Homework

Exercice 1. Let $N \in \mathbf{N}^*$, $k \in \mathbf{N}$ and $p \in [1, +\infty[$ be such that $k - \frac{N}{p} > 0$.

1. Justify that the linear maps

$$T_R : W^{k,p}(\mathbf{R}^N) \to L^\infty(\mathbf{R}^N), \quad u \mapsto \chi_{B(0,R)^c} u$$

are bounded uniformly with respect to $R > 0$. Note that in the foregoing definition, $\chi_{B(0,R)^c}$ denotes the characteristic function of the complement of the ball centered at 0 and of radius R.

2. Deduce that for any $u \in W^{k,p}(\mathbf{R}^N)$

$$\lim_{R \to \infty} \|u\|_{L^\infty(B(0,R)^c)} = 0\,.$$

Exercice 2. Let $N \in \mathbf{N}^*$, $t_0 \in \mathbf{R}$, $u \in L^1_{loc}(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $f \in L^1_{loc}(\mathbf{R} \times \mathbf{R}^N; \mathbf{R}^N)$ and $g \in L^1_{loc}(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$. We assume that

1. $\partial_t u + \mathrm{div}_x(f) = g$ in $\mathcal{D}'(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, that is, for any $\varphi \in \mathcal{C}_c^\infty(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$

$$\iint_{\mathbf{R} \times \mathbf{R}^N} (u\, \partial_t \varphi + f \cdot \nabla_x \varphi + g \varphi) = 0\,;$$

2. $t \mapsto u(t, \cdot)$ is continuous at t_0 in $\mathcal{D}'(\mathbf{R}^N; \mathbf{R})$, in the sense that there exists $u_0 \in \mathcal{D}'(\mathbf{R}^N; \mathbf{R})$ such that for any $\varphi_0 \in \mathcal{C}_c^\infty(\mathbf{R}^N; \mathbf{R})$

$$\lim_{\varepsilon \to 0} \|t \mapsto \left(\int_{\mathbf{R}^N} u(t, \cdot) \varphi_0 - \langle u_0; \varphi_0 \rangle \right) \|_{L^\infty((t_0 - \varepsilon, t_0 + \varepsilon))} = 0\,,$$

where $\langle \cdot; \cdot \rangle$ denotes duality pairing.

Prove that for any $\varphi \in \mathcal{C}_c^\infty(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$

$$\iint_{(t_0, \infty) \times \mathbf{R}^N} (u\, \partial_t \varphi + f \cdot \nabla_x \varphi + g \varphi) = -\langle u_0; \varphi(t_0, \cdot) \rangle\,.$$

Hint : use as a test function $(t, x) \mapsto \chi_\varepsilon(t)\, \varphi(t, x)$ with $\chi_\varepsilon : t \mapsto \varepsilon^{-1} \chi((t - t_0)/\varepsilon)$, where $\chi : \mathbf{R} \to \mathbf{R}$ is a smooth nondecreasing function such that $\chi_{|(-\infty,0]} \equiv 0$ and $\chi_{|[1,\infty)} \equiv 1$.

Exercice 3. In the present problem, the norm on products is always chosen as the ℓ^2-norm. Let $N \in \mathbf{N}^*$. The Fourier transform on $\mathcal{S}'(\mathbf{R}^N)$, denoted as $\mathcal{F}$, is normalized in such a way that

$$\mathcal{F}u_0(\xi) := \int_{\mathbf{R}^N} \mathrm{e}^{-\mathrm{i}\,\xi \cdot x}\, u_0(x)\, \mathrm{d}x\,, \qquad \xi \in \mathbf{R}^N\,,$$

when $u_0 \in L^1(\mathbf{R}^N)$.

For $t \in \mathbf{R}$, we define

$$S(t) : \mathcal{S}'(\mathbf{R}^N; \mathbf{R}^2) \to \mathcal{S}'(\mathbf{R}^N; \mathbf{R}^2)\,, \quad \mathbf{U} \mapsto \mathcal{F}^{-1} \left(\begin{pmatrix} \cos(t\, \|\cdot\|) & \dfrac{\sin(t\, \|\cdot\|)}{\|\cdot\|} \\ -\|\cdot\| \sin(t\, \|\cdot\|) & \cos(t\, \|\cdot\|) \end{pmatrix} \mathcal{F}(\mathbf{U}) \right)\,.$$

1. Let $\mathbf{U}^{(0)} = (u_0^{(0)}, u_1^{(0)}) \in \mathcal{S}'(\mathbf{R}^N;\mathbf{R}^2)$ and set for $t \in \mathbf{R}$, $\mathbf{U}(t,\cdot) = (u_0(t,\cdot), u_1(t,\cdot)) := S(t)\mathbf{U}^{(0)}$.
   (a) Show that if $\mathbf{U}^{(0)} \in L^2(\mathbf{R}^N;\mathbf{R}^2)$ then $u_0 \in L^\infty_{\text{loc}}(\mathbf{R}; L^2(\mathbf{R}^N))$ and for any $t \in \mathbf{R}$,
$$\|u_0(t,\cdot)\|_{L^2(\mathbf{R}^N)} \le \|u_0^{(0)}\|_{L^2(\mathbf{R}^N)} + |t|\,\|u_1^{(0)}\|_{L^2(\mathbf{R}^N)}\,.$$
   (b) Show that for any $k \in \mathbf{N}^*$, if $\mathbf{U}^{(0)} \in H^k(\mathbf{R}^N) \times H^{k-1}(\mathbf{R}^N)$ then $\mathbf{U} \in L^\infty_{\text{loc}}(\mathbf{R}; H^k(\mathbf{R}^N) \times H^{k-1}(\mathbf{R}^N))$ and for any $t \in \mathbf{R}$,
$$\|\mathbf{U}(t,\cdot)\|_{\dot H^k(\mathbf{R}^N)\times \dot H^{k-1}(\mathbf{R}^N)} = \|\mathbf{U}^{(0)}\|_{\dot H^k(\mathbf{R}^N)\times \dot H^{k-1}(\mathbf{R}^N)}\,,$$
   where, for $\ell \in \mathbf{N}$,
$$\|\mathbf{V}\|_{\dot H^\ell(\mathbf{R}^N)} := \frac{1}{(2\pi)^{\frac{N}{2}}}\left\|\,\|\cdot\|^\ell\,\mathcal{F}(\mathbf{V})\right\|_{L^2(\mathbf{R}^N)}\,.$$
2. Let I be an interval containing 0, $\mathbf{U}^{(0)} = (u_0^{(0)}, u_1^{(0)}) \in H^1(\mathbf{R}^N)\times L^2(\mathbf{R}^N)$ and $F \in L^1_{\text{loc}}(I; L^2(\mathbf{R}^N))$. We define $\mathbf{U} \in L^\infty_{\text{loc}}(I; H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N))$ by
$$\mathbf{U}(t,\cdot) = \begin{pmatrix} u_0(t,\cdot) \\ u_1(t,\cdot)\end{pmatrix} := S(t)(\mathbf{U}^{(0)}) + \int_0^t S(t-s)\begin{pmatrix} 0 \\ F(s,\cdot)\end{pmatrix} \mathrm{d}s\,, \qquad t \in I\,.$$
   (a) Justify that $\mathbf{U} \in \mathcal{C}^0(I; H^1(\mathbf{R}^N)\times L^2(\mathbf{R}^N))$, $u_0 \in \mathcal{C}^1(I; L^2(\mathbf{R}^N))$ with $\partial_t u_0 = u_1$ and, for $t \in I$,
$$\|u_0(t,\cdot)\|_{L^2(\mathbf{R}^N)} \le \|u_0^{(0)}\|_{L^2(\mathbf{R}^N)} + |t|\,\|u_1^{(0)}\|_{L^2(\mathbf{R}^N)} + \left|\int_0^t |t-s|\,\|F(s,\cdot)\|_{L^2(\mathbf{R}^N)}\mathrm{d}s\right|\,,$$
$$\|\mathbf{U}(t,\cdot)\|_{\dot H^1(\mathbf{R}^N)\times L^2(\mathbf{R}^N)} \le \|\mathbf{U}^{(0)}\|_{\dot H^1(\mathbf{R}^N)\times L^2(\mathbf{R}^N)} + \left|\int_0^t \|F(s,\cdot)\|_{L^2(\mathbf{R}^N)}\mathrm{d}s\right|\,.$$
   (b) Prove that if moreover $\mathbf{U}^{(0)} \in H^2(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ and $F \in L^1_{\text{loc}}(I; H^1(\mathbf{R}^N)) \cap \mathcal{C}^0(I; L^2(\mathbf{R}^N))$ then $u_0 \in \bigcap_{\ell\in\{0,1,2\}} \mathcal{C}^\ell(I; H^{2-\ell}(\mathbf{R}^N))$, $u_0(0,\cdot) = u_0^{(0)}$, $\partial_t u_0(0,\cdot) = u_1^{(0)}$ and $\partial_t^2 u_0 = \Delta_x u_0 + F$.
   (c) Prove that u_0 is the unique function such that $u_0 \in \mathcal{C}^1(I; L^2(\mathbf{R}^N) - \textit{weak})$, $u_0(0,\cdot) = u_0$, $\partial_t u_0(0,\cdot) = u_1^{(0)}$, and $\partial_t^2 u_0 = \Delta_x u_0 + F$ in $\mathcal{D}'(\mathring{I} \times \mathbf{R}^N)$.
   In the foregoing, $\mathcal{C}^1(I; L^2(\mathbf{R}^N) - \textit{weak})$ denotes the set of functions $v \in L^\infty_{\text{loc}}(I; L^2(\mathbf{R}^N))$ such that, for any $\varphi \in L^2(\mathbf{R}^N)$, the function $I \to \mathbf{R}$, $t \mapsto \int_{\mathbf{R}^N} v(t,\cdot)\,\varphi$ is $\mathcal{C}^1$.
3. Let $k \in \mathbf{N}$ be such that $k > N/2$, $f \in \mathcal{C}^{k+1}(\mathbf{R})$ with $f(0) = 0$.
   (a) Prove that, for any $M > 0$, there exists $T_M > 0$ such that for any $\mathbf{U}^{(0)} = (u_0^{(0)}, u_1^{(0)}) \in H^k(\mathbf{R}^N) \times H^{k-1}(\mathbf{R}^N)$ with $\|\mathbf{U}^{(0)}\|_{H^k(\mathbf{R}^N)\times H^{k-1}(\mathbf{R}^N)} \le M$ there exists a unique $u \in \bigcap_{\ell\in\{0,1\}} \mathcal{C}^\ell([-T_M, T_M]; H^{k-\ell}(\mathbf{R}^N))$ such that
$$u(0,\cdot) = u_0^{(0)}\,, \qquad \partial_t u(0,\cdot) = u_1^{(0)}\,, \qquad \text{and} \qquad \partial_t^2 u = \Delta_x u + f(u) \text{ in } \mathcal{D}'((-T_M, T_M)\times\mathbf{R}^N)\,.$$
   (b) Show that if $\mathbf{U}^{(0)} = (u_0^{(0)}, u_1^{(0)}) \in H^k(\mathbf{R}^N) \times H^{k-1}(\mathbf{R}^N)$, $T > 0$ and
$$u \in \left(\bigcap_{\ell\in\{0,1\}} \mathcal{C}^\ell([0,T); H^{k-\ell}(\mathbf{R}^N))\right) \cap L^\infty([0,T)\times\mathbf{R}^N)$$
   are such that, for any $0 \le t < T$,
$$\begin{pmatrix} u \\ \partial_t u\end{pmatrix}(t,\cdot) = S(t)(\mathbf{U}^{(0)}) + \int_0^t S(t-s)\begin{pmatrix} 0 \\ f(u(s,\cdot))\end{pmatrix}\mathrm{d}s\,,$$
   then $(u, \partial_t u) \in L^\infty([0,T); H^k(\mathbf{R}^N) \times H^{k-1}(\mathbf{R}^N))$.