta(\mathbf{R}^2))$ for some $2 < r' < +\infty$, by interpolation.

Thus (b_k) converges in $L^p_w(\mathbf{R}^2)$ and $(\widetilde{w}_k)$ in $L^2_w(\mathbf{R}^2)$, locally uniformly in time. This implies at once that (b_k) and $(\widetilde{w}_k)$ also converges in $L^2(\mathbf{R}^2)$. Now by interpolation

– since (b_k) is bounded in $H^{s+2}(\mathbf{R}^2)$, (b_k) converges in $H^{\eta'}(\mathbf{R}^2)$, for any $0 < \eta' < s+2$

– since $(\widetilde{w}_k)$ is bounded in $H^s(\mathbf{R}^2)$, $(\widetilde{w}_k)$ converges in $H^{\eta''}(\mathbf{R}^2)$, for any $0 < \eta'' < s$.

These properties enable us to take the limit in the sequence of equations.

Note that we recover the regularity on the limit by a mere application of Fatou's lemma.

- *Uniqueness.* We obtain a bound similar to (66) for the difference of two solutions. Then Gronwall lemma gives the result.

■

4.2 Asymptotic behavior

We now state the asymptotic part of **Theorem 1**. Note that under the hypotheses of **Theorem 1**, the following assumptions are fulfilled.

Theorem 2 *Let $\alpha \in \mathbf{R}$. For any $0 < \gamma < \frac{1}{2}$, there exist $\varepsilon_0 > 0$ and $K, K' > 0$ such that if $(b, \widetilde{w})$ is a solution of (10) with initial data $(b_0, \widetilde{w}_0)$, such that*

$$|\, b_0 \,|_{w,2} \le \varepsilon_0 \,, \quad |\, b_0 \,|_{w,\infty} \le \varepsilon_0 \,,$$

and for $t > 0$, $\quad |\, \widetilde{w}(t) \,|^2_{w,2} + \int_0^t |\, \nabla \widetilde{w} \,|^2_{w,2} \;\le\; \varepsilon_0$,

$$|\, b(t) \,|_{w,2} \quad \le \quad K \,|\, b_0 \,|_{w,2}\, e^{-\frac{t}{2}} \quad , \quad |\, b(t) \,|_{w,\infty} \quad \le \quad K \,|\, b_0 \,|_{w,\infty} \quad ,$$

then, for $t > 0$, $|\, \widetilde{w}(t) \,|_{w,2} \le K'\, e^{-\gamma t}\ (|\, \widetilde{w}_0 \,|_{w,2} + |\, b_0 \,|_{w,2})$.

Proof. Let $0 < \gamma < \gamma' < \frac{1}{2}$. Let us reformulate (44):

$$\int_{\mathbf{R}^2} G^{-1}\, \widetilde{w}\, \mathcal{L}\, \widetilde{w} \;\le\; -\gamma' \,|\, \widetilde{w} \,|^2_{w,2} - (\frac{1}{2} - \gamma') \big(\frac{1}{3} \,|\, \nabla \widetilde{w} \,|^2_{w,2} + \frac{1}{2} \,|\, \frac{|\xi|}{4}\, \widetilde{w} \,|^2_{w,2} \big) \;. \quad (67)$$

Then we deal with the other terms of the vorticity equation as we did to obtain estimate (43), except for the pressure term and

$$|\int_{\mathbf{R}^2} G^{-1}\, \widetilde{w}\, \widetilde{v} \cdot \nabla \widetilde{w}| \;\le\; C \,|\, \widetilde{w} \,|_{w,2} \,(|\, \widetilde{w} \,|^2_{w,2} + |\, \nabla \widetilde{w} \,|^2_{w,2}) \quad (68)$$

obtained thanks to estimate (18) and Sobolev's embeddings.

We treat the pressure term as follows:

$$|\int_{\mathbf{R}^2} G^{-1}\,\widetilde{w}\,\operatorname{div}(b\,\nabla^\perp\Pi)| \;\le\; C\,|\,G^{\frac12}\,\nabla(G^{-1}\,\widetilde{w})\,|_2\,|\,G^{-\frac12}\,b\,\nabla^\perp\Pi\,|_2$$

with

$$\begin{aligned}|\,b\,\nabla^\perp\Pi\,|_{w,2} \le C\,\Big(&|\,b_0\,|_{w,2}\,e^{-\frac{t}{2}}|\,\alpha\,(1+b)\,\triangle v^G - \alpha^2\,(v^G\cdot\nabla)\,v^G\,|_\infty \\ &+|\,b_0\,|_{w,\infty}\,|\,(1+b)\,\triangle\widetilde{v} - \alpha\,((v^G\cdot\nabla)\,\widetilde{v} + (\widetilde{v}\cdot\nabla)\,v^G) - (\widetilde{v}\cdot\nabla)\,\widetilde{v}\,|_2\Big)\end{aligned}$$

and

$$\begin{aligned}|\,(1+b)\,\triangle\widetilde{v}\,|_2 &\le C\,(1+|\,b_0\,|_{w,\infty})\,|\,\nabla\widetilde{w}\,|_2\\ |\,(v^G\cdot\nabla)\,\widetilde{v}\,|_2 &\le C\,|\,v^G\,|_\infty\,|\,\widetilde{w}\,|_2\\ |\,(\widetilde{v}\cdot\nabla)\,v^G\,|_2 &\le C\,|\,\nabla v^G\,|_4\,|\,\widetilde{v}\,|_4 \;\le\; C\,|\,\widetilde{w}\,|_{w,2}\\ |\,(\widetilde{v}\cdot\nabla)\,\widetilde{v}\,|_2 &\le C\,|\,\widetilde{v}\,|_\infty\,|\,\nabla\widetilde{v}\,|_2 \;\le\; C\,\big(|\,\widetilde{w}\,|_{w,2}+|\,\nabla\widetilde{w}\,|_2\big)\,|\,\widetilde{w}\,|_2\,.\end{aligned}$$

This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\big(|\,\widetilde{w}\,|_{w,2}^2\big) + \gamma|\,\widetilde{w}\,|_{w,2}^2 \;+\; C\,(|\,\nabla\widetilde{w}\,|_{w,2}^2 + |\,|\xi|\,\widetilde{w}\,|_{w,2}^2) \;\le\; C\,|\,b_0\,|_{w,2}^2\,e^{-t} \qquad (69)$$

which integrating gives our result, since $2\,\gamma < 1$. ∎

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