Note on the stability of viscous roll-waves

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Abstract

In this note, we announce a complete classification of stability of periodic roll-wave solutions of the viscous shallow-water equations, from their onset at Froude number $F \approx 2$ up to the infinite-Froude limit. For intermediate Froude numbers, we obtain numerically a particularly simple power-law relation between F and the boundaries of the region of stable periods, that appears potentially useful in hydraulic engineering applications. In the asymptotic regime $F \to 2$ (onset), we provide an analytic expression of the stability boundaries whereas in the limit $F \to \infty$, we show that roll-waves are always unstable. *To cite this article: C. R. Mecanique.*

Résumé

Note sur la stabilité des roll-waves visqueuses. Les "roll-waves" sont des ondes progressives périodiques hydrodynamiques, modélisées comme des solutions des équations de Saint Venant. Dans cette note, nous annonçons une classification complète des roll-waves stables de leur apparition à F (le nombre de Froude) proche de 2 à $F \to \infty$. Pour les nombres de Froude intermédiaires, nous avons mené une étude numérique des critères de stabilité spectrale. Dans le régime asymptotique $F \to 2$, nous donnons une expression analytique des limites de stabilité alors que pour $F \to \infty$, nous montrons que les roll-waves sont toujours instables. *Pour citer cet article : C. R. Mecanique .*

Keywords: fluid mechanics; shallow water flows; roll waves
Mots-clés : Mécanique des fluides; écoulements peu profonds ; roll-waves

1. Introduction

In this note, we announce the classification in [1, 2, 3] of spectral stability of roll-wave solutions of the "viscous" St. Venant equations for inclined shallow-water flow, taking into account drag and viscosity. Written in nondimensional Eulerian form, the shallow water equations for a thin film down an incline are

$$\partial_t h + \partial_x(hu) = 0, \qquad \partial_t(hu) + \partial_x\left(hu^2 + \frac{h^2}{2\,F^2}\right) = h - |u|u + \nu\partial_x(h\partial_x u), \tag{1}$$

where F is the Froude number and $\nu = Re^{-1}$ is the inverse of the Reynolds number. Here $h(x,t)$ denotes the fluid height whereas $u(x,t)$ is the fluid velocity averaged with respect to height. The terms h and

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[1]Research of B.B. was partially supported by the National Science Foundation under grant DMS-1400872.
[2]Research of M.J. was partially supported by the National Science Foundation under grant DMS-1211183.
[3]Research of L.M.R. was partially supported by the ANR project BoND ANR-13-BS01-0009-01.
[4]Research of K.Z. was partially supported by the National Science Foundation under grant DMS-0300487.

$|u|u$ on the right hand side of the second equation model, respectively, gravitational force and turbulent friction along the bottom. Roll-waves are well-known hydrodynamic instabilities of (1), arising in the region $F > 2$ for which constant solutions, corresponding to parallel flow, are unstable. They are commonly found in man-made conduits such as aqueducts and spillways, and have been reproduced in laboratory flumes [4]. However, up until now, there has been no complete rigorous stability analysis of viscous St. Venant roll-waves either at the linear (spectral) or nonlinear level.

Roll-waves may be modeled as periodic wave train solutions of (1). In [2], it was proved for a large class of viscous conservation laws and under suitable spectral assumptions that periodic wave trains are nonlinearly stable (in a spatially-modulated sense). In [3, 5] this nonlinear analysis has been extended to encompass all periodic wave train solutions of the shallow water system (1) that satisfy those spectral assumptions. The main issue then is the verification of such assumptions. Here, we provide a complete description of the set of stable roll-waves of (1): for each Froude number $F > 2$, we exhibit (either theoretically or numerically) the range of spatial periods where stable roll-waves are found. To our knowledge, this is the first complete result of stability in the case of shallow water equations. However, let us mention the study in [6]: there, the authors studied the *modulational stability* of Dressler *inviscid* roll-waves. A set of modulation equations is derived by assuming that the parameters which encode the roll-waves slowly vary in time and space: lack of hyperbolicity of the modulation equations is expected to provide a sufficient criterion for spectral instability of roll-waves under special kinds of large scale perturbations.

In Section 2, we introduce the spectral problem and recall the spectral assumptions that have to be verified in order to obtain nonlinear stability of periodic waves. In Section 3, we consider the intermediate Froude number regime $2 \leq F \leq 100$. We find a dramatic transition around $F \approx 2.3$ from the small-F description of stability to a remarkably simple power-law description of surfaces bounding from above and below regions in parameter space corresponding to stable waves. These surfaces eventually intersect, yielding instability for all sufficiently large F. In Section 4, we focus on two asymptotic regimes: $F \to 2$ (onset) and $F \to \infty$. As $F \to 2$, the shallow water equations reduce to a generalized Kuramoto-Sivashinsky equation and we obtain asymptotic analytic formula for the stability boundaries. As $F \to \infty$, we exhibit a non-trivial regime and an asymptotic model which admits only unstable roll-waves, indicating the instability of roll-waves for sufficiently large F.

2. Formulation of the spectral problem

As the full nonlinear theory is given in Lagrangian coordinates of mass [5, 3], for the sake of consistency we rewrite the viscous shallow water system (1) as

$$\partial_t \tau - \partial_x u = 0, \quad \partial_t u + \partial_x \left(\frac{\tau^{-2}}{2\, F^2} \right) = 1 - \tau\, u^2 + \nu \partial_x (\tau^{-2} \partial_x u), \tag{2}$$

where $\tau := 1/h$ and x denotes now a Lagrangian marker rather than a physical location $\tilde{x}$, satisfying the relations $d\tilde{x}/dt = u(\tilde{x}, t)$ and $d\tilde{x}/dx = \tau(\tilde{x}, t)$.There is a one-to-one correspondence between periodic waves of the Lagrangian and Eulerian forms. It also holds for the spectral problem in its Floquet-by-Floquet description; see [7]. Thus there is no loss of information in choosing to work with the Lagrangian form. Now we introduce the spectral problem. Denote by $(\bar{\tau}, \bar{u}, \bar{c})$ a particular periodic traveling (roll-wave) solution of (2) of period X. Linearizing (2) about $(\bar{\tau}, \bar{u})$ in the co-moving frame $(x - \bar{c}t, t)$ and seeking modes of the form $(\tau, u)(x, t) = e^{\lambda t} (\tau, u)(x)$, one obtains

$$\begin{aligned} &(u + \bar{c}\tau)' = \lambda\, \tau, \\ &\nu(\bar{\tau}^{-2} u')' = (\lambda + 2\bar{u}\bar{\tau}) u - \left(\left(\frac{\bar{\tau}^{-3}}{F^2} - 2\bar{\tau}^{-3}\bar{u}' \right) \tau' + \bar{c} u' \right) + \left(\bar{u}^2 - \left(\frac{\bar{\tau}^{-3}}{F^2} - 2\bar{\tau}^{-3}\bar{u}' \right)' \right) \tau, \end{aligned} \tag{3}$$

where primes denote differentiation with respect to x. Setting $v = (\tau, u)^T$, the spectral problem (3) may be written as $Lv = \lambda v$ where L is a differential operator with *periodic coefficients*. By Floquet theory, one has that $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ (the spectrum of L acting on $L^2(\mathbb{R})$) if and only if there are $\xi \in [-\pi/X, \pi/X)$

and $w \in L^2_{per}([0,X])$ (a function of period X) such that $L_\xi\, w = \lambda w$, where L_ξ is the corresponding Bloch operator defined via $(L_\xi w)(x) := e^{-i\xi x} L\left[e^{i\xi \cdot} w(\cdot)\right](x)$. Consequently, the spectrum may be decomposed into countably many curves $\lambda(\xi)$ of $L^2_{\rm per}([0,X])$-eigenvalues of the operators L_ξ. Roll-waves are proved to be nonlinearly stable under the following *diffusive spectral stability conditions*:

1. (D1) $\sigma_{L^2(\mathbb{R})}(L) \subset \{\lambda \in \mathbb{C} \,|\, \Re(\lambda) < 0\} \cup \{0\}$.
2. (D2) There exists a $\theta > 0$ such that for all $\xi \in [-\pi/X, \pi/X)$, $\sigma_{L^2_{per}([0,X])}(L_\xi) \subset \{\lambda \,|\, \Re(\lambda) \leq -\theta\xi^2\}$.
3. (D3) $\lambda = 0$ is an eigenvalue of L_0 with generalized eigenspace $\Sigma_0 \subset L^2_{per}([0,X])$ of dimension 2.

Note that for each fixed period the set of periodic waves with given period is at least two-dimensional since the system is invariant by translation and one of the equations is a conservation law hence introduces a free constant of integration in the profile equations. This implies that the derivative of the profile with respect to the phase lies in the kernel of L_0 and that by differentiating with respect to the additional parameter one obtains a Jordan chain of heigth 2 over the former derivative so that the generalized eigenspace Σ_0 is in any case at least of dimension 2. For a complete discussion of the significance of these stability conditions, see [2]. In order to locate the spectrum, we introduce the Evans function $E_{SV}(\lambda,\xi)$. Write (3) as a first order differential system by setting $Z = (\tau, u, \bar{\tau}^{-2}u')^T$: $Z' = A(\cdot,\lambda)Z$. Denoting the resolvent matrix $R(\cdot,\lambda)$ associated to this system, it follows that $\lambda \in \sigma_{L^2_{per}([0,X])}(L_\xi)$ if and only if λ satisfies $E_{SV}(\lambda,\xi) := \det\left(R(X,\lambda) - e^{i\xi\, X}\mathrm{Id}_{\mathbb{R}^3}\right) = 0$.

3. Numerical Study: Spectral stability for intermediate F

In this section, we report on numerical investigations of (D1), (D2) and (D3) in the regime $2 \leq F \leq 100$ (see Figure 1) that is relevant for hydraulic engineering applications [4, 8]. We exhibit a simple description of the stability region and find that for sufficiently large Froude numbers, stable roll-waves do not exist. Our investigation roughly consists of two steps. To determine the global picture of spectrum of a linear X-periodic operator L, we use Hill's method, a Galerkin based truncation procedure which is implemented into STABLAB [9]. However, this method is not sufficient to study the spectrum near the origin and thus to verify hypothesis (D1), (D2) and (D3). For that purpose, we used the evaluation of the Evans function and its derivatives on contours to determine the coefficients and estimate the error terms in the expansion of E_{SV}.

A suitable parameterization, available for all Froude numbers, is given by (q, X), where $q = -\bar{c}\bar{\tau} - \bar{u}$ is the *total outflow* and X is the period. In [1] we have gathered numerous pieces of evidence leading to the clear picture that from F near $2.5 - 3$ and onward, stability is determined by simple relations

$$c_1^- \log F + c_2^- \log q + c_3^- \log X + c_4^- \log \nu \geq d^- \text{ and } c_1^+ \log F + c_2^+ \log q + c_3^+ \log X + c_4^+ \log \nu \leq d^+$$

with higher and higher accuracy as F increases, for some universal constants $c_j^\pm$ and $d^\pm$. Constants providing the lower and upper stability boundary are given approximately by $c_3^\pm = 1$ and

$$c_1^- = 0.69, \quad c_2^- = -3.5, \quad c_4^- = 0.18, \quad d^- = -0.11,$$

and

$$c_1^+ = 0.79, \quad c_2^+ = -1.7, \quad c_4^+ = 0.76, \quad d^+ = 2.2$$

respectively. In Figure 1 we illustrate this simple rule by providing one slice of the stability diagram obtained by enforcing the arbitrary constraint $q = 0.4F$.

4. Stability in the limits $F \to 2$ and $F \to \infty$

We now consider the two asymptotic regimes $F \to 2^+$ and $F \to \infty$. We provide an analytical description of the stability region in the limit $F \to 2^+$. The limit $F \to \infty$ is studied by a combination of asymptotic expansion and numerical simulations on the limit problems; roll-waves are always unstable there.

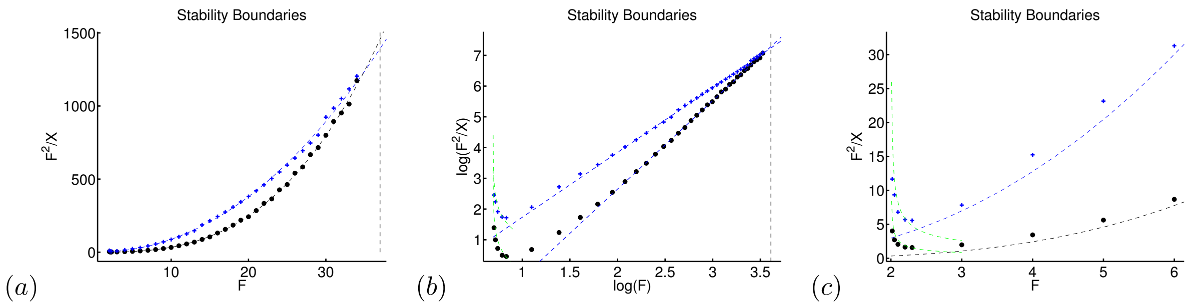

Figure 1: Lower and upper stability boundaries for $\nu = 0.1$, restricted to the slice $q = 0.4F$. Solid dots show numerically observed boundaries. Pale dashes indicate approximating curves given by (a) (upper) $F^2/X = e^{0.087}F^{2.88}$ and (lower) $F^2/X = e^{-2.97}F^{2.83}$, (b) (upper) $\log(F^2/X) = 2.88\log(F) + 0.087$ and (lower) $\log(F^2/X) = 2.83\log(F) - 2.97$. Pale dotted curves (Green in color plates) indicate theoretical boundaries as $F \to 2^+$. (c) Small- to large-F transition.

4.1. Stability of roll-waves at onset $F \to 2^+$

We focus on the onset of roll-waves at $0 < F - 2 \ll 1$. Various weakly nonlinear models have been derived depending crucially on the scaling between time and space and mildly on the precise form of the diffusion term and on whether or not a vanishing viscosity regime is under consideration. In the Korteweg-de Vries regime $(\xi, \tau) = (\delta(x - 3t/2), \delta^3 t)$ and in the small amplitude limit $h = 1 + \delta^2\,\tilde{h}(\xi, \tau)$, one obtains the generalized Kuramoto-Sivashinsky equation [10] (up to additional rescaling):

$$\partial_\tau \tilde{h} + \tilde{h}\partial_\xi \tilde{h} + \varepsilon \partial_\xi^3 \tilde{h} + \delta\left(\partial_\xi^2 \tilde{h} + \partial_\xi^4 \tilde{h}\right) = 0, \quad \varepsilon > 0. \tag{4}$$

The spectral and nonlinear stability of periodic traveling waves of (4) in the limit $\delta \to 0$ is fully described in [11] and a companion paper [12]. The classification of stable periodic wave can be extended to the shallow water equations (1) as follows.

Proposition 4.1. *For $\delta = \sqrt{F-2}$ sufficiently small, uniformly for δX on compact sets, periodic traveling waves of (2) are stable for (Lagrangian) periods $X \in \frac{\nu^{1/2}}{\tau_0^{5/4}\delta}(X_l, X_r)$ and unstable for $X \in \frac{\nu^{1/2}}{\tau_0^{5/4}\delta}[X_{min}, X_l)$ and $X \in \frac{\nu^{1/2}}{\tau_0^{5/4}\delta}(X_r, X_{max}]$ where $X_{min} \approx 6.284$, $X_l \approx 8.44$, $X_r \approx 26.1$ and $X_{max} \approx 48.3$.*

We recall that our numerical experiments suggests that this asymptotic description is relevant up to $F \approx 2.3$.

We do not expect other stability regimes when δ is sufficiently small. Indeed, both in the regime $(\xi, \tau) = (\delta^{-1}(x - 3t/2), \delta^{-1}t)$ and in the regime $(\xi, \tau) = (x - 3t/2, \delta t)$ amplitude equations have been derived from the shallow water equations indicating that periodic waves are always unstable [13], [14]. Numerical observations support this expectation.

4.2. Infinite-Froude number limit

To consider now the infinite-Froude number limit $F \to \infty$, we introduce a suitable rescaling in the equations and profiles with the requirements that (i) the limiting system ($F \to \infty$) be nontrivial and (ii) the limit be a regular perturbation. This results in a one-parameter family of rescalings indexed by $\alpha \geq -2$, given by $\tau = aF^\alpha$, $u = bF^{-\alpha/2}$, $c = c_0 F^{-1-3\alpha/2}$, $X = X_0 F^{-1/2-5\alpha/4}$ and $q = q_0 F^{-\alpha/2}$ where $a, b : \mathbb{R} \to \mathbb{R}$ and c_0, X_0, q_0 are real constants. Under this rescaling, we find that X-periodic traveling wave solutions of (2) correspond to X_0-periodic solutions to the rescaled profile equation

$$a'' = (-a^2/c_0 k_0^2 \nu)\big(k_0 a' F^{-3/2-3\alpha/4}(c_0^2 - 1/a^3) - 1 + a(q_0 - c_0 F^{-1} a)^2 - 2c_0 k_0^2 \nu (a')^2/a^3\big), \tag{5}$$

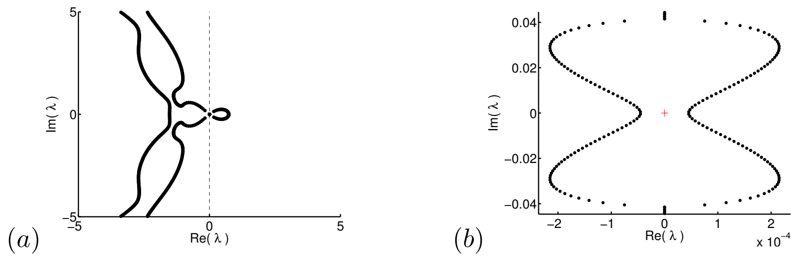

Figure 2: In (a) and (b) we plot a numerical sampling of the (unstable) spectrum corresponding to the $F \to \infty$ limiting spectral problems for the cases $\alpha = -2$ and $\alpha > -2$, respectively, for a representative periodic stationary solution of the appropriate limiting profile equation.

where $b = -q_0 - c_0 F^{-1} a$. Noting that the behavior of $F^{-3/2-3\alpha/4}$ as $F \to \infty$ depends on whether $\alpha = -2$ or $\alpha > -2$, one obtains two classes of limiting profile equations as $F \to \infty$. An additional rescaling $Fb = \check{b}$ and $F^{1/2+\alpha/4}\lambda = \Lambda$ yields the associated spectral problem

$$\Lambda a - c_0 k_0 a' - k_0 \check{b}' = 0; \qquad \frac{\Lambda \check{b} - c_0 k_0 \check{b}' - k_0 (a/\bar{a}^3)'}{F^{3/2+3\alpha/4}} = -\frac{2}{F}\bar{a}\bar{b}\check{b} - \bar{b}^2 a + \nu k_0^2 (\check{b}'\bar{a}^2 + 2c_0 \bar{a}' a/\bar{a}^3)', \tag{6}$$

where (a, b) denotes the perturbation of the underlying state $(\bar{a}, \bar{b})$. Observe that for $\alpha > -2$ the limiting profile equations, selection principles, and spectral problems are independent of the specific value of α. Noting that (6) is, again by design, a *regular perturbation* of the appropriate limiting spectral problem as $F \to \infty$, the following sufficient *instability condition* is obtained using standard perturbation techniques.

Proposition 4.2. *For all $\alpha \geq -2$, the profiles of (5) converging as $F \to \infty$ to solutions of the appropriate limiting profile equation, are* spectrally unstable *if the appropriate limiting spectral problem about the limiting profiles admit $L^2(\mathbb{R})$-spectrum in Λ with positive real part.*

We have investigated the stability of the limiting spectral problems numerically in both the cases $\alpha = -2$ and $\alpha = 0$; recall that the results for $\alpha = 0$ in fact hold for all $\alpha > -2$. This numerical study strongly indicates that, in both cases, all periodic solutions of the appropriate limiting profile equations are *spectrally unstable* and hence spectrally stable periodic traveling wave solutions of the viscous St. Venant system (2) do not exist for sufficiently large Froude numbers; see Figure 2.

5. Conclusions and Perspectives

We have provided a complete stability diagram in the plane (F, q, X, ν), F being the Froude number, q the total discharge rate, X the period and ν the Reynolds number. For various parametrizations of the problem, we found that for each $F \in [0, F^*)$ (for some $F^* < \infty$), $\nu > 0$ and q fixed in some (F,ν)-dependent interval, there exist $X_{min}(F, q, \nu)$ and $X_{max}(F, q, \nu)$ such that X-periodic roll-waves are stable if $X \in (X_{min}(F, q, \nu), X_{max}(F, q, \nu))$. Generically, the transition to instability for $X \approx X_{min}(F, q, \nu)$ is due to a loss of hyperbolicity of the Whitham modulation equations. On the other hand, the transition to instability for $X \approx X_{max}(F, q, \nu)$ is due to the crossing of a pair of eigenvalues far from the origin and is thus undetectable by similar criteria.

For the moment we are unable to provide any explanation, even of heuristic type, for the appearance of simple power laws governing the intermediate Froude number stability. Yet we expect that an inspection of the large Reynolds number limit $\nu \to 0$, which is the object of ongoing work, could shed some light on these phenomena. Note that this limit is a singular perturbation limit hence its analysis is expected to be far more involved than the large Froude number limits of the foregoing section.

Up to now, we have considered only viscous shallow water equations with turbulent friction terms. It is an interesting and physically relevant problem to extend our results to more realistic turbulent shallow water models such as (a viscous version of) the one derived in [8] which accurately reproduces Brock's experiments on turbulent roll-waves [4]. Another physically relevant problem is to consider laminar roll-waves as found e.g. in [15]. In this case, we would have to take into account surface tension effects as they play there an important role.

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