Université de Rennes Fall Semester 2026-2027

Spectral Theory Master 2-Fundamental Mathematics

Homework

Exercice 1.

1. Let A and B be operators from a Banach space X to a Banach space Y.
   Prove that if A is onto, B is one-to-one and $A \subset B$ then $A = B$.
2. Let A be a densely defined operator on a complex Hilbert space H.
   Prove that if $z \in \mathbf{C}$ is such that $z\,\mathrm{Id}_H - A$ is onto then $\overline{z}\,\mathrm{Id}_H - A^*$ is one-to-one.
3. Let A be a closed, densely defined operator from a complex Hilbert space H_0 to a complex Hilbert space H_1.
   (a) Prove that
   $$H_0 \times H_1 \;=\; \{(\varphi, A\varphi); \varphi \in D_A\} \oplus \{(-A^*\psi, \psi); \psi \in D_{A^*}\}\ .$$
   (b) Deduce that $\mathrm{Id}_{H_0} + A^*A$ is onto.
   (c) Deduce that A^*A is self-adjoint.

Exercice 2. For any open interval I and $1 \leq p \leq \infty$, we denote $W^{1,p}(I)$ the space of L^p functions v whose derivative in the sense of distributions v' may be identified with an L^p function (also denoted v') and we endow it with $\|v\|_{W^{1,p}(I)} := \|v\|_{L^p(I)} + \|v'\|_{L^p(I)}$.
We recall that, for any $1 \leq p \leq \infty$,

— $W^{1,p}(\mathbf{R})$ is a Banach space and that it is continuously embedded in $L^\infty(\mathbf{R}) \cap \mathcal{C}^0(\mathbf{R})$, with for some $C_p > 0$ and any $v \in W^{1,p}(\mathbf{R})$,
$$\|v\|_{L^\infty(\mathbf{R})} \leq C_p\, \|v\|_{L^p(\mathbf{R})}^{1-\frac{1}{p}}\, \|v'\|_{L^p(\mathbf{R})}^{\frac{1}{p}}\,;$$
— for any open bounded interval I, $W^{1,p}(I)$ is compactly embedded in $L^\infty(I) \cap \mathcal{C}^0(I)$.

Let $1 \leq p \leq \infty$, $\beta \in L^p(\mathbf{R};\mathbf{R}) + L^\infty(\mathbf{R};\mathbf{R})$ and $\alpha \in L^\infty(\mathbf{R};\mathbf{R})$ such that there exists $c_0 > 0$ such that $\alpha \geq c_0$ almost everywhere.
We define $A_{\alpha,\beta}$ on $L^p(\mathbf{R};\mathbf{C})$ with domain $W^{1,p}(\mathbf{R};\mathbf{C})$ by $A_{\alpha,\beta} v := \alpha v' + \beta v$.

1. Prove that $A_{\alpha,\beta}$ is well-defined and that it is closed.
2. Prove that if $\tilde\beta \in L^p(\mathbf{R};\mathbf{R}) + L^\infty(\mathbf{R};\mathbf{R})$ is such that $\lim_{\pm\infty} \tilde\beta = 0$, then $\sigma_{\mathrm{ess}}(A_{\alpha,\beta}) = \sigma_{\mathrm{ess}}(A_{\alpha,\beta+\tilde\beta})$.
3. Prove that if $M \in L^1_{\mathrm{loc}}(\mathbf{R})$ is such that $\overline{\lim}_{\pm\infty} M < 0$ then there exist positive K_0 and K_1 such that for any $(x_0, x_1) \in \mathbf{R}^2$ satisfying $x_0 \leq x_1$,
   $$\int_{x_0}^{x_1} M \leq K_0 - K_1(x_1 - x_0)\,.$$
4. Assume that there exist
   $$\beta_{+\infty} := \lim_{+\infty} \beta \in \mathbf{R}\,, \qquad\qquad \beta_{-\infty} := \lim_{-\infty} \beta \in \mathbf{R}\,.$$

(a) Prove that if $\beta_{-\infty} < \beta_{+\infty}$ and $\lambda \in \mathbf{C}$ is such that $\mathrm{Re}(\lambda) \in (\beta_{-\infty}, \beta_{+\infty})$ then λ is an eigenvalue of $A_{\alpha,\beta}$.

(b) Prove that if $\lambda \in \mathbf{C}$ is such that $\mathrm{Re}(\lambda) > \max(\{\beta_{-\infty}; \beta_{+\infty}\})$ then $\lambda \notin \sigma(A_{\alpha,\beta})$ and that the corresponding resolvent is given by, for any $f \in L^p(\mathbf{R})$ and any $x \in \mathbf{R}$,

$$R_\lambda(A_{\alpha,\beta})(f)(x) = \int_x^\infty e^{-\int_x^y \frac{\lambda-\beta}{\alpha}} \frac{f(y)}{\alpha(y)} \, \mathrm{d}y \, .$$

(c) Prove that if $\lambda \in \mathbf{C}$ is such that $\mathrm{Re}(\lambda) < \min(\{\beta_{-\infty}; \beta_{+\infty}\})$ then $\lambda \notin \sigma(A_{\alpha,\beta})$.

(d) When $\beta_{-\infty} < \beta_{+\infty}$, deduce $\sigma(A_{\alpha,\beta})$.

(e) Prove that if $\beta_{-\infty} > \beta_{+\infty}$, then any $\lambda \in \mathbf{C}$ such that $\mathrm{Re}(\lambda) \in (\beta_{+\infty}, \beta_{-\infty})$ is an eigenvalue of $A'_{\alpha,\beta}$, the adjoint of $A_{\alpha,\beta}$.

(f) When $\beta_{-\infty} > \beta_{+\infty}$, deduce $\sigma(A_{\alpha,\beta})$.

(g) By introducing Riesz sequences, compute $\sigma(A_{\alpha,\beta})$ when β is constant.

(h) Deduce $\sigma(A_{\alpha,\beta})$ when $\beta_{-\infty} = \beta_{+\infty}$.